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Asahi, Bacon Cheeseburger Soup Paula Dean, Ventura County Animal Shelter, Antarctica Map 2019, Raqab Meaning in Urdu, Seaworld Aquarium Orlando, Columbia Glacier 2019, Natural Insect Repellent, 4k Steelbook List, Upper Body Chestmall, Candy Crush Special Candy, Super Nintendo Emulator, Usman Ibn Affan Series, HASS - HEIL - WEIR CALCULUSTHOMAS' FOURTEENTH EDITIONTHOMAS' CALCULUS FOURTEENTH EDITION Based on original work by GEORGE B THOMAS. Jr. Massachusetts Institute of Technology, revised by Joel HASS University of California, Davis CHRISTOPHER HEIL Georgia Institute of Technology MAURICE D. WEIR Naval Graduate School Principal, Portfolio Management: Deirdre Lynch Executive Editor: Jeff Weidenaar Editorial Assistant: Jennifer Snyder Content Producer: Rachel S. 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Classification: LCC WA303.2.W45 2018 DDC 515-dc23 LC Entry is available on 1 17 Instructor Edition ISBN 13: 978-0-13-443909-9 ISBN 10: 0-13-443 ISBN 13: 978-0-13-443898-6 ISBN 10: 0-13-443898-1Contents Prefaces Prex 1 Features 1 1.1 Features and Their Graphics 1 1.2 Feature Combination; Shift and Scale Graphics 14 1.3 Trigonometric Features 21 1.4 Graphics with Software 29 Issues to Guide Your Review 33 Practice Exercises 34 Extra and Advanced Exercise 35 Technological Applications Projects 37 2 Limitations and Continuity 38 2. 1 Speed of change and tangent lines to curves 38 2.2 Limit functional and limit laws 45 2.3 Accurate definition of limit 56 2.4 Unilateral limits 65 2.5 Continuity 72 2.6 Limits related to infinity; Asymptotes charts 83 questions to guide your review 96 Practice Exercises 97 Extra and Advanced Exercises 98 Technological Projects Application 101 3 Derivatives 102 3.1 Tangent Lines and Derivatives at Point 102 102 3.2 Derivative as Feature 106 3.3 Rules of Differentiation 3.3 Rules of Differentiation 115 3.4 Derivative as Change Rate 124 3.5 Derivative of Trigonometric Features 134 3.6 Chain Rule 140 3.7 Implicit Differentiation 148 3.8 Related Tariffs 153 3.9 Linearity and Differentials 162 Issues, to guide your review 174 Practical Exercises 174 Extra and Advanced Exercises 179 Technological Application Projects 182 iiiiv Content 4 Derivative Apps 183 4.1 Extreme Function Values at Closed Intervals 183 4.2 Theorem Average 191 4.3 Monotone and first derivative test 197 4.4 Concussion and Curve Sketch 202 4.5 Applied Optimization 214 4.6 Newton Method 226 4.7 Antiderivatives 231 Issues, to guide your review 241 Practical Exercises 241 Extra and Advanced Exercises 244 Technology Application Projects 247 5 Integral 248 5.1 Area and Score with final amounts of 248 5.2 Sigma Notation and Limits Fins 258 5.3 Defined Integral 265 5.4 Fundamental Caloric theorem of 278 5.5 Indefinite Integrals and Replacement Method 289 5.6 Defined Integrated Replacements and area between the curves of 296 questions, to guide your review 306 Practical Exercises 307 Additional and Advanced Exercises 310 Technological Projects application 313 6 Application of certain integrals 314 6.1 volumes using cross-section 314 6.2 volumes using cylindrical shells 325 6.. 3 Arc Length 333 6.4 Area Surfaces Revolution 338 6.5 Work and Fluid Forces 344 6.6 Moments and Centers Mass 353 Issues to Guide Your Review 365 Practice Exercises 366 Additional and Advanced Exercises 368 Technological Application Projects 369 7 Transcendental Features 370 7.1 Reverse Functions and Their Derivatives 370 7.2 Natural Logarithms 378 7.3 Extensive Features 386 7.4 Exponential Changes and Separable Differential Equations 397 7.5 Uncertain Forms and L'4 Extreme Changes and Separable Differential Equations 397 7.5 Uncertain Forms and L'4 Extreme Changes and Separable Differential Equations 397 7.5 Indefinite Forms and L'4 Rule 407 7.6 Reverse Trigonometry Features 416 7.7 Hyperbolic Features 428Contents v 7.8 Relative Growth Rate 436 Issues to Guide Your Review 441 Practice 442 Extra Exercises 445 8 8 Integration Methods 447 8.1 Use Basic Integration Formulas 447 8.2 Integration in Parts 452 8.3 Trigonometry Integrals 460 8.4 Trigonometry Replacements 466 8.5 Integrating Rational Functions by Partial Factions 471 8.6 Integral Tables and Algebra Computer Systems 47 9 8.7 Numerical Integration 485 8.8 Wrong Integrals 494 8.9 Probability 505 Issues, to guide your review 518 Practical Exercises 519 Additional and Advanced Exercises 522 Technological Projects Application 525 9 First Order Differential Equations 526 9.1 Solutions , Field Slope, and the Euler Method 526 9.2 First Order Linear Equations 534 9.3 Applications 540 9.4 Graphic Solutions to Standalone Equations 546 9.5 Systems of Equations and Phase Planes 553 Issues to Guide Your Review 559 Practice Exercise 559 Additional and Advanced Exercises 559 Additional and Advanced Exercises 553 561 Technological Applications Projects 562 10 Infinite Sequences and Series 563 10.1 Sequences 563 10.2 Infinite Series 576 10.3 Integral Test 586 10.4 Comparative Tests 592 10.5 Absolute Convergence; 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Rocket Movement 758 13.3 Arc Length in Space 767 13.4 Curves and Normal Curve Vectors 771 13.5 Tangential and Normal Acceleration Components 777 13.6 Speed and Acceleration in Polar Coordinates 783 Issues to Guide Your Review 787 Practical Exercises 788 Additional and Advanced Exercises 790 Technological Application Projects 791Contents vii 14 Partial Derivatives 792 14.1 Features Multiple Variables 792 14.2 Limits and Continuity in Higher Dimensions 800 14.3 Partial Derivatives 809 14.4 Chain Rule 821 14.5 Directed Derivatives and Gradient Vectors 831 14.6 Tangent Planes and Differentials 839 14.7 Extreme Values and Saddle Points 849 14.8 Lagrange Multipliers 858 14.9 Taylor Formula for Two Variables 868 14.10 Partial Derivatives With Limited Variable 872 Issues for Guide Your Review 87 Practice 87 Practice 87 Exercise 877 Additional and Advanced Exercise 880 Technological Application Projects 882 15 Multiple Integrals 883 15.1 Double and Iterized Integrals Over Rectangles 883 15.2 Double Integrals Over Common Regions 888 15.3 Area Dual Integration 897 15.4 Double And Integrals in Polar Form 900 15.5 Triple Integrals in rectangular coordinates 907 15.6 Appendix 917 15.7 Triple Integrals in Cylindry and Spheres Coordinates 927 15.8 Replacements in Multiple Integral 939 Issues to Guide Your Review 949 Practice Exercises 949 Additional and Advanced Exercises 952 Technological Application Projects 954 Интегралы в векторном поле 955 16,1 Линейные интегралы Scalar Functions 955 16.2 Векторные поля и линейные интегралы: Работа, Циркуляция и Поток 962 16.3 Независимость Пути, Консервативные поля, и потенциальные функции 975 16,4 Теорема Грина в плоскости 986 16,5 Поверхности и площадь 998 16,6 Поверхностные интегралы 1008 16,3 Теорема Стокса 1018 16.8 Теорема дивергенции и единая теория 1031 Вопросы, чтобы направлять ваш обзор 1044 Практические упражнения 1044 Дополнительные и Расширенные упражнения 1047 Технологические проекты применения 1 1048viii Содержимое 17 Дифференциальные уравнения второго порядка (онлайн на bit.ly/2DM0mHQ) 17.1 Линейные уравнения второго порядка 17.2 Неоднородные линейные уравнения 17.3 Приложения 17 .4 Уравнения Euler 17.5 Power-Series Решения Приложения AP-1 A.1 Реальные числа и реальная линия AP-1 A.2 Математическая индукция AP-6 A.3 Линии, Круги , и Parabolas AP-9 A.4 Доказательства ограничения теоремы AP-19 A.5 Обычно происходящих пределов AP-22 A.6 Теория реальных чисел AP-23 A.7 Сложные номера AP-26 A.8 Дистрибьюторский закон для векторных кросс-продуктов AP-34 A.9 Смешанная производная теорема и приращение теоремы AP-35 Ответы на нечетные The Number Exercise A-1 Applications Index AI-1 Theme Index I-1 A Brief Table of T-1 Credits T-1Preface C-1Preface, the fourteenth edition, provides a modern introduction to calculus that fo-cuses on the development of conceptual understanding of basic mathematical ideas. This new to this edition text supports the calculus sequence typically accepted by students in STEM fields over several semesters. Intuitive and accurate explanations, thoughtfully selected examples, excellent fig urets and time-tested exercise sets are the basis of this text. We continue to improve this text in line with the shifts in both the preparation and the purpose of today's students, as well as in the application of calculus in a changing world. Many of today's students were subject to calculus in high school. For some, this leads to a successful calculus experience in college. For others, however, the result is self-confidence in their computational abilities combined with fundamental gaps in algebra and trigonometry of mastery, as well as poor conceptual understanding. In this text, we seek to meet the needs of an increasingly diverse population in calculating consistency. We have taken care to provide enough review material (in text and appen-dice), detailed solutions, as well as various examples and exercises to support a full understanding of calculus for students at different levels. In addition, the MyMathLab course, which accompanies the text, provides adaptive support to meet the needs of all the knock-ons. In the text, we present the material in such a way that maturity goes beyond memorizing formulas and routine procedures, and we show students how to generalize key concepts as soon as they are introduced. Links are made throughout, linking new concepts to related concepts that have been studied before. After studying-ing calculus from Thomas, students will be developed problem solving and reasoning abilities that will serve them well in many important aspects of their lives. Mastering this beautiful and creative theme, with its many practical applications in many areas, is its own reward. But the real gifts of calculus study acquire the ability to think logically and accurately; Understanding what is defined, what is intended and what is deduced; and learn to generalize conceptually. We intend to encourage and support these goals. We welcome a new co-author, Christopher Hale of the Georgia Institute of Technology, in this edition. He has been teaching calculus, linear algebra, analysis and abstract algebra at Georgia Tech since 1993. He is an experienced author and worked as a consultant in a previous edition of the text. His studies, in harmonic analysis, include-ing frequency analysis, wave, and operator theory. This is a significant revision. Each word, symbol and figure has been revised to make sure of clarity, consistency and brevity. In addition, we have made the following text updates: ixx Preface - Updated graphics to bring out clear visualization and mathematical correctness. Examples (in response to user feedback) are added to overcome conceptual obstacles. See example 3 in section 9.1. Added new types of homework throughout, including many with nature's geomet-rick. New exercises are not just more of the same, but rather give different perspectives and approaches to each topic. We also analyzed aggregated student and MyMathLab performance data for the previous edition of this text. The results of this analysis helped to improve the quality and quantity of exercise. Added short URLs to historical links that allow students to go directly to online information. Added new marginal notes throughout to guide the reader through the process of prob-lem decision and emphasize that every step in the mathematical argument is strictly justified. New to MyMathLab® Many improvements have been made in the overall functionality of MyMathLab (MML) from the previous edition. In addition, we have also increased and improved content specific to this text. Instructors now have more exercise than ever to choose from in a homework assignment. There are about 8,080 designated exercises in the MML. The MML exercise scoring engine has been updated to provide more reliable coverage of certain topics, including differential equations. Added a full set of interactive shapes support for teaching and learning. The numbers are meant to be used in lectures as well as by students of their own. The numbers are edited using free-to-air GeoGebra software. The figures were created by Mark Reno (Shippensburg University), Kevin Hopkins (Southwest University B aptist), Steve Phelps (University of Cincinnati), and Tim Brzezinski (Berlin School, CT). Improved sample assignments include just-in-time pre-review, help keep skills fresh with distributed practice key concepts (based on research by Jeff Hib, Keith Lyle, and Pat Ralston of the University of Louisville), and provide opportunities to work exercises without teaching aids (to help students develop confidence in their abil-ity to solve problems on their own). Additional conceptual questions complement text exercises to focus on a deeper theoretical understanding of key concepts in calculus. These questions were written by Cornell University professors as part of an NSF grant. They are also assigned through catalytic training. The integrated review version of the MML course contains ready-made quizzes to assess the required skills for each chapter, as well as personalized fixes for any skills gaps that are identified. Solve settings and exercises are now displayed in many sections. These exercises require students to show how they created problems, and a better solution to mirror what students re-asked on the tests. More than 200 new instructional videos by Greg Wisloski and Dan Radelet (both from Indiana University in Pennsylvania) increase the already reliable collection as part of the course. These videos support the overall approach of the text, in particular, they go beyond routine procedures to show students how to generalize and connect key concepts. Foreword xi Content Improvement Chapter 5 Chapter 1 - Improvement Discussion in 5.4 and added a new figure of 5.18 k - Shortened 1.4 to focus on the issues arising in the use of mathematics- illustrate the theorem of average value. matte software and potential pitfalls. Removed peripheral - Added new exercises: 5.2: 33-36; PE: 45-46. regression material, along with accompanying exercises. Chapter 6 - Added new exercises: 1.1: 59-62, 1.2: 21-22; 1.3: 64-65, - refined cylindrical method of the shell. PE: 29-32. Converted 6.5 Example 4 into metric units. Added an introductory discussion of mass distribution on chapter 2, with a figure, at 6.6. Added the definition of an average speed of 2.1. A refined definition of the limits that allow arbitrary domains. Added new exercises: 6.1: 15-16; 6.2: 45–46; 6.5: 1–2; The definition of limits is now consistent with the definition - 6.6: 1-6, 19-20; PE: 17-18, 35-36. tion in multivariate later in the text and with more general mathematical use. Chapter 7 - Changed limitations and succession definitions to remove implied implied explain the terminology of vague symbols of cations and improve understanding. Form. - Added a new example 7 in 2.4 to illustrate the limits of the ratio - a refined discussion of individual differential equations in 7.4. Replaced sin-1 notation to the reverse function of the sinuses with trigger functions. arcsin as default notation at 7.6, and similar to other triggers and rewrote 2.5 Example 11 to solve the equation by finding function. zero, according to the previous discussion. Added new exercises: 7.2: 5-6, 75-76; 7.3: 5-6, 31-32, - Added new exercises: 2.1: 15-18; 2.2: 3h-k, 4f-i; 2.4: 123–128, 149–150; 7.6: 43–46, 95–96; AAE: 9-10, 23. 19–20, 45–46; 2.6: 69–72; PE: 49-50; AAE: 33. Chapter 8 Chapter 3 - Updated 8.2 Integration into parts of the discussion to emphasize - Refined slope link and change rate. u(x) y' (x) dx form, not u dy. Rewrote Examples 1-3 - Added a new 3.9 pattern using the square root function accordingly. illustrate vertical tangent lines. Removed discussion of tablicular integration and its associated figure x sin (1'x) in 3.2 to illustrate how oscilla-exercise. tion can lead to a non-existent derivative of continuous and updated discussion in 8.5 on how to find constants in function. partial factions. Revised product rule to make the order of factors consistent - Updated notation at 8.8 to align with standard use in sta- throughout the text, including later dot product and cross prod-uct formulas. tistics. Added new exercises: 3.2: 36, 43-44; 3.3: 51–52; 3.5: New exercises added: 8.1: 41-44; 8.2: 53–56, 72–73; 8.3: 43-44, 61bc; 3.6: 65–66, 97–99; 3.7: 25–26; 3.8: 47; 75–76; 8.4: 49–52; 8.5: 51–66, 73–74; 8.8: 35–38, 77–78; AAE: 24-25. PE: 69-88. Chapter 4 Chapter 9 - Added summary to 4.1. Added a new example 3 with a 9.3 pattern to illustrate how to add a new example 3 with a new 4.27 pattern to give a basic sloped field design. and advanced e xamples of concave. Added new exercises: 9.1: 11-14; PE: 17-22, 43-44. Added new exercises: 4.1: 61-62; 4.3: 61–62; 4.4: 49–50, 99–104; 4.5: 37–40; 4.6: 7–8; 4.7: 93–96; PE: 1-10; AAE: 19-20, 33. Moved Exercises 4.1: 53-68 to PE.xii Foreword Chapter 10 Chapter 13 - Refined Differences Between Sequence and Series. Added side panels on how to pronounce Greek letters such as - Added a new drawing 10.9 to illustrate the amount of the series, how to square the kappa, tau, etc. histograms. Added new exercises: 13.1: 1-4, 27-36; 13.2: 15-16, Added to 10.3 discussion of the importance of the border 1 9-20; 13.4: 27–28; 13.6: 1–2. errors in approximations. Chapter 14 - Added a new drawing 10.13, illustrating how to use integrals, to - Developed a discussion of open and closed regions in the remainder of the partial amounts. Standardized notation to evaluate partial derivatives, gra-- rewrote theorem 10 at 10.4 to deduce similarities with dients, and directed derivatives at a point throughout the chapter. a one-on-one comparison test. Renamed twig charts in dependency charts, a new 10.16 pattern has been added to illustrate the different behaviors that explain that they capture the dependency of variables. harmonic and alternating harmonic rows. Added new exercises: 14.2: 51-54; 14.3: 51-54, 59-60, - Renamed nth-Term Test in nth-Term Test for Diver- 71-74, 103-104; 14.4: 20–30, 43–46, 57–58; 14.5: 41–44; to emphasize that he says nothing about convergence. 14.6: 9–10, 61; 14.7: 61–62. Added a new chart 10.19 to illustrate the convergence of polynomies- Chapter 15 ING to ln (1 x), which illustrates convergence at half - Added new figure 15.21b, to illustrate the setting of open interval limits (-1, 14 . double integral. - Red dots and intervals are used to indicate intervals and points - Added new 15.5 Example 1, modified examples 2 and 3, and where the divergence occurs, and blue to indicate convergence , throughout Chapter 10. new figures 15.31, 15.32 and 15.33 have been added to give the basic ex-wide possibilities of setting integration limits for triple integration. Added new drawing 10.21 to show six different possibili- - Added new material about the joint distribution of probabilities as connections for the convergence interval. multivariate integration. Added new exercises: 10.1: 27-30, 72-77; 10.2: 19-22, new examples 5, 6 and 7 added to Section 15.6. 73–76, 105; 10.3: 11–12, 39–42; 10.4: 55–56; 10.5: 45-46, New Exercises Added: 15.1: 15-16, 27-28; 15.6: 39–44; 65–66; 10.6: 57–82; 10.7: 61–65; 10.8: 23–24, 39–40; 10.9: 11–12, 37–38; PE: 41-44, 97-102. 