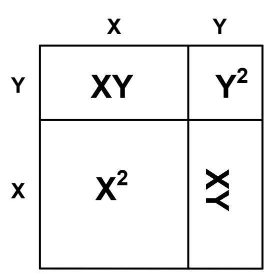
The Air Force Research Laboratory

The Handbook of Essential Mathematics

Formulas, Processes, and Tables
Plus Applications in Personal Finance



$$(X + Y)^2 = X^2 + 2XY + Y^2$$

Compilation and Explanations: John C. Sparks

Editors: Donald D. Gregory and Vincent R. Miller

The Handbook of Essential Mathematics

Air Force Publication 2006

Forward

Wright-Patterson Air Force Base (WPAFB) has enjoyed a lengthy and distinguished history of serving the Greater-Dayton community in a variety of ways. One of these ways is through the WPAFB Educational Outreach (EO) Program, for which the Air Force Research Laboratory (AFRL) is a proud and continuous supporter, providing both technical expertise (from over 2000 practicing scientists and engineers) and ongoing resources for the various programs sponsored by the WPAFB Educational Outreach. The mission of the WPAFB EO program is

To form learning partnerships with the K-12 educational community in order to increase student awareness and excitement in all fields of math, science, aviation, and aerospace; ultimately developing our nation's future scientific and technical workforce.

In support of this mission, the WPAFB EO aspires to be the best one-stop resource for encouragement and enhancement of K-12 science, math and technology education throughout the United States Air Force. It is in this spirit that AFRL offers The Handbook of Essential Mathematics, a compendium of mathematical formulas and other useful technical information that will well serve both students and teachers alike from early grades through early college. It is our sincere hope that you will use this resource to either further your own education or the education of those future scientists and engineers so vital to preserving our cherished American freedoms.

LESTER MCFAWN, SES
Executive Director
Air Force Research Laboratory

Introduction

Formulas! They seem to be the bane of every beginning mathematics student who has yet to realize that formulas are about structure and relationship—and not about memorization. Granted, formulas have to be memorized; for, it is partly through memorization that we eventually become 'unconsciously competent': a true master of our skill, practicing it in an almost effortless, automatic sense. In mathematics, being 'unconsciously competent' means we have mastered the underlying algebraic language to the same degree that we have mastered our native tongue. Knowing formulas and understanding the reasoning behind them propels one towards the road to mathematical fluency, so essential in our modern high-tech society.

The Handbook of Essential Mathematics contains three major sections. Section I, "Formulas", contains most of the mathematical formulas that a person would expect to encounter through the second year of college regardless of major. In addition, there are formulas rarely seen in such compilations, included as a mathematical treat for the inquisitive. Section I also includes select mathematical processes, such as the process for solving a linear equation in one unknown, with a supporting examples. Section II, "Tables", includes both 'pure math' tables and physical-science tables, useful in a variety of disciples ranging from physics to nursing. As in Section I, some tables are included just to nurture curiosity in a spirit of fun. *In Sections I and II, each formula and table is enumerated for easy referral.* Section III, "Applications in Personal Finance", is a small textbook within a book where the language of algebra is applied to that everyday financial world affecting all of us throughout our lives from birth to death. *Note: The idea of combining mathematics formulas with financial applications is not original in that my father had a similar type book as a Purdue engineering student in the early 1930s*.

I would like to take this opportunity to thank Mr. Al Giambrone—Chairman of the Department of Mathematics, Sinclair Community College, Dayton, Ohio—for providing required-memorization formula lists for 22 Sinclair mathematics courses from which the formula compilation was partially built.

John C. Sparks
March 2006

Dedication

The Handbook of Essential Mathematics is dedicated to all
Air Force families

O Icarus...

I ride high...
With a whoosh to my back
And no wind to my face,
Folded hands
In quiet rest—
Watching...O Icarus...
The clouds glide by,
Their fields far below
Of gold-illumed snow,
Pale yellow, tranquil moon
To my right—
Evening sky.

And Wright...O Icarus...
Made it so—
Silvered chariot streaking
On tongues of fire leaping—
And I will soon be sleeping
Above your dreams...

August 2001: John C. Sparks

100th Anniversary of Powered Flight
1903—2003

Table of Contents

Section I: Formulas with *Select Processes*

Table of Contents cont

Table of Contents cont

Table of Contents cont

Section II: Tables

Section III: Applications in Personal Finance

Table of Contents cont

Appendices

Section I

Formulas with
Select Processes

Index to Processes

1. Algebra

1.1. What is a Variable?

In the fall of 1961, I first encountered the monster called x in my high-school freshman algebra class. The letter x is still a monster to many, whose real nature has been confused by such words as *variable* and *unknown*: perhaps the most horrifying description of x ever invented! Actually, x is very easily understood in terms of a language metaphor. In English, we have both proper nouns and pronouns where both are distinct and different parts of speech. Proper nouns are specific persons, places, or things such as John, Ohio, and Toyota. Pronouns are nonspecific persons or entities such as he, she, or it.

To see how the concept of pronouns and nouns applies to algebra, we first examine arithmetic, which can be thought of as a precise language of quantification having four action verbs, a verb of being, and a plethora of proper nouns. The four action verbs are addition, subtraction, multiplication, and division denoted respectively by $+,-,\cdot,\div$. The verb of being is called *equals* or *is*, denoted by $=$. Specific numbers such as 12, 3.4512, $23\frac{3}{5}$, $\frac{123}{769}$, 0.00045632, -45, , serve as the arithmetical equivalent to proper nouns in English. So, what is x? x is merely a *nonspecific number*, the mathematical equivalent to a pronoun in English. English pronouns greatly expand our capability to describe and inform in a general fashion. Hence, pronouns add increased flexibility to the English language. Likewise, mathematical pronouns—such as x, y, z, see **Appendix B** for a list of symbols used in this book—greatly expand our capability to quantify in a general fashion by adding flexibility to our language of arithmetic. Arithmetic, with the addition of x, y, z and other mathematical pronouns as a new part of speech, is called algebra.

<u>In Summary</u>: Algebra can be defined as a generalized arithmetic that is much more powerful and flexible than standard arithmetic. The increased capability of algebra over arithmetic is due to the inclusion of the mathematical pronoun x and its associates y, z, etc. A more user-friendly name for *variable* or *unknown* is *pronumber*.

1.2. Field Axioms

The field axioms *decree* the fundamental operating properties of the real number system and provide the basis for all advanced operating properties in mathematics. Let $a, b \& c$ be any three real numbers (pronumbers). The field axioms are as follows.

Properties	Addition +	Multiplication ·
Closure	$a+b$ is a unique real number	$a \cdot b$ is a unique real number
Commutative	$a+b=b+a$	$a \cdot b = b \cdot a$
Associative	$(a+b)+c=$ $a+(b+c)$	$(ab)c=$ $a(bc)$
Identity	$0 \Rightarrow a+0=a$	$1 \Rightarrow a \cdot 1 = a$
Inverse	$a \Rightarrow a+(-a)=0$ $\Rightarrow (-a)+a=0$	$a \neq 0 \Rightarrow a \cdot \frac{1}{a}=1$ $\Rightarrow \frac{1}{a} \cdot a = 1$
Distributive *or Linking Property*	$a \cdot (b+c) = a \cdot b + a \cdot c$	
Transitivity	$a=b \& b=c \Rightarrow a=c$ $a>b \& b>c \Rightarrow a>c$ $a<b \& b<c \Rightarrow a<c$	
Note: $ab=a(b)=(a)b$ are alternate representations of $a \cdot b$		

1.3. Divisibility Tests

Divisor	Condition That Makes it So
2	The last digit is 0,2,4,6, or 8
3	The sum of the digits is divisible by 3
4	The last two digits are divisible by 4
5	The last digit is 0 or 5
6	The number is divisible by both 2 and 3
7	The number formed by adding five times the last digit to the "number defined by" the remaining digits is divisible by 7**
8	The last three digits are divisible by 8
9	The sum of the digits is divisible by 9
10	The last digit is 0
11	11 divides the number formed by subtracting two times the last digit from the " " remaining digits**
12	The number is divisible by both 3 and 4
13	13 divides the number formed by adding four times the last digit to the " " remaining digits**
14	The number is divisible by both 2 and 7
15	The number is divisible by both 3 and 5
17	17 divides the number formed by subtracting five times the last digit from the " " remaining digits**
19	19 divides the number formed by adding two times the last digit to the " " remaining digits**
23	23 divides the number formed by adding seven times the last digit to the " " remaining digits**
29	29 divides the number formed by adding three times the last digit to the remaining digits**
31	31 divides the number formed by subtracting three times the last digit from the " " remaining digits**
37	37 divides the number formed by subtracting eleven times the last digit from the " " remaining digits**
**These tests are iterative tests in that you continue to cycle through the process until a number is formed that can be easily divided by the divisor in question.	

1.4. Subtraction, Division, Signed Numbers

1.4.1. Definitions:

Subtraction: $a - b \equiv a + (-b)$

Division: $a \div b \equiv a \cdot \frac{1}{b}$

1.4.2. Alternate representation of $a \div b$: $a \div b \equiv \frac{a}{b}$

1.4.3. Division Properties of Zero

Zero in numerator: $a \neq 0 \Rightarrow \frac{0}{a} = 0$

Zero in denominator: $\frac{a}{0}$ is *undefined*

Zero in both: $\frac{0}{0}$ is *undefined*

1.4.4. Demonstration that division-by-zero is *undefined*

$\frac{a}{b} = c \Rightarrow a = b \cdot c$ for all real numbers a

If $\frac{a}{0} = c$, then $a = 0 \cdot c \Rightarrow a = 0$ for all real numbers a, an algebraic impossibility

1.4.5. Demonstration that attempted division-by-zero leads to erroneous results.

Let $x = y$; then multiplying both sides by x gives

$x^2 = xy \Rightarrow$

$x^2 - y^2 = xy - y^2 \Rightarrow$

$(x - y)(x + y) = y(x - y)$

Dividing both sides by $x - y$ where $x - y = 0$ gives

$x + y = y \Rightarrow 2y = y \Rightarrow 2 = 1$.

The last equality is a false statement.

1.4.6. Signed Number Multiplication:

$(-a) \cdot b = -(a \cdot b)$

$a \cdot (-b) = -(a \cdot b)$

$(-a) \cdot (-b) = (a \cdot b)$

1.4.7. Table for Multiplication of Signed Numbers: the *italicized words* in the body of the table indicate the resulting sign of the associated product.

Multiplication of $a \cdot b$		
Sign of a	Sign of b	
	Plus	Minus
Plus	*Plus*	*Minus*
Minus	*Minus*	*Plus*

1.4.8. Demonstration of the algebraic reasonableness of the laws of multiplication for signed numbers. In both columns, both the middle and rightmost numbers decrease in the expected logical fashion.

$(4) \cdot (5) = 20$	$(-5) \cdot (4) = -20$
$(4) \cdot (4) = 16$	$(-5) \cdot (3) = -15$
$(4) \cdot (3) = 12$	$(-5) \cdot (2) = -10$
$(4) \cdot (2) = 8$	$(-5) \cdot (1) = -5$
$(4) \cdot (1) = 4$	$(-5) \cdot (0) = 0$
$(4) \cdot (0) = 0$	$(-5) \cdot (-1) = 5$
$(4) \cdot (-1) = -4$	$(-5) \cdot (-2) = 10$
$(4) \cdot (-2) = -8$	$(-5) \cdot (-3) = 15$
$(4) \cdot (-3) = -12$	$(-5) \cdot (-4) = 20$
$(4) \cdot (-4) = -16$	$(-5) \cdot (-5) = 25$
$(4) \cdot (-5) = -20$	$(-5) \cdot (-6) = 30$

1.5. Rules for Fractions

Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions with $b \neq 0$ and $d \neq 0$.

1.5.1. Fractional Equality: $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

1.5.2. Fractional Equivalency: $c \neq 0 \Rightarrow \frac{a}{b} = \frac{ac}{bc} = \frac{ca}{cb}$

1.5.3. Addition (like denominators): $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

1.5.4. Addition (unlike denominators):

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+cb}{bd}$$

Note: bd is the common denominator

1.5.5. Subtraction (like denominators): $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$

1.5.6. Subtraction (unlike denominators):

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{cb}{bd} = \frac{ad-cb}{bd}$$

1.5.7. Multiplication: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

1.5.8. Division: $c \neq 0 \Rightarrow \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

1.5.9. Division (missing quantity): $\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$

1.5.10. Reduction of Complex Fraction: $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$

1.5.11. Placement of Sign: $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$

1.6. Partial Fractions

Let $P(x)$ be a polynomial expression with degree less than the degree of the factored denominator as shown.

