Chapter 9

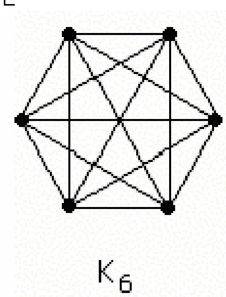
Use the following to answer questions 1-5:

In the questions below find an ordered pair, an adjacency matrix, and a graph representation for the graph.

1. K_6.

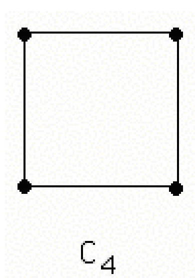
   Ans: Vertices = {1,2,3,4,5,6}, Edges = $\{\{a,b\} \mid 1 \le a \le 6, 1 \le b \le 6, a \ne b\}$;

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$



2. C_4.

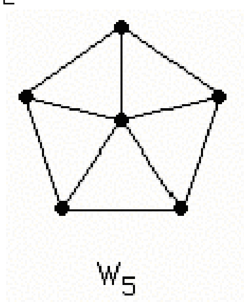
   Ans: Vertices = {1,2,3,4}, Edges = {{1,2},{2,3},{3,4},{4,1}}; $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.



3. W_5.

Ans: Vertices = {1,2,3,4,5,6}, Edges $= \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,2\}\}$;

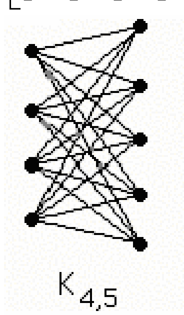
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$



4. $K_{4,5}$.

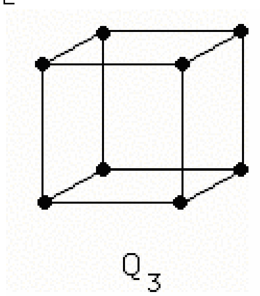
Ans: Vertices = $\{a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4,b_5\}$, Edges = $\{\{a_i,b_j\} \mid i = 1,2,3,4, j = 1,2,3,4,5\}$;

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$



5. Q_3.
Ans: Vertices = {(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)},
Edges = $\{\{(a_1,a_2,a_3),(b_1,b_2,b_3)\} \;:\; |a_1-b_1|+|a_2-b_2|+|a_3-b_3|=1\}$;

$$\begin{bmatrix} 0&1&1&0&1&0&0&0 \\ 1&0&0&1&0&1&0&0 \\ 1&0&0&1&0&0&1&0 \\ 0&1&1&0&0&0&0&1 \\ 1&0&0&0&0&1&1&0 \\ 0&1&0&0&1&0&0&1 \\ 0&0&1&0&1&0&0&1 \\ 0&0&0&1&0&1&1&0 \end{bmatrix}.$$



Use the following to answer questions 6-46:

In the questions below fill in the blanks.

6. K_n has ________ edges and ________vertices.
Ans: $n(n-1)/2$, n.

7. $K_{m,n}$ has ________ edges and ________vertices.
Ans: mn, $m+n$.

8. W_n has ________ edges and ________vertices.
Ans: $2n$, $n+1$.

9. Q_n has ________ edges and ________ vertices.
Ans: $n2^{n-1}$, 2^n.

10. The length of the longest simple circuit in K_5 is ________.
Ans: 10.

11. The length of the longest simple circuit in W_{10} is ________.
Ans: 15.

12. The length of the longest simple circuit in $K_{4,10}$ is ________.
Ans: 40.

13. List all positive integers n such that C_n is bipartite ________.
Ans: n even.

14. The adjacency matrix for $K_{m,n}$ has ________ columns.
Ans: $m + n$.

15. The adjacency matrix for K_n has ________ 1s and ________ 0s.
Ans: $n(n - 1)$, n.

16. There are ________ 0s and ________ 1s in the adjacency matrix for C_n.
Ans: $n^2 - 2n$, $2n$.

17. The adjacency matrix for Q_4 has ________ entries.
Ans: 256.

18. The incidence matrix for W_n has ________ rows and ________ columns.
Ans: $n + 1$, $2n$.

19. The incidence matrix for Q_5 has ________ rows and ________ columns.
Ans: 32, 80.

20. There are ________ non-isomorphic simple undirected graphs with 5 vertices and 3 edges.
Ans: 4.

21. There are ________ non-isomorphic simple digraphs with 3 vertices and 2 edges.
Ans: 4.

22. There are ________ non-isomorphic simple graphs with 3 vertices.
Ans: 4.

23. List all positive integers n such that K_n has an Euler circuit. ________
Ans: n odd.

24. List all positive integers n such that Q_n has an Euler circuit. ________
Ans: n even.

25. List all positive integers n such that W_n has an Euler circuit. ________
Ans: None.

26. Every Euler circuit for K_9 has length ________.
Ans: 36.

27. List all positive integers n such that K_n has a Hamilton circuit. ________
Ans: All n except $n = 2$.

28. List all positive integers n such that W_n has a Hamilton circuit. ________
Ans: All n.

29. List all positive integers n such that Q_n has a Hamilton circuit. ________
Ans: All n except $n = 1$.

30. List all positive integers m and n such that $K_{m,n}$ has a Hamilton circuit. ________
Ans: $m = n > 1$.

31. Every Hamilton circuit for W_n has length ________.
Ans: $n + 1$.

32. List all positive integers n such that K_n has a Hamilton circuit but no Euler circuit. ________
Ans: n even ($\neq 2$).