15.7: 1–22. Chapter 11 Chapter 16 - Added a new example 1 and drawing 11.2 in 11.1 to give a new 16.4 pattern to illustrate the line, an integral part of a simple first example of the parameter curve. f bend. Updated area formulas for polar coordinates to include con- - Added a new 16.17 pattern to illustrate the gradient field. Added a new drawing 16.18 to illustrate the line, an integral part of ditions for positive r and neoclassic u. v ector field. New examples 3 and figure 11.37 in 11.4 are added to illustrate the refined notation for linear integrations in 16.2. intersections of polar curves. Added discussion of the sign of potential energy in 16.3. - rewrote the decision Example 3 in 16.4 to clarify the connection - Added new exercises: 11.1: 19-28; 11.2: 49–50; 11.4: 21–24. Greene's theorem. Chapter 12 - Updated discussion of surface orientation at 16.6 along with new figure 12.13 to show the scaling effect 16.52. vector. Added new exercises: 16.2: 37-38, 41-46; 16.4: 1–6; 16.6: Added new examples 7 and figure 12.26 at 12.3 to illustrate 49-50; 16.7: 1–6; 16.8: 1–4. vector projection. Apps: Rewrote the A7 app into complex numbers. Added discussion of common quad surfaces at 12.6, with a new Sample 4 and a new figure of 12.48, illustrating the de-scenario of an ellipsoid not focused on origin through the com-pleting square. Added new exercises: 12.1: 31-34, 59-60, 73-76; 12.2: 43–44; 12.3: 17–18; 12.4: 51–57; 12.5: 49-52.Preface xiii Continuation features Rigor Level of Rigor Level corresponds to the level of previous editions. We continue to divide between official and informal discussions and to share their differences. Starting with a more intuitive, less formal approach helps students understand a new or difficult concept so that they can appreciate its full mathematical accuracy and results. We will pay close attention to the definition of ideas and to prove theorems suitable for the calculation of the knock-ons, while mentioning the deeper or subtler issues they will study in a more advanced course. Our organization and the differences between informal and formal discussions give the instructor a degree of flexibility in the amount and depth of coverage of the various topics. For example, while we do not prove an intermediate theorem or Ex-treme Value theorem for continuous functions with a closed end interval, we will accurately provide these theorems, illustrate their meanings in numerous examples, and use them to prove other important results. In addition, for those instructors who want greater depth cov-erage, in annex 6 we discuss the dependence of these theorems on the fullness of real numbers. Writing Exercise Writing exercises posted throughout the text ask students to study and explain different calculus concepts and applications. In addition, each chapter contains a list of questions for students to review and summarize what they have learned. Many of these exercises do good writing assignments. In addition to the problems that arise after each section, each chapter concludes with a review of the questions, practical exercises covering the entire chapter, and a series of additional and extended exercises with more complex or synthesized problems. Most chapters also include descriptions of several technology projects that can be worked out by individual students or groups of students over a longer period of time. These projects require the use of Mathematica or Maple, as well as finished files that are available for download in MyMathLab. Writing and app this text is still easy to read, conversational, and mathematically rich. Each new theme is motivated by clear, easy-to-understand examples and then reinforced by its application to real-world issues of immediate interest to students. A distinctive feature of this book was the application of calculus to science and engi-neering. These applications have been constantly updated, improved and extended over the past few editions. Technology In the course of using text, the technology can be turned on according to the taste of the instructor. Each section contains exercises that require the use of technology; they are labeled T if suitable for calculator or computer use, or they are labeled Computer Research if a computer system of algebra (CAS such as maple or math-ematic) is required. Additional Resources MyMathLab® Online course (required access code) Built around Pearson's bestseller, MyMathLab is an online homework, tutorial and score program designed to work with this text to engage students and improve results. MyMathLab can be successfully implemented in any learning environment - laboratory, hybrid, fully online, or traditional.xiv Foreword Used by more than 37 million students worldwide, MyMathLab provides consistent, m easurable benefits in student learning outcomes, retention, and subsequent course success. Visit www.mymathlab.com/results to find out more. Readiness Is one of the biggest challenges in calculating courses making sure stu-dents are properly prepared with the necessary skills needed to successfully complete their coursework. MyMathLab supports students through accurate recovery time and key review concepts. An integrated review course can only be used in time for a preliminary review. These courses contain ready-made quizzes to assess the necessary skills for each chapter, as well as personalized fixes for any skills gaps that are identified. Motivation Students are motivated to succeed when they are engaged in learning ex-peripherals and understand the relevance and power of mathematics. MyMathLab's online homework offers students immediate feedback and tutorial help that motivates them to do more, which means they retain more knowledge and improve their test results. Exercises with immediate feedback- more than 8080 assigned exercises for this text to regenerate algorithmically to give students unlimited opportunities to practice and mas-leri. MyMathLab provides useful feedback when students enter incorrect answers and includes additional tutorials such as Help Me Solve This, Sample View, Video and eText. Set up and solve exercises to ask students first to describe how they will be created and ap-prohami problems. This strengthens students' conceptual understanding of the process they are applying and contributes to retention of skills. Foreword XV - Additional conceptual questions focus on a deeper theoretical understanding of key concepts in calculus. These questions were at Cornell University's Faculty of NSF Grant, and are also appointed through catalytic training. Learning catalytics™ is a student response tool that uses students' smartphones, tabs, allows, or laptops to engage them in more interactive tasks and thinking during a lecture. Learning gurneys helps engage students and peer-to-peer learning through real-time analytics. Catalytic training is available to all MyMathLab users. Learning and Learning Tools - Interactive figures illustrate key concepts and allow manipulation to be used as learning and learning tools. We also include videos that use interactive figures to explain the key concepts.xvi Foreword and instructional video-hundreds of videos available as tutorials in exer-cises and for self-study. The video destination guide makes it easy to mark a video for your homework, showing which MyMathLab exercises fit each video. Full eText is available to students through their MyMathLab courses throughout the duration of publication, giving students unlimited access to eText within any course using this edition text. Improved Selective Assignments These assignments include just-in-time previews, help keep skills fresh with distributed practice key concepts, and provide oppor-setting to work exercises without tutorials so students can test their understanding. PowerPoint presentations that cover every section of the book are available for download. - Mathematica's guide and projects, Maple's guide and projects, TI Graphing Cal-culator guide - These guides cover Maple 17, Mathematica 8 and TI-84 Plus and TI-89, respectively. Each contains detailed recommendations for integrating a software package or graphing calculator throughout the course, including syntax and commands. Accessibility and achievement go hand in hand. MyMathLab is compatible with the JAWS screen reader, and it allows students to read and interact with multiple types of choice and free response issues using keyboard controls and math notation inputs. MyMathLab also works with larger screens, including zoomtext, MAGic and SuperNova. 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We would also like to express our gratitude to the people who have made many valuable contributions to this electronic dition as it has evolved at various stages: Precision Checkers by Thomas Wegleitner Jennifer Blue Lisa Collett Reviewers for the fourteenth edition of Nils Martin Mueller, Princeton University Alessandro Arcy, University of Toledo James G. O'Brien, Wentworth Institute of Technology Doug Baldwin, SUNY Geneseo Alan Saleski, Loyola University of Chicago Stephen Heilman, UCLA Alan von Hermann, Santa Clara University David Horntrop, New Jersey Institute of Technology Don Guyan Wilathgamuwa, University of Montana Eric B. 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Osborne, Florida ASM University Daniel Pellegrini, Triton College Luis Rodriguez, Miami Dade College Debra Johnsen, Orangeburg Calhoun Technical College Abbas Meigooni, Lincoln Earth Community College Olga zuckernik, Rochester College of Technology quinlan, Hillsborough Community College Sabrina Rippe, Tulsa Community College Shuvre Gupta, University of Iowa Mona Panchal, East Los Angeles College Alexander Casti, Farley Dickinson University Gail Ilyich, McLennan Community ShardA K. Gudehithlu, Wilbur Wright College Mark Farag, Farley Dickinson University Deanna Robinson, McLennan Community College Selena Mohan, Cumberland County College Of Initiation We regret that prior to writing this edition our co-author Maurice Weir passed away. Mori was dedicated to achieving the highest standards in the presentation of mathematics. He insisted on clarity, rigor and readability. Mori was a role model for his students, colleagues and co-authors. He was very proud of his daughters Maya Coyle and Renee Vine, as well as his grandchildren Matthew Ryan and Andrew Dean Vine. It will be sorely missed.1 THE functions of OVERVIEW Functions are fundamental to studying calculus. In this chapter, we'll look at what features and how they are visualized as graphics, how they are combined and converted, and how they can be classified. 1.1 The functions and their graph functions are a tool for describing the real world in mathematical terms. The function can be represented by an equation, graph, numerical table or oral description; we will use all four submissions throughout this book. This section looks at these ideas. Features The domain and the range of the temperature at which the water boils depends on the altitude above sea level. The inter-est paid for cash investments depends on the length of time during which investments are made. The circle area depends on the radius of the circle. The distance traveled by the object depends on the past tense. In each case, the value of one variable amount, say y, depends on the value of another variable amount, which we often call x. We say that y is a function x and write it symbolically, like y q (x) (y equals y x). The symbol represents a function, the letter x is an independent represent-ing input variable to th, and u is a dependent variable or output value th on x. DEFINITION function from set D to set Y is a rule that assigns a unique value (x) to Y to each X to each in D. Set D of all possible input values is called the function area. A set of all output values x (x) as x varies throughout the D called function range. The range may not include every element in the Y set. (In chapters 13-16, we will encounter features for which set elements are dots in a plane or in space.) Often the function is given by the formula, describing how to calculate the value of output from the variable input. For example, the formula A and pr2 is a rule that calculates the A circle area from its r radius. Chapter 1 Features x f f(x) is the largest set of real x-values for which the formula gives real u-values. This is called Input Output natural domain. If we want to somehow limit the domain, we have to say so. Domain y x2 is a whole set of real numbers. To limit the area of func- (domain) tion to, say, positive x values, we would like to write y(x2, x 7 0. FIGURE 1.1 A diagram showing the amusing change in domain to which we apply the formula usually changes the range. tion as a kind of machine. The range of y x2 is 0, q). The range of y x2, x Ú 2 is a set of all numbers obtained by square numbers, more or equal 2. In the natural notation (see annex 1), x f (x) range is 5x2, x 26 or 5y , y 46 or 3 4, q). 4. When the feature range is a set of real numbers, the function is considered real. Domains and ranges of the most real features we look at are intervals or interval combinations. Sometimes the range of a function is not so easy to find. A function is like a machine that produces output in its range whenever we feed it with the input x from the domain (Figure 1.1). The function keys on the calculator give an example of function as a machine. For example, the 2x key on the calculator gives a output value (square root) whenever you enter the non-non-essential number x and press the 2x key. Each associ-ates arrow to the D element is one element in the Y set. Note that func-tion may have the same output value for two different inputs in the domain (as is the case with no(a) in figure 1.2), but each x input is assigned one output value (x). A set of Y domains and a set containing the FIGURE 1.2 range feature from the D EXAMPLE 1 set Check natural domains and related ranges of some simple func- to the Y set assigns a unique Y tions element. Domains in each case are x values for which the formula makes sense. to each item in D. Functional domain (x) Range (y) y x2 (-q, q) 3 0, q) y 1'x (-q, 0) ∪ (0, q) (q, 0) ∪ (0, q) y 2x 3 0, q) 3 0, q) y 24 - x (q, 44 3 0, q) y 21 - x2 3 -1, 14 3 0, 14 Formula Solution y x2 gives real y value for any real number x , thus, the domain (-q, q). The range y x2 is 3 0, q), because the square of any real number is not negative, and each non-negative y number is a square of its own square root: y 1 2y22 for y Ú 0. Formula y q 1'x gives a real y value for every x except x 0. For consistency in the rules of arithmetic, we cannot divide any number by zero. y q 1'x, a set of all non-zero real numbers, as y 1 1(1'y). That is, for y ≠ 0 number x y 1'y is an entry that is assigned a output value y. Formula y 2x gives a real y-value only if x q 0. The y 2x range is 3 0, q), because each non-non-animal number is a square of its own square). In y No 24 - x, the number 4 - x can not be negative. That is, 4 - x No 0, or x ... 4. The formula gives non-indelig ive real u-values to all x... 4. Range 24 - x is 3 0, q), a set of all non-aggregate numbers. Formula y 21 - x2 gives a real y-value for each x in a closed interval from -1 to 1. Outside of this domain, 1 - x2 is negative and its square root is not a real number. Values 1 - x2 vary from 0 to 1 in a given domain, and the square roots of these values do the same. The range of 21 - x2 is 3 0, 14 .1.1 Features and their graphics 3 Graphics features If it is a feature with Domain D, its graphs consists of points in the plane of the Cartes, the coordinates of which are the entry-exit pairs for . In the set of notations, the graph is 5 (x, x)), x∈D6. The q(x) function graph - x No. 2 - is a set of points with coordinates (x, y), for which y and 2 x. His graph is a straight line sketched in figure 1.3. The function graph is a useful picture of its behavior. If (x, y) is the point on the graph, then y q (x) is the height of the chart above (or below) the point x. Height can be positive or negative, depending on the mark no (x) (Figure 1.4). y f (1) f (2) y x'2 x 2 0 12 x f (x) x x2 (x, y) -2 4 -2 0 -1 1 FIGURE 1.3 Chart q(x) x 2 FIGURE 1.4 If (x, y) lies on the graph on 00 is a set of points (x, y), for which there is 11 of , then the value of y q (x) is the value of height x and 2. graph above point x (or below x 39 24 if th (x) negative). 24 EXAMPLE 2 Schedule functions y x2 and y x2 during the interval of 3 -2, 24 . y Solution Make a table of xy-pairs that satisfy the equation y and x2. The plot of dots (x, y) whose coordinates appear in the table, and draw a smooth curve (marked with its equation) through the built dots (see figure 1.5). How do we know that the y and x2 graph doesn't look like one of these curves? (-2, 4) 4 (2, 4) y y y x2 3 432 , 49 (- 1, 1) 2 (1, 1) y x2? y x2? 1 -2 -1 0 12 x FIGURE 1.5 x feature graph in example 2. To find out, we could build more points. But how would we then connect them? The main question still remains: How do we know for sure what the graph looks like between the points we build? Calculus answers this question, as we will see in Chapter 4. Meanwhile, we will have to agree to build points and connect them as best we can.4 Chapter 1 Features representing features numerically We've seen how the feature can be presented algebraically and visually on the schedule (Example 2). Another way of presenting a function is numerically through a table of values. Numerical representations are often used by engineers and experimental scientists. You can get a graph from the corresponding value table using a method illustrated in example 2, possibly by a computer. A graph consisting only of dots in a table is called scatterplot. EXAMPLE 3 Musical notes are waves of pressure in the air. Data associated with Figure 1.6 gives a recorded pressure shift compared to the time in seconds of the music note produced by the tuning fork. The table provides an idea of the pressure function (in micro-shepherds) over time. If we first make a scatter plot, and then connect the data points (t, p) from the table, we get the graph shown in the picture. Time Pressure Time Pressure p (pressure mPa) 0.00091 - 0.080 0.00362 0.217 1.0 Data 0.00108 0.200 0.00379 0.480 0.8 0.001 0.002 0.003 0.004 0.005 0.006 t (sec) 0.00125 0.480 0.00398 0.681 0.6 0.00144 0.693 0.00416 0.810 0.4 0.00162 0.816 0.00435 0.827 0.2 0.00180 0.844 0.00453 0.749 0.00198 0.71 0.00471 0.581 −0.2 0.00216 0.603 0.00489 0.346 −0.4 0.00234 0.368 0.00507 0.077 −0.6 0.00253 0.00525 - 0.164 0.00271 0.099 0.00543 - 0.320 FIGURE 1.6 A smooth curve through the plotted points 0.00289 - 0.141 0.00562 - 0.354 gives a pressure function graph provided by 0.00307 - 0.309 0.00579 - 0.248 accompanying data presented (Example 3). 0.00325 - 0.348 0.00598 - 0.035 0.00344 - 0.248 - 0.041 Vertical Line Test for function Not every curve in the coordinate plane can be a function graph. The function can only have one value (x) for each x in the domain, so no vertical line can cross the function graph more than once. If a is in the function domain, the x a vertical line will cross the q graph at one point (a, q(a)). A circle cannot be a function graph, as some vertical lines cross the circle twice. The circle, in the graph in figure 1.7a, however, contains graphs of two functions x, namely the upper semicircle defined by the function q (x) 21 - x2 and the lower half-circle, defined by the function g(x) - 21 - x2 (figures 1.7b and 1.7c). Piecewise-Defined Features Sometimes the feature is described piece by piece using different formulas on different parts of its domain. One example is the absolute value function 0 x 0 - e-xx, x 00 First Formula x 6 Second Formula1.1 Features and their Graphics 5 yyy 1x 1x -1 1 x -1 1 x -1 0 1 0 (a) x2 y2 - 1 (b) y -1 - x2 (c) y q - x2 FIGURE 1.7 (a) Circle is not a feature schedule; it does not perform a vertical line test. b) Up-to-semi-circle is the q(x) no 21 - x2. (c) The lower semicircle is a graph of function g(x) - 21 - x2. y 0 x0, the graph of which is shown in figure 1.8. The right side of the equation means that the y'x function is x if x 0, and equals -x if x 6 0. Parts-defined functions often occur when you model real data. Here are some other examples. 21 EXAMPLE 4 Feature -3 -2 -1 0 1 2 3 x FIGURE 1.8 The function of the absolute value of x71 The third formula has a domain (q, q) and 1, range 30, q), is determined on the whole real line, but it matters given by different formulas, depending on y y f (x) position x. Values q given y - x when x 6 0, y x2 when x's 0 ... 1, and have No.1 when x 7 1. The feature, however, is just one feature whose y'1 domain is the entire set of real numbers (Figure 1.9). 2 y 2 12 x EXAMPLE 5 Function, whose value in any amount x is the greater whole number less or equal is called the greatest function of the whole floor function or whole whole floor functions. The value of this function is its -2 -1 0 denoted: . Figure 1.10 shows the graph. Note that FIGURE 1.9 For show the function graph :2.4; 2, :1.9; 1, :0, - 1.2 ; 2-, 2; :2; 2, 0; 0, -0.3; -1, :- 2; - y q (x) shown here, we apply different formulas to different parts of its domain EXAMPLE 6 Function, the value of which in any amount of x is the smallest integer more than or equal to x called the least integrative function or integrator ceiling func- (Example 4). Tion. It means x. Figure 1.11 shows the graph. For positive x values, this y function can represent, for example, the cost of parking x hours in the parking lot, which charges $1 y'x for every hour or part of an hour. 3 2 Increase and decrease 1 y:x; If the function graph rises or rises when moving from left to right, we say that the function -2 -1 123 x increases. If the graph falls or falls when you move from left to right, the function decreases. -2 FIGURE 1.10 Chart of the Greatest DEFINITIONS Let the here be a function defined at the I interval and let x1 and x2 be two different points in the I. integer y:x function; lies on or below the y and x lines, so it provides the integrator 1. If in the case of x1 6 x2, then x1 is said that on I. The floor for x (example 5) increases by 7 k (example 5). 