1.6.1. Two Distinct Linear Factors:

$$\frac{P(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

The numerators A, B are given by

$$A = \frac{P(a)}{a-b}, B = \frac{P(b)}{b-a}$$

1.6.2. Three Distinct Linear Factors:

$$\frac{P(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

The numerators A, B, C are given by

$$A = \frac{P(a)}{(a-b)(a-c)}, B = \frac{P(b)}{(b-a)(b-c)},$$

$$C = \frac{P(c)}{(c-a)(c-b)}$$

1.6.3. N Distinct Linear Factors:

$$\frac{P(x)}{\prod_{i=1}^{n}(x-a_i)} = \sum_{i=1}^{n} \frac{A_i}{x-a_i} \quad \textit{with} \quad A_i = \frac{P(a_i)}{\prod_{\substack{j=1 \\ j\neq i}}^{n}(a_i - a_j)}$$

1.7. Rules for Exponents

1.7.1. Addition: $a^n a^m = a^{n+m}$

1.7.2. Subtraction: $\frac{a^n}{a^m} = a^{n-m}$

1.7.3. Multiplication: $(a^n)^m = a^{nm}$

1.7.4. Distributed over a Simple Product: $(ab)^n = a^n b^n$

1.7.5. Distributed over a Complex Product: $(a^m b^p)^n = a^{mn} b^{pn}$

1.7.6. Distributed over a Simple Quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

1.7.7. Distributed over a Complex Quotient: $\left(\frac{a^m}{b^p}\right)^n = \frac{a^{mn}}{b^{pn}}$

1.7.8. Definition of Negative Exponent: $\frac{1}{a^n} \equiv a^{-n}$

1.7.9. Definition of Radical Expression: $\sqrt[n]{a} \equiv a^{\frac{1}{n}}$

1.7.10. Definition when No Exponent is Present: $a \equiv a^1$

1.7.11. Definition of Zero Exponent: $a^0 \equiv 1$

1.7.12. Demonstration of the algebraic reasonableness of the definitions for a^0 and a^{-n} via successive divisions by 2. Notice the power decreases by 1 with each division.

$$16 = 32 \div 2 = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$$
$$8 = 16 \div 2 = 2 \cdot 2 \cdot 2 = 2^3$$
$$4 = 8 \div 2 = 2 \cdot 2 = 2^2$$
$$2 = 4 \div 2 \equiv 2^1$$
$$1 = 2 \div 2 \equiv 2^0$$

$$\tfrac{1}{2} = 1 \div 2 = \tfrac{1}{2^1} \equiv 2^{-1}$$
$$\tfrac{1}{4} = \left[\tfrac{1}{2}\right] \div 2 = \tfrac{1}{2^2} \equiv 2^{-2}$$
$$\tfrac{1}{8} = \left[\tfrac{1}{4}\right] \div 2 = \tfrac{1}{2^3} \equiv 2^{-3}$$
$$\tfrac{1}{16} = \left[\tfrac{1}{8}\right] \div 2 = \tfrac{1}{2^4} \equiv 2^{-4}$$

1.8. Rules for Radicals

1.8.1. Basic Definitions: $\sqrt[n]{a} \equiv a^{\frac{1}{n}}$ and $\sqrt[2]{a} \equiv \sqrt{a} \equiv a^{\frac{1}{2}}$

1.8.2. Complex Radical: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

1.8.3. Associative: $(\sqrt[n]{a})^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$

1.8.4. Simple Product: $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$

1.8.5. Simple Quotient: $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

1.8.6. Complex Product: $\sqrt[n]{a}\sqrt[m]{b} = \sqrt[nm]{a^m b^n}$

1.8.7. Complex Quotient: $\dfrac{\sqrt[n]{a}}{\sqrt[m]{b}} = \sqrt[nm]{\dfrac{a^m}{b^n}}$

1.8.8. Nesting: $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$

1.8.9. Rationalizing Numerator for $n > m$: $\dfrac{\sqrt[n]{a^m}}{b} = \dfrac{a}{b\sqrt[n]{a^{n-m}}}$

1.8.10. Rationalizing Denominator for $n > m$: $\dfrac{b}{\sqrt[n]{a^m}} = \dfrac{b\sqrt[n]{a^{n-m}}}{a}$

1.8.11. Complex Rationalization Process:

$$\frac{a}{b+\sqrt{c}} = \frac{a(b-\sqrt{c})}{(b+\sqrt{c})(b-\sqrt{c})} \Rightarrow$$

$$\frac{a}{b+\sqrt{c}} = \frac{a(b-\sqrt{c})}{b^2-c}$$

Numerator: $\dfrac{a+\sqrt{c}}{b} = \dfrac{a^2-c}{b(a-\sqrt{c})}$

1.8.12. Definition of Surd Pairs: If $a \pm \sqrt{b}$ is a radical expression, then the associated surd is given by $a \mp \sqrt{b}$.

1.9. Factor Formulas

1.9.1. Simple Common Factor: $ab + ac = a(b + c) = (b + c)a$

1.9.2. Grouped Common Factor:

$$ab + ac + db + dc =$$
$$(b + c)a + d(b + c) =$$
$$(b + c)a + (b + c)d =$$
$$(b + c)(a + d)$$

1.9.3. Difference of Squares: $a^2 - b^2 = (a + b)(a - b)$

1.9.4. Expanded Difference of Squares:

$$(a + b)^2 - c^2 = (a + b + c)(a + b - c)$$

1.9.5. Sum of Squares: $a^2 + b^2 = (a + bi)(a - bi)$ *i complex*

1.9.6. Perfect Square: $a^2 \pm 2ab + b^2 = (a \pm b)^2$

1.9.7. General Trinomial:

$$x^2 + (a + b)x + ab =$$
$$(x^2 + ax) + (bx + ab) =$$
$$(x + a)x + (x + a)b =$$
$$(x + a)(x + b)$$

1.9.8. Sum of Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

1.9.9. Difference of Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

1.9.10. Difference of Fourths:

$$a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) \Rightarrow$$
$$a^4 - b^4 = (a - b)(a + b)(a^2 + b^2)$$

1.9.11. Power Reduction to an Integer:

$$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$$

1.9.12. Power Reduction to a Radical:

$$x^2 - a = (x - \sqrt{a})(x + \sqrt{a})$$

1.9.13. Power Reduction to an Integer plus a Radical:

$$a^2 + ab + b^2 = (a + \sqrt{ab} + b)(a - \sqrt{ab} + b)$$

1.9.14. Quadratic Trinomial Factoring Process

Let $ax^2 + bx + c$ be a quadratic trinomial where the three coefficients a, b, c are integers.

Step 1: Find integers M, N such that

$$\begin{aligned} M + N &= b \\ M \cdot N &= ac \end{aligned}.$$

Step 2: Substitute for b ,

$$ax^2 + bx + c =$$
$$ax^2 + (M + N)x + c$$

Step 3: Factor by Grouping (**1.9.2**)

$$ax^2 + Mx + Nx + c =$$
$$(ax^2 + Mx) + \left(Nx + \frac{M \cdot N}{a} \right) =$$
$$ax\left(x + \frac{M}{a} \right) + N\left(x + \frac{M}{a} \right) =$$
$$\left(x + \frac{M}{a} \right)(ax + N) \therefore$$

Note: if there are no pair of integers M, N with both $M + N = b$ and $M \cdot N = ac$ then the quadratic trinomial is prime.

Example: Factor the expression $2x^2 - 13x - 7$.

$$\overset{1}{\mapsto}: MN = 2 \cdot (-7) = -14 \,\&\, M + N = -13 \Rightarrow$$
$$M = -14, N = 1$$
$$\overset{2}{\mapsto}: 2x^2 - 3x - 7 =$$
$$2x^2 - 14x + x - 7$$
$$\overset{3}{\mapsto}: 2x(x - 7) + 1 \cdot (x - 7) =$$
$$(x - 7)(2x + 1)$$

1.10. Laws of Equality

Let $A = B$ be an algebraic equality and C, D be any quantities.

1.10.1. Addition: $A + C = B + C$

1.10.2. Subtraction: $A - C = B - C$

1.10.3. Multiplication: $A \cdot C = B \cdot C$

1.10.4. Division: $\frac{A}{C} = \frac{B}{C}$ provided $C \neq 0$

1.10.5. Exponent: $A^n = B^n$ provided n is an integer

1.10.6. Reciprocal: $\frac{1}{A} = \frac{1}{B}$ provided $A \neq 0, B \neq 0$

1.10.7. Means & Extremes: $\frac{C}{A} = \frac{D}{B} \Rightarrow CB = AD$ if $A \neq 0, B \neq 0$

1.10.8. Zero Product Property: $A \cdot B = 0 \Leftrightarrow A = 0$ or $B = 0$

1.10.9. The Concept of Equivalency

When solving equations, the Laws of Equality—with the exception of **1.10.5**, which produces equations with extra or 'extraneous' solutions in addition to those for the original equation—are used to manufacture equations that are *equivalent* to the original equation. Equivalent equations are equations that have identical solutions. However, equivalent equations are not identical in appearance. The goal of any equation-solving process is to use the Laws of Equality to create a succession of equivalent equations where each equation in the equivalency chain is algebraically simpler than the preceding one. The final equation in the chain should be an expression of the form $x = a$, the no-brainer form that allows the solution to be immediately determined. In that algebraic mistakes can be made when producing the equivalency chain, the final answer must always be checked in the original equation. When using **1.10.5**, one must check for extraneous solutions and delete them from the solution set.

1.10.10. Linear Equation Solution Process

Start with the general form $L(x) = R(x)$ where $L(x)$ and $R(x)$ are first-degree polynomial expressions on the left-hand side and right-hand side of the equals sign.

Step 1: Using proper algebra, independently combine like terms for both $L(x)$ and $R(x)$

Step 2: Use **1.10.1** and **1.10.2** on an as-needed basis to create an equivalent equation of the form $ax = b$.

Step 3: use either **1.10.3** or **1.10.4** to create the final equivalent form $x = \frac{b}{a}$ from which the solution is easily deduced.

Step 4: Check solution in original equation.

Example: Solve $4\{3[7(y-3)+9]+2(y-9)\}-1=5(y-1)-3$.

$$\overset{1}{\mapsto}: 4\{3[7(y-3)+9]+2(y-9)\}-1=5(y-1)-3 \Rightarrow$$
$$4\{3[7y-21+9]+2y-18\}-1=5y-5-3 \Rightarrow$$
$$4\{3[7y-12]+2y-18\}-1=5y-8 \Rightarrow$$
$$4\{21y-36+2y-18\}-1=5y-8 \Rightarrow$$
$$4\{23y-54\}-1=5y-8 \Rightarrow$$
$$92y-216-1=5y-8 \Rightarrow$$
$$92y-217=5y-8$$

$$\overset{2}{\mapsto}: 92y-217=5y-8 \Rightarrow$$
$$92y-5y-217=5y-5y-8 \Rightarrow$$
$$87y-217=-8 \Rightarrow$$
$$87y-217+217=-8+217 \Rightarrow$$
$$87y=209$$

$$\overset{3}{\mapsto}: 87y=209 \Rightarrow$$
$$y=\frac{209}{87} \therefore$$

$\overset{4}{\mapsto}$: Check the final answer $y=\frac{209}{87}$ in the original equation

$$4\{3[7(y-3)+9]+2(y-9)\}-1=5(y-1)-3.$$

1.11. Laws of Inequality

Let $A > B$ be an algebraic inequality and C be any quantity.

1.11.1. Addition: $A + C > B + C$
1.11.2. Subtraction: $A - C > B - C$

1.11.3. Multiplication:
$$C > 0 \Rightarrow A \cdot C > B \cdot C$$
$$C < 0 \Rightarrow A \cdot C < B \cdot C$$

1.11.4. Division:
$$C > 0 \Rightarrow \frac{A}{C} > \frac{B}{C}$$
$$C < 0 \Rightarrow \frac{A}{C} < \frac{B}{C}$$

1.11.5. Reciprocal: $\frac{1}{A} < \frac{1}{B}$ provided $A \neq 0, B \neq 0$

Similar laws hold for $A < B$, $A \leq B$, and $A \geq B$. When multiplying or dividing by a negative C, one must reverse the direction of the original inequality sign. Replacing each side of the inequality with its reciprocal also reverses the direction of the original inequality.

1.11.6. Linear Inequality Solution Process

Start with the general form $L(x) > R(x)$ where $L(x)$ and $R(x)$ are as described in **1.10.10**. Follow the same four-step process as that given in **1.10.10** modifying per the checks below.

- ✓ Reverse the direction of the inequality sign when multiplying or dividing both sides of the inequality by a negative quantity.

- ✓ Reverse the direction of the inequality sign when replacing each side of an inequality with its reciprocal.

- ✓ The final answer will have one the four forms $x > a$, $x \geq a$, $x < a$, and $x \leq a$. One must remember that in of the four cases, x has infinitely many solutions as opposed to one solution for the linear equation.

1.12. Order of Operations

Step 1: Perform all power raisings in the order they occur from left to right

Step 2: Perform all multiplications and divisions in the order they occur from left to right

Step 3: Perform all additions and subtractions in the order they occur from left to right

Step 4: If parentheses are present, first perform steps 1 through 3 *on an as-needed basis* within the innermost set of parentheses until a single number is achieved. Then perform steps 1 through 3 (*again, on an as-needed basis*) for the next level of parentheses until all parentheses have been systematically removed.

Step 5: If a fraction bar is present, simultaneously perform steps 1 through 4 for the numerator and denominator, treating each as totally-separate problem until a single number is achieved. Once single numbers have been achieved for both the numerator and the denominator, then a final division can be performed.

1.13. Three Meanings of 'Equals'

1. **Equals** is the mathematical equivalent of the English verb "is", the fundamental verb of being. A simple but subtle use of equals in this fashion is $2 = 2$.
2. **Equals** implies an equivalency of naming in that the same underlying quantity is being named in two different ways. This can be illustrated by the expression $2003 = MMIII$. Here, the two diverse symbols on both sides of the equals sign refer to the same and exact underlying quantity.
3. **Equals** states the product (either intermediate or final) that results from a process or action. For example, in the expression $2 + 2 = 4$, we are adding two numbers on the left-hand side of the equals sign. Here, addition can be viewed as a process or action between the numbers 2 and 2. The result or product from this process or action is the single number 4, which appears on the right-hand side of the equals sign.

