Vector Arithmetic and Geometry

In applied mathematics and physics and engineering, vectors often have two components to represent for example planar motion or – more likely – have three components to represent the three-dimensional world. In this document we consider some of the geometrical properties and arithmetic of vectors. For futher reinforcement or development, an Excel spreadsheet carrying out the vector operations and including vector operations in VBA is available[1], alternatively a set of Fortran codes for the same purpose are also available[2] and a set of Matlab/Freemat/Octave codes are available.

Vectors

The physical meaning of a vector is that it is a quantity that has both magnitude and direction. This is often diagrammatically-represented by an arrow, its angle representing the direction and its length representing its magnitude. A vector in a two-dimensional system a vector can be resolved into two perpendicular components; one in the x-direction and one in the y-direction. A vector in a three dimensional system a vector can be resolved into three perpendicular components; one in the x, y and z-directions.

Vector addition and subtraction for the physical vectors considered in this document follow the same rules as in matrix arithmetic[3]; it simply involves the component-wise addition or subtraction.

Example 1

$$\begin{pmatrix}1\\5\end{pmatrix}+\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$

and

$$\begin{pmatrix}7\\1\\3\end{pmatrix}-\begin{pmatrix}2\\-2\\5\end{pmatrix}=\begin{pmatrix}5\\3\\-2\end{pmatrix}.$$

Vectors are often written in bold or are underlined and in this document we use the former (the latter is often used in handwriting where it is more difficult to express and distinuish bold characters). For example we may write $\boldsymbol{a}=\begin{pmatrix}1\\5\end{pmatrix}$. Some of the properties and vectors are outlined in this document. A more thorough coveage can be found in *Cartesian components of vectors*[4]

[1] GEOM.xlsm spreadsheet of vector operations and user-guide
[2] Fortran codes for vector geometry: 2D- GEOM2D.FOR and 3D GEOM3D.FOR and test codes GEOM2D_T.FOR and GEOM3D_T.FOR
[3] Matrix Arithmetic
[4] Mathcentre: *Cartesian components of vectors* document and video

Points and Vectors

The most natural method of representing a point is to use Cartesian coordinates[5]. In two dimensions any point in the plane can be represented by two co-ordinates, usually an *x*-co-ordinate and a *y*-coordinate. For example the point P_1=(2,4) has an *x*-co-ordinate equal to '2' and a *y*-coordinate equal to '4'. If P_2=(3,9) is another point then the line connecting P_1 to P_2 is a vector, having magnitude (size or length of the line) and direction (the direction that follows the line from P_1 to P_2), and it written $\overline{P_1P_2}$ (athough a variety of other notations exist, such as the bar being replaced by an arrow).

An alternative common notation is through utilising the unit directional vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, where $\hat{\boldsymbol{\imath}}$ is the unit directional vector in the *x*-direction and $\hat{\boldsymbol{\jmath}}$ is the unit directional vector in the *y*-direction;

$$\hat{\boldsymbol{\imath}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \hat{\boldsymbol{\jmath}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Example 2

Let P_1=(2,4) and P_2=(3,9) be points on the *x*-*y* plane.

The vector represents 3-2=1 units in the *x*-direction and 9-4=5 units in the *y*-direction, and can therefore be written

$$\overline{P_1P_2} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

We may also write $\overline{P_1P_2} = \hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}}$.

In three dimensions any point can be respresented by three coordinates an *x*-coordinate a *y*-coordinate and a *z*-coordinate. In three dimensions $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ are the unit directional vectors:

$$\hat{\boldsymbol{\imath}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \hat{\boldsymbol{\jmath}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \hat{\boldsymbol{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Example 3

Let P_1=(1,-3,2) be a point in 3D (P_1 has an *x*-co-ordinate equal to '1' a *y*-coordinate equal to '-3' and a *z*-coordinate equal to '2'). Similarly, if P_2=(3,-5,7) is another point then the line connecting P_1 to P_2 is a vector and it written $\overline{P_1P_2}$ and

$$\overline{P_1P_2} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} = 2\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}.$$

[5] Cartesian Coordinates

The magnitude of a vector

The size or magnitude of a two-dimensional vector is defined by: $|\boldsymbol{a}| = \left|\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right| = \sqrt{{a_1}^2 + {a_2}^2}$, which results on the application of Pythagoras' theorem[6]. This also extends to three dimensions: $|\boldsymbol{a}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$. In terms of vector norms the the magnitude of a vector is equivalent to its 2-norm[7].