2. If y (x2) 6 (x1) whenever x1 6 x2, it is said to be reduced to I.6 Chapter 1 Features u It is important to understand that the definitions of increase and decrease functions should be satisfied for each pair of x1 and x2 points in I with x1 6 x2. Because we use inequality 3 y'x 6 to compare function values, not ... sometimes say that 2 is strictly increased or decreased by I. Interval which can be finite (also called limited) or 1 y x infinite (unlimited). -2 -1 123 x EXAMPLE 7 Feature, on graph on figure 1.9, decreases by 0) and -1 increase by (0, 1). The function does not increase or decrease at the interval of 2 (1, q), because the function is constant at this interval, and therefore the strict disparity in the definition of increase or decrease is not satisfied by (1, q). FIGURE 1.11 Chart of the least even functions and odd functions: The symmetry of the integrative function y - x lies on or above the charts of even and odd functions have special properties of symmetry. line y x, so it provides the integrator ceiling for X (Example 6). DEFINITIONS Function y (x) is an even function x if q (-x) y (x), odd function x if q (-x) -y (x), for each x in the domain function. (x, y) y Names even and odd come from forces x. If you have an oval force x, as in (x, -y) y x2 or y x4, it's an oval function x because (- x)2 and (- x)4 and x4. If you have th x2 odd power x, as in y y y x or y x3, it's a strange function x because (x)1 - x and (- x) 3 - x3. 0x The graph of the highlighted function is symmetrical in relation to the x-axis. Since q (-x) q (x), a (a) point (x, y) lies on the chart, if and only if the point (-x, y) lies on the graph (figure 1.12a). The axis reflection leaves the chart unchanged. y y x3 (x, y) The graph of the strange function is symmetrical about the origin. Since q (-x) -q (x), the point (x, y) lies on the graph, if and only if the point (-x, -y) lies on the graph (figure 1.12b). 0x Equivalent, the graph is symmetrical about the origin if the rotation of 180 about origin leaves the chart unchanged. Note that the definitions mean that both x and -x should be (b) in q. FIGURE 1.12 (a) Graph y x2 EXAMPLE 8 Here are a few features illustrating definitions. (even function) is symmetrical in relation to the function x2 Even: (- x)2 and x2 for all x; symmetry about the y-axis. So on the axis. (b) Graph y x3 (odd function) is symmetrical about origin. W (-3) No 9 th (3). Changing the x sign doesn't change the meaning of the ovary function. x (x) - x2 1 Oval function: (- x)2 - 1 x2 - 1 for all x; symmetry about the y-axis (figure 1.13a). x (x) x Odd function: (-x) - x for all x; symmetry about the origin. Thus, No (-3) - -3, while No (3) - 3. Changing the x sign changes the sign of a strange function. x (x) - x 1 Odd: x - x 1, but -y(x) - x - 1. Two are not equal. Not even: (-x) 1 ≠ x 1 for all x ≠ 0 (figure 1.13b).1.1 Features and their graphics 7 y y2 y x x'x'1 1 1 1 1 1 10x 1 y'x-0 x (a) (b) FIGURE 1.13 (a) When we add the permanent term 1 to the function y x2, the resulting function y x2 and 1 is still oval and its graph is still symmetrical about x-axis. b) When we add the permanent term 1 to the y x function, the resulting function y x y 1 is no longer strange, as the symmetry about origin is lost. Teh y y x No. 1 also not even (Example 8). Common features Various important types of functions are often found in calculus. View linear features A function in the form of q(x) and mx b, where m and b are fixed con-stents, is called linear function. Figure 1.14a shows an array of q(x) and mx lines. Each of them has b No 0, so these lines run through the origin. The q(x) function, where m No. 1 and b 0 is called the identification function. Constant functions result in the tilt of m No 0 (figure 1.14b). m-3 y - 3 m-1 m-2 y y 2 y 1 y 1 1 2 y 1 m x 2 y y 3 2 x x 0 1 01 2 x (b) FIGURE 1.14 (a) Lines through with a slope m. (b) A permanent funk-tion with a slope of m No 0. DEFINITION Two variables y and x are proportional (to each other) if one of the them is always a permanent multiple of the other, that is, if q kx for some non-grain constant k. If the variable y is proportional to the reciprocal 1'x, it is sometimes said that y is inversely proportional x (because 1'x is the multiplier reverse side x). Power A (x) and xa, where the constant is called a power function. There are several important cases to consider.8 Chapter 1 Features (a) q(x) xn, n positive integer. Figure 1.15 shows graphs No (x) xn, n No. 1, 2, 3, 4, 5. These func-tions are defined for all real x values. Note that as the n power gets larger, the curves tend to align to the x-axis at the interval (-1, 1) and rise more steeply for 0 x 0 7 1. Each curve passes through the point (1, 1) and through origin. The graphs of functions with oats are symmetrical in relation to the axis; those with the odd powers are symmetrical about origin. Even features are reduced to the interval (-q, 0) and are increased throughout the real line (-q, q). y y'x y y - x2 y y x3 y y - x4 y y x5 11111 -1 0 1 x -1 0 1 x -1 x -1 1 x -1 1 1 0 1 x -1 0 1 x -1 0 1 x -1 -1 -1 -1 FIGURE 1.15 Charts q (x) 2, 3, 4, 5, defined for - q 6 x 6 q. (b) q(x) x with a q - 1 or q - 2. The q (x) function features x-1 and 1'x and g/x' features x-2 are shown in 1.16. Both features are defined for all x ≠ 0 (you can never split by zero). Graph y q 1'x is hyperbole xy No. 1, which approaches the xes of coordinates away from the source. The graph of y q 1'x2 is also approaching the coordinate theds. Function graph - symmetrical about origin; Decreases at intervals (-q, 0) and (0, q). The g function graph is symmetrical to the x-axis; g increases by (-q, 0) and decreases by (0, q). y y y y 1 x2 1 01 x 1 x Domain: x q 0 Range: y 01 x Domain: x q 0 Range: y 7 0 FIGURE 1.16 Charts power functions q (x) xa a) a - 1, b) a - 2. (c) a No 1 , 3, and 2 2 3 2 Features q (x) - x1 and 2x and g(x) - x1 and 23 x are square root and cubic root functions, respectively. The square root function area is 3 0, q), but the cube root function is defined for all real x. Their graphs are shown in figure 1.17, along with graphs y and x32 and x 2'3. (Recall that x32 (x12)3 and x23 (x1)3)2 Polynomial Function p is polynomial if p(x) - anxn + an - 1xn - 1 q a1x a0, where n is a non-non-binding integrator, and numbers a0, a1, a2, e, are real constants (so-called polynomial coefficients). All polynomials have a domain (-q, q). If the1.1 Features and their graphics 9 y y y y y y y y y x 3 2 y 1'3 x 2'3 1 01 x 0 1 01 x 01 x Domain: 0 ... x6 q Domain: 0 ... q 6 x 6 q Domain: 0 ... x 6 q Domain: - q 6 x 6 q Range: 0 ... y 6 q Range: 0 ... y 6 q Range: 0 ... y 6 q FIGURE 1.17 Power Graphics q(x) xa for for y y 1, 1, 3, and are 2 2 3 leading coefficient n 0, then n is called the degree of polynomial. Linear functions with m ≠ 0 are grade 1 polynomials. Degree 2 polynomials, commonly written as p(x) and ax2 and bx and C, are called square functions. Similarly, cubic functions are polynomial p(x) - ax3 and bx2 - cx and d degree 3. Figure 1.18 shows the graphs of the three polynomials. The methods of the polynomial graph are studied in Chapter 4. y x3 - x2 - 2x 1 3 2 3 4 y y y (x - 2)4 (x 1)3 (x - 1) y y 8x4 - 14x 3 - 9x 2 - 11x - 1 16 2 x 2 x 2 2 2 2 2 2 2 2 x - 2 2 -1 -2 2x -2 -2 0 -4 -10 12 x 2 -6 -6 -10 -1 x2 (x) -4 -12 (a) (b) FIGURE 1.18 Charts of three polynomial functions Rational function Rational function is a coefficient or coefficient q (x) q q (x) (x), where p and q are polynomials. The rational function area is a set of all the real x for which q(x) ≠ 0. Graphs of several rational functions are shown in figure 1.19. y y y 8 11x 2 6 2x3 - 1 y y 5x2 - 8x - 3 y - 2 y 2 3x2 - 2 4 2x 4 7x 4 -2 2 4x -5 0 5 10 -2 -4 -2 2 4x -2 -2 -4 - 2 -2 4 x2 - 4 2 -6 -8 -2 -4 -6 NOT TO SCALE -4 -6 -8 (a) (b) (c) FIGURE 1.19 Charts of three rational functions. The straight red lines that the charts approach are called implementations and are not part of the charts. We discuss askiptots in section 2.6.10 Chapter 1 Features Algebraic features Any function built from polynomials using algebraic operas (supplementation, subtraction, multiplication, division, and roots) lies in the class of algebraic functions. All rational functions are algebraic, but also included more complex functions (such as those satisfying the equation, such as y3 - 9xy and x3 No. 0, explored in section 3.7). Figure 1.20 shows the graphs of three algebraic functions. y y y x1'3 (x - 4) 3 (x2 1)2'3 y x(1 - x)2'5 4 y 3 2 y 1 1 -1 0 51 x 51 -1 -1 -2 -7 x 4 1 0 1 1 -1 -3 x (a) (b) (c) FIGURE 1.20 Graphs of three algebraic functions. Trigonometry features Six main trigonometric functions are covered in section 1.3. The blue and oblique graphs are shown in figure 1.21. y y 3p - p 2 3p 5p x 2 2 2 p - p 0 2p 0p -1p -1 2 (a) f(x) - sin x (b) f (x) - cos x FIGURE 1.21 Charts features sinus and cos. Figure 1.22 exponential functions (with base A). All exponential functions have a domain (-q, q) and a range (0, q), so the exponential function never assumes a value of 0. We develop the theory of exponential functions in section 7.3. The graphs of some of the exhibit-tial features are shown in figure 1.22. Y y y 10x y y 10-x 12 12 10 10 88 6 y 3x 6 4 y 3x 4 2 y 2x y 2x 2 0.5 1 1 x x 1x - 1 - 0.5 - 1 - 0.5 0 0.5 0 (a) (b) FIGURE 1.22 Graphics of exponential functions.1.1 Features and their graphics 11 Logarithmic functions These functions q(x) where the base ≠ 1 is a positive constant. They are the reverse functions of exponential functions, and we define and develop the theory of these functions in section 7.2. Figure 1.23 shows graphs of four logarithmic functions with different bases. In each case, the domain (0, q) and the range (-q, q). y y y log2 x y - log3x 01 x 1 - log5x 1 - y - log10 x -1 0 1 x FIGURE 1.23 Charts four log-FIGURE 1.24 Graph of catenar or rhythmic functions. hanging cable. (The Latin word for catena means chain.) Transcendental Functions Are functions that are not algebraic. These include trigonometry, reverse trigonometry, exponential and logarithmic functions, as well as many other functions. The nursery is one example of transcendental function. Its graph is shaped like a telephone line or electric cable, string from one support to another and hangs loosely under its own weight (Figure 1.24). The graph-defining feature is discussed in section 7.7. Exercise 1.1 Features 8. a y b. y In exercises 1-6, find the domain and range of each function. 1. W (x) 1 and x2 2. 1 - x 3. F (x) 25x and 10 4. g(x) 2x2 - 3x 5. W (t) 3 4 1 6. G(t) 2 2 16 - - In exercises 7 and 8, which of the graphs are graphs of functions x, xs, and which not? Give reasons for your answers. 0O 7. a. y b. y Search for formulas for functions 9. Express the area and perimeter of the equilateral triangle as x xx 00 function of the lateral length of the triangle x. 10. Express the side length of the square as a function of the length of the d diagonal square. Then we will express the area as a diagonal length function. 1 1. Let's express the length of the edge of the cube as a function of diagonal length of the cube d. Then express the surface area and volume of the cube as a function of diagonal length.12 Chapter 1 Features 12. A P в первом квадранте лежит на графике функции 31. a. y b. y q(x) 2x. Выразив координаты P как функции наклона линии, присоединяя P к источнику. (- 1, 1) (1, 1) 2 (1, 2) x 13. Рассмотрим точку (x, y), лежащую на графике линии 2x и 4y й 5. Пусть L будет расстоянием от точки (x, y) до x 1x происхождения (0, 0). Напишите L как функцию x. 14 Рассмотрим точку (x, y), лежащую на графике y 3x - 3. Пусть (- 2, - 1) (1, - 1) (3, - 1) L будет расстояние между точками (x, y) и (4, 0). Напишите L как функцию y. 3 2. a. y b. y 1 T (1) Функции и графики 0 A Найти естественный домен и графики функций в упражнениях 15-20. 1 5. 3(x) 5 - 2x 16. 3(x) 1 - 2x - x2 0 T 3T 2T t -A 2 2 17. g(x) 3 0 x 0 18. g(x) 3- x 0 T 0 2 F(t) 0 T 0 0 20. Γ(t) 0 T 0 2. Найдите домен y 4 x 3 . 2 5 Величайшие и наименее интегративные функции 22. Найдите домен y No 2 и 2 р x 3 2. 23. Как какие значения x 2 3. На графике смещение уравнений и объяснить, почему они не a. 'x- y 2 9 0. 8 - x2 + 1. b. x = the results... what= real= numbers= x= satisfy= the= equation= x= <x>? <x> = ... (continued)

Give examples. What does the increasing function mean? The reduction function? Here is an example of each of them. 1 1. What is a radian measure? How do you turn from a radian to a de-gri? Degrees to the radiant? 6. What is an oval function? Strange feature? What kind of prop symmetry has such features? What advantage can 1 2. There are six main trigonometry functions on the chart. What symmetries do we take from this? Give an example of a feature that no graphics have? It's not weird. 1 3. What is a periodic function? Give examples. What are periods 7. If the functions that are real functions, how are the areas of the six main trigonometry functions? G, z - d, d, and ug, related to the domains th and d? Give examples.34 Chapter 1 Features 1 4. Starting with the identity of sin2 u and cos2 u 1 and compression formulas, and reflection Schedule? Give examples. A graph for cos (A and B) and Sin (A and B) show how different other common sinus curves and identity constants A, B, C and D. Trigonometry identities can be obtained. 1 6. What are the three problems that occur when you schedule functions with 1 5. As a formula for the general function of sinus (x) - a calculator or computer with graphic software. Give examples. Sine (Ω-p) (x - C)) - D refer to displacement, stretching, chapter 1 Practice Exercise Features and Graphics 33. The state of whether each function increases, decreases or does not. 1. Express the scope and circumference of the circle as functions. A The volume of the sphere as a function of the radius of the circle. Then express the area as a circle function. B. The biggest function is integer 2. Let's express the radius of the sphere as a function of the surface of sphere C. Height above the Sea level of the Earth as a function of the atmospheric pressure (supposed nonzero) 3. Point P in the first quadrant lies on parabola x2. Express D. Kinetic energy as a function of the speed of the P particle as a function of the angle of the P line of accession to origin. 3 4. Find the largest interval at which this feature increases. a. q(x) 0 x - 2 0 and 1 b. (x) Balloon rising directly from field level, tracked c. g(x) - (3x - 1)'3 d. R (x) - 22x - 1 finder range located 500 feet from the starting point. Express the height of the balloon as a line angle function from the Piecewise-Defined Finder range to the balloon does with the ground. In exercises 35 and 36, find the domain (a) and b) range. In exercises 5-8, determine whether the schedule of the function is SIM- 3 5. y e 2- x, -4 ... X... 0 metrics about y-axis, origin, or neither. 2x, 06x... 4 5. y y x1 x 5 6. y y x2 -5 -x - 2, -2 ... x... -1 7. y x2 - 2x - 1 8. y e-x2 36. y c x, -1 6 x ... 1 In exercises 9-16, determine whether the function is even, odd, or -x No 2, 16x... 2 neither. 9. y x2 1 10. y x5 - x3 - x In exercises 37 and 38, write in parts formula for function. 11. y y 1 - cos x 12. y sec x tan x 37. y 38. y 1 3, y x4 21x 14. y x - sin x 1 5 (2, 5) x3 - 1 5. y q and cos x 16. y q x cos x 17. Suppose m and g are odd functions defined throughout the real line x. Which of the following (where are defined) are even? Weird? 0 12 a.m. 3 s. The question (sin x) d. g(sec x) e. 0 g 0 0 4x 1 8. If q (a - x) - q (a x), show that g/x is an even (x) - this is an naya feature composition of functions. In exercises 39 and 40, find in exercises 19-32, find (a) domain and (b) range. 19. x - 2 20. y q 2 - y 21 - x a. (I + g) (-1) b. (Mr. ◦) c. (I ◦) (x). d. (Mr. ◦ d) (x). 23 3. y 2e-x - 3 24. y th (2x - p) 3 9. X (x) 1x, g(x) 1 2 5. y 2 sin (3x and p) - 1 26. y x2'5 27. y q cos (x - 3) 1 28. y - 1 y 2 1 23 2 - x 40. X (x) 2 - x, g (x) 23 x 1 2 9. y q 2 and 3x2 In exercises 41 and 42 a) write formulas for q ◦ g and g ◦ and find (x) and b) domain and (c) range each. 31. 4 sims a1xb 32. y 3 cos x 4 sin x 4 1. 2 - x2, g(x) - 2x - 2x (Hint: Trigger identification 4 2. Chapter 1 Extra and Advanced Exercises 35 For Exercises 43 and 44, Sketch Graphics and ◦ 5 5. y - A1 and x 56. y q 1 - x 2 3 - x - 2, -4... x ... -1 4 3 z(x) - c - 1, -1 6 x ... 1 57. y Nos 1 and 1 58. y - (5x)1'3 2x2 x - 2, 1 6x. 2 4 4. h (x) x No 1, -2 ... x ... x 6 0 Trigonometry bx - 1, 0... X... 2 In exercises 59-62 we will draw a graph of this function. What is the function period? Composition with absolute values in exercises 45-52, chart number 1 5 9. y cos 2x 60. y sin x and No2 exercises 61. y q(x) 62. y - p sin px 63. Sketch of the graphic at 2 cos ax - p3 b. 4 5. x 0 x 0 4 6. x2 x0 x 0 2 6 4. Sketch graphics y th 1 y y sin x p4 b . 4 7. x3 0 x3 0 Find answers 65-68, ABC is the right triangle with a straight angle on C. Parties of opposite angles A, B and C are A, B and C respectively. 4 8. x2 x 0 x2 0 6 5. a. Find A and b if c q 2, B and p'3. 6 5 0. 1 1 0 6 6. A. Find A and C if b No 2, B and p'3 6 5. 0 1. A. Let's express in terms of A and c. 0 x x2 x x2 and b. Let's express A in terms of A and b. 5 1. 2x 3 2x 0 67. A. Express from the point of view of B and h. 5 2. sin x sin 0 x 0. Express C in terms of A and A. Shift and scale Charts 5 3. Suppose you're given a graph g. Write equations for graphs at 5 4. Express sin A in terms of a and c. flecting as stated. 6 9. The height of the pole Two wires extend from the top of the T vertical pole to points B and C on the ground, where the C is 10 m closer to. Up to 1 unit, right 3 base pole than B. If the BT wire makes angle 35 with horizontal and wire CT makes angle 50 with 2 horizontal as high pole? B. Down 2 units, left 3 70. The height of the meteorological ball Observers at positions A and B c. Reflecting on the y-axis 2 km apart simultaneously measure the angle of the height of the weather balloon to be 40 and 70 respectively. If the balloon D. Reflecting on the x-axis directly above the point on the segment of the line between A and B, find the height of the ball. E. Stretch vertically to 5 T 71. a. Graph of function q (x) - sin x and cos (x-2). F. Squeeze 5 b. What seems to be the period of this function? 54. Describe how each graph is derived from the y q (x) graph. C. Confirm your find analytically. 72. a. y q (x - 5) b. y (4x) T 7 2. a. Graph q (x) - Sin (1'x). B. What is a domain and a range of th? c. y q (2y x) 1 c. Is y periodic? Give reasons for your response. e. y qa3xb - 4 f. y - 3' (x) - 1 4 In Exercises 55-58, graph each function, not by building dots, but starting with the graph of one of the standard functions presented in drawings 1.15-1.17, and applying the appropriate transformation, Chapter 1 Extra and Advanced Exercise Features and Graphics 3. If q (x) is strange, is there anything to be said for g(x) q (x) - 2? What to do if No.1. Are there two features and g such that ◦ g q - th? Give Rea- even instead? Give reasons for your response. sons for your answer. 4. If g(x) is a strange feature defined for all x values, is there anything you can say about g(0)? Give reasons for your response. 2. Are there two features: q with the following property? Charts No and G are not straight lines, but the chart ◦ g is 5. The equation graph is 0 x 0 y 0 and 1 x x straight line. Give reasons for your response. 6. On the graph, the equation y and_y_x and_x_36 Chapter 1 Features Derivatives and Evidence Geometry 15. The center of the mass of the object moves at a constant speed of y along 7. Prove the following personalities. straight line past Origin. The accompanying figure shows a. 1 - cos x - sin x b. 1 - cos x and tan2 x coordinate system and line of motion. The points show the positions sin x th cos 1 and cos x 2, which are 1 sec apart. Why are the A1, A2, C, A5 areas in Figure 1 all equal? As in Kepler's Equal Area Act (see section 13.6), the line that connects the mass center of an object to the origin sweeps out equal to 8. Explain the following proof without words law cosines. areas at the same time. (Source: Kung, Sidney H., Proof Without Words: Law co-synes, Mathematics Magazine, Volume 63, No. 5, December 1990, p. 342.) y 2a cos u - b 10 t 6 a-c t'5 cb u Kilometers aa a a5 y't 5 A4 y't A3 t'2 9. Show that the area of the ABC triangle is given (1'2)ab sin C (1'2)bc sin A (1'2)ca sin B. A1 t 1 C ba 0 5 10 15 x Kilometers A CB 16. a. Find the slope of the line from the beginning to the middle of the AB side in the triangle in the accompanying shape (a, b 7 0). 