33. List all positive integers m and n such that $K_{m,n}$ has a Hamilton path but no Hamilton circuit. ________
Ans: $m = n + 1$ or $n = m + 1$.

34. The largest value of n for which K_n is planar is ________.
Ans: 4.

35. The largest value of n for which $K_{6,n}$ is planar is ________.
Ans: 2.

36. List all the positive integers n such that $K_{2,n}$ is planar. ________
Ans: All n.

37. The Euler formula for planar connected graphs states that ________.
Ans: $v - e + r = 2$.

38. If G is a connected graph with 12 regions and 20 edges, then G has ________ vertices.
Ans: 10.

39. If G is a planar connected graph with 20 vertices, each of degree 3, then G has ________ regions.
Ans: 12.

40. If a regular graph G has 10 vertices and 45 edges, then each vertex of G has degree ________.
Ans: 9.

41. The edge-chromatic number for $K_{2,5}$ = ________.
Ans: 5.

42. The vertex-chromatic number for $K_{7,7}$ = ________.
Ans: 2.

43. The vertex-chromatic number for C_{15} = ________.
Ans: 3.

44. The vertex-chromatic number for W_9 = ________.
Ans: 4 (if the infinite region is colored).

45. The region-chromatic number for W_9 = ________.
Ans: 4.

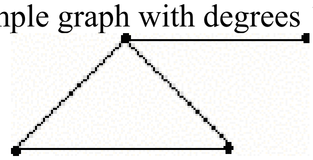
46. The vertex-chromatic number for K_n = ________.
Ans: n.

Use the following to answer questions 47-71:

In the questions below either give an example or prove that there are none.

47. A simple graph with 6 vertices, whose degrees are 2,2,2,3,4,4.
Ans: None. It is not possible to have one vertex of odd degree.

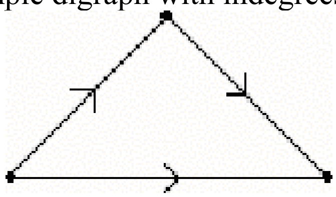
48. A simple graph with 8 vertices, whose degrees are 0,1,2,3,4,5,6,7.
Ans: None. It is not possible to have a vertex of degree 7 and a vertex of degree 0 in this graph.

49. A simple graph with degrees 1,2,2,3.
Ans: 

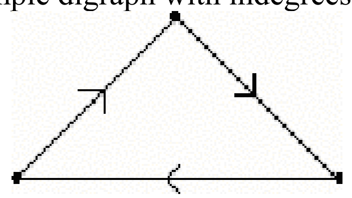
50. A simple graph with degrees 2,3,4,4,4.
Ans: None. It is not possible to have a graph with one vertex of odd degree.

51. A simple graph with degrees 1,1,2,4.
Ans: None. In a simple graph with 4 vertices, the largest degree a vertex can have is 3.

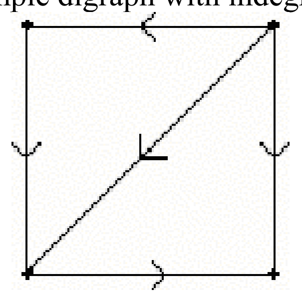
52. A simple digraph with indegrees 0,1,2 and outdegrees 0,1,2.

Ans: 

53. A simple digraph with indegrees 1,1,1 and outdegrees 1,1,1.

Ans: 

54. A simple digraph with indegrees 0,1,2,2 and outdegrees 0,1,1,3.

Ans: 

55. A simple digraph with indegrees 0,1,2,4,5 and outdegrees 0,3,3,3,3.
    Ans: None. In a simple graph with five vertices, there cannot be a vertex with indegree 5.

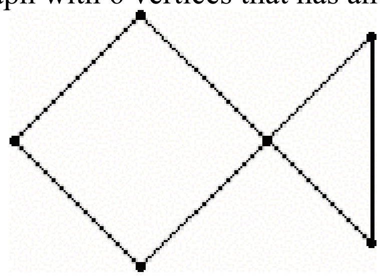
56. A simple digraph with indegrees 0,1,1,2 and outdegrees 0,1,1,1.
    Ans: None. The sum of the outdegrees must equal the sum of the indegrees.

57. A simple digraph with indegrees: 0,1,2,2,3,4 and outdegrees: 1,1,2,2,3,4.
    Ans: None. The sum of the outdegrees must equal the sum of the indegrees.