1.14. The Seven Parentheses Rules

1.14.1. Consecutive processing signs $+,-,\cdot,\div$ are separated by parentheses.

1.14.2. Three or more consecutive processing signs are separated by nested parenthesis where the rightmost sign will be in the innermost set of parentheses.

1.14.3. Nested parentheses are typically written using the various bracketing symbols to facilitate reading.

1.14.4. The rightmost processing sign and the number to the immediate right of the rightmost sign are both enclosed within the same set of parentheses.

1.14.5. Parentheses may enclose a signed or unsigned number by itself but never a sign by itself.

1.14.6. More than one number can be written inside a set of parentheses depending on what part of the overall process is emphasized.

1.14.7. When parentheses separate numbers with no intervening multiplication sign, a multiplication is understood. The same is true if just one plus or minus sign separates the two numbers and the parentheses enclose both the rightmost number and the separating sign.

1.14.8. Demonstrating the Seven Basic Parentheses Rules

- ✓ $5+-12$: Properly written as $5+(-12)$. **1.14.1**, **1.14.4**
- ✓ $5\cdot-12$: Properly written as $5\cdot(-12)$. **1.14.1**, **1.14.4**
- ✓ $5+--12$: Properly written as $5+[-(-12)]$. **1.14.1** thru **4**
- ✓ $5\cdot(-)12$: Incorrect per **1.14.5**
- ✓ $(5)\cdot(-12)$: Correct per **1.14.1**, **1.14.4**, **1.14.5**
- ✓ $-5\cdot12$: Does not need parentheses to achieve separation since the 5 serves the same purpose. Any use of parentheses would be optional
- ✓ $(-5)\cdot12$: The optional parentheses, though not needed, emphasize the negative 5 per **1.14.5**
- ✓ $-(5\cdot12)$: The optional parentheses emphasize the fact that the final outcome is negative per **1.14.5**, **1.14.6**

- ✓ $4(12)$: The *mandatory* parentheses indicate that 4 is multiplying 12. Without the parentheses, the expression would be properly read as the single number 412, **1.14.7**.
- ✓ $7(-5)$: The *mandatory* parentheses indicate that 7 is multiplying -5. Without the intervening parentheses, the expression is properly read as the difference $7-5$, **1.14.7**.
- ✓ $(-32)(-5)$: The *mandatory* parentheses indicate that -32 is multiplying -5. The expression $(-32)\cdot(-5)$ also signifies the same, **1.14.7**.

1.14.9. Demonstration of Use of Order-of-Operations with Parentheses Rules to Reduce a Rational Expression.

$$\frac{4(18-\{-8\}+2^3)+6\cdot 9}{2(9^2-8^2)}=$$

$$\frac{4(18-\{-8\}+8)+6\cdot 9}{2(81-64)}=$$

$$\frac{4([18+8]+8)+6\cdot 9}{2(17)}=$$

$$\frac{4(26+8)+6\cdot 9}{34}=$$

$$\frac{4(34)+6\cdot 9}{34}=$$

$$\frac{136+6\cdot 9}{34}=$$

$$\frac{136+54}{34}=$$

$$\frac{190}{34}=$$

$$\frac{2\times 95}{2\times 17}=\frac{95}{17}\therefore$$

1.15. Rules for Logarithms

1.15.1. Definition of Logarithm to Base $b > 0$:

$y = \log_b x$ if and only if $b^y = x$

1.15.2. Logarithm of the Same Base: $\log_b b = 1$

1.15.3. Logarithm of One: $\log_b 1 = 0$

1.15.4. Logarithm of the Base to a Power: $\log_b b^p = p$

1.15.5. Base to the Logarithm: $b^{\log_b p} = p$

1.15.6. Notation for Logarithm Base 10: $Logx \equiv \log_{10} x$

1.15.7. Notation for Logarithm Base e: $\ln x \equiv \log_e x$

1.15.8. Change of Base Formula: $\log_b N = \dfrac{\log_a N}{\log_a b}$

1.15.9. Product: $\log_b (MN) = \log_b N + \log_b M$

1.15.10. Quotient: $\log_b \left(\dfrac{M}{N} \right) = \log_b M - \log_b N$

1.15.11. Power: $\log_b N^p = p \log_b N$

1.15.12. Logarithmic Simplification Process

Let $X = \dfrac{A^n B^m}{C^p}$, then

$$\log_b (X) = \log_b \left(\frac{A^n B^m}{C^p} \right) \Rightarrow$$
$$\log_b (X) = \log_b \left(A^n B^m \right) - \log_b \left(C^p \right) \Rightarrow$$
$$\log_b (X) = \log_b \left(A^n \right) + \log_b \left(B^m \right) - \log_b \left(C^p \right) \Rightarrow$$
$$\log_b (X) = n \log_b (A) + m \log_b (B) - p \log_b (C) \therefore$$

Note: The use of logarithms transforms complex algebraic expressions where products become sums, quotients become differences, and exponents become coefficients, making the manipulation of these expressions easier in some instances.

1.16. Complex Numbers

1.16.1. Definition of the imaginary unit i: i is defined to be the solution to the equation $x^2+1=0$.

1.16.2. Properties of the imaginary unit i:
$i^2+1=0 \Rightarrow i^2=-1 \Rightarrow i=\sqrt{-1}$

1.16.3. Definition of Complex Number: Numbers of the form $a+bi$ where a,b are real numbers

1.16.4. Definition of Complex Conjugate: $\overline{a+bi}=a-bi$

1.16.5. Definition of Complex Modulus: $|a+bi|=\sqrt{a^2+b^2}$

1.16.6. Addition: $(a+bi)+(c+di)=(a+c)+(b+d)i$

1.16.7. Subtraction: $(a+bi)-(c+di)=(a-c)+(b-d)i$

1.16.8. Process of Complex Number Multiplication

$$(a+bi)(c+di)=$$

$$ac+(ad+bc)i+bdi^2=$$

$$ac+(ad+bc)i+bd(-1)$$

$$ac-bd+(ad+bc)i$$

1.16.9. Process of Complex Number Division

$$\frac{a+bi}{c+di}=$$

$$\frac{(a+bi)(\overline{c+di})}{(c+di)(\overline{c+di})}=$$

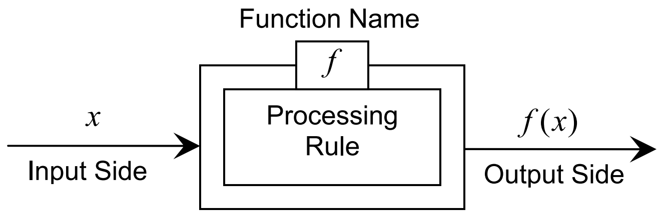
$$\frac{(a+bi)(c-di)}{(c+di)(c-di)}=$$

$$\frac{(ac+bd)+(bc-ad)i}{c^2-d^2}=$$

$$\frac{ac+bd}{c^2-d^2}+\left(\frac{bc-ad}{c^2-d^2}\right)i$$

1.17. What is a Function?

The mathematical concept called a *function* is foundational to the study of higher mathematics. With this statement in mind, let us define in a working sense the word *function*: A *function* is any process where numerical input is transformed into numerical output with the operating restriction that each unique input must lead to one and only one output.



The above figure is a diagram of the general function process for a function named f. Function names are usually lower-case letters, $f, g, h,$ etc. When a mathematician says, 'let f be a function', the entire input-output process—start to finish—comes into discussion. If two different function names are being used in one discussion, then two different functions are being discussed, often in terms of their relationship to each other. The variable x is the *independent or input* variable; it is independent because any specific input value can be freely chosen. Once a specific input value is chosen, the function then processes the input value via the processing rule in order to create the *output variable* $f(x)$, also called the *dependent variable* since the value of $f(x)$ is entirely determined by the action of the processing rule upon x. Notice that the complex symbol $f(x)$ reinforces the fact that output values are created by direct action of the function process f upon the independent variable x. Sometimes, a simple y will be used to represent the output variable $f(x)$ when it is well understood that a function process is indeed in place. Two more definitions are noted. The set of all possible input values for a function f is called the *domain* and is denoted by the symbol Df. The set of all possible output values is called the *range* and is denoted by Rf.

1.18. Function Algebra

Let f and g be functions, and let f^{-1} be the inverse for f

1.18.1. Inverse Property: $f[f^{-1}(x)] = f^{-1}[f(x)] = x$

1.18.2. Addition/Subtraction: $(f \pm g)(x) = f(x) \pm g(x)$

1.18.3. Multiplication: $(f \cdot g)(x) = f(x) \cdot g(x)$

1.18.4. Division: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}; \left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$

1.18.5. Composition: $(f \circ g)(x) = f[g(x)]$
$(g \circ f)(x) = g[f(x)]$

1.18.6. Process for Constructing Inverse Functions

Step 1: Start with $f(f^{-1}(x)) = x$, the process equality that must be in place for an inverse function to exist.

Step 2: Replace $f^{-1}(x)$ with y to form the equality $f(y) = x$.

Step 3: Solve for y in terms of x. The resulting y is $f^{-1}(x)$.

Step 4: Verify by the property $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.

1.18.7. Demonstration of 1.18.6:

Find $f^{-1}(x)$ for $f(x) = x^3 + 2$.

$\stackrel{1}{\mapsto}: f(f^{-1}(x)) = (f^{-1}(x))^3 + 2 = x$

$\stackrel{2}{\mapsto}: (y)^3 + 2 = x$

$\stackrel{3}{\mapsto}: (y)^3 + 2 = x \Rightarrow$

$y = \sqrt[3]{x-2}$

$\stackrel{4}{\mapsto}: f^{-1}(f(x)) = \sqrt[3]{(x^3+2)-2} = \sqrt[3]{x^3} = x$

$\stackrel{4}{\mapsto}: f(f^{-1}(x)) = \left(\sqrt[3]{(x-2)}\right)^3 + 2 = (x-2) + 2 = x$

1.19. Quadratic Equations & Functions

1.19.1. Definition and Discussion

A complete quadratic equation in standard form (*ready-to-be-solved*) is an equation having the algebraic structure $ax^2 + bx + c = 0$ where $a \neq 0, b \neq 0, c \neq 0$. If either $b = 0$ or $c = 0$, the quadratic equation is called incomplete. If $a = 0$, the quadratic equation reduces to a linear equation. All quadratic equations have exactly two solutions if complex solutions are allowed. Solutions are obtained by either *factoring* or by use of the *quadratic formula*. If, within the context of a particular problem complex solutions are not admissible, quadratic equations can have up to *two real solutions*. As with all real-world applications, the number of admissible solutions depends on context.

1.19.2. Quadratic Formula with Development:

$$ax^2 + bx + c = 0 \Rightarrow x^2 + \left(\frac{b}{a}\right)x = -\frac{c}{a} \Rightarrow$$

$$x^2 + \left(\frac{b}{a}\right)x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \Rightarrow$$

$$\left[x + \left(\frac{b}{2a}\right)\right]^2 = \frac{b^2 - 4ac}{4a^2} \Rightarrow$$

$$x + \left(\frac{b}{2a}\right) = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \Rightarrow$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \therefore$$

1.19.3. Solution of Quadratic Equations by Formula

To solve a quadratic equation using the quadratic formula—the more powerful of two common methods for solving quadratic equations—apply the following four steps.

Step 1: Rewrite the quadratic equation so it matches the standard form $ax^2 + bx + c = 0$.

Step 2: Identify the two coefficients and constant term $a, b, \& c$.
Step 3: Apply the formula and solve.
Step 4: Check your answer(s) in the original equation.

1.19.4. Solution Discriminator: $b^2 - 4ac$

$b^2 - 4ac > 0 \Rightarrow$ two real solutions

$b^2 - 4ac = 0 \Rightarrow$ one real solution of multiplicity two

$b^2 - 4ac < 0 \Rightarrow$ two complex (conjugates) solutions

1.19.5. Solution when $a = 0 \,\&\, b \neq 0$:

$$bx + c = 0 \Rightarrow x = \frac{-c}{b}$$

1.19.6. Solution of Quadratic Equations by Factoring

To solve a quadratic equation using the factoring method, apply the following four steps.

Step 1: Rewrite the quadratic equation in standard form
Step 2: Factor the left-hand side into two linear factors using the quadratic trinomial factoring process **1.9.14**.
Step 3: Set each linear factor equal to zero and solve.
Step 4: Check answer(s) in the original equation

Note: Use the quadratic formula when a quadratic equation cannot be factored or is hard to factor.

1.19.7. Quadratic-in-Form Equation: $aU^2 + bU + c = 0$ where U is an algebraic expression of varying complexity.

1.19.8. Definition of Quadratic Function:

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

1.19.9. Axis of Symmetry for Quadratic Function: $x = \frac{-b}{2a}$

1.19.10. Vertex for Quadratic Function: $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$

1.20. Cardano's Cubic Solution

Let $ax^3 + bx^2 + cx + d = 0$ be a cubic equation written in standard form with $a \neq 0$

Step 1: Set $x = y - \dfrac{b}{3a}$. After this substitution, the above cubic becomes $y^3 + py + q = 0$ where $p = \left[\dfrac{c}{a} - \dfrac{b^2}{3a^2}\right]$ and $q = \left[\dfrac{2b^2}{27a^3} - \dfrac{bc}{3a^2} + \dfrac{d}{a}\right]$

Step 2: Define $u \,\&\, v$ such that $y = u - v$ and $p = 3uv$

Step 3: Substitute for $y \,\&\, p$ in the equation $y^3 + py + q = 0$. This leads to $(u^3)^2 + qu^3 - \dfrac{p^3}{27} = 0$, which is quadratic-in-form in u^3.