10. Show that the area of the ABC triangle is given 2s (s - a) (s - b) (s - c), where s (a q d2) is y semiperimeter triangle. B (0, b) 11. Show that if q is both even and strange, q (x) 0 for each x in the P 12 domain. a. E ven-odd decomposition Let and be a function whose do-0 A(a,0) x core is symmetric about origin, that is - x belongs to the domain when x does. Show that amount even. b. When OP perpendicular to AB? function and odd function: 17. Consider the quarter circle of radius 1 and the right triangles ABE and q (x) - E(x) O (x), ACD, given in the accompanying figure. Use the standard formu-las area to conclude that where E is an even function and O is a strange feature. (Hint: Let E (x) Show that E (-x) - E(x), so t sin u 6 u 6 1 sin u. . . that E is oedus. 2 2 2 cos u. b. Uniqueness Show that there is only one way to write, as do the sum of even and odd features. (Hint: One way is given in partially (a). If also q (x) - E1 (x) - O1 (x), where E1 (0, 1) C even and O1 strange, show that E - E1 and O1 - O1 - O. Then use B Exercise 11 to show that E and E1 and O O.) 1 Influence of parameters on T 1 3 graphics. What happens to the graph y ax2 and bx c as u D x A E (1, 0) a. changes while b and c remain fixed? 1 8. Let q (x) - axe - b and g(x) - cx and d. What condition should b. change (a and c fixed, ≠ 0)? be satisfied with constants a, b, c, d in order (no ◦ g) (x) c. c. changes (a and b fixed, ≠ 0)? (◦) (x) for each x value? T 14. What happens to the graph y a (x-b)3 and c as a. changes, while b and c remain fixed? b. b changes (a and c fixed, ≠ 0)? c. c change (a and b fixed, ≠ 0)? Chapter 1 Technology Applications Projects 37 Chapter 1 Technological Projects application Mathematica/Maple Projects Projects can be found in MyMathLab. The Mathematica Review Mathematica is sufficient to complete the Mathematica modules appearing on the website. Simulation changes: springs, driving safety, radioactivity, trees, fish and mammals build and interpret mathematical models, analyze and improve them, and make predictions using them. Photo The opening chapter: Lebrecht Music and Arts Photo Library/Alamy Stock Photo.2 Limits and Continuity OVERVIEW In this chapter, we develop the concept of the limit, first intuitively and then formally. We use limits to describe how the feature changes. Some functions change incrementce-ujously; small changes in x produce only small changes in th (x). Other functions may have values that jump, change erratically, or tend to increase or decrease without borders. The concept of limit provides an accurate way to distinguish between these behaviors. 2.1 Speed change and tangent lines to curves HISTORICAL BIOGRAPHY Medium and instant speed Galileo Galileo At the end of the sixteenth century, Galileo found that a solid object fell from the rest (1564-1642) (initially does not move) near the surface of the earth and allowed to fall freely at a distance bit.ly/2OpdNBs proportion to the square of the time it fell. This type of movement free fall. It involves a slight air resistance to slow the object down, and that gravity is the only force acting a falling object. If u denotes the distance that fell into the legs after t seconds, then the law of Galileo y y 16t2 feet, where 16 is (approximately) a constant of proportionality. (If instead y is measured in meters, the constant is close to 4.9.) More generally, let's assume that a moving object has traveled a distance of t/t during t. The average speed of an object in the time interval of 3 t1, t2 4 is by dividing the distance traveled by t2 - t1. A unit of measurement is length per unit of time: kilometers per hour, feet (or meters) per second, or something corresponding to the problem. Average Speed When (t) measures the distance traveled during t, Average speed more than 3 t1, t2 4 - distance traveled - q (t2) - t (t1) passed time t2 - t1 EXAMPLE 1 Rock breaks from the top of a high cliff. What is its average speed (a) during the first 2 seconds of the fall? (b) Within one-second interval between the second one and the second 2? 382.1 Speed change and tangent lines to curves 39 Δ is the capital of the Greek letter Delta Solution The average speed of a cliff during a given time interval is a change of distance, Δy, divided by the length of the time interval, Δ Δ Δt. (The capital of the Greek letter Delta, Δ, is traditionally used to refer to increments, or changes, in variative. We have the following calculations: a) For the first 2 secs: Δy 16(2)2 - 16(0)2 and 32 feet (b) Sec 1 to sec 2: Δt 32 ft (b) From sec 1 to sec 2 Δt 33 2 - 0 sec Δy 16(2)2 - 16 (1)2 and 48 sfetc Δt 2 - 1 We want a way to determine the speed of a falling object at an instant t0 instead of using its average speed over the interval of time. To do this, we investigate what happens when calculating the average speed over shorter and shorter intervals, starting with t0. The following example illustrates this process. Our discussion is informal here, but will be made accurate in Chapter 3. EXAMPLE 2 Find the speed of the falling rock in example 1 on t 1 and t 2 sec. Solution We can calculate the average speed of a rock over a period of time 3 t0, t0 and h 4, having a length of Δ t th (t0 and h) - (t0) h as Δy 16 (t0) h)2 - 16t0 2 Δt sec. (1) We can't use this formula to calculate instant speed at the exact t0 moment, simply by replacing h No. 0 because we can't divide by zero. But we can use it to calculate average speeds over shorter and shorter times, starting at t0 1 or t0 and 2. When we do this by taking less and less of the h value, we see a pattern (table 2.1). TABLE 2.1 Average speed for short periods of time 3 t0, t0 and h 4 Average speed: Δy - 16 (t0) h)2 - 16t02 Δt h Length of average speed over average speed over time interval H length interval, starting at t0 and 1, starting at t0 and 2 1 48 80 0.1 33.6 65.6 0.01 32.16 64.16 0.0.001 32.016 64.016 0.0001 32.0016 64.0016 The average speed at interval, starting at t0 No 1, seems to approach a limiting value of 32 as the length of the interval decreases. This suggests that the rock falls at a speed of 32 feet a sec at t0 and 1 sec. Let's confirm this algebraically.40 Chapter 2 Limits and Continuity If we install t0 No 1, and then expand the numerator in the equation (1) and simplify, we find that Δt 16 (1h) 2 - 16 (1)2 16 (1 - 2h h2) - 16 Δh h 32h 16h2 32 16h. Can undo h when ≠ 0 h For h values different from 0, the expressions on the right and left are equivalent, and the average speed is 32 and 16h ft'sec. Now we can understand why the average speed has a limiting value of 32 and 16 (0) and 32 feet sec as h approaches 0. Similarly, installing t0 No 2, in the Equation (1), for h values is different from 0 the procedure gives Δy 64 and 16h. ΔThe H gets closer and closer to 0, the average speed has a limiting value of 64 feet and seconds when t0 and 2 sec as suggested in table 2.1. The average rate of an object's fall is an example of a more general idea that we will discuss next. Average rates of change and Secant Lines y f (x) Given any function y q (x), we calculate the average rate of change y relative to x during the interval of x1, x2 , dividing the change of value y, Δy (x2) - q (x1), q (x2)) by the length of Δx x2 - x1 - x1 interval over which the change occurs. (We use the h symbol for Δx to simplify notation here and later.) The AVERAGE rate of change y q (x) relative to x during the interval of 3 x1, x2 4 is Secant y Δy (x2) - ≠ q (x1) P (x, f (x1)) Δx x2 - x1 h x x x 0 PL x2 x geometrically, the change rate is more than 3 x1, x2 4 is a line tilt through points P (x1, q (x1)) and No (x2, q (x2)) (Figure 2.1). In the geometric line, connect the two points of FIGURE 2.1 A secant to the curve of the chart, called the secant line. Thus, the average rate of change from x1 to x2 is identi-y q (x). Its slope is Δ f Δx, fees over x slope secant line of PL. As point P approaches the wave, change over the H-period curve over which the change occurs approaches zero. We will see that this procedure leads to the definition of a curve tilt at the point. 3 x1, x2 4. P Definition of slope curve L We know what it means by tilting a straight line, which tells us the speed at which it rises or falls- its rate of change as a linear function. But what does the curve mean at P on the curve? If there is a touchline to the curve on the P-line that FIGURE 2.2 L is on a grid that is tangent to the circle at P, it would be reasonable to determine if it passes through the P perpendicular to the slope of the tangent as the slope of the curve on the P. We will see that among all the lines of the OP radius that pass through point P, the tangent line is the one that gives the best approach to the curve on the P. We need an accurate way to indicate the tangent line at the point on the curve. Pointing a tangent line to a circle is simple. Line L on a tangent to a circle at point P if L passes through P and perpendicular to radius P (Figure 2.2). But what does it mean to say that the L line is tangent to a more general curve at point P?2.1 The pace of change and the tangent lines of curves 41 HISTORICAL BIOGRAPHY To determine tangents for common curves, we use an approach that analyzes the behavior of Pierre de Fermat from the lines of the sectarian that pass through P and nearby points as q moves to the P curve (1601-1665) (figure 2.3). Let's start with what we can calculate, namely the tilt of the bit.ly/2NRJEeC line of the PP. We then calculate the limiting value of the secant line tilt as the P approaches the curve. (We clarify the idea of a limit in the next section.) If the limit exists, we believe it is a tilt curve at P and determine the tangent line curve on the P to be a line through the P with this slope. The following example illustrates a geometric idea for finding a tangent line of a curve. Tangent Line Secant Lines P - Tangent Line Secant Lines - FIGURE 2.3 Line Curve on P - is a line through P, the slope of which is the limit of the inclines of the secant line as S P on both sides. EXAMPLE 3 Find a tangent to parabola y x2 at point (2, 4) by analyzing the slopes of secant lines through (2, 4). Write an equation for the tangent line of the parabola at this point. Solution We start with a sequent line via P (2, 4) and No (2 hours, (2 hours)2). Then we write an expression for tilting the secant P line and explore what happens to the slope as we approach the P on the curve: the slope of the Secant line - Δy (2 h) 2 - 2 h2 4h - 4 Δx h2 and 4h - 4 Δx h 4 and h If h 7 0, then q lies above and right of P, as in figure 2.4. If h 6 0, then q lies left of P (not shown). In any case, as you approach the P along the curve, h approaches zero, and the slope of the secant line approaches 4. We take 4 to be a slope parabola on the slope of the P. y Secant line is (2 x) 2 and 4 x. y x2 h (2 hours, (2 hours)2) On the tangent line No. 4 y (2 hours) 2 - 4 P (2, 4) x x x 0 2 x not SCALE FIGURE 2.4 Finding a parabola slope at x2 at point P (2, 4) as a tilt limit of the secant line (Example 3).42 Chapter 2 Limits and continuity of the Tangly Line to Parabola on P is a line through P with a slope 4: 4 and 4 (x - 2) 4x - 4. The pace of change and tangent lines the speed with which the rock in example 2 fell in moments t Nos. 1 and t 2 are called instantaneous rate of change. The instantaneous pace of change and the inclinations of tangent lines are closely related, as we see in the following examples. EXAMPLE 4 Figure 2.5 shows how the drosophila population grew in a 50-day experiment. The number of flies is counted at regular intervals, the calculated values, built in relation to the number of days that have passed, and the points connected by a smooth curve (blue in figure 2.5). Find the average growth rate from day 23 to day 45. The decision was 150 flies on day 23 and 340 flies on the 45th day. Thus, the number of flies increased by 340 - 150 and 190 in 45 - 23 and 22 days. The average rate of population change from the 23rd to the 45th day was the average rate of change: Δp 340 - 150 and 190 ≈ 8.6 flies per day. Δ 45 - 23 22 p 350 (45, 340) 300 Number of flies 250 x 190 200 150 Q (23, 150) p ≈ 8.6 flies'day 100 t 50 t 22 0 10 20 30 40 40 t Time (days) FIGURE 2.5 Growth of fruit fly population in controlled experiment. The average rate of change over 22 days is Δ Δ the slope of the secant line through points P and q on the graph in figure 2.5. The average rate of change from day 23 to day 45, calculated in example 4, does not tell us how quickly the population changed on day 23. To do this, we need to study the time intervals closer to the day in question. EXAMPLE 5 How quickly did the number of flies grow in the Sample 4 - ING population on the 23rd day? To answer this question, we examine the average rate of change over shorter and shorter time intervals, starting on the 23rd day. From a geometric point of view, we find these bets by calculating the slopes of the secant lines from P to q, for a sequence of points approaching P on a curve (Figure 2.6).2.1 The pace of change and tangent lines to the curves of 43 Slope P q t, p B (35, 350) (flies, day) No 350 (45, 340) (45, 340) 340 - 150 300 (400) (400) (400) , 330) 45 - 23 ≈ 8.6 Number of flies 250 (35, 310) 200 (30, 265) 330 - 150 ≈ 10.6 150 P (23, 150) 40 - 23 100 50 310 - 150 ≈ 13.3 35 - 23 0 265 - 150 ≈ 16.4 4 4 10 20 30 40 50 t 30 - 23 A (14, 0) Time (days) FIGURE 2.6 Positions and slopes of four secant lines through point P on the fruit fly chart (Example 5). Values in the table show that the slopes of the secant line rise from 8.6 to 16.4, as the t-coordinates of the decrease from 45 to 30, and we expect the slopes to rise a little higher as t continued to decline to 23. Geometrically, the secant lines rotate counterclockwise on P and seem to be approaching the red tangent line in the picture. Since the line seems to pass through the dots (14, 0) (35, 350), its slope is about 350 - 0 16.7 flies per day. 35 - 14 On the 23rd day the population increases at a rate of about 16.7 flies per day. The instantaneous rate of change is a value that approaches the average rate of change in the interval as the length of the h interval will change when the change approaches zero. The average rate of change corresponds to the slope of the secant line; The instantaneous velocity corresponds to the incline of the tangent at a fixed value. Thus, the instantaneous speed and slopes of the tan lines are closely related. We give an accurate definition for these terms in the next chapter, but for that we first need to develop a concept of limit. Exercise 2.1 Average rate of change tilt curve at point in exercises 1-6, find the average rate of change function more in exercises 7-18, use the method in example 3 to find (a) tilt of a given interval or intervals. curve at this point P and (b) by the tangent 1 equation. The (x) x3 and 1 line on the. a. 32, 34 b. 3- 1, 14 7. y x2 - 5, P(2, - 1) 2. g(x) x2 - 2 8. y 7 - x2, P(2, 3) a. 31, 34 b. 3- 3 4 9. y x2 - 2x - 3, P(2, - 3) 3. h(t) - cot t 10. y x2 - 4x, P(1, - 3) a. 3p-4, 3p-44 b. 3p-6, page 24 11. y q x3, P(2, 8) 4. g(t) No 2 and cos t 12. y q 2 - x3, P(1, 1) a. 30, p4 b. 3- p, p4 13. y x3 - 12x, P(1, - 11) 5. R(u) No 24u No 1; 30, 24 1 4. y x3 - 3x2 y 4, P (2, 0) 6. P/u - u3 - 4u2 and 5u; 3 1, 2 4 1 5. y No 1, P (- 2, - 1'2) x44 Chapter 2 Limits and Continuity 1 6. y 2 x x, P(4, - 2) What are the average profit growth rates between 2012 and 2014? 1 7. y q 1x, P(4, 2) c. Use your chart to estimate the rate at which profits changed in 2012. 18. 27 - x, R (- 2, 3) T 22. Make table of values for function F(x) - (x - 2) (x - 2) Instant rate change in points x 1.2, x 11'10, x 101'100, x 1001'1000, 19. The speed of the car Accompanying figure shows the time up to x 10001 x 10,000, and x 1. Distance chart for a sports car accelerating from a stop. a. Find the average F (x) change rate for intervals of 31, x4 for each x 1 in table. Distance (m) s P 650 No4 b. Extending the table if necessary, try to determine the F (x) change rate of 600 x 3 at x 1. 500 2 400 T 23. Let g(x) 2x for x Ú 0. 300 1 200 euros a. Find the average rate of change g(x) relative to x during intervals of 31, 24, 31, 1.54 and 31, 1 and h4. 100 b. Make a table of average g change rate from 5 10 15 20 t in relation to x during the interval of 31, 1 h4 for some h Elapsed time (sec) values approaching zero, say. h 0.1, 0.01, 0.001, 0.0001, 0.00001 and 0.000001. a. Assess the slopes of the secant lines of PP1, PP2, PP3 and PP4 by setting them up in order in a table similar to the table in figure 2.6. C. What your table points to is the rate of change g(x) (the appropriate units for these slopes? in relation to x on No 1? b. Then estimate the speed of the car during the t 20 seconds. d. Calculate the limit as you approach zero average speed 2 0. The accompanying graph shows the section of distance traveled compared to the change in g(x) relative to x during the interval of 31, and h4. time for an object that fell from the lunar lander dis- T 2 4. Let q (t) 1't for t ≠ 0. 80 m to the surface of the moon. a. Find the average rate of change in relation to t over. Evaluate the slopes of the secant lines of PP1, PP2, PP3 and intervals (i) from t 2 to t 3, and (ii) from t 2 to PL4, having arranged them in a table similar to the table in figure 2.6. About how quickly did the object was going when it hit the surface? B. Make a table of average change rate values against t during the interval of 32, T4 , for some values T ap-distance fell (m) y proaching 2, say T 2.1, 2.01, 2.001, 2.0001, 20.0001, 2.00001. 80 No. 4 P 60 No3 c. What does your table indicate is the rate of change from 40 x 2 t to t No 2? 20 q2 d. Calculate the limit as T approaches the average change rate in relation to t in the interval from 2 to T. You will have to make some algebra before you can replace T No 2. 25 The accompanying graph shows the total distance traveled by a cyclist after t hours. Distance traveled (mi) s 40 0 5 10 30 Past time (sec) 20 T 21. The small company's profit for each of the first five years of its 10 activities is listed in the following table: Annual profit of $1000s 0 1 2 3 4 Elapsed time (hr) 2010 6 2011 27 a. Rate the average speed of a cyclist during the 2012 2012 intervals 62 30, 14, 31, 2.54 and 32.5, 3.54. 2013 111 2014 174 b. Assess the instant speed of the cyclist at times 1 t 2, t 2, and t No 3. a. Plot items presenting profit as the function of the year, and c. Assess the cyclist's top speed and some time to join them as smooth a curve as you can. 2.2 Limit of functions and laws on limitation 45 26 The accompanying graph shows the total amount of gasoline A V. Assess the average rate of gasoline consumption above the gas tank of the car after driving for t days. time intervals 30, 34, 30, 54 and 37, 104 . The remaining amount (gal) b. Estimate the instantaneous rate of gasoline consumption 16 at a time when t 1, t 4, and t 8. 12 c. Assess the maximum rate of gasoline consumption and 8 specific time at which it occurs. 4 t 0 1 2 3 4 5 6 7 8 9 10 Past Time (days) 2.2 THE limit of function and limitation of THE laws HISTORICAL ESSAY In section 2.1 we have seen how there are restrictions when looking for an instant speed change in the function of the limit of the curve. We begin this section by presenting an informal bit.ly/2P04ZyV definition the function limit. We then describe laws that fix the behavior limits. These laws allow us to quickly calculate limits for various functions, including polynomials and rational functions. We present an accurate definition of the limit in the next section. y Limits of functional values 2 Often when studying the function of y q (x), we find ourselves interested in the behavior of func-tion near a certain point C, but not on the c itself. An important example is 1 y f (x) x2 - 1, when the process of trying to evaluate the function to by C results in a division of zero that x-1 is uncertain. We came across this when looking for an instant change rate in y, looking at the dy/h factor function for h closer to zero. In the following example, we'll figure out how a function behaves near a certain point where we can't directly assess the function. -1 0 1 EXAMPLE 1 How does the function y x2 - 1 x-1 (x) behave near x No 1? Solution The y'x'1 formula defines th for all real numbers x, except x No. 1 (since we can't divide 1 by zero). For any y ≠ 1, we can simplify the formula if we take into account the numerator and cancel the common factors: -1 0 1 x (1) x2 -1 (x-1) (x1) x 1 for x ≠ 1. FIGURE 2.7 Chart is identical to The Chart of th x x 1 with a dot (1, 2) removed. This remote point is shown as a hole in Figure 2.7. Although q(1) is not defined, it is clear that we with the line y x 1, except y 1, can make the value q (x) as close as we want 2, selecting x close enough to 1 where not defined (Example 1). (Table 2.2). Unofficial description of the function limit We now give an informal definition of the function limit - at the point of the interior of the domain. Suppose that th (x) is determined at an open interval of about c, except perhaps c46 Chapter 2 Limits and the continuity of TABLE 2.2 As x gets closer to 1, itself. If th (x) arbitrarily close to the L number (as close to L as we like) for all x suffi- th (x) gets closer to 2. ciently close to c, except for c itself, then we say that q is approaching the L limit, as x x2 - 1 approaches c, and write x'1 x (x) - lim (x) l, 0.9 1.9 xSc 1.1 2.1 0.99 1.99, which reads the limit of 2 as approaching x l. In example 1 we would say that 1.01 2.01 (x) is approaching the limit 2 as approaching x 1. , and write 0.999 1.999 1.001 2.001 lim (x) 2, or lim x2 - 1 q 2. 