Example 4

The vector $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ has magnitude $\left|\begin{pmatrix} 1 \\ 5 \end{pmatrix}\right| = \sqrt{1^2 + 5^2} = \sqrt{26} = 5.099\ (3\text{d.p.})$. .

Examples 5

The magnitude of the vector $\begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$ is $\left|\begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}\right| = \sqrt{2^2 + (-2)^2 + 5^2} = \sqrt{33} = 5.745\ (3\text{d.p.})$.

The magnitude of the vector $\begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$ is $\left|\begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}\right| = \sqrt{7^2 + 1^2 + 3^2} = \sqrt{59} = 7.681\ (3\text{d.p.})$.

Unit vectors

A vector is said to be a unit vector is its magnitude is one. For example the directonal vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ are unit vectors.

Example 6

The vector $\begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$ has magnitude $\left|\begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}\right| = \sqrt{0.8^2 + 0.6^2} = \sqrt{0.64 + 0.36} = \sqrt{1} = 1$ and hence it is a unit vector. .

Example 7

The magnitude of the vector $\frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ is $\frac{1}{7}\left|\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}\right| = \frac{1}{7}\sqrt{2^2 + 3^2 + 6^2} = \frac{1}{7}\sqrt{4 + 9 + 36} = \frac{1}{7}\sqrt{49} = 1$ and hence it is a unit vector.

---

[6] Trigonometry
[7] Vector Norm and Normalisation

A vector can be transformed into a unit vector with the same direction but with unit magnitude by dividing the components by the vector's magnitude; for any vector $\boldsymbol{a} \neq \mathbf{0}, \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$ is a unit vector.

Example 8

From Example 4 it was found that the vector $\begin{pmatrix}1\\5\end{pmatrix}$ has magnitude $\sqrt{26}$. The vector $\frac{1}{\sqrt{26}}\begin{pmatrix}1\\5\end{pmatrix} = \begin{pmatrix}0.1961\\0.9806\end{pmatrix}$ has magnitude $\left|\begin{pmatrix}0.1961\\0.9806\end{pmatrix}\right| =$

$\sqrt{0.1961^2 + 0.9806^2} = \sqrt{0.03845 + 0.9615} = \sqrt{1} = 1$ (working to 4 d.p.).

Distance between two points

The geometrical distance between two points P_1 and P_2 is the magnitude of the vector P_1P_2; $|P_1P_2|$.

Example 9

Let P_1=(2,4) and P_2=(3,9) be points on the x-y plane. Following on from Example 1, the vector linking the two points is $\overline{P_1P_2} = \begin{pmatrix}1\\5\end{pmatrix}$. From Example 4 it was shown that $\left|\begin{pmatrix}1\\5\end{pmatrix}\right| = \sqrt{26} = 5.099$ (3d.p.) and hence this is also the distance between the two points.

Example 10

Following on from Example 3, P_1=(1,-3,2) and P_2=(3,-5,7) are points in three dimensions and $\overline{P_1P_2} = \begin{pmatrix}2\\-2\\5\end{pmatrix}$. In Example 5 it was shown that is $\left|\begin{pmatrix}2\\-2\\5\end{pmatrix}\right| = \sqrt{33} = 5.745$ (3d.p.), and hence this is the distance between the points.

Scalar or Dot Product

The scalar or dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written $\boldsymbol{a}.\boldsymbol{b}$ is the sum of the component-wise products; $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2$ in two dimensions and $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$ in three dimensions.

Example 11

$$\begin{pmatrix}1\\5\end{pmatrix}.\begin{pmatrix}2\\-3\end{pmatrix} = 1\times 2 + 5\times(-3) = -13$$

and

$$\begin{pmatrix}7\\1\\3\end{pmatrix}.\begin{pmatrix}2\\-2\\5\end{pmatrix} = 7\times 2 + 1\times(-2) + 3\times 5 = 27.$$

The dot product of two vectors in the same direction is equal to the product of their magnitudes. The dot product of two perpendicular vectors is zero.