58. A simple graph with 6 vertices and 16 edges.
    Ans: None. The largest number of edges in a simple graph with six vertices is 15.

59. A graph with 7 vertices that has a Hamilton circuit but no Euler circuit.
    Ans: W_6.

60. A graph with 6 vertices that has an Euler circuit but no Hamilton circuit.

Ans: 

61. A graph with a Hamilton path but no Hamilton circuit.
Ans: $K_{1,1}$.

62. A graph with a Hamilton circuit but no Hamilton path.
Ans: None. Every Hamilton circuit is a Hamilton path.

63. A connected simple planar graph with 5 regions and 8 vertices, each of degree 3.
Ans: None. The graph would have 12 edges, and hence $v - e + r = 8 - 12 + 5 = 1$, which is not possible.

64. A graph with 4 vertices that is not planar.
Ans: None. The largest such graph, K_4, is planar.

65. A planar graph with 10 vertices.
Ans: C_{10}.

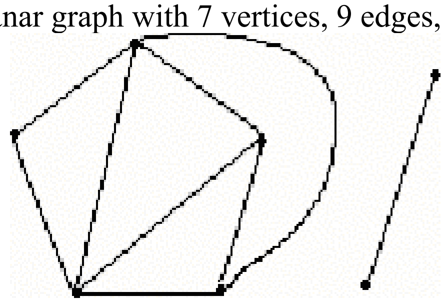
66. A graph with vertex-chromatic number equal to 6.
Ans: K_6.

67. A graph with 9 vertices with edge-chromatic number equal to 2.
Ans: C_9 with one edge removed.

68. A graph with region-chromatic number equal to 6.
Ans: None. The 4-color theorem rules this out.

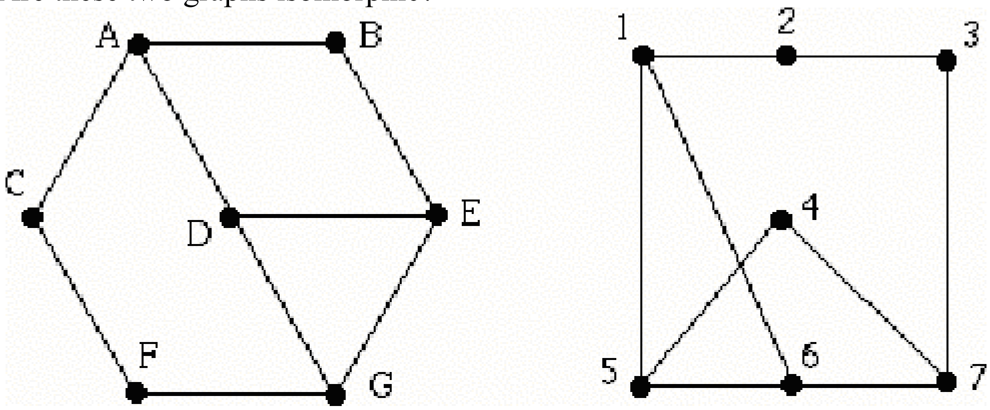
69. A planar graph with 8 vertices, 12 edges, and 6 regions.
Ans: Q_3.

70. A planar graph with 7 vertices, 9 edges, and 5 regions.

Ans: 

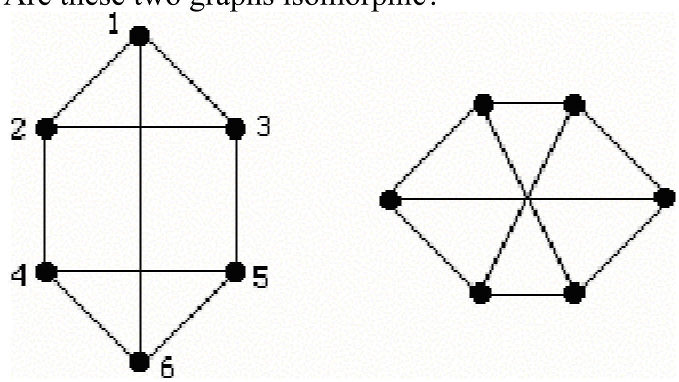
71. A bipartite graph with an odd number of vertices that has a Hamilton circuit.
Ans: None. Any bipartite Hamilton graph must have an even number of vertices.