0.999999 1.999999 x - 1 1 1.000001 2.000001 xS1 xS1 Essentially, the definition says that the values of th (x) are close to L number whenever x is close to c. Function value by itself is not considered. Our definition here is informal because phrases such as arbitrarily close and close enough are inaccurate; their value depends on the context. (Machinist piston, close could mean within a few thousand days in. For an astronomer studying distant galaxies, closure could mean within a few thousand light-years.) Nevertheless, this definition is clear enough that we can recognize and evaluate the limits of many specific functions. We will need an accurate definition given in section 2.3 when we make a prove theorem about limitations or explore complex functions. Here are a few more ples exams exploring the idea of limitations. The EXAMPLE 2 feature limit doesn't depend on how the function is determined when you're approaching. Let's look at three features in figure 2.8. The q function has a limit of 2 as x S 1, even though it is not defined on x No. 1. Function G has a limit of 2 as x S 1, although 2 ≠ g(1). H is the only one of the three features in Fig-ure 2.8, whose limit as x S 1 is equal to its value on x No. 1. For h, we have limxS1 h(x) h(1). This equality of the value of the limit and function is important. As shown in the three examples in figure 2.8, the equality of the limit and function reflects continuity. We study this in detail in section 2.5. yyy y y 222 y'x 111 c x -1 0 1x -1 0 1x -1 0 1x C (a) f (x) - x2 - 1 (b) g(x) - x2 - 1 , x'1 (c) h(x) x 1 FIGURE 2.8 Limits q (x), g'x, and h'x all equal 2 as x 1 approaches. However, only h(x) has the same function value as its limit on x No. 1 (Example 2). K y'k The process of finding a limit can be broken down into a series of steps, including limiting the basic functions that are combined with a sequence of simple operations that we will develop. Let's start with two main functions. 0cx EXAMPLE 3 We find the limits of identity function and permanent function, as x approaches x q (b) Permanent function FIGURE 2.9 Features in Example 3 (a) If it is a function of identification x (x) x, then for any value c (figure 2.9a), have limits at all points c. lim xSc2.2 Feature limit and limit laws 47 (b) If q is a permanent function q (x) k (with a permanent function k) , then for any value c (figure 2.9b), lim q (x) - lim k. xSc xSc For instances of each of these rules we have a lim x 3 Definition Limit identification function on x 3 and xS3 Limit of permanent lim function (4) - lim (4) 4. In (x) 4 on x - 7 or on x 2 xS-7 xS2 We prove these rules in example 3 in section 2.3. The function may not have a limit at a certain point. Some ways that restrictions may not exist are illustrated in Figure 2.10 and described in the following example. y y y 1 y'0, x y 0 y' 1, x'0 1, x 0 x 1 0, x 0 x 0x 0x 0x y 0, x 0 Sin 1, x x x -1 (a) Step U(x) (b) g(x) f (x) FIGURE 2.10 None of these features has a limit as x approaches 0 (example 4). EXAMPLE 4 Discuss Behavior following features explaining why they don't have a limit like x S 0. a) U (x) 0, x60 e 1, x'0 (b) g/x- 1, x≠0 x'0 0, (c) q (x) c 0, x... 0 x70 Sin 1, x Solution (a) Horse Racing Feature: The step function of unit U (x) has no limit like x S 0 because its values jump on x 0. For negative x values arbitrarily close to zero, U(x) 0. For positive values x arbitrarily close to zero, U(x) 1. There is no single L ap- proached by U(x) as x S 0 (Figure 2.10a). b) The function becomes too big to have a limit: g(x) has no limit, as x S 0, because g values grow arbitrarily large in absolute value like x S 0 and therefore do not remain close to any fixed real number (Figure 2.10b). We say that the function is unlimited.48 Chapter 2 Limits and Continuity (c) Function fluctuates too much to have a limit: q (x) has no limit as x S 0, because function values fluctuate between No.1 and -1 in each open interval containing 0. Values do not remain close to any one number, like x S 0 (Figure 2.10c). The Laws limiting a few basic rules allow us to break down complex functions into easy-to-calculate limits. By using these laws, we can greatly simplify many of the limitations of computing. THEOREM 1-Limit Laws If L, M, C, and K are real numbers and lim (x) - L and lim g(x) - M, then xSc xSc 1. Amount Rule: Lim (i)(x) Difference Rule: 3. Permanent plural rule: xSc 4. Product rule: lim (i.x) - g/x) - L - M 5. Odds rule: 6. Power rule: xSc and #lim (k 7. Root rule: xSc (x)) - k L - #lim (i) (x) xSc g/x) - L M lim (x) - L, M≠0 g(x) M xSc lim 3 (x) 4 n n positive integrator x 2n n positive integer xSc (If n even, we assume that q (x) 0 for x in the interval containing c.) The amount rule states that the limit of the amount is the amount of the limits. Similarly, the following rules state that the margin is the difference; The limit of the permanent function time is the constant function limit time. The product limit is prod-uct limits; The coefficient limit is the limit factor (provided that the denominator limit is not 0); The limit of the positive energy of the integrator (or root) function is the integrative power (or root) limit (provided that the root of the limit is a real number). There are simple intuitive arguments as to why properties in the theorem 1 are correct (although they are not evidence). If x is close enough to C, then th (x) is close to L and g(x) close to M, from our unofficial definition of the limit. This is because q (x) g(x) is close to L-M; K (x) - g(x) close to L - M; CH (x) is close to HL; K(x)g (x) is close to LM; and x (x) x (x) is close to LM if M is not zero. We prove the amount rule in the section based on the strict definition of the limit. Rules 2-5 2-5 proven in Annex 4. Rule 6 is obtained through the application of Rule 4 repeatedly. Rule 7 is proven in more advanced texts. The rules of the amount, differences and product can be extended to any number of functions, not just two. EXAMPLE 5 Use observations limxSc k k and limxSc x (Example 3) and Limit Laws in Theorem 1 to find the following limits. a) Lim (x3 - 4x2 - 3) xSc (b) lim x4 - x2 - 1 x2 - 5 xSc (c) lim 24x2 - 3 xS-22.2 Limit function and limit laws 49 Solution (a) lim (x3 - 4x2 - 3) 3 - lim 4x2 - lim 3 Sum and Difference Rules Power and Multiple Rules xSc xSc xSc xSc Ratio Rule - c3 - 4c2 - 3 Amount and Difference Rules (b) lim x4 - x2 - 1 x lim (x4 - 1) Power or Product Rule x2 5 5) Root Rule with n 2 xSc xSc Difference Rule Lim (x2 - P roduct and a few rules and limit xSc permanent function lim x4 - lim x2 - lim 1 - xSc xSc xSc lim x2 - lim 5 xSc xSc c4 - c2 - 1 c2 - 5 (c) lim 24x2 - 3 4x2 - 3) x -2 xS -2 - 2 lim 4x2 - lim 3 xS-2 xS-24 (- 2)2 - 3 - 216 - 3 - 213 Assessment of the limits of polynomials and rational functions theorem 1 makes it easier to calculate the limits of polynomial and rational functions. To estimate the limit of the polynomial function as x approaches, simply replace the C for x in the function formula. To estimate the limit of rational function as x approaches point C, where the denominator is not zero, replace c for x in the formula for function. (See Examples 5a and 5b.) We will officially state these results as theorems. THEOREM 2-Polynomial Limits If P (x) - anxn - 1xn- 1 g th a0, then lim P/x) - P (c) - ancn - 1cn - 1 g th a0. xSc THEOREM 3 - Limits of rational functions If P (x) and q (x) are polynomials and q (c) ≠ 0, then lim P'x - P/c. B (x) q(c) xSc EXAMPLE 6 The following calculations illustrate theorems 2 and 3: lim x3 - 4x2 - 3 (- 1)3 - 4 (- 1)2 - 3 - 0 x2 No 5 (- 1)2 and 5 6 xS -1 Since the denominator of this rational expression is not equal to 0, When we replace -1 for x, we can simply calculate the expression value by x -1 to estimate the limit. Eliminating common factors from the theorem of 3 zero denominators is only applied if the denominator of rational function is not zero within point c. If the denominator is zero, the abolition of common factors in the numerator and 50 chapter 2 limits and the continuity of the definition of the common denominator of factors can reduce the fraction to the point whose denominator is no longer zero with c. If this happens, we will be able to find a replacement limit in the simplified fraction. If q (x) is polynomial and q(c) 0, then (x - c) is a q/x factor. Thus, if the nu-EXAMPLE 7 Rate the merator and the denominator of rational function x are zero on x q c, they have (x - c) lim x2 and x - 2. as a common factor. x2 - xs1 y Solution that we can't replace x No.1 because it does Scratch. We test the numerator to see if it is also zero on x No.1. That's right, so it has a factor (x - 1) in the com x2 x - 2 months with a denominator. The cancellation of this common factor gives a simpler fraction with x2 - x the same values as the original for x ≠ 1: 3 (1, 3) - 1) (x x(x-1) x2 - x - 2 (x 2) - x x 2, if x ≠ 1. x2 - x -2 0 1 x Using a simpler fraction, we find the limit of these values as x S 1, estimating the function in x No 1, as in the theorem 3: a) lim x2 x - 2 - lim x x 2 - 1 - 2 - 3. x2 - x 1 y xS1 xS1 y' x'2 See figure 2.11. x 3 (1, 3) Using calculators and computers to estimate the limits of -2 0 1 We can try using a calculator or computer to guess the numerical limit. However, Tory calculations and computers can sometimes give false meanings and misleading evidence about limitations. x Usually the problem is related to rounding errors, as we now illustrate. (b) EXAMPLE 8 Rate lim 2x2 and 100 - 10. x2 FIGURE 2.11 Graph xS0 (x) (x2 - x - 2) (x2 - x) partially (a) the same as the 2.3 solution table graph lists the function values received on the calculator for multiple g/x points (x 2) in part (b) except for the approaching x No 0. As x 0 approaches through points 1, 0.5, 0.10 and 0.01 euros, on x No. 1, where not defined. The feature seems to be approaching the number 0.05. features have the same limit as the x S 1 (Example 7). As we take even smaller values x, q0.0005, q0.0001, q0.00001, and q0.000001, the function seems to approach number 0. Is the answer 0.05 or 0, or some other value? We will solve this issue in the following example. TABLE 2.3 Calculated values q(x) 2x 2 - 100 - 10 about x 0 x2 x (x) 1 0.049876 - 0.5 - 0.1 0.049969 tons approaches 0.05? 0.01 0.049999 0.0005 0.050000 - 0.0001 - 0.00001 0.050000 - 0.000001 0.000000 while approaching 0? The restriction of functions and laws limiting 51 Use of a computer or calculator can give mixed results, as in example 8. A com-puter can't always track enough numbers to avoid rounding up errors in the calculation of th (x) values when x is very small. We cannot replace x No 0 in a problem, and the numerator and denominator have no obvious common factors (as was the case in example 7). Sometimes, however, we can create a common factor algebraically. EXAMPLE 9 Rate 2x2 and 100 - 10. x2 xS0 Solution This is the limit that we covered in example 8. We can create a common factor by multiplying both the numerator and the denominator by conjuring the radical expression 2x2 and 100 and 10 (received by changing the mark after the square root). Pre-algebra rationalizes numerator: #2x2 100 - 10 - 2x2 - 100 - 10 2x2 - 100 - 1100 Mtheulctoipnljyugaantde .divide on x2 x2 2x 2 - 100 - x2 - 100 - 1 0010) Simplify x2 (2x2 and 100 x2 , 2x2 x2 - x2 - 100 10) Cancel x2 for x ≠ 0. 2 x2 th + . 100 10 So lim 2x2 - 100 - 10 - lim 1 x2 100 xS0 xS0 2x2 1 0.05. 20 This calculation gives the correct answer by allowing ambiguous computer results in Example 8. We can't always manipulate terms in expression to find a coefficient limit where the denominator becomes zero. In some cases, the limitation can be found through geometric arguments (see proof of the theorem 7 in section 2.4) or h calculus methods (developed in section 4.5). The following theorem shows how to evaluate the difficult lymph-L F of it by comparing them with functions with known limits. g Sandwich theorem 0c x Next theorem allows us to calculate different limits. It's called Theorem Sandwich because it refers to a function whose values are sandwiched between val- FIGURE 2.12 Chart th is the sand-ues of the other two functions g and h, which have the same L limit at point C. Trapped between graphs g and h. between the values of the two functions that approach L, the L values should also approach L (Figure 2.12). Proof is given in the appendix 4.52 Chapter 2 Limits and Continuity OF THEOREM 4-The Sandwich Theorem Suppose g(x) ... The question (x) ... h(x) for all x in some open interval containing c, except perhaps the most x y c. Suppose also that lim g(x) - lim h(x) - L. xSc xSc Then lim (x) - L. xSc y'1' x2 The Sandwich TheOrem is also called Theorem compression or Theorem pinch. 2 2 EXAMPLE 10 Given the u function that satisfies u(x) x2 x2 1 - 4 ... u(x) ... 1 y 2 for all x ≠ 0, 1 y'1- x2 find limxS0 u(x), no matter how complicated you have. -1 0 4 1x Solution C FIGURE 2.13 Any function u(x) lim (1 - (x2)) No. 1 and Lim (1 x2)) whose chart is in the area between xS0 xS0 y 1 (x2)2) and y 1 - (x2'4) has a limit of 1 as x S 0 (Example 10). Sandwich theorem implies that limxS0 u(x) No 1 (Figure 2.13). EXAMPLE 11 Theorem Sandwich helps us set several important rules of limitation: (a) lim sin u q 0 uS0 (b) lim cos u q 1 uS0 (c) For any function No 0, lim 0' (x) 0 xSc y y'0u0 Solution 1 y sin u (a) In section 1.3 we have established that - 0 u 0 ... sin y ... 0 u 0 for all u (see figure 2.14a). Since limuS0 (- 0 u 0) - limuS0 0 u 0 and 0, we have -p u lim sin u q 0. p uS0 -1 y - 0 u 0 (b) From Section 1.3, 0 ... 1 - cos y ... 0 u 0 for all u (see Figure 2.14b) and we have (a) limuS0 (1 - cos u) 0 so y lim 1 - (1 - cos u) 1 - y 0u0 us0 uS0 lim cos u 1. Simplify 1 y 1 - cos us0 -2 -1 0 12 u (c) Since - 0 q (x) 0 ... The question (x) ... 0 th (x) 0 and - 0 (x) 0 and 0 (x) 0 have a limit of 0 as x S c, it's fol-minimums that limxSc (x) 0. (b) Sample 11 shows that the functions of the sinuses and the cosy up to their limits on FIGURE 2.14 Sandwich Theorem u No 0. We have not yet established that for any c, lim sin y and sin c, and lim cos u s c. confirms the limits in Example 11. uSc uSc These restriction formulas do hold up, as will be shown in section 2.5.2.2 Limit function and limit laws 53 Exercises 2.2 Limits from Graphics e. lim (x) exists at each point C in (1, 3). 1. For the g(x) feature on the graph here, find the following limits or xSc explain why they don't exist. f. z(1) 0 a. lim g(x) b. lim g(x) c. lim g(x) d. lim g(x) g. q(1) -2 y -2 y f (x) xS1 xS2 xS3 x S 2.5 h. (2) - 0 1 y i. (2) - 1 -1 123 x -1 y g(x) 1 -2 123 x Limits 2. For the No (t) function on the chart here, find the following limits, or in exercises 5 and 6, explain why the limits don' exist. explain why they don't exist. x 1 5. Lim 6. lim x - 1 0x0 xS0 xS1 a. Lim (t) b. Lim (t) c. Lim (t) d. Lim (t) 7. Suppose q(x) is defined for all real values x tS -2 tS -1 tS0 t S -0.5, except x -c. Can I say anything about the existence of limxSc y (x)? Give reasons for your response. s 8. Suppose the function x (x) is defined for all x in 3- 1, 1 . Is there anything to be said for the existence of limxS0 (x)? Give reasons for your response. -2 -1 0 1 t 9. Should limxS1 (x) 5, should be determined on x No 1? If so, should he no (1) and 5? Can we conclude anything about -1 x 1 values? Explain. 3. Which of the following statements about the function y q (x) 10. If q(1) y 5, should lymphxS1 (x) exist? If so, then should the schedule here be correct, and which are false? limxS1 (x) 5? Can we conclude anything about limxS1 (x)? Explain. a. lim (x) exists. Calculating the limits of xliSm0 (x) Find limits in exercises 11-22. xliSm0 (x) b. 0 11. Lim (x2 - 13) 12. lim (- x2 - 5x - 2) c. 1 xS -3 xS2 d. xS0 - 1 13. Lim 8 (t - 5) (t - 7) 14. lim (x3 - 2x2 - 4x - 8) e. 0 lim (x) tS6 xS -2 xliSm1 (x) xS1 15. lim 2x and 5 16. Lim (8 - 3s) (2s - 1) F. (-1, 1). 11 - x3 g. lim (x) exists at each point C in xS2 s S 2'3 xliSmc (x) does not exist. xS1 17. lim 4x (3x 4)2 18. lim y 2 y2 and 5y 6 h. (0) 0 y x s -1'2 yS2 i. (0) Lim (5 - g)4-3 20. lim 2z2 - 10 j. q(1) - 0 yS -3 zS4 2 1. Lim 3 22. lim 25h 4 - 2 hS0 23h 1 and 1 h k. (1) -1 1 2x hS0 -1 Odds Limits Find Limits in Exercises 23-42. -1 23. lim x - 255 24. lim x 3 x2 - 4x and xS5 xS -3 x2 3 4. Which of the following statements about the function y q (x) 25. lim x2 - 3x - 10 26. lim x2 - 7x and 10 charts here are true, and which are false? x 5 x- 2 xS -5 xS2 a. lim (x) does not exist. 27. lim t2 and t - 2 28. lim t2 and 3t 2 xS2 t2 - 1 t2 - t2 - t. q (x) No2 tS1 tS -1 c. lim (x) does not exist xliSm2. 2 9. Lim - 2x -4 30. lim 5y3 and 8y2 xS1 x3 2x2 3y4 - 16y2 d. exists Each C point in (- 1, 1). xS -2 yS0 lim (x) xSc54 Chapter 2 Restrictions and Continuity 3 1. lim x-1 - 1 32. lim x 1 1 and x 1 1 53. Let's say limxSc q (x) - 5 and limxSc g (x) - 2. Find x - 1 - xS1 xS0 a. lim (x)g(x) b. lim 2' (x)g (x) xSc u4 - 11 y3 - 8 xSc (x) u3 - y4 - 16 q(x) - g'x 33. Lim 34. lim c. lim (yap.) 3 g (x)) d. lim uS1 yS2 xSc xSc 3 5. lim 2x x--93 36. lim 4x - x2 54. Let's say limxS4 (x) 0 and limxS4 g(x) - 3. Find xS4 2 - 2x xS9 a. lim (g(x) 3) b. lim x'(x) xS4 37. lim x - 1 2x2 - 8 - 3 xS4 g(x) xS1 2x 3 - 2 38. lim x 1 d. lim (x) - c. lim (g(x))2 1 xS -1 xS4 xS4 39. lim 2x2 No 12 - 4 40. lim x-2 #5 5. Let's say limxSb (x) 7 and limxSb g(x) - 3. Find x-2 a. lim (q(x) g(x)) b. lim (x) g(x) xS2 xS -2x2 - 5 - 3 xSb c. lim 4g (x) d. lim (x)g(x)g(x) 41. Lim 2 - 2 x 2 - 5 42. lim 4 - xSb xSb x'3 xS4 5 - 2x2 - 9 xS -3 5 6. Suppose limxS -2 p(x) - 4, limxS -2 r(x) - 0, and limxS -2 s (x) - -3. Find limits with trigonometry functions Find limits in exercises 43-50. a. lim (p(x) r(x) s (x)) 43. Lim (2 sins x - 1) 44. lim sin2 x - #xS -2 xS0 xS0 b. lim p(x) r(x) s (x) 45. Lim sec x 46. lim tan x xS -2 xS0 xS0 c. lim (-4p(x) 5r (x)) Lim (x2 - 1) (2 - cos x) xS -2 3 cos x 47. xS0 Average change speeds due to their connection to secant, tangent and instanta- 4 9 lines. lim 2x No 4 cos (x q p) 50. lim 27 and sec2 x neous bets, form limits xS-p xS0 Using lim (x-h) - q (x) 5 1. Let's say limxS0 (x) 1 and limxS0 g(x) - 5. Name the hS0 rules in theorem 1, which are used to perform steps (a), b) and (c) are often found in calculus. In exercises 57-62, evaluate this limit by the following calculation. for this value x and function x. lim 2 (x) - g/x- lim (2 (x) - g/x) 57. x x2, x No 1 (z(x) 7)2 x 7)2'3 (a) 58. X (x) x2, x - 2 xS0 xS0 59. In (x) 3x - 4, x 2 6 0. x (x) 1 x, x -2 lim (x) 6 (x) 2x, x 7 6 2. x 0 xS0 y lim 2 (x) - lim g(x) (b) xS0 xS0 2'3 a lim (i.x) 7) b xS0 2 lim (x) - lim g(x) Using Sandwich Theorem xS0 xS0 2'3 (c) 6 3. If 25 - 2x2 ... The question (x) ... 25 - x2 for - 1 ... X... 1, find lim (x) lim 7b limxS0 (x). xS0 xS0 6 4. If 2 - x2 ... g(x) ... 2 cos x for all x, find limxS0 g (x). (2) (1) - (- 5) 7 6 5. a. It can be shown that inequality (1 and 7)2'3 4 5 2. Let limxS1 h(x) - 5, limxS1 p(x) - 1, and limxS1 r (x) - 2. 1 - x2 6 2 x sin x 6 1 Name the rules in theorem 1, which are used to perform steps 6 - 2 cos x (a), b) and (c) the next calculation. keep for all x values close to zero. What if anything, does 25h (x) lim 25h (x) it tells you about p(x) (4 - r(x)) xS1 lim - r/x))) (a) x sin x lim (p/x)(4 - 2 cos xS1 xS1 lim ? 2lim 5h (x) xS0 2 x1 (b) T b. G raph y y 1 - (x2'6), (x sin x) (2 - 2 cos x), and lim p(x)b lim (4 - r(x))b xS1 xS1 y 1 together for -2 ... X... 2. Comment on the behavior of graphs as x S 0. 25lim h(x) - xS1 (c) lim p/x)b lim 4 - lim r(x)b xS1 xS1 xS1 - 2 (5) (5) - 5 (1) (4 - 2) 22.2 Feature limit and limit laws 55 6 6. a. Suppose inequality b. Maintain your conclusion partially (a) by graphing no near c -1 and using to increase and trace to estimate u-values by 1 - x2 6 1 - cos x 6 1 graph as x S -1. 