Example 12

$$\begin{pmatrix}1\\5\end{pmatrix}.\begin{pmatrix}2\\10\end{pmatrix} = 1\times 2 + 5\times 10 = 52$$

and

$$\begin{pmatrix}1\\1\\0\end{pmatrix}.\begin{pmatrix}2\\-2\\0\end{pmatrix} = 1\times 2 + 1\times(-2) + 0\times 0 = 0.$$

In general

$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

where θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. For further information on the properties of the scalar product see *The Scalar Product*[8]. In the following examples the angle between two vectors in two dimensions and three dimensios is calculated and the result is compared to the angle obtained through the application of the cosine formula to a triangle[9].

[8] Mathcentre: *The Scalar Product* document and video
[9] Mathcentre: *Triangle Formulae* document and video

Example 13

The cosine of the angle between the vectors $\boldsymbol{a} = \binom{1}{5}$ and $\boldsymbol{b} = \binom{2}{-3}$ is $\frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{\binom{1}{5}.\binom{2}{-3}}{\left|\binom{1}{5}\right|\left|\binom{2}{-3}\right|} = \frac{-13}{\sqrt{26}\sqrt{13}} = \frac{-1}{\sqrt{2}}$. Hence the angle is $\frac{3\pi}{4}$ or 135^0.

Let us compare this with the angle that is obtained by the cosine rule. The length of the vector $\boldsymbol{a}$ is $|\boldsymbol{a}| = \sqrt{26}$, the length of the vector $\boldsymbol{b}$ is $|\boldsymbol{b}| = \sqrt{13}$ and the length of the remaining vector is $|\boldsymbol{b} - \boldsymbol{a}| = \sqrt{65}$. Applying the cosine rule:

$$\left(\sqrt{65}\right)^2 = \left(\sqrt{26}\right)^2 + \left(\sqrt{13}\right)^2 - 2\sqrt{26}\sqrt{13}\cos\theta\ .$$

Hence

$$65 = 26 + 13 - 2\sqrt{26}\sqrt{13}\cos\theta$$

and

$$\cos\theta = \frac{-13}{\sqrt{26}\sqrt{13}}\text{ , as before .}$$

Example 14

The cosine of the angle between the vectors $\boldsymbol{a} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$ is $\frac{\begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}.\begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}}{\left|\begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}\right|\left|\begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}\right|} = \frac{27}{\sqrt{33}\sqrt{59}} = 0.6119$. Hence the angle is 0.9123 radians (4d.p.) or

52.27^0 (2 d.p.).

Let us compare this with the angle that is obtained by the cosine rule. The length of the vector $\boldsymbol{a}$ is $|\boldsymbol{a}| = \sqrt{33}$, the length of the vector $\boldsymbol{b}$ is $|\boldsymbol{b}| = \sqrt{59}$ and the length of the remaining vector is $|\boldsymbol{b} - \boldsymbol{a}| = \sqrt{38}$. Applying the cosine rule:

$$\left(\sqrt{38}\right)^2 = \left(\sqrt{33}\right)^2 + \left(\sqrt{59}\right)^2 - 2\sqrt{33}\,\sqrt{59}\cos\theta\ .$$

Hence

$$38 = 33 + 59 - 2\sqrt{33}\,\sqrt{59}\cos\theta$$

and

$$\cos\theta = \frac{27}{\sqrt{33}\sqrt{59}}\text{ , as before .}$$

Vector or Cross Product

The cross product of two vectors results in a vector that is perpendicular to the plane of the two original vectors. The cross product therefore only makes sense in three dimensions (in practical setting). Useful definitions and uses of the cross product are outlined in this sections, for a more thorough coverage see *The Vector Product*[10].

For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the cross product is written $\boldsymbol{a} \times \boldsymbol{b}$ and is defined as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

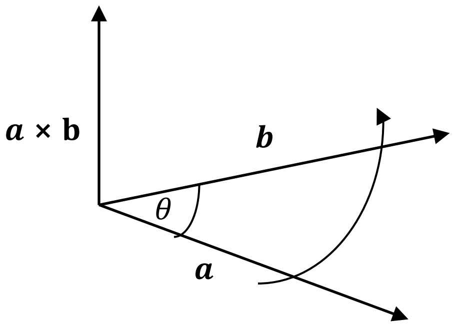
using the notation of the method for finding the determinant of a 3×3 matrix[11], or by

$$\boldsymbol{a} \times \boldsymbol{b} = \hat{\boldsymbol{\imath}}(a_2 b_3 - a_3 b_2) + \hat{\boldsymbol{\jmath}}(a_3 b_1 - a_1 b_3) + \hat{\boldsymbol{k}}(a_1 b_2 - a_2 b_1).$$

A further definition of the vector product is as follows

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta \, \hat{\boldsymbol{n}},$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\hat{\boldsymbol{n}}$ is a unit vector that is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ (or perpendicular to the plane occupied by $\boldsymbol{a}$ and $\boldsymbol{b}$), as illustrated in the following diagram.