Step 4: Use the quadratic formula **1.19.3** to solve for u^3

$$u^3 = \frac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2}$$

Step 5: Solve for $u \,\&\, v$ where $v = \dfrac{p}{3u}$ to obtain

$$u = \sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2}} \;\&\; v = -\sqrt[3]{\frac{-q - \sqrt{q^2 + \frac{4}{27}p^3}}{2}}$$

Step 6: Solve for x where $x = y - \dfrac{b}{3a} \Rightarrow x = u - v - \dfrac{b}{3a}$

1.21. Theory of Polynomial Equations

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ be a polynomial written in standard form.

The Eight Basic Theorems

1.21.1. Fundamental Theorem of Algebra: Every polynomial $P(x)$ of degree $N \geq 1$ has at least one solution x_0 for which $P(x_0) = 0$. This solution may be real or complex (i.e. has the form $a + bi$).

1.21.2. Numbers Theorem for Roots and Turning Points: If $P(x)$ is a polynomial of degree N, then the equation $P(x) = 0$ has up to N real solutions or *roots*. The equation $P(x) = 0$ has exactly N roots if one counts complex solutions of the form $a + bi$. Lastly, the graph of $P(x)$ will have up to $N - 1$ turning points (which includes both relative maxima and minima).

1.21.3. Real Root Theorem: If $P(x)$ is of odd degree having all real coefficients, then $P(x)$ has at least one real root.

1.21.4. Rational Root Theorem: If $P(x)$ has all integer coefficients, then any rational roots for the equation $P(x) = 0$ must have the form $\frac{p}{q}$ where p is a factor of the constant coefficient a_0 and q is a factor of the lead coefficient a_n. *Note: This result is used to form a rational-root possibility list.*

1.21.5. Complex Conjugate Pair Root Theorem: Suppose $P(x)$ has all real coefficients. If $a + bi$ is a root for $P(x)$ with $P(a + bi) = 0$, then $P(a - bi) = 0$.

1.21.6. Irrational Surd Pair Root Theorem: Suppose $P(x)$ has all rational coefficients. If $a + \sqrt{b}$ is a root for $P(x)$ with $P(a + \sqrt{b}) = 0$, then $P(a - \sqrt{b}) = 0$.

1.21.7. Remainder Theorem: If $P(x)$ is divided by $(x-c)$, then the remainder R is equal to $P(c)$. *Note: this result is extensively used to evaluate a given polynomial $P(x)$ at various values of x.*

1.21.8. Factor Theorem: If c is any number with $P(c)=0$, then $(x-c)$ is a factor of $P(x)$. This means $P(x)=(x-c)\cdot Q(x)$ where $Q(x)$ is a new, reduced polynomial having degree one less than $P(x)$. The converse $P(x)=(x-c)\cdot Q(x) \Rightarrow P(c)=0$ is also true.

The Four Advanced Theorems

1.21.9. Root Location Theorem: Let (a,b) be an interval on the x axis with $P(a)\cdot P(b)<0$. Then there is a value $x_0 \in (a,b)$ such that $P(x_0)=0$.

1.21.10. Root Bounding Theorem: Divide $P(x)$ by $(x-d)$ to obtain $P(x)=(x-d)\cdot Q(x)+R$. Case $d>0$: If both R and all the coefficients of $Q(x)$ are positive, then $P(x)$ has no root $x_0>d$. Case $d<0$: If the roots of $Q(x)$ alternate in sign—with the remainder R "in sync" at the end—then $P(x)$ has no root $x_0<d$. *Note: Coefficients of zero can be counted either as positive or negative—which ever way helps in the subsequent determination.*

1.21.11. Descartes' Rule of Signs: Arrange $P(x)$ in standard order as shown in the title bar. The number of positive real solutions equals the number of coefficient sign variations or that number decreased by an even number. Likewise, the number of negative real solutions equals the number of coefficient sign variations in $P(-x)$ or that number decreased by an even number.

1.21.12. Turning Point Theorem: Let a polynomial $P(x)$ have degree N. Then the number of turning points for a polynomial $P(x)$ can not exceed $N-1$.

1.22. Determinants and Cramer's Rule

1.22.1. Two by Two Determinant Expansion:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

1.22.2. Three by Three Determinant Expansion:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} =$$

$$a(ei - fh) - b(di - fg) + c(dh - eg) =$$

$$aei - ahf + bfg - bdi + cdh - ceg$$

1.22.3. Cramer's Rule for a Two-by-Two Linear System

Given $\begin{matrix} ax + by = e \\ cx + dy = f \end{matrix}$ with $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

Then $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{D}$ and $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{D}$

1.22.4. Cramer's Rule for a Three-by-Three Linear System

Given $\begin{matrix} ax + by + cz = j \\ dx + ey + fz = k \\ gx + hy + iz = l \end{matrix}$ with $D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0$

Then $x = \dfrac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{D}, y = \dfrac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{D}, z = \dfrac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{D}$

1.22.5. Solution Types in $x_i = \frac{Dx_i}{D}$

$Dx_i = 0, D \neq 0 \Rightarrow x_i = 0$

$Dx_i = 0, D = 0 \Rightarrow x_i$ has infinite solutions

$Dx_i \neq 0, D \neq 0 \Rightarrow x_i$ has a unique solution

$Dx_i \neq 0, D = 0 \Rightarrow x_i$ has no solution

1.23. Binomial Theorem

Let n and r be positive integers with $n \geq r$.

1.23.1. Definition of $n!$: $n! = n(n-1)(n-2)...1$,

1.23.2. Special Factorials: $0! = 1$ and $1! = 1$

1.23.3. Combinatorial Symbol: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

1.23.4. Summation Symbols:

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + a_3 + a_4 + ... + a_n$$

$$\sum_{i=k}^{n} a_i = a_k + a_{k+1} + a_{k+2} + a_{k+3} ... + a_n$$

1.23.5. Binomial Theorem: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$

1.23.6. Sum of Binomial Coefficients when $a = b = 1$:

$$\sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i = (1+1)^n = 2^n$$

1.23.7. Formula for the $(r+1)th$ Term: $\binom{n}{r} a^{n-r} b^r$

1.23.8. Pascal's Triangle for $\binom{n}{r}: n = 10$

```
                    1
                  1   1
                1   2   1
              1   3   3   1
            1   4   6   4   1
          1   5  10  10   5   1
        1   6  15  20  15   6   1
      1   7  21  35  35  21   7   1
    1   8  28  56  70  56  28   8   1
  1   9  36  84 126 126  84  36   9   1
1  10  45 120 210 252 210 120  45  10   1
```

1.24. Arithmetic Series

1.24.1. Definition: $S = \sum_{i=0}^{n}(a+ib)$ where b is the common increment

1.24.2. Summation Formula for S: $S = \frac{(n+1)}{2}[2a+nb]$

1.25. Geometric Series

1.25.1. Definition: $G = \sum_{i=0}^{n} ar^i$ where r is the common ratio

1.25.2. Summation Formula for G:

$$G = \sum_{i=0}^{n} ar^i \Rightarrow rG = \sum_{i=0}^{n} ar^{i+1} \Rightarrow$$

$$G - rG = \sum_{i=0}^{n} ar^i - \sum_{i=0}^{n} ar^{i+1} = a - ar^{i+1} \Rightarrow$$

$$G = \frac{a(1-r^{i+1})}{1-r}$$

1.25.3. Infinite Sum Provided $0 < r < 1$: $\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$

1.26. Boolean Algebra

In the following tables, the propositions $p \,\&\, q$ are either True (T) or False (F).

1.26.1. Elementary Truth Table:

$and = \wedge : or = \vee : negation = \sim : implies = \Rightarrow, \Leftrightarrow$							
p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	F	F
F	F	T	T	T	F	T	T

1.26.2. Truth Table for the Exclusive Or $\overset{e}{\vee}$:

p	q	$p \overset{e}{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

1.26.3. Modus Ponens: Let $p \Rightarrow q \,\&\, p = T$. Then, $q = T$.

1.26.4. Chain Rule: Let $p \Rightarrow q \,\&\, q \Rightarrow r$. Then $(p \Rightarrow r) = T$.

1.26.5. Modus Tollens:
Let $p \Rightarrow q \,\&\, q = F$. Then $(\sim q \Rightarrow \sim p) = T$.

1.26.6. Fallacy of Affirming the Consequent:
Let $p \Rightarrow q \,\&\, q = T$ .Then $(q \Rightarrow p) = F$.

1.26.7. Fallacy of Denying the Antecedent:
Let $p \Rightarrow q \,\&\, p = F$. Then $(\sim p \Rightarrow \sim q) = F$.

1.26.8. Disjunctive Syllogism for the Exclusive Or:
Let $p \overset{e}{\vee} q = T \,\&\, q = F$. Then $p = T$

1.26.9. Demonstration that the English double-negative in the slang expression "I don't got none" actually affirms the opposite of what is intended.

Step	Phrase	Comment
1	I do not have any	The original proposition p as intended
1	I do have *none*	Assume $p = T$
2	I do *not* have *none*	Negation of $p : (\sim p) = F$
3	I don't have none	Proper contracted form of 3: $(\sim p) = F$
4	I don't got none	Slang version of 3
5	I have some	Logical consequence of 3: $(\sim p) = F \Rightarrow \sim (\sim p) = T$

1.27. Variation or Proportionality Formulas

1.27.1. Direct: $y = kx$

1.27.2. Inverse: $y = \frac{k}{x}$

1.27.3. Joint: $z = kxy$

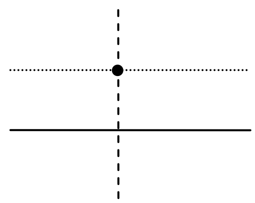
1.27.4. Inverse Joint: $z = \frac{kx}{y}$

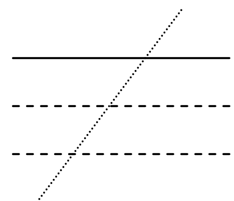
1.27.5. Direct to Power: $y = kx^n$

1.27.6. Inverse to Power: $y = \frac{k}{x^n}$

2. Geometry

2.1. The Parallel Postulates





2.1.1. Let a point reside outside a given line. Then there is exactly one line passing through the point parallel to the given line.

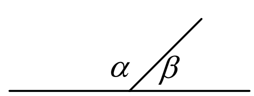
2.1.2. Let a point reside outside a given line. Then there is exactly one line passing through the point perpendicular to the given line.

2.1.3. Two lines both parallel to a third line are parallel to each other.

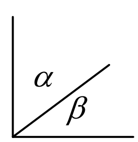
2.1.4. If a transverse line intersects two parallel lines, then corresponding angles in the figures so formed are congruent.

2.1.5. If a transverse line intersects two lines and makes congruent, corresponding angles in the figures so formed, then the two original lines are parallel.

2.2. Angles and Lines



$\alpha + \beta = 180^0$



$\alpha + \beta = 90^0$

2.2.1. Complimentary Angles: Two angles α, β with $\alpha + \beta = 90^0$.

2.2.2. Supplementary Angles: Two angles α, β with $\alpha + \beta = 180^0$

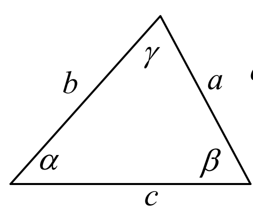
2.2.3. Linear Sum of Angles: The sum of the two angles α, β formed when a straight line is intersected by a line segment is equal to 180^0

2.2.4. Acute Angle: An angle less than 90^0

2.2.5. Right Angle: An angle exactly equal to 90^0

2.2.6. Obtuse Angle: An angle greater than 90^0

2.3. Triangles



$$\alpha + \beta + \gamma = 180^0$$

2.3.1. Triangular Sum of Angles: The sum of the three interior angles α, β, γ in any triangle is equal to 180^0

2.3.2. Acute Triangle: A triangle where all three interior angles α, β, γ are acute

2.3.3. Right Triangle: A triangle where one interior angle from the triad α, β, γ is equal to 90^0

2.3.4. Obtuse Triangle: A triangle where one interior angle from the triad α, β, γ is greater than 90^0

2.3.5. Scalene Triangle: A triangle where no two of the three side-lengths a, b, c are equal to another

2.3.6. Isosceles Triangle: A triangle where exactly two of the side-lengths a, b, c are equal to each other

2.3.7. Equilateral Triangle: A triangle where all three side-lengths a, b, c are identical $a = b = c$ or all three angles α, β, γ are equal with $\alpha = \beta = \gamma = 60^0$

2.3.8. Congruent Triangles: Two triangles are congruent (equal) if they have identical interior angles and side-lengths.

2.3.9. Similar Triangles: Two triangles are similar if they have identical interior angles.

2.3.10. Included Angle: The angle that is between two given sides

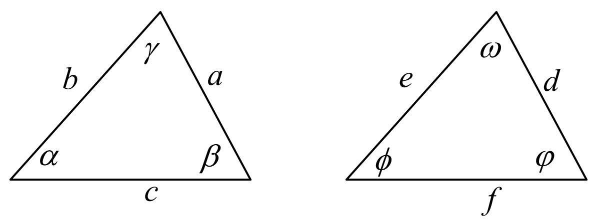
2.3.11. Opposite Angle: The angle opposite a given side

2.3.12. Included Side: The side that is between two given angles

2.3.13. Opposite Side: The side opposite a given angle

2.4. Congruent Triangles

Given the congruent two triangles as shown below



2.4.1. Side-Angle-Side (**SAS**): If any two side-lengths and the included angle are identical, then the two triangles are congruent.