72. Are these two graphs isomorphic?



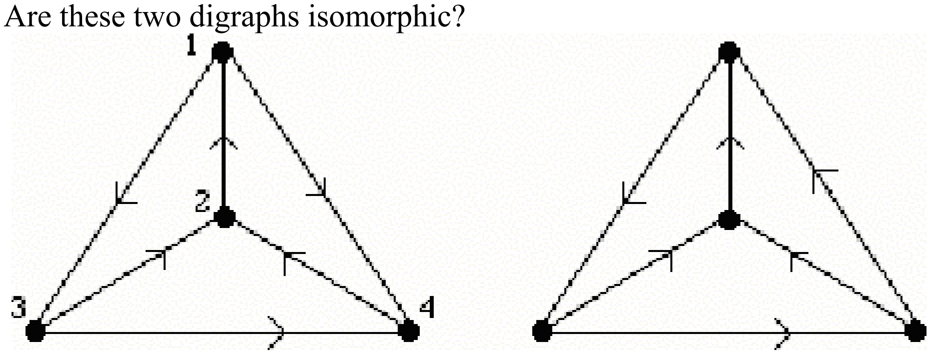
Ans: The graphs are isomorphic: A–7, B–4, C–3, D–6, E–5, F–2, G–1.

73. Are these two graphs isomorphic?



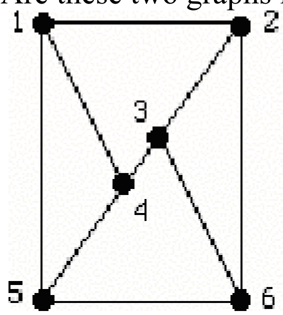
Ans: The graphs are not isomorphic: the graph on the left is planar, but the graph on the right is isomorphic to $K_{3,3}$.

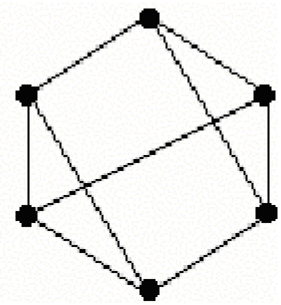
74. Are these two digraphs isomorphic?



Ans: The digraphs are isomorphic: label the center vertex 4, the top vertex 2, the left vertex 1, and the right vertex 3.

75. Are these two graphs isomorphic?





Ans: The graphs are isomorphic: label the graph clockwise from the top with 2,3,6,5,4,1.

76. Suppose you have a graph G with vertices $v_1, v_2, \ldots, v_{17}$. Explain how you would use the adjacency matrix A to find

(a) The number of paths from v_5 to v_3 of length 12.

(b) The length of a shortest path from v_5 to v_3.

Ans: (a) Use the 5,3-entry of A^{12}. (b) Examine the 5,3-entry of $A, A^2, A^3, \ldots, A^{16}$. The smallest positive integer i such that the 5,3-entry of A^i is not zero is the length of a shortest path from v_5 to v_3. If the 5,3-entry is always zero, there is no path from v_5 to v_3.

77. A simple graph is *regular* if every vertex has the same degree.

(a) For which positive integers n are the following graphs regular: C_n, W_n, K_n, Q_n?

(b) For which positive integers m and n is $K_{m,n}$ regular?

Ans: (a) All $n \geq 3$, $n = 3$, all $n \geq 1$, all $n \geq 0$. (b) $m = n$.

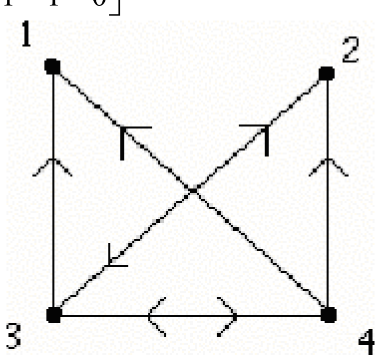
78. If a simple graph G has v vertices and e edges, how many edges does $\overline{G}$ have?

Ans: $\frac{v(v-1)}{2} - e$.

79. Draw the digraph with adjacency matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

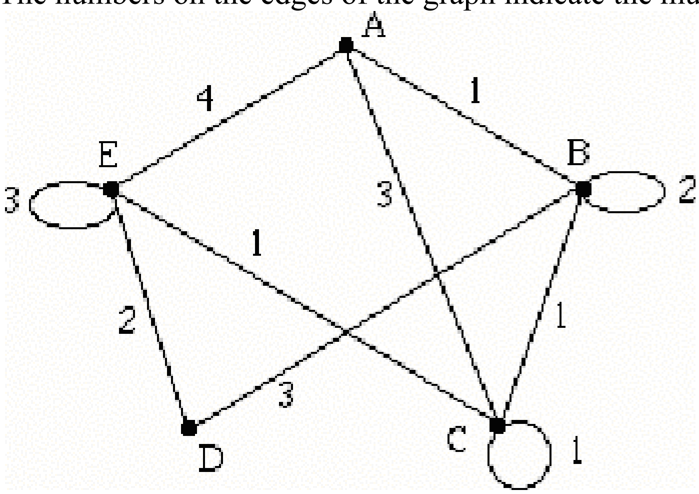
Ans:



80. Draw the undirected graph with adjacency matrix

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

Ans: The numbers on the edges of the graph indicate the multiplicities of the edges.