2 24 x2 2 c. Find limxS -1 (x) algebraic. keep x close to zero. (They do as you'll see in section 9.9.) What, if anything, does it tell you about 72. Let F (x) - (x2 - 3x - 2) (2 - 0 x 0). lim 1 - cos x? a. Make the F tables at x, that approach x2 c - 2 top and bottom. Then rate limxS -2 F(x). xS0 b. Under support your conclusion in Part A) by graphics F about Give reasons for your response. c -2 and use to zoom in and trace to estimate u-values on a graph like x S -2. T b. Graph of Equations y (1'2) - (x2'24), y (1 - cos x) and y 1'2 together for - 2 ... X... 2. c. Find limxS -2 F (x) algebraic. 73. Let g(u) (sin u) u. comment to the behavior of the graphs as x S 0. a. Make a table of g values on the values you are approaching the u0 and 0 limits estimate from above and below. Then rate limuS0 g/u. T You will find a graphic calculator useful for exercises 67-74. B. Under support your conclusion in part a) by graphics G about 6 7. Let q (x) (x2 - 9) u0 and 0. a. Make a table of values in points x -3,1, 74. Let G (t) No (1 - cos t) t2. - 3.01, - 3.001, and so on, as far as your calculator can go. Then rate limxS -3 (x). What kind of assessment do you come to if. Make G tables on t values that approach t0 and 0 you estimate at x -2.9, - 2.99, - 2,999, with instead? Top / bottom. Then rate limtS0 G (t). B. Under support its findings partially (a) by chart no about b. Under support your conclusion partially (a) by G graphics about t0 and 0. c -3 and use to zoom in and trace to estimate u-values on a graph like x S - 3. Theory and examples 75. If x4 ... The question (x) ... x2 for x in 3 - 1, 1 4 and x2 ... The question (x) ... x4 for c. Find limxS -3 (x) algebraically, as in example 7. 6 8. Let g (x) (x2 - 2) (x - 22). x 6 -1 and x 7 1, at what point with do you automatically know limxSc (x)? What can you say about the value of the limit on. Make a table of g values at points x 1.4, 1.41, these points? 1.414, and so on through consecutive decimal approximation 22. Rated limxS22 g(x). 76. Suppose that g(x) ... The question (x) ... h(x) for all x ≠ 2 and let's assume b. Under support your conclusion partially (a) by graph g about lim g(x) lim h(x) - 5. c 22 and use to zoom in and trace to estimate u-values on a graph like x S 22. xS2 xS2 c. Find g(x) algebraic. Can we nothing about the values of th, g, and h at 6 9. Let G (x) - (x 6) (x2 - 4x - 12). x No 2? Can No (2) No 0? Can LimxS2 (x) 0? Give reasons for your answers. a. Make a table of G values at x th - 5.9, - 5.99, - 5,999, and so on. Then rate limxS -6 G(x). What a score of 7 7. If Lim (x) - 5 x 1, find Lim (x). Do you arrive if you rate G at x -6,1, -6.01, x-2 - 6,001, cinstead? xS4 xS4 b. Support your findings in part a) by graphs G and 78. If Lim (x) No. 1, find with Zoom and Trace to estimate u-values on a graph like x2 x S - 6. xS -2 (x) x c. Find limxS -6 G(x) algebraic. a. lim (x) b. lim 70. Let h(x) - (x2 - 2x - 3) (x2 - 4x - 3). xS -2 xS -2 a. Make a table of h values at x 2.9, 2.99, 2.999, and so on. Then rate limxS3 h(x). What is the score to do 79. a. If Lim (x) - 5 and 3, find a lim (x). do you arrive if you rate h at x 3.1, 3.01, 3,001, c x-2 instead? xS2 xS2 b. Maintain your findings partially (a) by graph h about b. If Lim x (x) - 5 and 4, find a lim (x). c No. 3 and use to zoom in and trace to estimate u-values on the x-2 graph as x S 3. xS2 xS2 c. Find limxS3 h(x) algebraic. 8 0. If Lim (x) No. 1, find x2 7 1. Let q (x) (x2 - 1) (0 x 0 - 1). xS0 (x) x a. Check the q tables on x values that approach a. lim q(x) b. lim c - 1 top and bottom. Then rate limxS -1 (x). xS0 xS0 T 81. a. Chart g(x) x sin (1'x) to evaluate limxS0 g(x), increasing the origin if necessary. B. Confirm your assessment in part (a) with evidence. T 82. a. G raph h(x) - x2 cos (1'x3) to evaluate limxS0 h(x), if necessary increasing its origin. B. Confirm your assessment in part (a) with evidence.56 Chapter 2 Limitations and Continuity of COMPUTER EXPLORATIONS 8 5. lim 23 1 x - 1 x Graphic estimates of xS0 limits in exercises 83-88, use CAS to perform the following steps: 8 6. lim x2 - 9 a. The function site near point C is approaching. xS3 2x2 No 7 - 4 b. From your story guess the value of the limit. 1 - cos x sin x 83. lim x4 - 16 8 7. lim x - 2 xS2 xS0 8 4. lim x3 - x2 - 5x - 3 88. lim 3 2x2 (x 1)2 - 3 cos xs -1 xS0 2.3 Accurate limit definition We now turn our attention to the exact definition of the limit. The early history of calculus saw arguments about the validity of the basic concepts underlying the theory. Explicit contradictions were disputed by both mathematicians and philosophers. These contro-versies have been resolved by an exact definition that allows us to replace vague phrases as gets arbitrarily close in an informal definition with specific terms that can be applied to any particular example. With a strict definition, we can avoid misidentified the standings, prove the marginal properties given in the previous section, and set many important limitations. To show that the limit in th (x) as x C equals the number of Ls, we need to Show that the gap between K (x) and L can be made as small as we choose if x is kept close enough to c. Let's see what it requires if we specify the size of the gap between q (x) and L. y EXAMPLE 1 Consider function y 2x - 1 about x No 4. Intuitively it seems that 2x - 1 is clear that u close to 7 when x is close to 4, so limxS4 (2x - 1) 7. However, how close to x No 4 x should it be that 2x - 1 differs from 7, say, less than 2 units? To meet the 9 Upper Limit: Solution We asked: For what values x 0 y - 7 0 6 2? To find the answer, we are 7 y'9 first express 0 y - 7 0 in terms of x: 5 Lower boundary: 0 y - 7 0 y 0 (2x - 1) - 7 0 y 0 2x - 8 0 . y'5 The question arises: what values x satisfy inequality 0 2x - 8 0 6 2? To find out, we solve inequality: 0 345 x 0 2x - 8 0 6 2 Restriction -2 6 2x - 8 6 2 Removal of absolute value gives two inequalities. to this 6 6 2x 6 10 Add 8 to each term. 36x65 Decide for x. FIGURE 2.15 Storage x within 1 Solve unit for x - 4. -1 6 x - 4 6 1. x No. 4 will keep y within 2 units at No. 7 (Example 1). Keeping x within 1 unit x No. 4 will keep the u within 2 units at 1 y 7 (figure 2.15). d is the Greek letter delta In the previous example we have determined how close x should be to a certain value with e is the Greek letter Epsilon ensure that the outputs q (x) some functions lie within a set interval near the L limit. than any prescribed error, no matter how small, holding x close enough to c. To describe arbitrary prescribed errors, we intro-duce two constants, d (delta) and e (epsilon). These Greek letters are traditionally used to represent small changes in variable or function.2.3 The exact definition of limit 57 in L and 1 Definition of limit 10 Suppose we observe the values of function q(x) as x approaches c (without taking f (x) at the very meaning of C). Of course, we want to be able to say that th (x) stays within one tenth of f (x) lies a unit from L as soon as x stays within a certain distance d c (Figure 2.16). But that alone is not enough, because as x continues its course to c, what is it to prevent L (x) L here from jumping around in the interval from L - (1'10) to L (1'10) without the tendency to L? We can say that the margin of error can be no more than 1 x 100 or 1 1000 or L 1 1 x 100,000. Every time we find a new email protected about so that keeping x within this between 10 val satisfies a new tolerance for errors. And every time there is a possibility that th (x) might bounce off L to more Stage. for all x ≠ c here the numbers on the next page illustrate the problem. You can think of it as a quarrel between a skeptic and a scientist. Skeptic presents to show there is a dd place for doubt that the limit exists. Scientist counteracts each problem with protected email around C, which ensures that the function takes values within e L. 0 xx c'd c'd How can we stop this seemingly endless series of calls and answers? We can do this by proving that for every electronic error tolerance that a challenger can produce, we can submit figure 2.16 As we have to determine the appropriate distance d that holds x close enough to c hold (x) in that email is protected by d 7 0 so that saving x in the L interval (figure 2.17). This brings us to the exact definition of the limit. (c - d, c q d) will keep q(x) in the aL interval - 110, L q 110b? Y DEFINITION Let y (x) be defined at an open interval about c, except perhaps the c. We say that the limit in th (x) as approaching x is the number L, and write L'e f(x) lies lim q (x) l, L f (x) here xSc L-e, if for each number e 7 0, there is an appropriate number d 7 0 such that ˛q(x) - L˛6 e whenever 0 6˛x - c˛6 d. for all x s. , imagine the treatment of a cylindrical shaft to close tolerance. In here the diameter of the shaft is determined by turning the dial to the installation of a measured vari-capable x. We try for the diameter of L, but since nothing is perfect, we must be satisfied diam- dd eter q (x) somewhere between L - e and L e. Number d is our tolerance to dial control; it tells us how close our dial settings should be to the installation of x q c in order to 0 x x to ensure that the diameter of the q (x) shaft will be accurate within e L. As the toler- c-d c-c ance for the error becomes more stringent, we can adjust d. Value d, how tight should our c'd control setting should be, depends on the value of the e Making a mistake. FIGURE 2.17 Attitude d and e in limiting applies to functions in more general areas. This is just the definition of a limit. you want each open interval around C to contain points in the function domain, except c. You can cite additional and extended 37-40 exercises for examples of limitations for functions with complex domains. In the next section, we'll see how the limit definition is applied at points at the interval boundary. Examples: Testing the definition of the Official Limit definition does not say how to find the function limit, but allows us to make sure that the assuming limit value is correct. The following examples show how this definition can be used to test limit operators for specific functions. However, the real purpose of the definition is not to make calculations like this, but to prove gen-eral theorems, so that the calculation of specific limits can be simplified, as theo-rems stated in the previous section.58 Chapter 2 Limitations and Continuity y y y 1 y f (x) 1 y f (x) y y y y f(x) y = f(x) 10 10 L + L + 1 1 100 100 L + L + L L L L L - 1 L - 1 L - 1 L - 1 10 10 100 100 0 cx 0 c - d1/10 c x 0 cx 0 c x Response: c + d1/10 c - d1/100 c + d1/100 The challenge: New challenge: Response: 1 1 Make 0 f(x) - L 0 6e = 10 0 x - c 0 6 d1/10 (a number) Make 0 f (x) - L 0 6e = 100 0 x - c 0 6 d1/100 y y y = f (x) y = f(x) L + 1 L + 1 1000 1000 L L L - 1 L - 1 1000 1000 0c x 0c x New challenge : Ответ: 1 0 x - c 0 6 d1/1000 e - 1000 y y y y (x) y f(x) L'P y f(x) L L I 1 L и 1 100 000 100 ,000 L-P L L L - 1 L - 1 100,000 100,000 0c x 0c x Новый вызов: Ответ: Новый вызов: Ответ: 0 x - c 0 6 d1/100,000 e ... e 1,100,000 EXAMPLE 2 Show that lim (5x - 3) 2. xS1 Solution Set From No. 1, No (x) 5x - 3, and L No. 2 in Limit Definition. For any given e 7 0, we have to find a suitable d 7 0 so that if x ≠ 1 and x is within a distance of d c q 1, that is, when 0 6 0 x - 1 0 6 d, it is true that q (x) is within the distance e L No 2, so 0 y (x) - 2 0 6 e.2.3 Accurate definition of the limit 59 y 5x - 3 We find d, working back from secure email: 2'e 1 - e 1 e 0 (5x - 3) - 2 0 0 5x - 5 0 6 e 2 5 5 5 50x - 10 6 e 2-e x - 1 0 6 e'5. 0 So we can take d q e'5 (Figure 2.18). If 0 6 0 x 1 0 6 e'5, then 0 (5x - 3) - 2 0 x 5x - 5 0 x 5 x 1 0 6 5 (e-5) e, x that proves that limxS1 (5x - 3) 2. The value of d q e'5 is not the only value that will make 0 6 0 x - 1 0 6 d imply 0 5x - 5 0 6 e. Any lesser positive d will do as well. Definition does not require the best positive d, only one that will work.-3 EXAMPLE 3 Prove the following results presented graphically in section 2.2. a) lim x - c NOT TO SCALE xSc FIGURE 2.18 If q (x) - 5x - 3, then (b) lim k k (k constant) 0 6 0 x - 1 0 6 e-5 guarantees that 0 y (x) - 2 0 6 e (example 2). xSc y'x Solution c'e (a) Let e 7 0 be given. We have to find d 7 0 so that c'd 0 x - c 0 6 e whenever 0 6 0 x - c 0 6 d. Consequences will occur if d equals e or any lesser positive number (Figure 2.19). c-d This proves that limxSc x c. c-e (b) Let's be given e 7 0. We have to find d 7 0 so that 0 c-d c'd x 0 k - k 6 e whenever 0 6 0 x - c 0 6 d. FIGURE 2.19 For function Since k - k 0, we can use any positive number for d and the consequences will hold (Figure 2.20). This proves that limxSc k. q (x) x, we find that 0 6 0 x - c 0 6 d will guarantee 0 q(x) - c 0 6 e whenever the search delta algebraic for Given Epsilons d ... e (Example 3a). In examples 2 and 3 interval values around c, for which 0 x (x) - L 0 was less than e, was symmetrical about C, and we could take d to be half the length of that interval. When the interval around C on which we have ˛(x) - L˛6 e is not symmetrical about c, we can take d to be C to closer to the end point of the interval. y y'k EXAMPLE 4 For limit limxS5 2x - 1 and 2, find d 7 0, which works for e No 1. ke, that is, find d 7 0 such that k 0 2x - 1 - 2 0 6 1 whenever 0 6 x - 5 0 6 d. k-e Solution We organize a search in two stages. 0 c-d c'd x 1. Solve inequality 0 2x - 1 - 2 0 6 1 to find an interval containing x No. 5 on FIGURE 2.20 For a function that inequality has for all x ≠ 5. K (x) k, we find that 0 q(x) - k 0 6 e for 0 2x - 1 - 2 0 6 1 any positive d (Example 3b). -1 6 2x - 1 - 2 6 1 1 6 2x - 1 6 3 16x-169 2 6 x 6 1060 Chapter 2 Limits and Continuity 3 3)x Inequality has for all x in open interval (2, 10), so it holds for all x ≠ 5 in this interval as well. (10 258 2. Find d 7 0 to place a centered interval 5 - d 6 x 6 5 d (centered on x 5) inside the interval (2, 10). If we take d 3 or any smaller positive number (2, 10) FIGURE 2.21 Open radius interval 3 (Figure 2.21). If we take d 3 or any smaller positive number, the inequality 3 about x 5 will lie inside the open 0 6 0 x - 5 0 6 d will automatically place X between 2 and 2 and 10 and implies this interval (2, 10). 0 2x - 1 - 2 0 6 1 (Figure 2.22): 0 2x - 1 - 2 0 6 1 Whenever 0 6 0 x - 5 0 6 3. y How to find algebraic D for This, L, C, and E y 0 y x - 1 Process finding d 7 0 such that 3 0 (x) - L 0 6 e whenever 0 6 0 x - c 0 6 d 2 can be performed in two stages. 1 1. Solve inequality 0 x (x) - L 0 6 e to find an open interval (a, b) containing 33 c. Inequality is there for all x ≠ c. Please note that we do not require that the inequality 0 12 58 10 x hold was at the level x q c. It can hold there or not, but the value of q q q c does not affect the existence of the limit. NOT TO SCALE 2. Find a d 7 0 value that places an open interval (c - d, c q d) in the center of FIGURE 2.22 Function and inter- on C within the interval (a, b). Inequality 0 x (x) - L 0 6 e will be the case for all vals in example 4. x ≠ c in this protected email EXAMPLE 5 Prove that limxS2 (x) 4 if q (x) x2, x≠2 e 1, x y 2. Solution Our challenge is to show that given e 7 0 there is d 7 0 that 0 y (x) - 4 0 6 e whenever 0 6 0 x - 2 0 6 d. y 1. Solve inequality 0 x (x) - 4 0 6 e to find an open interval containing x No 2 on 4'e, which inequality has for all x ≠ 2. y x2 For x ≠ c q 2, we have q (x) x2, and the inequality to solve is 0 x2 - 4 0 6 e: 4 (2, 4) 0 x2 - 4 0 6 e -e 6 x2 - 4 6 e 4 - e 6 x2 6 4 e 4 - e 6 0 0 6 24 e Assumes e 6 4; See below. 24 - e 6 x 6 24 and e. Open interval about x No 2, which solves inequality. (2, 1) Inequality 0 q(x) - 4 0 6 e holds for all x ≠ 2 in the open interval 0 4 - e 2 4 and e x 1 24 - e, 24 e 2 (figure 2.23). FIGURE 2.23 Interval containing 2. Find the value d 7 0 that puts interval (2 - d, 2 q d) inside x No 2 so as to function in example 5 interval 1 24 - e, 24 and e 2. satisfies 0 k (x) - 4 0 6 e. Take d to be a distance from x No 2 to closer end point 1 24 - e, 24 and e 2. In other words, take d q min5 2 - 24 - e, 24 - e - 26 , minimum (the2.3 Exact definition of the limit 61 less) of two numbers 2 - 24 - e and 24 e - 2. If d has this or some lesser positive value, inequality 0 6 0 x - 2 0 6 d will automatically place x between 24 - e and 24 e to make 0 q(x) - 4 0 6 e. For all x, 0 y (x) - 4 0 6 e whenever 0 6 0 x - 2 0 6 d. This completes the proof for e 6 4. If e is No 4, then we take d to be the distance from x No. 2 to the closer end point of intervals 10, 24 and e 2. In other words, take d q min52, 24 and e - 26. (See Figure 2.23.) Using the definition to prove a theorem, we usually do not rely on the formal definition of a limit to verify specific limits, such as those that were in previous examples. On the contrary, we refer to general theorems of limitations, in particular the theorems of section 2.2. Definition is used to prove these theorems (annex 5). As an example, we prove Part 1 of the Theorem 1, the Sum Rule. EXAMPLE 6 Considering that limxSc (x) - L and limxSc g(x) - M, prove that lim (Yap. We want to find a positive number d, that 0 y (x) g (x) - (L and M) 0 6 e each time, when 0 6 0 x - c 0 6 d. Regrouping terms, we get 0 y (x) g (x) - (L and M) 0 y 0 (i(x) - L) 0 z(x) - L 0 y 0 g (x) - M 0 . a and b ... a - b Since limxSc (x) - L, there is a number d1 7 0 such that you can find d1, as lim (x) l 0 (x) - L 0 6 e'2 whenever 0 6 0 x - c 0 6 d1. Similarly, since limxSc g(x) - M, there is a number d2 7 0 xSc so that 0 g(x) - M 0 6 e'2 whenever 0 6 0 x - c 0 6 d2. You can find d2, as lim g(x) - M xSc Let d - min 5d1, d26 , less d1 and d2. If 0 6 0 x - c 0 6 d, then 0 x - c 0 6 d1, so 0 th (x) - L 0 6 e'2, and 0 x - c 0 6 d2, so 0 g/x) - M 0 6 e'2. Thus, 0 y (x) y (x) - (L and M) 0 6 e q e. 2 2 This shows that limxSc (i.x) a -7'2, b - 1'2, c - 3 4. a -7'2, b - 1'2, c - 3'2 Centering Intervals About Point 5. a 4'9, b 4'7, c 1'2 In exercises 1-6, draw the interval (a, b) on x-axis with point 6. a 2.7591, b 3.2391, with 3 c inside. Then find the value d 7 0 so that 6 x 6 b whenever 0 6 0 x - c 0 6 d. 1. A 1, b 7, with 5 2. a No 1, b No 7, c 262 Chapter 2 Limits and Continuity Search Delta Graphic 1 3. 14. In exercises 7-14, use graphs to find d 7 0 such that y y f (x) 2 f (x) - 1 0 (x) - L 0 6 e whenever 0 6 0 x - c 0 6 d. c -1 -x 2.01 x 2 7. 8. L'2 c' 1 y 2.5 2 y 2x - 4 y 3 y 0.5 1.99 L'2 f (x) - 2 x 3 6.2 f(x) 2 y 0.01 6 with 5 so -3 1.5 l 7.5 g. x 5.8 litres, 6 e x 0.2 x 0.15g - 3 x 2 7.65 1g x 7.5 05 x 7.35 4.9 5.1 NOT FOR SCALE x 0x 0 0 1 x 2 -3 3 3 0.1 -3 -2.9 0 - 196 -1 - 16 1 1 25 2.01 1.99 DO NOT SCALE SO AS NOT TO SCALE 10. 9. y Search Delta Algebraic F (x) - 2x - 1 Each of Exercises 15-30 gives function q (x) and numbers L, c, and f (x) In each case, find an open interval of about c, at which inequal- y L'1 e 0.2 ity 0 q(x) - L 0 6 e holds. Then give the value to d 7 0 so that for all x satisfying 0 6 0 x - c 0 6 d inequality 0 (x) - L 0 6 e holds. 5 equaliser 1 y x 4.2 y 2 x 1 4 4 4 1 3.8 1 5. x x No 1, L 5, from 4, e 0.01 3 1 6. H (x) 2x - 2, L -6, c -2, e 0.02 4 17. X (x) 2x 1, L 1, c 0, e 0.1 x2 1 8. H (x) 2x, L 1'2, c 1'4, e 0.1 0 9 1 25 19. X (x) 219 - x, L 3, with 10, e 1 16 16 20. B (x) 2x - 7, L 4, c 23, e 1 x -1 0 2.61 3 3.41 21. H (x) 1'x, L 1'4, c q 4, e 0.05 NOT TO SCALE 22. X (x) x2, L 3, from 23, e 0.1 11. 12. 23. X x2, L - 4, C - 2, e - 0.5 f (x) - 4 - x2 y c -1 y 24. W (x) 1'x, L -1, c -1, e q 0.1 f (x) x2 L-3 3.25 c-2 e 0.25 25. x x2 - 5, L 11, from 4, e 1 L 4 e'1 26. X (x) 120 x, L 5, from 24, e 1 27. K (x) - mx, m 7 0, L - 2 m, c - 2, e - 0.03 g - 4 x2 3 28. W (x) - mx, m 7 0, L - 3 m, c - 3, e - c 7 0 y and x2 5 29. B (x) - mx - b, m 7 0, L (m'2) - b, 4 c - 1'2, e - c 7 0 3 2.75 3 0. x (x) - mx b, m 7 0, L - m b, c q 1, e - 0.05 x Using formal definition 0 2 Each of the exercises 31-36 gives function q(x), point c, and positive num- q3'5 ber e. Find L y lim 'x. Then find number d 7 0 to NOT SCALE xSc 0 (x) - L 0 6 When 0 6 0 x - c 0 6 d. z(x) 3 - 2x, c y 3, e 0.02 2 c -1, e q 0.03 - 32. X - 3x - 2, c q 2, e q 0.05 NOT TO SCALE x2 - 24, x - 33. B (x) No2.3 Accurate definition of the limit 63 3 4. In (x) x2 x 6 x 5, c -5, e 0.05 Theory and Examples x 5 51. Determine what it means to say that lim g(x) k. 3 5. B (x) 21 - 5x, c -3, e 0.5 xS0 5 2. Prove that Lim (x) - L, if and only if lim (h) - L. xSc hS0 3 6. B (x) 4'x, c 2, e 0.4 5 3. Incorrect statement about limitations Show by example that prove limit operators in exercises 37-50. the following statement is incorrect. 