Example: $b \,\&\, \alpha \,\&\, c = e \,\&\, \phi \,\&\, f$

2.4.2. Angle-Side-Angle (**ASA**): If any two angles and the included side are identical, then the two triangles are congruent.

Example: $\alpha \,\&\, c \,\&\, \beta = \phi \,\&\, f \,\&\, \varphi$

2.4.3. Side-Side-Side (**SSS**): If the three side-lengths are identical, then the triangles are congruent.

Example: $b \,\&\, c \,\&\, a = e \,\&\, f \,\&\, d$

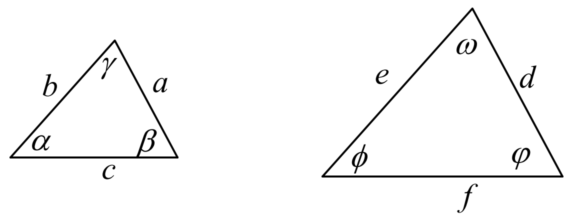
2.4.4. Three Attributes Identical: If any three attributes—side-lengths and angles—are equal with at least one attribute being a side-length, then the two triangles are congruent. These other cases are of the form Angle-Angle-Side (**AAS**) or Side-Side-Angle (**SSA**).

Example (**SSA**): $b \,\&\, a \,\&\, \beta = e \,\&\, d \,\&\, \varphi$

Example (**AAS**): $\alpha \,\&\, \beta \,\&\, a = \phi \,\&\, \varphi \,\&\, d$

2.5. Similar Triangles

Given the two similar triangles as shown below



2.5.1. Minimal Condition for Similarity: If any two angles are identical (**AA**), then the triangles are similar.

Suppose $\alpha = \phi \;\&\; \beta = \varphi$

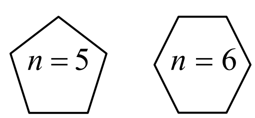
Then $\alpha + \beta + \gamma = 180^0 \;\&\; \phi + \varphi + \varpi = 180^0 \Rightarrow$
$\alpha = 180^0 - \beta - \gamma = 180^0 - \varphi - \varpi = \phi$

2.5.2. Ratio laws for Similar Triangles: Given similar triangles as shown above, then $\frac{b}{e} = \frac{c}{f} = \frac{a}{d}$

2.6. Planar Figures

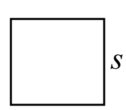
A is the planar area, P is the perimeter, n is the number of sides.

2.6.1. Degree Sum of Interior Angles in General Polygon:
$D = 180^0[n-2]$

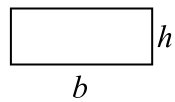


$n = 5 \Rightarrow D = 540^0$
$n = 6 \Rightarrow D = 720^0$

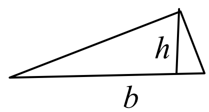
2.6.2. Square: $A = s^2 : P = 4s$, s is the length of a side



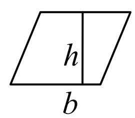
2.6.3. Rectangle: $A = bh : P = 2b + 2h$, $b \& h$ are the base and height



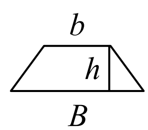
2.6.4. Triangle: $A = \frac{1}{2}bh$, $b \& h$ are the base and altitude



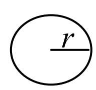
2.6.5. Parallelogram: $A = bh$, $b \& h$ are the base and altitude



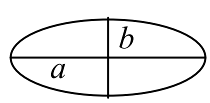
2.6.6. Trapezoid: $A = \frac{1}{2}(B + b)h$, $B \& b$ are the two parallel bases and h is the altitude



2.6.7. Circle: $A = \pi r^2 : P = 2\pi r$ where r is the radius, or $P = \pi d$ where $d = 2r$, the diameter.



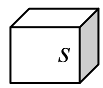
2.6.8. Ellipse: $A = \pi ab$; $a \& b$ are the half lengths of the major & minor axes



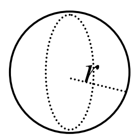
2.7. Solid Figures

A is total surface area, V is the volume

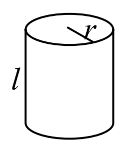
2.7.1. Cube: $A = 6s^2 : V = s^3$, s is the length of a side



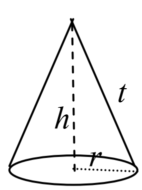
2.7.2. Sphere: $A = 4\pi r^2 : V = \frac{4}{3}\pi r^3$, r is the radius



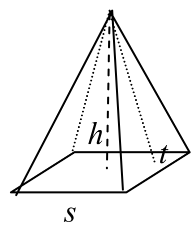
2.7.3. Cylinder: $A = 2\pi r^2 + 2\pi rl : V = \pi r^2 l$, $r \,\&\, l$ are the radius and length



2.7.4. Cone: $A = \pi r^2 + 2\pi rt : V = \frac{1}{3}\pi r^2 h$, $r \,\&\, t \,\&\, h$ are the radius, slant height, and altitude

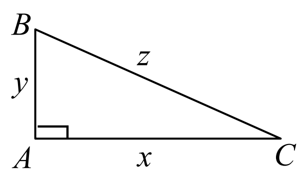


2.7.5. Pyramid (square base): $A = s^2 + 2st : V = \frac{1}{3}s^2h$, $s \,\&\, t \,\&\, h$ are the side, slant height, and altitude

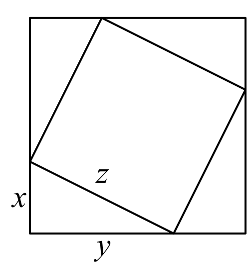


2.8. Pythagorean Theorem

2.8.1. Statement: Let a right triangle ΔABC have one side $\overline{AC}$ of length x, a second side $\overline{AB}$ of length y, and a hypotenuse (long side) $\overline{BC}$ of length z. Then $z^2 = x^2 + y^2$



2.8.2. Traditional Algebraic Proof: Construct a big square by bringing together four congruent right triangles.



The area of the big square is given by

$A = (x + y)^2$, *or equivalently by*

$A = z^2 + 4\left(\frac{xy}{2}\right)$.

Equating:

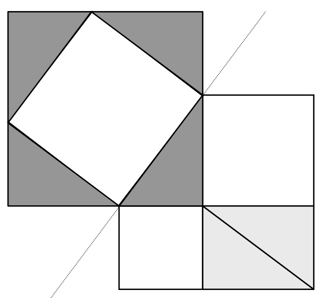
$(x + y)^2 = z^2 + 4\left(\frac{xy}{2}\right) \Rightarrow$

$x^2 + 2xy + y^2 = z^2 + 2xy \Rightarrow$.

$x^2 + y^2 = z^2 \Rightarrow$

$z^2 = x^2 + y^2 \therefore$

2.8.3. Visual (Pre-Algebraic) Pythagorean Proof:



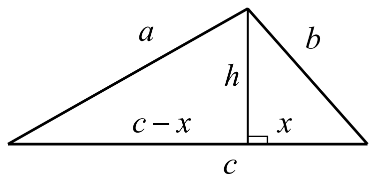
The idea is to observe that the two five-sided irregular polygons on either side of the dotted line have equivalent areas. Taking away three congruent right triangles from each area leads to the desired Pythagorean equality.

2.8.4. Pythagorean Triples: Positive integers L, M, N such that $L^2 = M^2 + N^2$

2.8.5. Generating Formulas for Pythagorean triples: Let m, n with $m > n > 0$ be integers. Then $M = m^2 - n^2$, $N = 2mn$, and $L = m^2 + n^2$

2.9. Heron's Formula

Let $s = \frac{1}{2}(a+b+c)$ be the semi-perimeter of a general triangle and A be the internal area enclosed by the same.



2.9.1. Heron's Formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$

2.9.2. Derivation Using Pythagorean Theorem:

$\overset{1}{\mapsto}$: Create two equations for the unknowns h and x

$E_1 : h^2 + x^2 = b^2$

$E_2 : h^2 + (c-x)^2 = a^2$

$\overset{2}{\mapsto}$: Subtract E_2 from E_1 and solve for x

$x^2 - (c-x)^2 = b^2 - a^2 \Rightarrow$

$2cx - c^2 = b^2 - a^2 \Rightarrow$

$$x = \frac{c^2 + b^2 - a^2}{2c}$$

$\overset{3}{\mapsto}$: Substitute the value for x into E_1

$$h^2 + \left[\frac{c^2 + b^2 - a^2}{2c}\right]^2 = b^2$$

$\overset{4}{\mapsto}$: Solve for h

$$h = \sqrt{\frac{4c^2b^2 - \left[c^2 + b^2 - a^2\right]^2}{4c^2}} \Rightarrow$$

$$h=\sqrt{\frac{\left\{2cb-\left[c^2+b^2-a^2\right]\right\}\left\{2cb+\left[c^2+b^2-a^2\right]\right\}}{4c^2}}\Rightarrow$$

$$h=\sqrt{\frac{\left\{a^2-\left[c-b\right]^2\right\}\left\{\left[c+b\right]^2-a^2\right\}}{4c^2}}\Rightarrow$$

$$h=\sqrt{\frac{\{a+b-c\}\{a+c-b\}\{c+b-a\}\{c+b+a\}}{4c^2}}$$

$\overset{5}{\mapsto}$: Solve for area using $A=\frac{1}{2}ch$.

$$A=\tfrac{1}{2}c\sqrt{\frac{\{a+b-c\}\{a+c-b\}\{c+b-a\}\{c+b+a\}}{4c^2}}\Rightarrow$$

$$A=\sqrt{\frac{\{a+b-c\}\{a+c-b\}\{c+b-a\}\{c+b+a\}}{16}}$$

$\overset{6}{\mapsto}$: Substitute $s=\dfrac{a+b+c}{2}$ and simplify.

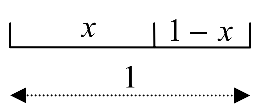
$$A=\sqrt{\left\{s-\frac{2c}{2}\right\}\left\{s-\frac{2b}{2}\right\}\left\{s-\frac{2a}{2}\right\}\{s\}}\Rightarrow$$

$$A=\sqrt{(s-c)(s-b)(s-a)s}\Rightarrow$$

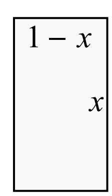
$$A=\sqrt{s(s-a)(s-b)(s-c)}\ \therefore$$

2.10. Golden Ratio

2.10.1. Definition: Let $p=1$ be the semi-perimeter of a rectangle whose base and height are in the proportion shown, defining the Golden Ratio ϕ. Solving for x leads to $\phi=1.6181$.

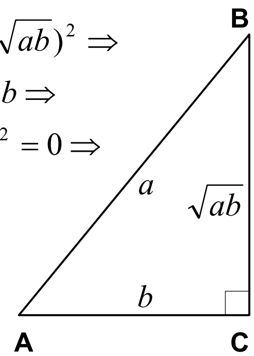


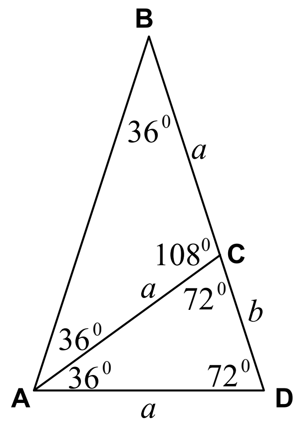
$$\frac{1}{x}=\frac{x}{1-x}=\phi$$



2.10.2. Golden Triangles: Triangles whose sides are proportioned to the Golden Ratio. Two examples are shown below.

$$a^2 = b^2 + (\sqrt{ab})^2 \Rightarrow$$
$$a^2 = b^2 + ab \Rightarrow$$
$$a^2 - ab - b^2 = 0 \Rightarrow$$
$$a = \phi b$$





2.11. Distance and Line Formulas

Let (x_1, y_1) and (x_2, y_2) be two points where $x_2 > x_1$.

2.11.1. 2-D Distance Formula: $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

2.11.2. 3-D Distance Formula: For the points (x_1, y_1, z_1) and (x_2, y_2, z_2), $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

2.11.3. Midpoint Formula: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Line Formulas

2.11.4. Slope of Line: $m = \frac{y_2 - y_1}{x_2 - x_1}$

2.11.5. Point/Slope Form: $y - y_1 = m(x - x_1)$

2.11.6. General Form: $Ax + By + C = 0$

2.11.7. Slope/Intercept Form: $y = mx + b$ where $\left(\frac{-b}{m}, 0\right)$ and $(0, b)$ are the x and y Intercepts:

2.11.8. Intercept/Intercept Form: $\frac{x}{a}+\frac{y}{b}=1$

where $(a,0)$ and $(0,b)$ are the x and y intercepts

2.11.9. Slope Relationship between two Parallel Lines L_1 and L_2 having slopes m_1 and m_2: $m_1 = m_2$

2.11.10. Slope Relationship between two Perpendicular Lines L_1 and L_2 having slopes m_1 and m_2: $m_1 = \frac{-1}{m_2}$

2.11.11. Slope of Line Perpendicular to a Line of Slope m: $\frac{-1}{m}$

2.12. Formulas for Conic Sections

2.12.1. General: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

2.12.2. Circle of Radius r Centered at (h,k):

$(x-h)^2 + (y-k)^2 = r^2$

2.12.3. Ellipse Centered at (h,k): $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

I) If $a > b$, the two foci are on the line $y = k$ and are given by $(h-c,k)$ & $(h+c,k)$ where $c^2 = a^2 - b^2$.