81. Suppose *G* is a graph with vertices *a*,*b*,*c*,*d*,*e*,*f* with adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

(where alphabetical order is used to determine the rows and columns of the adjacency matrix). Find

(a) the number of vertices in *G*.
(b) the number of edges in *G*.
(c) the degree of each vertex.
(d) the number of loops.
(e) the length of the longest simple path in *G*.
(f) the number of components in *G*.
(g) the distance between vertex *a* and vertex *c*.

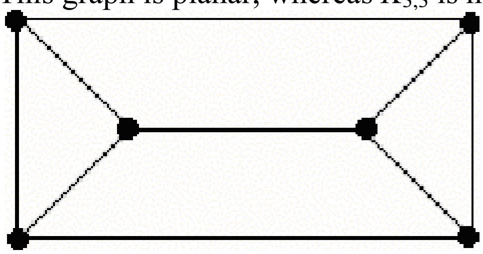
Ans: (a) 6. (b) 9. (c) 2,4,2,3,4,3. (d) 0. (e) 9 (*G* has an Euler circuit). (f) 1. (g) 3.

Use the following to answer questions 82-84:

In the questions below a graph is a *cubic* graph if it is simple and every vertex has degree 3.

82. Draw a cubic graph with 7 vertices, or else prove that there are none.
    Ans: None, since the number of vertices of odd degree must be even.

83. Draw a cubic graph with 6 vertices that is not isomorphic to $K_{3,3}$, or else prove that there are none.
    Ans: This graph is planar, whereas $K_{3,3}$ is not.



84. Draw a cubic graph with 8 edges, or else prove that there are none.
    Ans: None. If $e = 8$, then $3v = 2e = 16$, which is not possible.

85. In K_5 find the number of paths of length 2 between every pair of vertices.
    Ans: 3.

86. In K_5 find the number of paths of length 3 between every pair of vertices.
    Ans: 13.

87. In K_5 find the number of paths of length 6 between every pair of vertices.
    Ans: 819.

88. In $K_{3,3}$ let *a* and *b* be any two adjacent vertices. Find the number of paths between *a* and *b* of length 3.
    Ans: 9.

89. In $K_{3,3}$ let *a* and *b* be any two adjacent vertices. Find the number of paths between *a* and *b* of length 4.
    Ans: 0.

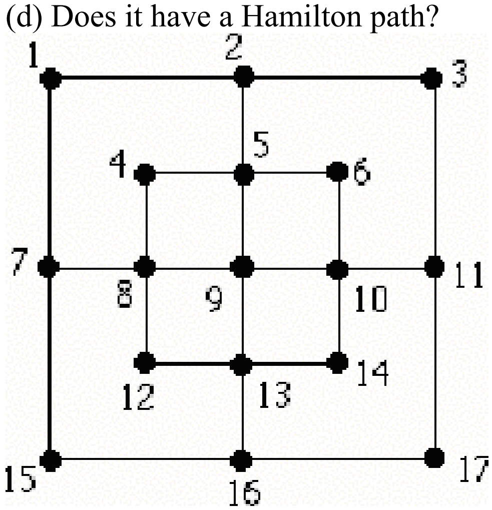
90. In $K_{3,3}$ let *a* and *b* be any two adjacent vertices. Find the number of paths between *a* and *b* of length 5.
    Ans: 81.

91. How many different channels are needed for six television stations (*A*,*B*,*C*,*D*,*E*,*F*) whose distances (in miles) from each other are shown in the following table? Assume that two stations cannot use the same channel when they are within 150 miles of each other?

	A	*B*	*C*	*D*	*E*	*F*
A	–		175	100	50	100
B	85	–	125	175	100	130
C	175	125	–	100	200	250
D	100	175	100	–	210	220
E	50	100	200	210	–	100
F	100	130	250	220	100	–

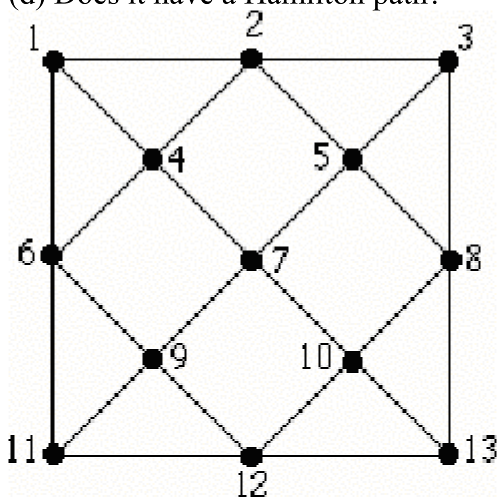
Ans: 4. Stations *A*, *B*, *E*, and *F* require different channels. Stations *C* and *A* can be assigned the same channel. Stations *D* and *B* can be assigned the same channel.

92. Consider the graph shown.
(a) Does it have an Euler circuit?
(b) Does it have an Euler path?
(c) Does it have a Hamilton circuit?
(d) Does it have a Hamilton path?