Note that the direction of $\boldsymbol{a} \times \boldsymbol{b}$, as defined above, is ambiguous. The direction is 'upward' if the movement from $\boldsymbol{a}$ to $\boldsymbol{b}$ is in the counter-clockwise direction, as illustrated in the diagram.

---

[10] Mathcentre: The Vector Product document and video
[11] Mathcentre: *Determinants* document

Example 15

Let $\boldsymbol{a} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$ be two vectors,

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 2 & -2 & 5 \\ 7 & 1 & 3 \end{vmatrix}$$

$$= \hat{\boldsymbol{\imath}}((-2) \times 3 - 5 \times 1) + \hat{\boldsymbol{\jmath}}(5 \times 7 - 2 \times 3) + \hat{\boldsymbol{k}}(2 \times 1 - (-2) \times 7)$$

$$= -11\hat{\boldsymbol{\imath}} + 29\hat{\boldsymbol{\jmath}} + 16\hat{\boldsymbol{k}} = \begin{pmatrix} -11 \\ 29 \\ 16 \end{pmatrix}.$$

To show that $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$, let us find the dot products. [Note the dot product of two perpendicular vectors is zero.]

$$(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{a} = \begin{pmatrix} -11 \\ 29 \\ 16 \end{pmatrix}.\begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} = (-11) \times 7 + 29 \times 1 + 16 \times 3$$

$$= -77 + 29 + 48 = 0.$$

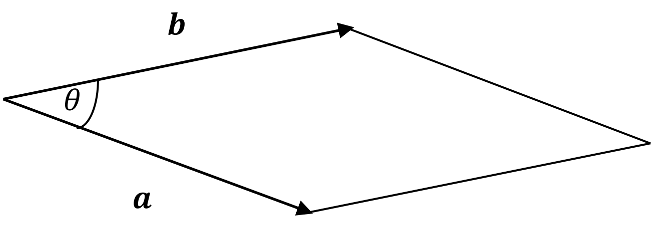
$$(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{b} = \begin{pmatrix} -11 \\ 29 \\ 16 \end{pmatrix}.\begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} = (-11) \times 2 + 29 \times (-2) + 16 \times 5$$

$$= -22 - 58 + 80 = 0.$$

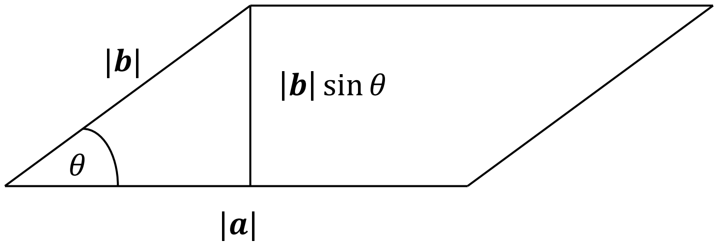
From the equation above, the magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is equal to $|\boldsymbol{a}||\boldsymbol{b}| \sin\theta$

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta,$$

which is also equal to the area of the parallelogram illustrated in the following diagram.



In order to show this let us view the parallelogram as follows, with the vector $\boldsymbol{a}$ viewed on a horizontal axis and the vectors replaced by their lengths $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$.

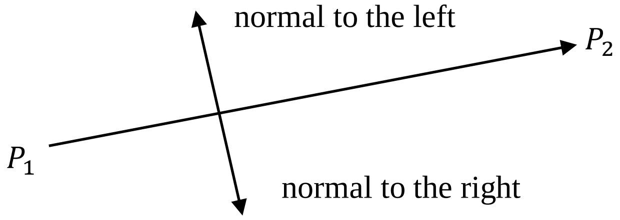


The height of the parallelogram is $|\boldsymbol{b}|\sin\theta$ and hence the area of the parallelogram is $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$.