II) If $b > a$, the two foci are on the line $x = h$ and are given by $(h,k-c)$ & $(h,k+c)$ where $c^2 = b^2 - a^2$.

2.12.4. Hyperbola Centered at (h,k):

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ *or* $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$

I) When $\frac{(x-h)^2}{a^2}$ is to the left of the minus sign, the two foci are on the line $y = k$ and are given by $(h-c,k)$ & $(h+c,k)$ where $c^2 = a^2 + b^2$.

II) When $\frac{(y-k)^2}{b^2}$ is to the left of the minus sign, the two foci are on the line $x = h$ and are given by $(h, k-c)$ & $(h, k+c)$ where $c^2 = b^2 + a^2$.

2.12.5. Parabola with Vertex at (h,k) and Focal Length p:

$(y-k)^2 = 4p(x-h)$ *or* $(x-h)^2 = 4p(y-k)$

I) For $(y-k)^2$, the focus is $(h+p, k)$ and the directrix is given by the line $x = h - p$.

II) For $(x-h)^2$, the focus is $(h, k+p)$ and the directrix is given by the line $y = k - p$.

2.12.6. Transformation Process for Removal of xy Term in the General Conic Equation

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$:

Step 1: Set $\tan(2\theta) = \frac{B}{A-C}$ and solve for θ.

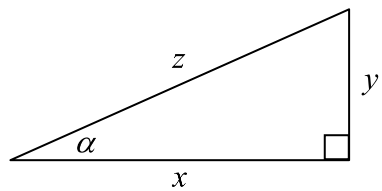
Step 2: let $\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}$

Step 3: Substitute the values for x, y obtained in Step 2 into $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Step 4: Reduce. The final result should be of the form $A'(x')^2 + C'(y')^2 + D'(x') + E'(y') + F' = 0$.

3. **Trigonometry**

3.1. Basic Definitions of Trigonometric Functions & Trigonometric Inverse Functions



Let the figure above be a right triangle with one side of length x, a second side of length y, and a hypotenuse of length z. The angle α is opposite the side of length. The six trigonometric functions—where each is a function of α —are defined as follows:

Z :	**Arbitrary**	**Z is 1**	**Inverse when Z is 1**
3.1.1.	$\sin(\alpha)=\dfrac{y}{z}$	$\sin(\alpha)=y$	$\sin^{-1}(y)=\alpha$
3.1.2.	$\cos(\alpha)=\dfrac{x}{z}$	$\cos(\alpha)=x$	$\cos^{-1}(x)=\alpha$
3.1.3.	$\tan(\alpha)=\dfrac{y}{x}$	$\tan(\alpha)=\dfrac{y}{x}$	$\tan^{-1}\left(\dfrac{y}{x}\right)=\alpha$
3.1.4.	$\cot(\alpha)=\dfrac{x}{y}$	$\cot(\alpha)=\dfrac{x}{y}$	$\cot^{-1}\left(\dfrac{x}{y}\right)=\alpha$
3.1.5.	$\sec(\alpha)=\dfrac{z}{x}$	$\sec(\alpha)=\dfrac{1}{x}$	$\sec^{-1}\left(\dfrac{1}{x}\right)=\alpha$
3.1.6.	$\csc(\alpha)=\dfrac{z}{y}$	$\csc(\alpha)=\dfrac{1}{y}$	$\csc^{-1}\left(\dfrac{1}{y}\right)=\alpha$

Note: $\sin^{-1}$ *is also known as* $\arcsin$. *Likewise, the other inverses are also known as* $\arccos, \arctan, arc\cot, arc\sec$ *and* $arc\csc$.

3.2. Fundamental Definition-Based Identities

3.2.1. $\csc(\alpha) = \dfrac{1}{\sin(\alpha)}$

3.2.2. $\sec(\alpha) = \dfrac{1}{\cos(\alpha)}$

3.2.3. $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$

3.2.4. $\cot(\alpha) = \dfrac{\cos(\alpha)}{\sin(\alpha)}$

3.2.5. $\tan(\alpha) = \dfrac{1}{\cot(\alpha)}$

3.3. Pythagorean Identities

3.3.1. $\sin^2(\alpha) + \cos^2(\alpha) = 1$

3.3.2. $1 + \tan^2(\alpha) = \sec^2(\alpha)$

3.3.3. $1 + \cot^2(\alpha) = \csc^2(\alpha)$

3.4. Negative Angle Identities

3.4.1. $\sin(-\alpha) = -\sin(\alpha)$

3.4.2. $\cos(-\alpha) = \cos(\alpha)$

3.4.3. $\tan(-\alpha) = -\tan(\alpha)$

3.4.4. $\cot(-\alpha) = -\cot(\alpha)$

3.5. Sum and Difference Identities

3.5.1. $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

3.5.2. $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

3.5.3. $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

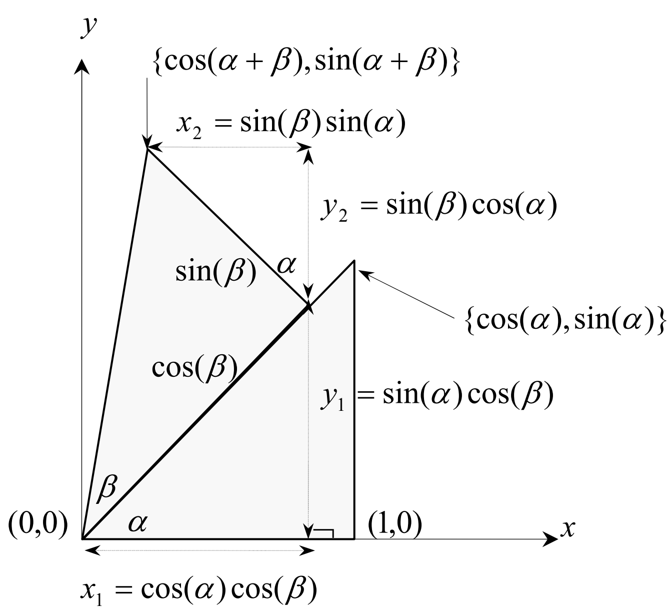
3.5.4. $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

3.5.5. $\tan(\alpha+\beta)=\dfrac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

3.5.6. $\tan(\alpha-\beta)=\dfrac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

3.5.7. Derivation of Formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$:

In the figure below, each coordinate of the point $\{\cos(\alpha+\beta),\sin(\alpha+\beta)\}$ is decomposed into two components using *both definitions* for the sine and cosine in **3.1.1.** and **3.1.2.**



From the figure, we have

$$\cos(\alpha+\beta)=x_1-x_2 \Rightarrow$$
$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\;\therefore$$

$$\sin(\alpha+\beta)=y_1+y_2 \Rightarrow$$
$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)\;\therefore$$

3.6. Double Angle Identities

3.6.1. $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$

3.6.2. $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

3.6.3. $\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$

3.6.4. $\tan(2\alpha) = \dfrac{2\tan(\alpha)}{1-\tan^2(\alpha)}$

3.7. Half Angle Identities

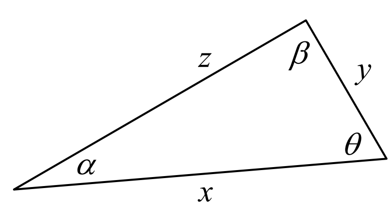
3.7.1. $\sin(\dfrac{\alpha}{2}) = \pm\sqrt{\dfrac{1-\cos(\alpha)}{2}}$

3.7.2. $\cos(\dfrac{\alpha}{2}) = \pm\sqrt{\dfrac{1+\cos(\alpha)}{2}}$

3.7.3. $\tan(\dfrac{\alpha}{2}) = \pm\sqrt{\dfrac{1-\cos(\alpha)}{1+\cos(\alpha)}} = \dfrac{\sin(\alpha)}{1+\cos(\alpha)} = \dfrac{1-\cos(\alpha)}{\sin(\alpha)}$

3.8. General Triangle Formulas

Applicable to all triangles: right and non-right



3.8.1. Sum of Interior Angles: $\alpha + \beta + \theta = 180^0$ (also **2.3.1.**)

3.8.2. Law of Sines: $\dfrac{\sin(\alpha)}{y} = \dfrac{\sin(\beta)}{x} = \dfrac{\sin(\theta)}{z}$

3.8.3. Law of Cosines:

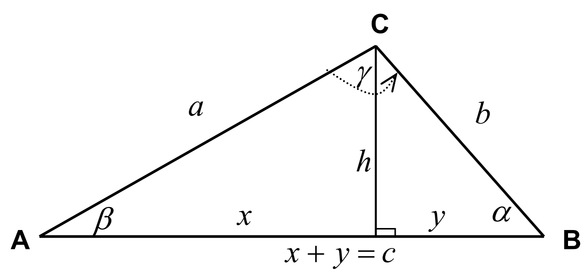
a) $y^2 = x^2 + z^2 - 2xz\cos(\alpha)$
b) $x^2 = y^2 + z^2 - 2yz\cos(\beta)$
c) $z^2 = x^2 + y^2 - 2xy\cos(\theta)$

3.8.4. Area Formulas for a General Triangle:

a) $A = \frac{1}{2}xz\sin(\alpha)$
b) $A = \frac{1}{2}yz\sin(\beta)$
c) $A = \frac{1}{2}xy\sin(\theta)$

3.8.5. Derivation of Law of Sines and Cosines:

Let ΔABC be a general triangle and drop a perpendicular from the apex as shown.



For the Law of Sines we have

$$\overset{1}{\mapsto}: \frac{h}{b} = \sin(\alpha) \Rightarrow h = b\sin(\alpha)$$

$$\overset{2}{\mapsto}: \frac{h}{a} = \sin(\beta) \Rightarrow h = a\sin(\beta)$$

$$\overset{3}{\mapsto}: b\sin(\alpha) = a\sin(\beta) \Rightarrow \frac{b}{\sin(\beta)} = \frac{a}{\sin(\alpha)} \therefore$$

The last equality is easily extended to include the third angle γ.

For the Law of Cosines we have $h = b\sin(\alpha)$.

$\overset{1}{\mapsto}$: Solve for y and x in terms of the angle α

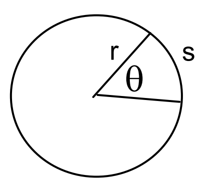
$$\frac{y}{b} = \cos(\alpha) \Rightarrow y = b\cos(\alpha) \Rightarrow$$
$$x = c - y = c - b\cos(\alpha)$$

$\overset{2}{\mapsto}$: Use the Pythagorean Theorem to complete the derivation.

$$x^2 + h^2 = a^2 \Rightarrow$$
$$[c - b\cos(\alpha)]^2 + [b\sin(\alpha)]^2 = a^2 \Rightarrow$$
$$c^2 - 2bc\cos(\alpha) + b^2\cos^2(\alpha) + b^2\sin^2(\alpha) = a^2 \Rightarrow$$
$$c^2 - 2bc\cos(\alpha) + b^2 = a^2 \Rightarrow$$
$$a^2 = c^2 + b^2 - 2bc\cos(\alpha) \therefore$$

Similar expressions can be written for the remaining two sides.

3.9. Arc and Sector Formulas



3.9.1. Arc Length s: $s = r\theta$

3.9.2. Area of a Sector: $A = \frac{1}{2}r^2\theta$

3.10. Degree/Radian Relationship

3.10.1. Basic Conversion: $180^0 = \pi$ radians

3.10.2. Conversion Formulas:

From	To	Multiply by
Radians	Degrees	$\frac{180^0}{\pi}$
Degrees	Radians	$\frac{\pi}{180}$

3.11. Addition of Sine and Cosine

$a\sin\theta + b\cos\theta = k\sin(\theta+\alpha)$ where

$$k = \sqrt{a^2+b^2}$$

$$\alpha = \sin^{-1}\left[\frac{b}{\sqrt{a^2+b^2}}\right]$$

or

$$\alpha = \cos^{-1}\left[\frac{a}{\sqrt{a^2+b^2}}\right]$$

3.12. Polar Form of Complex Numbers

3.12.1. $a+bi = r(\cos\theta + i\sin\theta)$ where

$$r = \sqrt{a^2+b^2}\ ,\theta = Tan^{-1}\left[\frac{b}{a}\right]$$

3.12.2. Definition of $re^{i\theta}$: $re^{i\theta} = r(\cos\theta + i\sin\theta)$

3.12.3. Euler's Famous Equality: $e^{i\pi} = -1$

3.12.4. De-Moivre's Theorem: $(re^{i\theta})^n = r^n e^{in\theta}$ or

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos[n\theta] + i\sin[n\theta])$$

3.12.5. Polar Form Multiplication:

$$r_1 e^{i\alpha} \cdot r_2 e^{i\beta} = r_1 \cdot r_2 e^{i(\alpha+\beta)}$$

3.12.6. Polar Form Division: $\dfrac{r_1 e^{i\alpha}}{r_2 e^{i\beta}} = \dfrac{r_1}{r_2} e^{i(\alpha-\beta)}$

3.13. Rectangular to Polar Coordinates

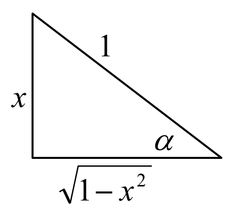
$$(x, y) \Leftrightarrow (r, \theta)$$

$$x = r\cos\theta, y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}(y/x)$$

3.14. Trigonometric Values from Right Triangles

In the right triangle below, let $\sin^{-1}(x) = \alpha \Rightarrow \sin(\alpha) = x = \dfrac{x}{1}$.