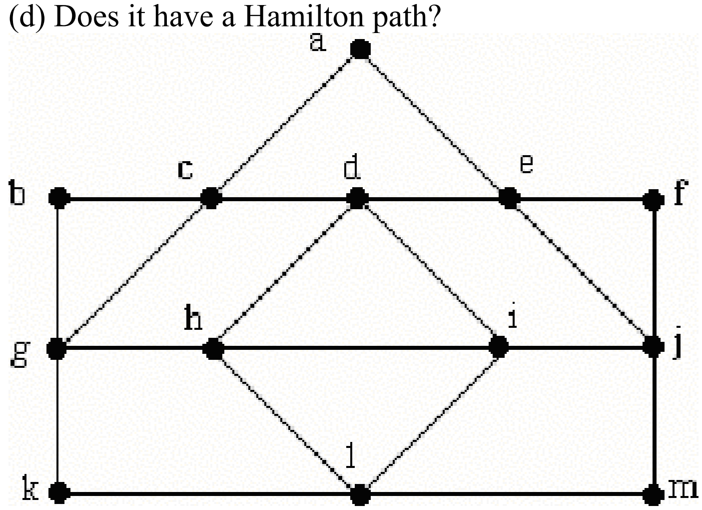
Ans: (a) No. (b) No. (c) No. (d) No.

93. Consider the graph shown below.
(a) Does it have an Euler circuit?
(b) Does it have an Euler path?
(c) Does it have a Hamilton circuit?
(d) Does it have a Hamilton path?



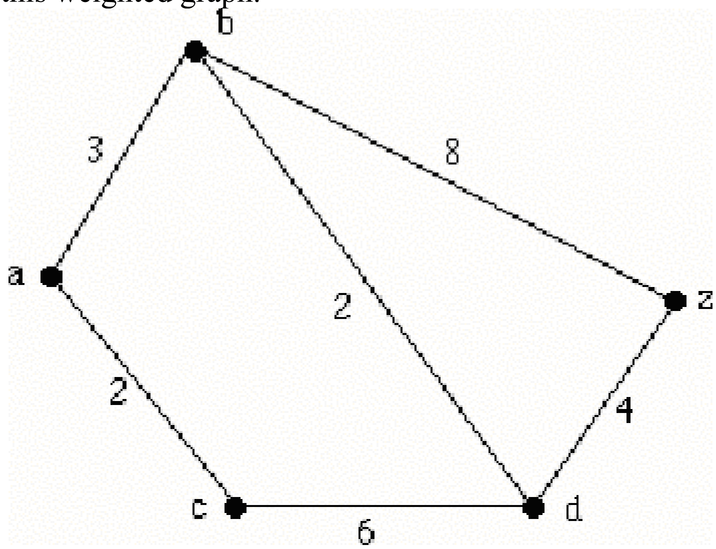
Ans: (a) No. (b) No. (c) Yes. (d) Yes.

94. Consider the graph shown below.
(a) Does it have an Euler circuit?
(b) Does it have an Euler path?
(c) Does it have a Hamilton circuit?
(d) Does it have a Hamilton path?



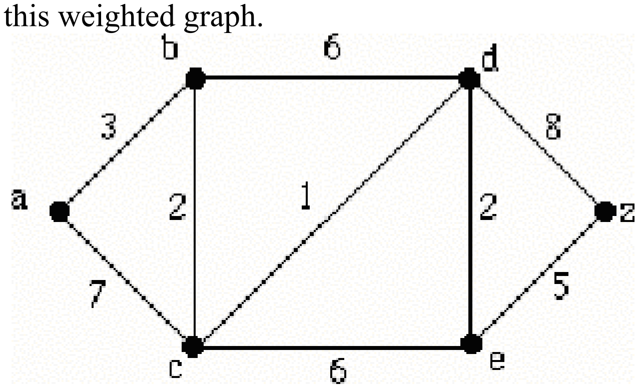
Ans: (a) Yes. (b) Yes. (c) No. (d) Yes.

95. Use Dijkstra's Algorithm to find the shortest path length between the vertices *a* and *z* in this weighted graph.



Ans: First iteration: distinguished vertices *a*; labels *a*:0, *b*:3, *c*:2, *d*,*z*:∞; second iteration: distinguished vertices *a*,*c*; labels *a*:0, *b*:3, *c*:2, *d*:8, *z*:∞; third iteration: distinguished vertices *a*,*b*,*c*, labels *a*:0, *b*:3, *c*:2, *d*:5, *z*:11; fourth iteration: distinguished vertices *a*,*b*,*c*,*d*, labels *a*:0, *b*:3, *c*:2, *d*:5, *z*:9. Since *z* now becomes a distinguished vertex, the length of a shortest path is 9.