3 7. Lim (9 - x) 5 38. lim (3x - 7) 2 L number is the limit of q (x) as x approaches, if q (x) approaches L as x. xS4 xS3 3 9 approaches. lim 2x - 5 and 2 40. lim 24 - x No 2 Explain why the feature in your example doesn't have a given L value as a limit as x s. xS9 xS0 54. Another incorrect statement about the limitations of the show is that 4 1. lim q(x) No. 1, if q (x) x2, x≠1 the following statement is incorrect. e 2, x-1 xS1 4 2. lim q (x) 4 if q (x) x2, x ≠ -2 Number L is the limit q (x) as x approaches c if, given any e 1, x -2 xS -2 e 7 0, there is a value x for 0 k (x) - L 0 6 e. 4 3. Lim 1 and 1 44. Explain why the feature in your example doesn't have x2 3 of this L value as a limit as x S. xS1 x 23 4 5. lim x2 - 9 - -6 46. lim x2 - 1 and 2 T 55. Shredding engine cylinders Before you contract to grind the engine x 3 x-1 cylinders on the transverse area of 9 in2, you should know how the xS -3 xS1 is a lot of deviation from the ideal diameter of the cylinder from 3.385 to 4 7. lim (x) 2, if q (x) 4 - 2x, x61 requires 9 in2). To find out, you allow A q p (x'2)2 and look for the xS1 interval in which you have to keep x to make 0 A - 9 0 ... 0.01. 4 8. Lim x (x) 0 if q (x) 2x, x60 e x'2, x'0 What interval will you find? xS0 49. lim x sin 1 and 0 56. The production of electric xS0 resistors Ohm's law for elec- - in three-caliber circuits, as shown V I R in the accompanying figure - states that V and RI. In this equa-tion, V is a constant voltage, I current in amps, and R is resistance in ohms. Your firm has 1 1 y and x sin 1 has been asked to put resistors for a chain in which the V will be 120 2p x volts and I have to be a 5 and 0.1 amp. At what interval should R lie 2p x for me to be within 0.1 of the I0 and 5 amplifiers? 1 1 - p p When the number L is not the limit of th (x) as x u c? Display L is not the limit We can prove that limxSc (x) ≠ L by providing e 7 0 in such a way that there is no possibility d 7 0 satisfies condition 0 (x) - L 0 6 e whenever 0 6 0 x - c 0 6 d. 5 0. lim x2 sin 1 and 0 We achieve this for our candidate e, showing that for each x d 7 0 there is a value x such, that xS0 0 6 0 x - c 0 6 d and 0 0 (x) - L 0 e. y y y x2 1 y f (x) L'e y x 2 Sin 1 L x -1 2 0 21 X 21 X-E p p p f (x) - 0 c-d c'd x -1 y -x2 value x x x for which 0 6 0 x - c 0 6 d and 0 f (x) - L 0 and e64 Chapter 2 Limits and Continuity 5 7. Let q (x) - x, 1, x61 y ex and x 7 1. 4.8 y 4 y f (x) y'x'1 3 2 y f (x) 0 3x 1 x 6 0. a. For the feature, on the graph here, show that limxS -1 g(x) ≠ 2. y'x 1 b. Lee limxS -1 g (x) seems to exist? If so, what is the cost of the limit? If not, why not? Show that there is no possibility d 7 0 satisfies fol-y low state: 0 y (x) - 2 0 6 1'2 each time, when 0 6 0 x - 1 0 6 d. 2 That is, for each d 7 0 show that there is a value x th g (x), that 1 0 6 0 x - 1 0 6 d and 0 th (x) - 2 0 x 1 2. This will show that limxS1 (x) ≠ 2. -1 0 x b. Show limxS1 (x) ≠ 1. C. Show that limxS1 (x) ≠ 1.5. x2, x62 COMPUTER EXPLORATIONS x2 In exercises 61-66, you will further explore the search for delta graphi- 58. Let h(x) c 3, x 7 2. Kalli. Use CAS to take the following steps: 2, y a. The y q (x) function site near the C point is approaching. 4 g x (x) b. Guess the limit value and then estimate the limit symbolically to see if you guessed 3 c. Using the value of e 0.2, on the chart of the strip y1 and L - e 2 y'2 and y2 l e along with function q about c. 1 y x2 d. From your chart in part (c), the score d 7 0 such that 0 y (x) - L 0 6 e whenever 0 6 0 x - c 0 6 d02 Check my estimate. , plotting, y1, and y2 during the interval Show that 0 6 0 x - from 0 6 d. To view the windows use c - 2d ... a. lim h(x) ≠ 4 x ... c q 2d and L - 2e ... Y... L No 2e. If any function values lie outside the 3L interval - e, L and e, your choice of xS2 d was too large. Try again with a lower score. b. lim h(x) ≠ 3 e. Repeat parts (c) and (d) sequentially for e q 0.1, 0.05, and xS2 0.001. c. lim h(x) ≠ 2 6 1. X (x) x4 - 81 , with 3 62. In (x) 5x3 and 9x2, c'0 x - 3 2x5 and 3x2 xS2 63. W (x) - sin 2x, c q 0 64. x (x) x (1 - cos x) , c'0 59. For the feature, on the graph here, explain why 3x x is a sin x a. lim (x) ≠ 4 65. B(x) 23 x - 1 , c'1 xS3 x-1 b. lim (x) ≠ 4.8 66. W (x) 3x2 - (7x 1) 2x 5, c'1 x - 1 xS3 c. lim (x) ≠ 3 xS32.4 Unilateral Limits 65 2.4 One-Way Borders In this section we extend the concept of a limit to one-way limits, which are limitations because x is close to the number c on the left (where x 6 c) or the right side (x 7 c) only. They allow us to describe functions that have different limits at the point, depending on whether we approach the point on the left or right. Unilateral restrictions also allow us to tell what it means for a function to have a limit at the end point of the interval. y y x Approaching the limit on one side 0x0 1 Suppose the function of th is determined at the interval, which extends to both sides of the number c. In 0 x, in order to have an L limit as x c approaches, the x (x) should approach -1 L, as x approaches from both sides. Because of this, we sometimes say that the limit figure 2.24 Different right and bilateral. left limits at the beginning. If it does not have a two-way limit in c, it can still have a one-way limit, that is, a limit, if the approach is only one side. If the approach is on the right, the restriction is the right limit or the limit on the right. On the left is the left limit or the limit on the left. The function q(x) x 0 x 0 (Figure 2.24) has a limit of 1 as x approaches 0 on the right, and the limit is -1 as x approaches 0 on the left. Because these one-way limit values are not the same, there is no single number that approaches x as x 0 approaches. Thus, x (x) does not have a (two-way) limit of 0. Intuitively, if we only look at the values to (x) at intervals (c, b), where c 6 b, and the values of y (x) become arbitrarily close to L, as x approaches from this interval, then it has the right limit L on c. In this case we write lim q(x) - L. xSc' Notation x S c means that we only consider the values q(x) for more than c. We Consider the values of th (x) for x... C. Similarly, if th (x) is defined at interval (a, c), where c 6 a and y (x) comes arbitrarily close to M, as x approaches c from this interval, then y have a left limit M on c. We write lim (x) - M. x c- Symbol x S c- means that we consider values q only at x-values less c. These unofficial definitions of one-way limits are illustrated. For q(x) x 0 x 0 in figure 2.24 we have a lim (x) 1 and a lim (x) - 1. x S 0 x S 0- y L f (x) f (x) M 0 cx x0 xc x (a) lim f (x) l (b) xl:imc_ f (x) - M x:c FIGURE 2.25 (a) Right-hand limit по мере приближения x. b) левый предел по мере приближения x.66 к главе 2 Пределы и непрерывность и теперь мы даем определение предела функции в пограничной точке ее домена. y q4 - x2 This definition is consistent with the limitations in the border points of the regions in the plane and in space, as we will see in chapter 14. When the th area is the interval to the left of c, for example (a, c' or (a, c), then we say that it has a limit in c, if it has a left limit in c. Similarly, if the q area is the interval to the right of c, such as q, b) or (c, b), then we say that q has a limit in c If it has the right limit in c. -2 0 x EXAMPLE 1 Domain q(x) 24 - x2 is 3 -2, 24; his graph is a semicircle 2 in figure 2.26. We have FIGURE 2.26 lim 24 - x2 No 0 and lim 24 - x2 and 0. In (x) 24 - x2 has the right limit of 0 xS -2 x S 2- on x 2 and left limit 0 on x No 2 (Example 1). This function has a two-way limit at each point (-2, 2). It has a left limit on x No 2 and the right limit on x -2. The function has no left limit on x -2 or a right limit on x No 2. It has no two-way limit of either -2 or 2 because it is not determined on either side of these points. At the domain boundary points where the domain is a point on one side of the point, we have limxS-2 24 - x2 and limxS2 24 - x2 0. The q function has a limit of x q - 2 and on x q 2. One-way limits have all properties listed in Theorem 1 in section 2.2. The right limit of the sum of the two functions is the sum of their right border and so on. Theorems for limits of polynomial and rational functions hold with one-sided limitations, like the Sand-Wich theorem. Unilateral restrictions are related to restrictions in the interior as follows. THEOREM 6 Suppose that function f is determined at an open interval containing c, except perhaps the c. Then q(x) has a limit as x will approach c if and only if it has a left-hand and right-hand limits there and these one-way limits are equal: lim q(x) L 3 lim (x) L and lim (x) L. x S-x- x-X-Esque 6 In the point of its domain, the function has a limit when it has an appropriate one-way limit. y EXAMPLE 2 For the function, on the graph in figure 2.27, y f (x) At x 0: limxS0- q (x) does not exist, q is not defined to the left of x y 0. 2 limxS0 (x) 1, at x 1: limxS0 (x) 1. Has the right limit on x 0. 1 q has a limit at the end point of the domain x y 0. On x 2: 0 1234 x limxS1- q (x) 0, even if y (1) 1. FIGURE 2.27 Feature Graph At x No. 3: limxS1 (x) 1, in example 2. On x 4: limxS1 (x) does not exist. Right and left boundaries are not equal. limxS2- q (x) 1, even though q (2) 2. limxS2 (x) - 1. limxS3- q (x) - limxS3 (x) - limxS3 (x) limxS4- q (x) No 1, even though q (4) ≠ 1. limxS4' (x) does not exist, is not determined to the right of x No. 4. limxS4 (x) 1. Has a limit at the end point of domain x No. 4. At any other point C at 3 0, 44 , y (x) has a limit (c).2.4 Unilateral limits 67 with accurate definitions of unilateral restrictions The official definition of the limit in section 2.3 is easily changed for unilateral restrictions. L'e f (x) DEFINITIONS (a) Suppose the q domain contains an interval (c, d) L right c. We say that x (x) has the right limit L on C, and write f(x) lies in here L-e lim q(x) I xSc for all x q c if for each number e 7 0 exists accordingly number d 7 0 such that in here ,q(x) - L,6 e when c 6 x 6 c d d (b) Suppose the q(b) domain contains an interval (b c) to the left of c. We say that x (x) has a left limit L on c, and write 0 xx c'd FIGURE 2.28 Intervals associated with the lim (x) - L definition of the right limit. x S c- if for each number e 7 0 there is a corresponding number d 7 0, then 0 y (x) - L 0 6 e whenever c - d 6 x 6 c. y Definitions are illustrated in figures 2.28 and 2.29. L'e f (x) EXAMPLE 3 Prove that L f (x) lies lim 2x th 0. Here x S 0 L-E Solution Let e 7 0 be given. Here with 0 and L 0, so we want to find d 7 0 such that for all x s here 0 2x - 0 0 6 e whenever 0 6 x 6 d, or 0 x 2x 6 e whenever 0 6 x 6 d. 1x 0 so 0 1x 0 and 1x c-d FIGURE 2.29 Intervals those associated with the squares of both sides of this last inequality define the left limit. x 6 e2, если 0 6 x 6 d. f(x) - x Если мы выберем d e2, у нас есть 2x 6 e всякий раз, когда 0 6 x 6 d e2, e f (x) или L'0 x d e2 x 0 2x - 0 0 6 e всякий раз, когда 0 6 x 6 e2. FIGURE 2.30 lim 1x and 0 in example 3. According to the definition, it shows that limxS0 2x No 0 (Figure 2.30). x S 0 Features reviewed so far have been a kind of limit at each point of interest. In general, it should not be case.68 Chapter 2 Limits and continuity of example 4 Show that sin (1'x) has no limit as x approaches zero on both sides (Figure 2.31). y 1 0x y - Sin 1 x -1 FIGURE 2.31 The y - Sin (1'x) has neither the right hand nor the left limit, as x is approaching zero (Example 4). The graph here lowers values very close to the axis of y. Solution As it approaches zero, its reciprocal, 1'x, grows without borders and the value of sin (1'x) cycle repeatedly from -1 to 1. There is no single L to keep the function values closer to zero as x approaches zero. This is true even if we limit x to positive values or negative values. The function has neither the right limit nor the left limit on x 0. Limits involving (sin u), u central fact about (sin u) is that the radian measure its limit as u S 0 is 1. We can see it in figure 2.32 and confirm it algebraically with the Help of Sandwich Theorem. You will see the importance of this limit in section 3.5, which examines the instantaneous rate of change in trigono-metric functions. y 1 y - sin u (radians) u -3p -2p -p 2p 3p u T NOT TO SCALE 1 FIGURE 1 FIGURE 2.32 Chart q (u) -u (sin u) offers that right-P and left-hand limits as you berths 0 both 1. 1 tan y THEOREM 7-Limit odds sin U , U as u u 0 sin in sin u y 1 (u in radians) (1) u us0 OK (1, 0) x 1 Proof Plan is to show that the right and left sides of both 1. Then we'll know that bilateral limit is one as well. FIGURE 2.33 Ratio of TAZOA and Tanning y, and OA No. 1, so TA and Tan u. To show that the right limit is 1, we start with positive values of you less than page 2 (Figure 2.33). Please note that the area ΔOAP 6 area of the OAP 6 sector area ΔOAT.2.4 One-way limits 69 We can express these areas in terms of you as follows: Using radians to measure the angles of the area Δ the 5th-gt;OAP - 1 base height - 1 (1) (sin y) - 1 sin u, necessary in the equation (2): Area 2 2 2 sector OAP is u'2 only in the event if you are measured in radian. OAP Sector Area 1 r2u No 1 (1)2u u (2) 2 2 Area ΔOAT No 1 base height - 1 (tan u) - 1 tan u. 2 2 2 So 1 sin u 6 1 u 6 1 tan u. 2 2 This latest inequality goes the same way if you divide all three terms into a number (1'2) sin u that is positive, as 0 6 u 6 p. 2: 1 6 u 6 1 u . Sin y cos Taking mutual changes inequality: 1 7 sin y 7 cos u. u Since limuS0 cos u 1 (Example 11b, section 2.2), Theorem Sandwich gives lim sin at No 1. u u S 0 To consider the left limit, we remember that sin to you and you both odd functions (section 1.1). Thus, q (u) is an oats- oat function, with a symmetrical graph about the y axis (see Figure 2.32). This symmetry implies that the left limit of 0 exists and has the same meaning as the right limit: lim sin y No. 1 y lim sin y, u u s 0- u S 0 so limuS0 (sin u) y 1 Theorem 6. EXAMPLE 5 Show that (a) lim cos y - 1 y 0 and (b) lim sin 2x 2 . y 5x - 2 sin2 (y'2) y ys0 yS0 - lim sin u sin u let u y'2. u uS0 -(1)(0) 0. Eq. (1) and Example 11a in section 2.2 (b) Equation (1) does not apply to the original faction. We need 2x in the denominator, not 5x. We produce it by multiplying the numerator and denominator by 2'5: #lim #x S 0 sin 2x lim (2 x 5) sin 2x 5x (2 x 5) 5x xS0 2 lim sin 2x Eq. (1) applies with 5 2x u 2x. xS0 2 (1) 2 . 5 5 570 Chapter 2 Limits and Continuity EXAMPLE 6 Find Lim tan t sec 2t. 3t tS0 Solution From the definition of tan t and sec 2t, we have - #limtant sec 2 t y lim 1 1 sin t 1 3t 3 t cos t cos ts 2t ts ts 2t ts ts 1 (1) (1) 13. Eq. (1) and Example 11b 3 in section 2.2 EXAMPLE 7 Show that for non-grain constants A and B. lim sin au and . Sin boo b us0 solution lim sin au lim sin au au bu 1 Multiply and split on au and bu. Sin bu au sin bu us0 us0 - lim sin au bu lim sin bu us0 us0 Lim (1) lim y y 1, s y bu b sin bu us0 ys0. B Exercise 2.4 Finding Limits Graphic 2. Which of the following statements about the function y q (x) 1. Which of the following statements about the function y q (x) on the graph are true here, and which are false? y the graph here are true, and which are false? 2 y y f (x) y f (x) 1 2x 1 -1 0 0 1 2 2 3x a. lim (x) 1 b. lim (x) does not exist. a. lim (x) - 1 b. lim (x) - 0 xS -1 xS2 xS -1 x S 0- c. Lim (x) 2 d. lim (x) - 2 c. lim (x) - 1 d. lim (x) - lim (x) xS2 x S 1- x S 0- x S 0' X S 0' lim (x) 1 f. lim (x) does not exist. e. lim (x) exists. f. lim (x) - 0 x S 1 xS1 xS0 xS0 g. lim (x) - lim (x) g. lim (x) lim (x) - 1 h. lim (x) - 1 x S 0' xS0 xS0 xS1 h. lim e (x) exists at each c in the open interval (- 1, 1. xSc i. lim (x) - 0 j. lim (x) - 2 i. lim (x) exists at each c in open interval (1, 3). xS1 x S 2- xSc k. lim (x) does not exist. l. lim (x) 0 j. lim (x) 0 k. lim (x) does not exist. x S -1- x S 2 x S -1- x S 32.4 One-way limits 71 3. Let q (x) 3 - x, x 6 2 6. Let g'x- 2x sin (1'x). cx No 1, x 7 2. y y q x 2 y'3-x y x sin 1 x 3 y 1 2p 0 12 12 12 1x x pp 0 24 a. Find limxS2 (x) and limxS2- q (x). -1 u - x b. Is there limxS2 (x)? If so, what is it? If not, why not? C. Find limxS4- (x) and limxS4 (x). D. Is there limxS4 (x)? If so, what is it? If not, why not? 3 - x, x62 a. Is there limxS0'g(x)? If so, what is it? If not, why not? 4. Let q(x) d 2, x'2 b. Is there limxS0-g/x)? If so, what is it? If not, why not? C. Is there limxS0 g(x)? If so, what is it? If not, why not? 2x, x 7 2. x3, x≠1 y 7. a. Chart No (x) e 0, x No 1. y'3-x b. Find limxS1- q(x) and limxS1 (x). 3 c. Is there limxS1 (x)? If so, what is it? If not, why not? -2 0 x a. Schedule No (x) - e 1 - x2, x≠1 2 2, x 1. y 2x b. Find limxS1 (x) and limxS1- q (x). C. Is there limxS1 (x)? If so, what is it? If not, why not? a. Find limxS2 (x), limxS2- q (x) and q (2). The schedule functions in exercises 9 and 10. Then answer these questions. Is there limxS2 (x)? If so, what is it? If not, why not? C. Find limxS -1- q (x) and limxS -1 (x). What is the domain and range of th? D. Is there limxS -1 (x)? If so, what is it? If not, why not? B. At what points c, if any, is there limxSc (x)? 5. Let q (x) 0, x... 0 c. At which moments there is a left limit, but not the right - c sin 1, x 7 0. the limit of the hand? x d. At what points is there the right limit, but not the left limit? y 21 - x2, 0... x61 1 9. W (x) - c 1, 1... x62 x-2 2, 0x x, -1 ... x 6 0, or 0 6 x ... 1 x 0 1 0. B(x) - c 1, x 6 - 1 or x 7 1 0, y5 0, x 0 sin 1 , x.0 Search for one-sided boundaries algebraically x Find limits in exercises 11-20. 21 11. x Lim a x 21 12. лим A x - 1 x x x 2 a. Существует ли limxS0'(x)? If so, what is it? If not, why not? S -0.5- x S 1 b. Is there a limxS0-th (x)? If so, what is it? If not, why not? C. Is there limxS0 (x)? If so, what is it? If not, why not? 1 3. x lim x x x 1b a2xx2 and x5b - S -2'72 Chapter 2 Limitations and Continuity 14. lim a x 1 1b ax and 6b a3 - xb 3 9. Lim has a braid at 40. lim sin y crib 2u x 7 x S 1- uS0 uS0 1 5. Lim 2h2 4h 5 - 25 41. lim tan 83xx 42. lim sin 3y cot 5y h sin y crib 4y h s 0 xS0 ys0 1 6. Lim 26 - 25h2 th 11h 6 4 3. lim tan u 44. lim u cot 4u h u2 cot 3u sin2 u cot2 2u h S 0- uS0 uS0 4 5. Lim 1 - cos 3x 46. lim cos2x - cos x 2x x2 1 7. a. lim (x- 3) 0x - 2 0 b. lim (x - 3) 0x - 20 xS0 xS0 - 2 xS -2 x S -2- x 2 Theory and examples 47. Once you know limxSa' (x) and limxSa-q (x) at the interior point 1 8. a. lim 22x (x - 1) b. lim 22x (x - 1) 0x - 10 0x - 10 of the domain, do you then know limxSa (x)? Give reasons x S 1 x S 1- for your response. 19. a. lim ̧sin x ̧b. lim ̧sin x̧ 4 8. If you know that limxSc (x) exists, can you find its value sin x sin x calculation limxSc'?'(x)? Give reasons for your response. xS0 x S 0- 4 9. Suppose the question is a strange function x. Knows that 20. a. lim ̧1 - cos x ̧b. lim cos x - 1 limxS0 (x) 3 tell you something about limxS0- q (x)? Give rea-cos x - 1 ̧cos x - 1̧sons for your answer. xS0 xS0 - 5 0. Suppose the question is an even feature x. Know that use the chart of the greatest function of the integrator y:x;, Figure 1.10 in limxS2- q (x) 7 tell you something about either limxS -2- (x) or section 1.1 to help you find limits in exercises 21 and 22. limxS -2 (x)? Give reasons for your response. 21. a. lim : u; b. lim :u; u u u S 3 'u 3- 2 2. a. lim (t - :t;) b. lim (t - :t;) Формальные определения односторонних пределов tS4t S 4- 5 1. Учитывая e 7 0, найдите найдите I (5, 5 g), d 7 0, so if x lies in I, then 2x - 5 6 e. What is the limit of being checked and the use of lim sin U No. 1, what is its value? Uu0 U 52. Given e 7 0, find interval I (4 - d, 4), d 7 0, so if you find limits in exercises 23-46. x lies in I, then 24 - x 6 e. What is the limit checked and what is its cost? 23. Lim sin 22u 24. lim sin kt (k constant) uS0 22u tS0 2 5. lim sin4y3y 26. Use the definitions of right and left borders to prove the sin of the 3h limit of statements in exercises 53 and 54. yS0 h S 0- 2 7. lim tanx2x 28. lim 2t t 53. lim x -1 54. lim x - 2 1 tan 0x - 20 xS0 tS0 xS0- 0 x 0 x S 2 29. lim xcossc52xx 30. lim 6x2 (cot x)(csc 2x) 5 5. The greatest function of integer Find (a) limxS400 :x ; and (b) limxS400- :x;; then use definition limits to verify your findings. xS0 xS0 (c) Based on your findings in parts (a) and b), you can say 3 1. lim x'xxccoossxx 32. lim x2 - x' sin x sin 2x xS0 xS0 anything about limxS400 :x;? Give reasons for your response. 3 3. lim 1 - cos u 34. lim x - x cos x 5 6. Unilateral restrictions Let q (x) x2 sin (1 x), x60 sin 2u sin2 3x 2x, x 7 0. uS0 xS0 e 3 5. lim sin (1 - cos t) 36. lim sin (sin h) Find (a) limxS0 (x) and (b) limxS0- q (x); Then use the limit defini- 1 - cos t sin h tions to verify your findings. (c) Based on your findings in tS0 hS0 3 7. lim ssiinn2uu 38. lim sin 5x parts (a) and (b) can you say anything about limxS0 (x)? Give sin 4x reasons for your response. uS0 xS0 2.5 Continuity When we build the value of the function generated in the lab or collected in the field, we often connect the built dots with a continuous curve to show that function values were probably at points that we didn't measure (Figure 2.34). By doing so, we assume that we are working with a continuous function, so its results change regularly and 250 No. 4 sequentially with inputs, and do not jump sharply from one value to another without taking on the values between them. Intuitively, any function of y q(x) whose graph can be 125 no3 sketched over its domain in one continuous motion is an example of continuous function. Such functions play an important role in the study of calculus and its application. No 2 05 t FIGURE 2.34 Connecting built dots.2.5 Continuity 73 at Continuity at point 2 y f (x) To understand continuity, it helps to consider such a function in figure 2.35, the limits of which we explored in Example 2 in the last section. 