Application: Normal to a line between two points in 2D

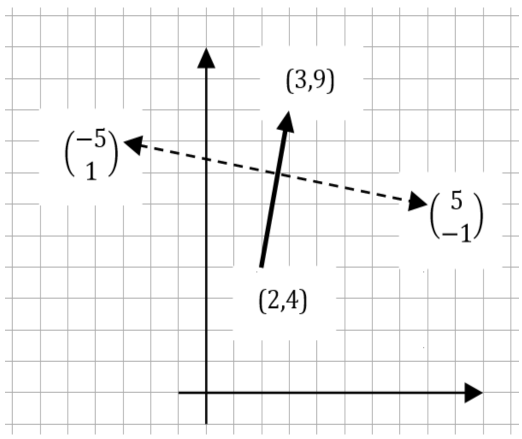
Consider the line joining two 2-points P_1 to P_2. Let $\boldsymbol{a}$ be the vector linking P_1 to P_2 ; $\boldsymbol{a} = P_1P_2$. The normal to the line is $\begin{pmatrix} -a_2 \\ a_1 \end{pmatrix}$ to the left of the line and $\begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}$ to the right. The unit normal to the left of the line is $\frac{1}{|\boldsymbol{a}|}\begin{pmatrix} -a_2 \\ a_1 \end{pmatrix}$ and to right it is $\frac{1}{|\boldsymbol{a}|}\begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}$.

Note that the normal may also be defined in the opposite direction, but in this case the normal lies to point to the left of the vector P_1P_2



Example 16

Let $P_1 = (2,4)$ and $P_2 = (3,9)$ be two points. The vector $\overline{P_1P_2}$ is equal to $\boldsymbol{a} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, as shown in Example 2. Hence the normal to P_1P_2 is $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$ to the left and $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ to the right,as illustrated in the following diagram.

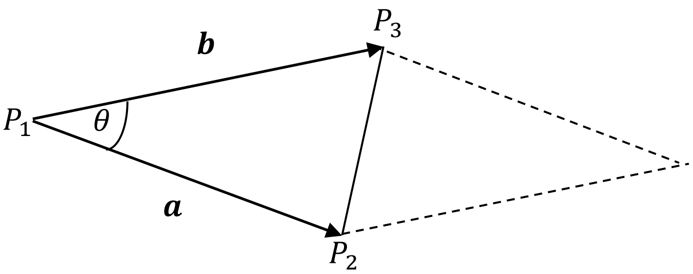


Application: Area of a triangle joining three points in 3D

Earlier, the following formula was stated

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta,$$

which is also equal to the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$. However, if we halve that area as follows, then it becomes the area of a triangle. Let $\boldsymbol{a}$ be the vector that joins two points $P_1$and P_2 so that $\boldsymbol{a} = \overline{P_1P_2}$ and let $\boldsymbol{b}$ be the vector that joins two points P_1 and P_3 so that $\boldsymbol{b} = \overline{P_1P_3}$, as illustrated in the following diagram.



Hence the area of the triangle joining the points P_1, P_2 and P_3 is $\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$.

Note that this is also equal to $\frac{1}{2}|\boldsymbol{a}||\boldsymbol{b}| \sin\theta$ and this fits in with the formula for the area of a triangle[12].

Example 17

In this example the area of the triangle with vertices P_1=(1,-3, 2), P_2=(3,-5, 7) and P_3=(8,-2, 5) is determined.

Let $\boldsymbol{a} = \overline{P_1P_2}$ and $\boldsymbol{b} = \overline{P_1P_3}$ then $\boldsymbol{a} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$.

Hence $\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} -11 \\ 29 \\ 16 \end{pmatrix}$, as shown in Example 14. The area of the triangle joining the three points is $\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = \frac{1}{2}\sqrt{(-11)^2 + 29^2 + 16^2}$

$= \frac{1}{2}\sqrt{121 + 841 + 256} = \sqrt{1218} = 17.45 \text{ (2 d.p.)}.$

In order to verify this result let us find $\frac{1}{2}|\boldsymbol{a}||\boldsymbol{b}| \sin\theta$. From Examples 5, $|\boldsymbol{a}| = \sqrt{33}$ and $|\boldsymbol{b}| = \sqrt{59}$ and from Example 14 $\theta = 52.27^0$. Hence

$\frac{1}{2}|\boldsymbol{a}||\boldsymbol{b}| \sin\theta = \frac{1}{2} \times 5.745 \times 7.681 \sin 52.27^o = 17.45 \text{ (2 d.p.)}.$

[12] Mathcentre: *Triangle Formulae* document