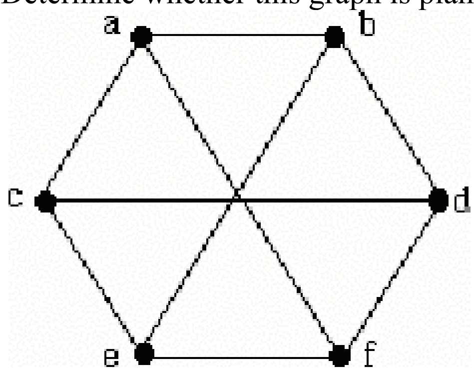
96. Use Dijkstra's Algorithm to find the shortest path length between the vertices *a* and *z* in this weighted graph.



Ans: First iteration: distinguished vertices *a*; labels *a*:0, *b*:3, *c*:7, *d*,*e*,*z*:∞; second iteration: distinguished vertices *a*,*b*; labels *a*:0, *b*:3, *c*:5, *d*:9, *e*,*z*:∞; third iteration: distinguished vertices *a*,*b*,*c*, labels *a*:0, *b*:3, *c*:5, *d*:6, *e*:11, *z*:∞; fourth iteration: distinguished vertices *a*,*b*,*c*,*d*; labels: *a*:0, *b*:3, *c*:5, *d*:6, *e*:8, *z*:14; fifth iteration: distinguished vertices *a*,*b*,*c*,*d*,*e*; labels *a*:0, *b*:3, *c*:5, *d*:6, *e*:8, *z*:13. Since *z* now becomes a distinguished vertex, the length of a shortest path is 13.

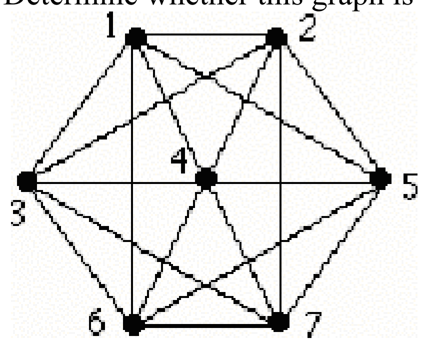
97. The Math Department has 6 committees that meet once a month. How many different meeting times must be used to guarantee that no one is scheduled to be at 2 meetings at the same time, if committees and their members are: C_1 = {Allen, Brooks, Marg}, C_2 = {Brooks, Jones, Morton}, C_3 = {Allen, Marg, Morton}, C_4 = {Jones, Marg, Morton}, C_5 = {Allen, Brooks}, C_6 = {Brooks, Marg, Morton}.
Ans: 5. Only C_4 and C_5 can meet at the same time.

98. Determine whether this graph is planar.



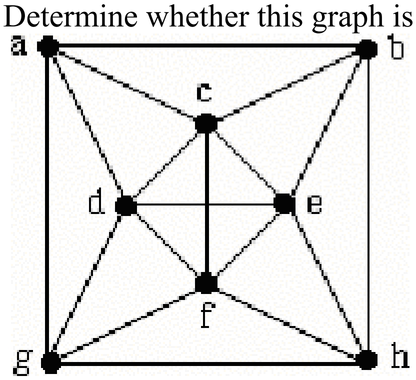
Ans: The graph is not planar. The graph is isomorphic to $K_{3,3}$.

99. Determine whether this graph is planar.



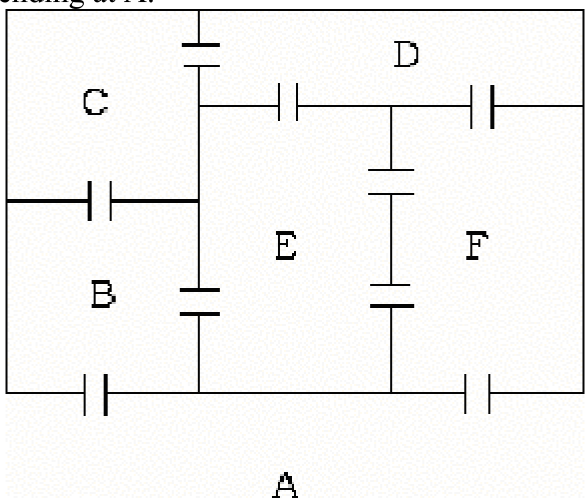
Ans: The graph is not planar. The graph contains a subgraph isomorphic to $K_{3,3}$, using {1,3,5} and {2,4,6} as the two vertex sets.

100. Determine whether this graph is planar.



Ans: The graph is not planar. The graph contains a subgraph homeomorphic to K_5, using vertices b,c,d,e,f.

101. The picture shows the floor plan of an office. Use graph theory ideas to prove that it is impossible to plan a walk that passes through each doorway exactly once, starting and ending at A.



Ans: Use vertices for rooms and edges for doorways. A walk would be an Euler circuit in this multigraph, which does not exist since *B* and *D* have odd degree.

102. Find the vertex-chromatic number, the edge-chromatic number, and the region-chromatic number for $K_{3,2}$.
Ans: vertex-chromatic number = 2; edge-chromatic number = 3; region-chromatic number = 3.