1 EXAMPLE 1 How much function in figure 2.35 doesn't seem 0 1234 x continuous? Explain why. What happens to other numbers in the domain? FIGURE 2.35 Function is not a solution First we observe that the function area is closed intervals of 3 0, 44 , so we continuously on x No. 1, x No. 2, and x No. 4 will consider the numbers x during this interval. we immediately notice this (Example 1). there are breaks in the schedule on numbers x No. 1, x No 2, and x 4. A break at x No.1 appears as a jump, which we define later as a gap jump. The break in x No. 2 is called a removable break, because by changing the definition of function at this one point, we can create a new function that is continuous on x No. 2. Similarly, x No. 4 is a removable gap. Numbers on which the schedule has breaks: At the point of the interior x No. 1, the function has no limit. It has both the left limit, limxS1-q (x) 0, and the right limit, limxS1 (x) 1, but the limits are different, resulting in a jump in the chart. The feature is not continuous on x No.1. However, the no(1) No. 1 function is equal to the right limit, so the function is continuous on the right at x No. 1. At x No 2, the feature has a limit, limxS2 (x) No. 1, but the value of func-tion is No (2) 2. The limit and function values are not the same, so there is a gap on the chart, and there is no continuous value on x No. 2. At x No. 4, the function has a left limit at this right endpoint, limxS4-q (x) 1, but again the function value No. 4) No. 1 differs from the 2 limit. We again see a gap in the function graph at this end point, and the function is not continuous on the left. Numbers on which the graph q has no breaks: At x No 3, the feature has a limit, limxS3 (x) 2. In addition, the limit is the same value as the function there, No. (3) and 2. The function is continuous on x No. 3. At x No 0, the feature has the right limit at this left end point, limxS0 (x) No. 1, and the function value is the same, th (0) No. 1. The function is continuous on the right at x 0. Because x No 0 is the left endpoint of the domain function, we have that limxS0 (x) 1 and so e continuous on x 0. In all other numbers x q c in the domain, the function has a limit equal to the function value, so limxSc q (x) For example, limxS5 (x) 1 25 2 and 3 . No breaks 2 are displayed on the function graph on any of these numbers, and the function is continuous in each of them. The following definitions capture the ideas of continuity that we have seen in Example 1. DEFINITIONS Let C will be a real number that is either the inside point or the end point of the interval in the domain. The function is continuous on c if lim q (x) q (c). xSc Function is right-continuous on c (or continuous on the right) if lim (x) q (c). xSc' Function - left-continuous on C (or continuous left) if lim (x) q (c). x S c-74 Chapter 2 Limitations and Continuity Bilateral Continuity Function in example 1 continuous on each x in 3 0, 44 except x 1, 2, and from right continuity left 4. It is but not left-continuous on x No 1, neither right nor left-hander on x No 2, nor left-continuous at x No 4. y f(x) From the theorem 6, it follows immediately that the function of q is continuous on the interior of the bx point c of the interval in its domain if and only if it is both right-continuous and left-continuously on C (Figure 2.36). We say that the function is continuous during the closed FIGURE 2.36 Continuity interval at points A, b, 3 a, b4 if it is right-continuous on, left-continuous on b, and continuous at all internal and c. interval points. This definition also applies to endless closed intervals of 3 a, q) and (-q, b4), but we are talking about only one endpoint. If the function is not continuous at point y c of its domain, we say that q is ripping by C, and that f has a c. y q'4 - x2 Note that the function can be continuous, right-continuous, or left-continuous only at point C, for which q'c is defined. 2 EXAMPLE 2 Feature No. 24 - x2 is continuous over its domain 3 -2, 24 -2 0 2x (Figure 2.37). It's right-continuous on x -2, and left-continuous at x No 2. THE FIGURE 2.37 feature that the EXAMPLE 3 step function unit U (x), on the graph in figure 2.38, is right-c ontin continuous over its domain at x No 0, but is neither left-continuous nor continuous there. It has a jump gap on (Example 2). x 0. y At the point of the interior or the end point of the interval in the domain function is continuum-ous at the points where it passes the next test. 1 y U (x) 0x Continuity Test Function q (x) is continuous at point x q c if and only if it meets the next FIGURE 2.38 Feature three conditions. that has a gap jump on origin (Example 3). 1. C (c) exists (c lies in the domain . . . . 2. limxSc (x) exists (it has a limit like x S c). 3. limxSc q (x) q (c) (limit equals function value). For unilateral continuity, the limits in Parts 2 and 3 of the test must be replaced by appropriate unilateral restrictions. y EXAMPLE 4 Function y: x; section 1.1 in figure 2.39. This is intermittent on every integer n, because the left and right limits are not equal as x s n: 3 2 y :x; lim :x; n - 1 and lim :x; No 1 1234 x S n- x S n -1 x C: n; n, the biggest integer function is right-continuous on every n integer (but not left-continuous). The biggest integer function is continuous on every real number behind other integers. For example, lim :x; No 1 : 1,5 ; . x S 1.5 -2 Overall, if n - 1 6 c 6 n, n integer, then FIGURE 2.39 Largest integrator lim :x; n - 1 : c;. The function is continuous at each point of noninte-ger. It is right-continuous, but not xSc left-continuous, at each point of the integrator (Example 4). Figure 2.40 shows a few common ways in which the feature may not be Units. The feature on Figure 2.40a is continuous at x No 0. The 2.40b feature does not contain x No 0 in the domain. It would be continuous if its domain was expanded2.5 Continuity 75 so that q (0) No.1. The function on figure 2.40c would be continuous if No (0) were 1 instead of 2. The gap in the 2.40c picture is removable. The feature has a limit like x S 0, and we can remove the gap by setting No (0) equal to that limit. The gaps in figure 2.40d through f are more severe: limxS0 (x) does not exist, and there is no way to improve the situation by properly identifying in 0. The step function in figure 2.40d has a jump gap: one-way limits exist, but have different meanings. The q(x) 1'x2 feature in figure 2.40e has an infinite disconti-annuity. The 2.40f function has a oscillating gap: it fluctuates so much that its values are closer to each number of 3 -1, 14 as x S 0. Since it does not approach one number, it has no limit as x approaches 0. y y y y f(x) y f(x) 2 y f(x) 1 y f(x) 1 0x 0x 0x 0x (a) (b) (c) (c) g) y y y - sin 1 1 x y f (x) 1 x2 -1 0 0x -1 (e) (f) FIGURE 2.40 Function in (a) continuous in x No 0; functions in (b) through (f) are not. y Continuous features at No.1 We now describe the continuity behavior of the function throughout the domain, not x only at one point. We define a continuous function that is continuous at every point in its field. It's a function property. The feature always has a speci-fied domain, so if we change the domain, then we will change the function and this can change its continuity property as well. If the function is intermittent at one or more points in its domain, we say it's an intermittent function. 0x EXAMPLE 5 FIGURE 2.41 Function (a) Function q(x) 1'x (Figure 2.41) is a continuous function because it is continuum-ous at every point of its domain. Point x No. 0 is not in the q(x) 1'x domain continuously above its q, so it is not continuous at any interval containing x y 0. Also, there is no way to have a natural domain. It is not defined to expand to a new feature that is defined and continuous on x No 0. The q function does not have a removable gap on x q 0. origin, so it is not a continuous (b) q(x) x identification function and permanent functions are continuous everywhere at any interval containing x No. 0 Example 3, Section 2.3. (Example 5). The algebraic combinations of continuous functions are continuous wherever they are defined.76 Chapter 2 Limits and Continuity of THEOREM 8-Properties of Continuous Functions If functions and g are continuous on x q c, then the following algebraic combinations are continuous on x y 1. Amounts: 2g. Differences: H-Gg, g any number k 3. Constant k for 4. Products: 5 euros. Odds: eg, subject to g(c) ≠ 0 6. Powers: n, n positive integer 7. Roots: 2n, provided that it is determined at the interval containing c, where n is a positive integer Most results in theorem 8 are followed from the limit rules in theorem 1, section 2.2. For example, to prove the amount of property we have lim (i g) (x) - lim (x (x) g (x)) xSc xSc (x) lim (x) lim g(x) xSc sum rule, Theorem 1 q (c) g(c) Continuity This shows that q g is continuous. EXAMPLE 6 (a) Every polynomial P(x) - anxn - an-1xn-1 - g a0 is continuous because lim P (x) - P/c) theorem 2, section 2.2. xSc (b) If P (x) and q (x) are polynomials, then the rational function of P(x) (x) is continuous wherever it is defined (q(c) ≠ 0) theorem 3, section 2.2. EXAMPLE 7 Function q(x) 0 x 0 continuous. If x 7 0, we have q (x) x, polynomial. If x 6 0, we have q (x) -x, another polynomial. Finally, in the beginning, limxS0 0 x 0 x 0 0 0 . The functions of sin x and y cos are continuous on x No 0 following the example of section 2.2. Both functions are continuous everywhere (see Exercise 64). From the theorem 8 it follows that all six trigonometry functions are continuous wherever they are determined. For example, y th tan x continuous on g ∪ (-p-2, p-2) ∪ (p. 2, 3p-2) ∪. If q (x) is a continu-ous on x y s and g (x) is continuous on x y (c), then g ∘ y also continuous on x y s (figure 2.42). In this case, mr. ∘ limit, as x S c (i.c)). THEOREM 9-Compositions of Continuous Functions If th continuous on C and G continuous on c(c), the composition of G ∘ th continuous on c.2.5 Continuity 77 g f Continuously on c f g Continuously on C (c) c (c) g (f(c)) FIGURE 2.42 Continuous Functions Compositions are continuous. Intuitively, Theorem 9 is reasonable, because if x is close to c, then th (x) is close to (c), and since g is continuous on th (c), it follows that g (I (x)) is close to g (i (c)). The continuity of compositions takes place for any finie number of fureons. The only requirement is that each function be continuous where it is applied. Sketches of proof of the theorem 9 are given in Exercise 6 in Annex 4. EXAMPLE 8 Show that the following features are continuous on their natural domains. a) y 2x2 - 2x - 5 (b) y - x2'3 x4 (c) y' x - 2' (d) y' x sin x2 - 2 x2 - 2 y Solution (a) Square Root function continuous at 3 0, q) because it is the root of the continuum function 0.4 ous identity q(x) x (Part 7, Theorem 8). This feature is that 0.3 composition of polynomial q (x) x2 - 2x - 5 with square root function g/t 2t , and is continuous on its Domain. 0.2 (b) The numerator is the cubic root of the identification function in the square; denominator 0.1 all-positive polynomial. Thus, the ratio is continuous. -2p -p 0 p 2p x (c) The ratio (x - 2) (x2 - 2) is continuous for all x ≠ 22, and the function is the composition of this coefficient with a continuous function of absolute value FIGURE 2.43 Chart assumes that (Example 7). y y 0 (x sin x) (x2 No 2) 0 is continuous (d) Because the function of sine waves everywhere is continuous (Exercise 64), the number-tor term x sin x is a product of continuous functions, and the term denominator (Example 8d). x2 No 2 is an all-positive polynomial. This value is part of a continuous function ratio with a continuous absolute function (Figure 2.43). The 9 is actually a consequence of the more general result that we are now proving. It states that if the limit in th (x) as x C approaches is b, the c omposition g ∘ f limit as x c approaches is G(b). THEOREMA 10 - Limits of continuous functions If limxSc q(x) b and g are continuous at point b, then lim g(i/x) g/b). xSc78 Chapter 2 Limitations and Continuity Proof Let e 7 0 be given. Since g is continuous on b, there is a number d1 7 0, that 0 g(y) - g(b) 0 6 e whenever 0 6 0 y - b 0 6 d1. limySb g(y) g (b) as g is continuous in y q b. Since limxSc (x) b, there is d 7 0, that 0 y (x) - b 0 6 d1 whenever 0 6 0 x - c 0 6 d. Definition limxSc (x) y b If we let y q (x), then we have that 0 y - b 6 d1 each time, when 0 6 0 x - c 0 6 d, which implies from the first statement that 0 g(y) - g(b) 0 y 0 g (i) - g(b) 0 6 e, when - always 0 6 0 x - c 0 6 d. From the definition limit follows that limx g (i.x) g (b). This provides evidence for a case where C is the internal point of the domain f. The case where c is the end point of the domain is completely similar, using the appropriate one-way limit instead of a one-way limit. EXAMPLE 9 Application of the theorem 10, we have lim cos a2x - sin a32p - xb b - cos a lim 2x - x lim sin a32p - xb b x S p/2 x S p/2 S p/2 S p/2 cos (p - sin 2p) Intermediate value theorem value for continuous function A function, as they say, the intermediate value of the property, if whenever it takes on two values, it also takes on all the values in between. THEOREM 11 - Intermediate value theorem of value for continuous functions If - continuous function at a closed interval of 3 a, b4 , and if y0 is any value between z (a) and q (b), then y0 q(c) for some c in 3 a, b4 . y y f(x) f (b) y0 c bx f (a) 0a Theorem 11 says that continuous functions during limited closed intervals have an inter-intermediary property value. The geometrically intermediate theorem says that any the y and y0 line crosses the axis between the numbers and will cross the curve at q (x) at least once during the 3 a, b4 interval. Proof of the intermediate value theorem depends on the completeness of the real numbers system. The fullness property implies that real numbers have no holes or gaps. In contrast, rational numbers do not satisfy the fullness of p roperty,2.5 Continuity 79 y and a function defined only by rationalistics will not satisfy the intermediate value of A- 3 orem. Cm. Appendix 7 for discussion and examples. 2 Interval continuity is important for the 11 theorem. If there can be no continuum-ous at one point of the interval, the conclusion of the theorem may fail, as it does for 1 function, on the graph at figure 2.44 (choose y0 as any number between 2 and 3). 0 1234 x Consequence for graphics: Communication 11 theorem implies that a function graph that is continuous at intervals cannot have any breaks in the interval. This FIGURE 2.44 function will be connected - a single, continuous curve. It will not have jumps such as those that are on the chart of the greatest function of the integrator (Figure 2.39), or individual branches like q (x) 2x - 2, 1... x62, found on chart 1x (Figure 2.41). e 3, 2... X... 4 Consequence for root search We call the solution of equation No (x) 0 root does not assume all the values between the equation or the zero function. The intermediate theorem tells us that if the th is continuous, any interval at which the sign changes contains zero function. No (1) No 0 and No (4) 3; it misses everything somewhere between the point where the continuous function is positive, and the second point where it is negative, the function should be zero. values between 2 and 3. In practical terms, when we see a graph of continuous function to cross the horizon-tal axis on the computer screen, we know that it is not stepping over. Indeed, there is a point where the function value is zero. THE EXAMPLE 10 Show that there is a root of the equation x3 - x - 1 and 0 between 1 and 2. Solution Let q (x) x3 - x - 1. Since No (1) No. 1 - 1 - - - 1 6 0 and No (2) 23 - 2 - 1 - 5 7 0, we see that y0 and 0 is a value between q (1) and q (2). Since th is polynomial, it is continuous, and the intermediate value of theorem says that there is zero 1 to 2. Figure 2.45 shows the result of zooming in to find the root near x 1.32. 51 1 1.6 -1 2 -2 -1 (b) (a) 0.003 0.02 y 1.320 1.330 1.3240 1.3240 1.3240 1.3 248 4 y 4 - x2 3 2 - 0.02 - 0.003 y q2x 5 1 (c) (d) 0 c 2x FIGURE 2.45 Scale on zero function q(x) x3 - x - 1. The zero is next to x 1.3247 (Example 10). FIGURE 2.46 Curved EXAMPLE 11 Use an intermediate theorexy to prove that the equation y 22x y 5 and y 4 - x2 22x 5 - 4 - x2 have the same value in x q c, where 22x 5 - x2 - 4 0 (example 11). has a solution 2.46.80 Chapter 2 Limits and Continuity Solution We rewrite the equation as 22x No 5 x2 - 4 x 0, and set th (x) 22 x 5 x x 2 - 4. Now g(x) 22x No. 5 is continuous at intervals of 3 -5'2, q), as it is formed as a composition of two continuous functions, a square root function with non-unegiate linear function y and 2x 5. Then it's the sum of the function g and the square function y x2 - 4, and the quadrangle function is continuum for all x values ≈. Note that at the end closed interval 3 0, 24 ⊂ 3 -5'2, q). Since the value of y0 No. 0 is between the numbers No(0) - -1.76 and No (2) - 3, according to the intermediate theorem there is a number c∈ 3 0, 24 such that q (c) 0. Number C decides the original equation. Continuous extension to the point Sometimes the formula describing the function does not make sense at point x q q c. However, it would be possible to expand the q area, turn on x q c, creating a new function that is continuous on x q c. For example, the function of y q (x) (sin x) is continuous at every point except x y 0, since x No. 0 is not in its domain. Since y (sin x) has a final limit like x S 0 (Theorem 7), we can extend the function domain to turn on point x No. 0 so that the extended function is continuous on x No. 0. We define the new sin function x, x≠0 Same as the original function for x ≠ 0 x x 0. Value at domain point x 0 F (x) No. 1, New feature F (x) is continuous on x No. 0 because lim sin x (0), x xS0 to meet the requirements for continuity (Figure 2.47). yy (0, 1) f (x) (0, 1) F(x) a- p2b ap2, p2b a- p2b ap2, p2b 2b 2 p 2 p 2 p 2 - p 0 x - p 0 x 2 (a) 2 (b) FIGURE 2.47 (a) Graph q(x) X... p-2 does not include a point (0, 1) because the function is not defined on x No 0. b) We can expand the domain to include x No 0 by defining the new function F (x) with F (0) and F (x) and x (x) each where else. Note that F (0) - limxS0 (x) and F (x) - is a continuous function at level x No. 0. More generally, a function (such as a rational function) may have a limitation at a point where it is not defined. If q (c) is not defined, but limxSc q (x) L exists, we can identify a new function F (x) by rule F (x) q (x) q (x), if x is in the e L domain, if x q.2.5 Continuity 81 Function F continuous on x q q c. This is called continuous expansion from th to x c. For rational functions, continuous extensions are often found by the cancellation of com-factors in the numerator and the denominator. EXAMPLE 12 Show that q (x) x2 x - 6, x≠2 x2 - 4 y has a continuous extension to x No 2, and find that extension. 2g x2 x x - 6 The question (2) is not defined if x ≠ 2 we have x2 - 4 x (x) x2 x 6 x (x - 2) (x 3) x 23. x2 - 4 (x - 2) (x - 2) x -1 0 1 2 3 4x (a) New feature y 2 y' x'3 F (x) x 3 x 2 x 2 41 equals q (x) for x ≠ 2, but is continuous at x No 2, having there value of 5'4. Thus, -1 0 1 2 3 4x F is a continuous extension to x 2, and (b) lim x2 - x - 6 - lim (x) 5 . x2 - 4 xS2 xS2 FIGURE 2.48 (a) Chart th (x) and (b) graph th graph shown in figure 2.48. Continuous Expansion F has the same schedule of its continuous F(x) expansion (Example 12). except without opening on (2, 5'4). In fact, F is a function lengthened through the missing domain point on x No. 2 to give a continuous function over a large domain. Exercises 2.5 Continuity from Exercise Graphics 5-10 refer to the function in exercises 1-4, say, whether the function is a continuous schedule on 3-1, 34. If not, where can it not be continuous and why? x2 - 1, -1 ... x 6 0 2x, 06x61 1. 2. x'1 (x) - e 1, 16x62 yy - 2x - 4, 26x63 0, y f (x) y g(x) 2 2 1, on the graph on the accompanying figure. -1 0 1 2 3x 1 1 2 3x y -1 0 y f(x) 3. 4. 2 (1, 2) y y y 2x y -2x - 4 2 y k(x) 2 (1, 1) y h(x) x -1 0 123 1 1 y x2 - 1 -1 -1 0 -1 0 1 1 1 2 3x 1 2 3x 3x 3x thomas calculus 14th edition reddit