Then

3.14.1. $\cos(\alpha) = \sqrt{1-x^2}$

3.14.2. $\tan(\alpha) = \dfrac{x}{\sqrt{1-x^2}}$

3.14.3. $\cot(\alpha) = \dfrac{\sqrt{1-x^2}}{x}$

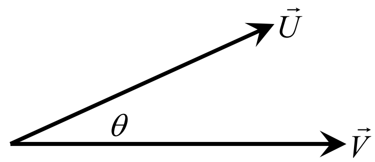
3.14.4. $\sec(\alpha) = \dfrac{1}{\sqrt{1-x^2}}$

3.14.5. $\csc(\alpha) = \dfrac{1}{x}$

4. Elementary Vector Algebra

4.1. Basic Definitions and Properties

Let $\vec{V} = (v_1, v_2, v_3)$, $\vec{U} = (u_1, u_2, u_3)$ be two vectors.



4.1.1. Sum and/or Difference: $\vec{U} \pm \vec{V}$

$\vec{U} \pm \vec{V} = (u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3)$

4.1.2. Scalar Multiplication: $(\alpha)\vec{U} = (\alpha u_1, \alpha u_2, \alpha u_3)$

4.1.3. Negative Vector: $-\vec{U} = (-1)\vec{U}$

4.1.4. Zero Vector: $\vec{0} = (0,0,0)$

4.1.5. Vector Length: $|\vec{U}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

4.1.6. Unit Vector Parallel to $\vec{V}$: $\frac{1}{|\vec{V}|}\vec{V}$

4.1.7. Two Parallel Vectors: $\vec{V} \parallel \vec{U}$ means there is a scalar c such that $\vec{V} = (c)\vec{U}$

4.2. Dot Products

4.2.1. Definition of Dot Product: $\vec{U} \bullet \vec{V} = u_1 v_1 + u_2 v_2 + u_3 v_3$

4.2.2. Angle θ Between Two Vectors: $\cos\theta = \frac{\vec{U} \bullet \vec{V}}{|\vec{U}||\vec{V}|}$

4.2.3. Orthogonal Vectors: $\vec{U} \bullet \vec{V} = 0$

4.2.4. Projection of $\vec{U}$ onto $\vec{V}$:

$$proj_{\vec{V}}(\vec{U}) = \left[\frac{\vec{U} \bullet \vec{V}}{|\vec{V}|^2}\right]\vec{V} = \left[\frac{\vec{U} \bullet \vec{V}}{|\vec{V}|}\right]\frac{\vec{V}}{|\vec{V}|} =$$

$$\left[|\vec{U}|\cos\theta\right]\frac{\vec{V}}{|\vec{V}|}$$

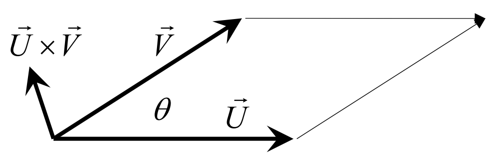
4.3. Cross Products

4.3.1. Definition of Cross Product: $\vec{U} \times \vec{V} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

4.3.2. Orientation of $\vec{U} \times \vec{V}$; Orthogonal to Both $\vec{U}$ and $\vec{V}$:

$$\vec{U} \bullet (\vec{U} \times \vec{V}) = \vec{V} \bullet (\vec{U} \times \vec{V}) = 0$$

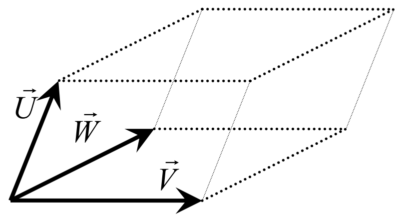
4.3.3. Area of Parallelogram: $A = |\vec{U} \times \vec{V}| = |\vec{U}||\vec{V}|\sin\theta$



4.3.4. Interpretation of the Triple Scalar Product:

$$\vec{U} \bullet (\vec{V} \times \vec{W}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The triple scalar product is numerically equal to the volume of the parallelepiped at the top of the next page



4.4. Line and Plane Equations

Given a point $\vec{P} = (a,b,c)$

4.4.1. Line Parallel to $\vec{P}$ Passing Through (x_1, y_1, z_1):

If (x, y, z) is a point on the line, then

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

4.4.2. Plane Normal to $\vec{P}$ Passing Through (x_1, y_1, z_1).

If (x, y, z) is a point on the plane, then

$$(a,b,c) \bullet (x - x_1, y - y_1, z - z_1) = 0$$

4.4.3. Distance D between a point & plane:

If a point is given by (x_0, y_0, z_0) and $ax + by + cz + d = 0$ is a plane, then

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

4.5. Miscellaneous Vector Equations

4.5.1. The Three Direction Cosines:

$$\cos\alpha = \frac{v_1}{|\vec{V}|}, \cos\beta = \frac{v_2}{|\vec{V}|}, \cos\gamma = \frac{v_3}{|\vec{V}|}$$

4.5.2. Definition of Work: constant force $\vec{F}$ along the path $\vec{PQ}$:

$$W = \vec{F} \bullet \vec{PQ} = |proj_{\vec{PQ}}(\vec{F})||\vec{PQ}|$$

5. Elementary Calculus

5.1. What is a Limit?

Limits are foundational to calculus and will always be so. Limits lead to results unobtainable by algebra alone.

So what is a limit? A limit is a numerical target, a target acquired and locked. Consider the expression $x \to 7$ where x is an independent variable. The arrow ($\to$) points to a target on the right, in this case the number 7. The variable x on the left is targeting 7 in a *modern smart-weapon sense*. This means x is moving, moving towards target, closing range, and programmed to merge eventually with the target. Notice that the quantity x is a true independent variable in that x has been launched and set in motion towards a target, a target that cannot escape from its sights. Independent variables usually find themselves embedded inside an algebraic (or transcendental) expression of some sort, which is being used as a processing rule for a function. Consider the expression $2x+3$ where the independent variable x is about to be sent on the mission $x \to -5$. Does the entire expression $2x+3$ in turn target a numerical value as $x \to -5$? A way to phrase this question using a new type of mathematical notation might be $\underset{x\to -5}{target}(2x+3) = ?$ Interpreting the notation, we are asking if the dynamic output stream from the expression $2x+3$ targets a numerical value in the modern smart-weapon sense as the equally-dynamic x targets the value -5. Mathematical judgment says yes; the output stream targets the value -7. Hence, we complete our new notation as $\underset{x\to -5}{target}(2x+3) = -7$. This explanation is reasonable except for one little problem: the word *target* is nowhere to be found in calculus texts. The traditional replacement (weighing in with 300 years of history) is the word *limit*, which leads to the following working definition:

Working Definition: A *limit* is a target in the modern *smart-weapon* sense. In the above example, we will write $\lim_{x\to -5}(2x+3) = -7$.

5.2. What is a Differential?

The differential concept is one of the two core concepts underlying calculus, limits being the other.

Wee is a Scottish word that means very small, tiny, diminutive, or minuscule. In the context of calculus, 'wee' can be used in similar fashion to help explain the concept of differential, also called an infinitesimal. To have a differential, we first must have a variable, x, y, z etc. Once we have a variable, say x, we can create a secondary quantity dx, which is called the differential of the variable x. What exactly is this dx, read 'dee x'? The quantity dx is a *very small, tiny, diminutive, or minuscule* numerical amount when compared to the original x. Moreover, it is the very small size of dx that makes it, by definition, a *wee* x. How small? In mathematical terms, the following two conditions hold:

$$0 < |xdx| << 1 \text{ and } 0 < \left|\frac{dx}{x}\right| << 1.$$

The two above conditions state $|dx|$ is small enough to guarantee that both its product and quotient with the original quantity x is still very small and much, much closer to zero than to one (the meaning of the symbol $<< 1$). Both inequalities imply that $|dx|$ is also very small when considered independently $0 < |dx| << 1$. Lastly, both inequalities state that $|dx| > 0$, which brings us to the following very important point*: although very small, the quantity dx is never zero.* One can also think of dx as the final h in a limit process $\lim_{h \to 0}$ where the process abruptly stops just short of target, in effect saving the rapidly vanishing h from disappearing into oblivion! Thinking of dx in this fashion makes the differential a prepackaged or frozen limit of sorts. Differentials are designed to be so small that second-order and higher terms involving differentials, such as $7(dx)^2$, can be totally ignored in associated algebraic expressions. This final property distinguishes the differential as a topic belonging to the subject of calculus.

5.3. Basic Differentiation Rules

5.3.1. Limit Definition of Derivative:

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

5.3.2. Differentiation Process Indicator: $[]'$

5.3.3. Constant: $[k]' = 0$

5.3.4. Power: $[x^n]' = nx^{n-1}$, n can be <u>any exponent</u>

5.3.5. Coefficient: $[af(x)]' = af'(x)$

5.3.6. Sum/Difference: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

5.3.7. Product: $[f(x)g(x)]' = f(x)g'(x) + g(x)f'(x)$

5.3.8. Quotient: $\left[\frac{f(x)}{g(x)} \right]' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

5.3.9. Chain: $[f(g(x))]' = f'(g(x))g'(x)$

5.3.10. Inverse: $[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}$

5.3.11. Generalized

Power: $[\{f(x)\}^n]' = n\{f(x)\}^{n-1} f'(x)$;

Again, n can be <u>any exponent</u>

5.4. Transcendental Differentiation

5.4.1. $[\ln x]' = \frac{1}{x}$

5.4.2. $[\log_a x]' = \frac{1}{x \ln a}$

5.4.3. $[e^x]' = e^x$

5.4.4. $[a^x]' = a^x \ln a$

5.4.5. $[\sin x]' = \cos x$

5.4.6. $[\sin^{-1}(x)]' = \dfrac{1}{\sqrt{1-x^2}}$

5.4.7. $[\cos x]' = -\sin x$

5.4.8. $[\cos^{-1}(x)]' = \dfrac{-1}{\sqrt{1-x^2}}$

5.4.9. $[\tan x]' = \sec^2 x$

5.4.10. $[\tan^{-1}(x)]' = \dfrac{1}{1+x^2}$

5.4.11. $[\sec x]' = \sec x \tan x$

5.4.12. $[\sec^{-1}(x)]' = \dfrac{1}{|x|\sqrt{x^2-1}}$

5.5. Basic Antidifferentiation Rules

5.5.1. Antidifferentiation Process Indicator: $\int$

5.5.2. Constant: $\int k dx = kx + C$

5.5.3. Coefficient: $\int af(x)dx = a\int f(x)dx$

5.5.4. Power Rule for $n \neq -1$: $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$

5.5.5. Power Rule for $n = -1$:

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$$

5.5.6. Sum:

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

5.5.7. Difference:

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

5.5.8. Parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

5.5.9. Chain: $\int f'(g(x))g'(x)dx = f(g(x)) + C$

5.5.10. Generalized Power Rule for $n \neq -1$:

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C$$

5.5.11. Generalized Power Rule for $n = -1$:

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C, n = -1$$

5.5.12. General Exponential: $\int e^{f(x)} f'(x)dx = e^{f(x)} + C$

5.6. Transcendental Antidifferentiation

5.6.1. $\int \ln x dx = x \ln x - x + C$

5.6.2. $\int e^x dx = e^x + C$

5.6.3. $\int xe^x dx = (x-1)e^x + C$

5.6.4. $\int a^x dx = \frac{a^x}{\ln a} + C$

5.6.5. $\int \cos x dx = \sin x + C$

5.6.6. $\int \sin x dx = -\cos x + C$

5.6.7. $\int \tan x dx = -\ln|\cos x| + C$

5.6.8. $\int \cot x dx = \ln|\sin x| + C$

5.6.9. $\int \sec x dx = \ln|\sec x + \tan x| + C$

5.6.10. $\int \sec x \tan x dx = \sec x + C$

5.6.11. $\int \sec^2 x dx = \tan x + C$

5.6.12. $\int \csc x dx = -\ln|\csc x + \cot x| + C$

5.6.13. $\int \csc^2 x dx = -\cot x + C$

5.6.14. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(\frac{x}{a}) + C$

5.6.15. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$

5.6.16. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln|\frac{x-a}{x+a}| + C$

5.7. Lines and Approximation

5.7.1. Tangent Line at $(a, f(a))$: $y - f(a) = f'(a)(x - a)$

5.7.2. Normal Line at $(a, f(a))$: $y - f(a) = \frac{-1}{f'(a)}(x - a)$

5.7.3. Linear Approximation: $f(x) \cong f(a) + f'(a)(x - a)$

5.7.4. Second Order Approximation:

$$f(x) \cong f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

5.7.5. Newton's Iterative Formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

5.7.6. Differential Equalities:

$$y = f(x) \Rightarrow dy = f'(x)dx$$

$$f(x + dx) = f(x) + f'(x)dx$$

$$F(x + dx) = F(x) + f(x)dx$$

5.8. Interpretation of Definite Integral

At least three interpretations are valid for the definite integral.

First Interpretation: As a processing symbol for functions, the definite integral $\int_a^b f(x)dx$ instructs the *operator* to start the process by finding $F(x)$ (*the primary antiderivative for* $f(x)dx$) and finish it by evaluating the quantity $F(x)|_a^b = F(b) - F(a)$. This interpretation is pure process-to-product with no context.