103. Find the vertex-chromatic number, the edge-chromatic number, and the region-chromatic number for K_4.
Ans: vertex-chromatic number = 4; edge-chromatic number = 3; region-chromatic number = 4.

104. Find the vertex-chromatic number, the edge-chromatic number, and the region-chromatic number for C_7.
Ans: vertex-chromatic number = 3; edge-chromatic number = 3; region-chromatic number = 2.

105. Find the vertex-chromatic number, the edge-chromatic number, and the region-chromatic number for Q_3.
Ans: vertex-chromatic number = 2; edge-chromatic number = 3; region-chromatic number = 3.

106. Find the vertex-chromatic number, the edge-chromatic number, and the region-chromatic number for W_5.
Ans: vertex-chromatic number = 4; edge-chromatic number = 5; region-chromatic number = 4 (assuming that the infinite region is colored).

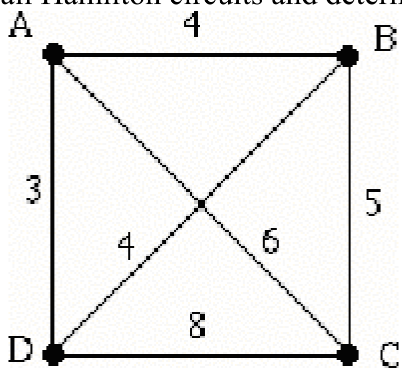
107. Give a recurrence relation for e_n = the number of edges of the graph K_n.
Ans: $e_n = e_{n-1} + n - 1$.

108. Give a recurrence relation for v_n = number of vertices of the graph Q_n.
Ans: $v_n = 2v_{n-1}$.

109. Give a recurrence relation for e_n = number of edges of the graph Q_n.
Ans: $e_n = 2e_{n-1} + 2^{n-1}$.

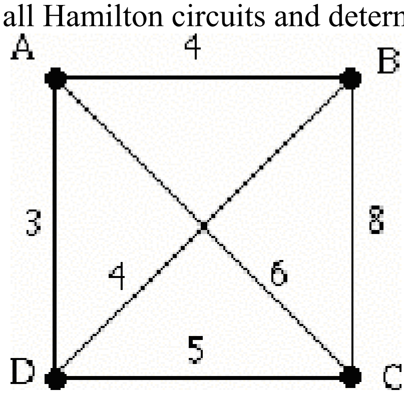
110. Give a recurrence relation for e_n = the number of edges of the graph W_n.
Ans: $e_n = e_{n-1} + 2$.

111. Solve the traveling salesman problem for the given graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



Ans: A–D–B–C–A (weight 18).

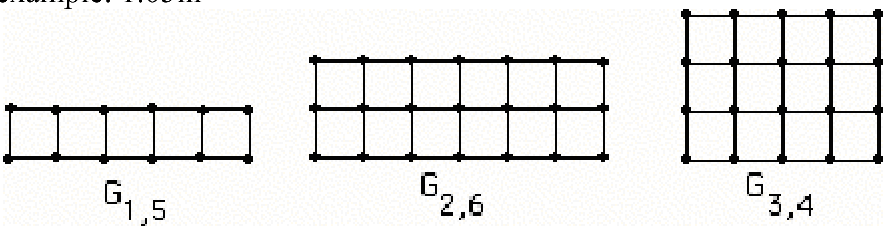
112. Solve the traveling salesman problem for the given graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



Ans: A–B–D–C–A (weight 19).

Use the following to answer questions 113-122:

In the questions below the *grid graph* $G_{m,n}$ refers to the graph obtained by taking an $m \times n$ rectangular grid of streets ($m \leq n$) with m north/south blocks and n east/west blocks. For example: 1.05in



113. Find a formula for the number of vertices of $G_{m,n}$.
Ans: $(m + 1)(n + 1)$.

114. Find a formula for the number of edges of $G_{m,n}$.
Ans: $n(m + 1) + m(n + 1)$.

115. Find a formula for the number of regions (including the infinite region) of $G_{m,n}$.
Ans: $mn + 1$.

116. For which positive integers m and n does $G_{m,n}$ have an Euler circuit?
Ans: $m = n = 1$.

117. For which positive integers m and n does $G_{m,n}$ have an Euler path but no Euler circuit?
Ans: $m = 1$, $n = 2$.

118. For which positive integers m and n does $G_{m,n}$ have a Hamilton circuit?
Ans: m or n odd.

119. For which positive integers m and n does $G_{m,n}$ have a Hamilton path but no Hamilton circuit?
Ans: m and n even.

120. Find the vertex-chromatic number for $G_{m,n}$.
Ans: 2.

121. Find the edge-chromatic number for $G_{m,n}$.
Ans: 4.

122. Find the region-chromatic number for $G_{m,n}$ (including the infinite face).
Ans: 3.