Second Interpretation: As a summation symbol for differential quantities, $\int_a^b f(x)dx$ signals to the *operator* that myriads of infinitesimal quantities of the form $f(x)dx$ are being continuously summed on the interval $[a,b]$ with the summation process starting at $x=a$ and ending at $x=b$. Depending on the context for a given problem, such as summing area under a curve, the differential quantities $f(x)dx$ and subsequent total can take on a variety of meanings. This makes continuous summing a powerful tool for solving real-world problems. The fact that continuous sums can also be evaluated by $\int_a^b f(x)dx = F(x)|_a^b = F(b)-F(a)$ is a *key consequence of the Fundamental Theorem of Calculus* (**5.9.**).

Third Interpretation: The definite integral $\int_a^b f(x)dx$ can be interpreted as a point solution $y(b)$ to any explicit differential equation having the general form $dy = f(x)dx : y(a)=0$. In this interpretation $\int_a^b f(x)dx$ is first modified by integrating over the variable subinterval $[a,z]\subset[a,b]$. This leads to $y(z)=\int_a^z f(x)dx = F(z)-F(a)$. Substituting $x=a$ gives the stated boundary condition $y(a)=F(a)-F(a)=0$ and substituting $x=b$ gives $y(b)=F(b)-F(a)=\int_a^b f(x)dx$. In this context, the function $y(z)=F(z)-F(a)$, as a unique solution to $dy = f(x)dx : y(a)=0$, can also be interpreted as a continuous running sum from $x=a$ to $x=z$.

5.9. The Fundamental Theorem of Calculus

Let $\int_a^b f(x)dx$ be a definite integral representing a continuous summation process, and let $F(x)$ be such that $F'(x) = f(x)$. Then, $\int_a^b f(x)dx$ can be evaluated by the alternative process

$$\int_a^b f(x)dx = F(x)\big|_a^b = F(b) - F(a).$$

Note: A continuous summation (or addition) process on the interval $[a,b]$ sums millions upon millions of consecutive, tiny quantities from $x = a$ to $x = b$ where each individual quantity has the general form $f(x)dx$.

5.10. Geometric Integral Formulas

5.10.1. Area Between two Curves for $f(x) \geq g(x)$ on $[a,b]$:

$$A = \int_a^b [f(x) - g(x)]dx$$

5.10.2. Area Under $f(x) \geq 0$ on $[a,b]$: $A = \int_a^b f(x)dx$

5.10.3. Volume of Revolution about x Axis Using Disks:

$$V = \int_a^b \pi[f(x)]^2 dx$$

5.10.4. Volume of Revolution about y Axis using Shells:

$$V = \int_a^b 2\pi x\,|\,f(x)\,|\,dx$$

5.10.5. Arc Length: $s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

5.10.6. Revolved Surface Area about x Axis:

$$SA_x = \int_a^b 2\pi \,|\, f(x) \,|\, \sqrt{1+[f'(x)]^2}\, dx$$

5.10.7. Revolved Surface Area about y Axis:

$$SA_x = \int_a^b 2\pi \,|\, x \,|\, \sqrt{1+[f'(x)]^2}\, dx$$

5.10.8. Total Work with Variable Force $F(x)$ on $[a,b]$:

$$W = \int_a^b F(x)dx$$

5.11. Select Ordinary Differential Equations (ODE)

5.11.1. First Order Linear: $\frac{dy}{dx} + f(x)y = g(x)$

5.11.2. Bernoulli Equation: $\frac{dy}{dx} = f(x)y + g(x)y^n$

5.11.3. ODE Separable if it reduces to: $g(y)dy = f(x)dx$

5.11.4. Falling Body with Drag: $-m\frac{dv}{dt} = -mg + kv^n$

5.11.5. Constant Rate Growth or Decay:

$$\frac{dy}{dt} = ky : y(0) = y_0$$

5.11.6. Logistic Growth: $\frac{dy}{dt} = k(L-y)y : y(0) = y_0$

5.11.7. Continuous Principle Growth:

$$\frac{dP}{dt} = rP + c_0 : P(0) = P_0$$

5.11.8. Newton's Law in One Dimension:

$$\frac{d}{dt}(mV) = \sum F$$

5.11.9. Newton's Law in Three Dimensions: $\frac{d}{dt}(m\vec{V}) = \sum \vec{F}$

5.11.10. Process for Solving a Linear ODE

Step1: Let $F(x)$ be such that $F'(x) = f(x)$

Step 2: Formulate the integrating factor $e^{F(x)}$

Step 3: Multiply both sides of $\dfrac{dy}{dx} + f(x)y = g(x)$ by $e^{F(x)}$

$$e^{F(x)}\left[\frac{dy}{dx}\right] + e^{F(x)} f(x)y = e^{F(x)} g(x) \Rightarrow$$

$$\frac{d}{dx}\left(ye^{F(x)}\right) = e^{F(x)} \cdot g(x)$$

Step 4: Perform the indefinite integration.

$$e^{F(x)} \cdot y = \int e^{F(x)} \cdot g(x)dx + C \Rightarrow$$

$$y = y(x) = e^{-F(x)} \cdot \left[\int e^{F(x)} \cdot g(x)dx\right] + Ce^{-F(x)} \therefore$$

5.12. Laplace Transform; General Properties

5.12.1. Definition: $L[f(t)] = \int_0^\infty f(t)e^{-st}dt \equiv F(s)$

5.12.2. Linear Operator Property:
$L[af(t) + bg(t)] = aF(s) + bG(s)$

5.12.3. Transform of the Derivative:
$L[f^{(n)}(t)] = s^n F(s) - s^{(n-1)} f(0) -$
$s^{(n-2)} f'(0) - ... - f^{(n-1)}(0)$

5.12.4. Derivative of the Transform: $F^{(n)}(s) = (-t)^n f(t)$

5.12.5. Transform of the Definite Integral:

$$L[\int_0^t f(\tau)d\tau] = F(s)/s$$

5.12.6. Transform of the Convolution:

$$\int_0^t f(\tau)g(t-\tau)d\tau \Leftrightarrow F(s)G(s)$$

5.12.7. First Shifting Theorem: $e^{at}f(t) \Leftrightarrow F(s-a)$

5.12.8. Transform of Unit Step Function $U(t-a)$ where $U(t-a)=0$ on $[0,a]$ and $U(t-a)=1$ on $(a,\infty]$. $U(t-a) \Leftrightarrow \dfrac{e^{-as}}{s}$

5.12.9. Second Shifting Theorem:

$f(t-a)U(t-a) \Leftrightarrow e^{-as}F(s)$

5.13. Laplace Transform: Specific Transforms

Entries are a one-to-one correspondence between $f(t)$ and $F(s)$.

5.13.1. $1 \Leftrightarrow 1/s$

5.13.2. $t \Leftrightarrow 1/s^2$

5.13.3. $t^n \Leftrightarrow n!/s^{(n+1)}$

5.13.4. $e^{at} \Leftrightarrow 1/(s-a)$

5.13.5. $te^{at} \Leftrightarrow 1/(s-a)^2$

5.13.6. $t^n e^{at} \Leftrightarrow n!/(s-a)^{n+1}$

5.13.7. $\sin(kt) \Leftrightarrow \dfrac{k}{s^2+k^2}$

5.13.8. $\sin^2(kt) \Leftrightarrow \dfrac{2k^2}{s(s^2+4k^2)}$

5.13.9. $t\sin(kt) \Leftrightarrow \dfrac{2ks}{(s^2+k^2)^2}$

5.13.10. $\cos(kt) \Leftrightarrow \dfrac{s}{s^2+k^2}$

5.13.11. $\cos^2(kt) \Leftrightarrow \dfrac{s^2+2k^2}{s(s^2+4k^2)}$

5.13.12. $t\cos(kt) \Leftrightarrow \dfrac{s^2-k^2}{(s^2+k^2)^2}$

5.13.13. $\sinh(kt) \Leftrightarrow \dfrac{k}{s^2-k^2}$

5.13.14. $\sinh^2(kt) \Leftrightarrow \dfrac{2k^2}{s(s^2-4k^2)}$

5.13.15. $t\sinh(kt) \Leftrightarrow \dfrac{2ks}{(s^2-k^2)^2}$

5.13.16. $\cosh(kt) \Leftrightarrow \dfrac{s}{s^2-k^2}$

5.13.17. $\cosh^2(kt) \Leftrightarrow \dfrac{s^2-2k^2}{s(s^2-4k^2)}$

5.13.18. $t\cosh(kt) \Leftrightarrow \dfrac{s^2+k^2}{(s^2-k^2)^2}$

5.13.19. $e^{at}\sin(kt) \Leftrightarrow \dfrac{k}{(s-a)^2+k^2}$

5.13.20. $e^{at}\sinh(kt) \Leftrightarrow \dfrac{k}{(s-a)^2-k^2}$

5.13.21. $\dfrac{e^{at}-e^{bt}}{a-b} \Leftrightarrow \dfrac{1}{(s-a)(s-b)}$

5.13.22. $e^{at}\cos(kt) \Leftrightarrow \dfrac{s-a}{(s-a)^2+k^2}$

5.13.23. $e^{at}\cosh(kt) \Leftrightarrow \dfrac{s-a}{(s-a)^2-k^2}$

5.13.24. $\dfrac{ae^{at}-be^{bt}}{a-b} \Leftrightarrow \dfrac{s}{(s-a)(s-b)}$

6. Money and Finance

6.1. What is Interest?

Interest affects just about every adult in America. If you are independent, own a car or a home or both, or have a credit card or two, you probably pay or have paid interest. So, what exactly is *interest*? *Interest is a rent charge for the use of money*.

As a rent charge for the use of housing accumulates over time, likewise, an interest charge for the use of money also accumulates over time. Interest is normally stated in terms of a *percentage interest rate* such as $8\frac{\%}{year}$. Just as velocity is a rate of distance accumulation (e.g. $60\frac{miles}{hour}$), percentage interest rate is a 'velocity' of percent accumulation. When driving in America, the customary units of velocity are *miles per hour*. Likewise, the customary units for interest rate are *percent per year*. The reader should be aware that other than customary units may be used in certain situations. For example, in space travel $7\frac{miles}{\sec}$ is used to describe escape velocity from planet earth; and, when computing a credit-card charge, a monthly interest rate of $1.5\frac{\%}{month}$ may be used. Both velocity and percentage interest rate need to be multiplied by time—specified in matching units—in order to obtain the total amount accumulated, either miles or percent, as in the two expressions $D = 75\frac{miles}{hour} \cdot 2\frac{1}{3}hours = 175 miles$ or $\% = 2\frac{percent}{month} \cdot 3\frac{1}{2}months = 7\,percent$.

Once the total accumulated interest is computed, it is then multiplied by the amount borrowed, called the principal P, in order to obtain the total accumulated interest charge I The total accumulated interest charge I, the principal P, the percentage-interest rate r (simply called the interest rate), and the time t during which a fixed principal is borrowed are related by the fundamental formula $I = \Pr t$. This basic formula applies as long as the principal P and the interest rate r remain constant throughout the duration of the accumulation time t.

For the remaining subsections in **6.0**, the following apply.

α : Annual growth rate as in the growth rate of voluntary contributions to a fund
A : Total amount gained or owed
D : Periodic deposit rate—weekly, monthly, or annually
D_i : Deposit made at the start of the i^{th} compounding period
FV : Future value
i : Annual inflation rate
L : Initial Lump Sum
M : Monthly payment
n : Number of compounding periods per year
P : Amount initially borrowed or deposited
PV : Present value
r : Annual interest rate
r_{eff} : Effective annual interest rate
SM : Total sum of payments
t : Time period in years for an investment
T : Time period in years for a loan

6.2. Simple Interest

6.2.1. Accrued Interest: $I = \Pr T$

6.2.2. Total repayment over T: $A = P + \Pr T = P(1 + rT)$

6.2.3. Monthly payment over $T : M = \dfrac{P(1 + rT)}{12T}$

6.3. Compound and Continuous Interest

6.3.1. Compounded Growth: $A = P(1 + \frac{r}{n})^{nt}$

6.3.2. Continuous Growth: $A = Pe^{rt}$

6.3.3. Annually Compounded Inflation Rate i: $A = P(1 - i)^t$

6.3.4. Continuous Annual Inflation Rate i: $A = Pe^{-it}$

Note: inflation rate can be mathematically treated as a negative interest rate, thus the use of the negative sign in ***6.3.3*** *and* ***6.3.4****.*

6.4. Effective Interest Rates

6.4.1. Simple Interest: $r_{eff} = \sqrt[T]{1 + rT} - 1$

6.4.2. Compound Interest: $r_{eff} = (1 + \frac{r}{n})^n - 1$

6.4.3. Continuous Interest: $r_{eff} = e^r - 1$

6.4.4. Given P, A, T : $r_{eff} = \sqrt[T]{\frac{A}{P}} - 1$

6.5. Present-to-Future Value Formulas

6.5.1. Compound Interest:

$$FV = PV(1 + \tfrac{r}{n})^{nt} \Leftrightarrow PV = \frac{FV}{(1 + \frac{r}{n})^{nt}}$$

6.5.2. Annual Compounding with r_{eff} :

$$FV = PV(1 + r_{eff})^t \Leftrightarrow PV = \frac{FV}{(1 + r_{eff})^t}$$

6.5.3. Constant Annual Inflation Rate with Yearly Compounding: Replace r_{eff} with $-i$ in **6.5.2**.

6.5.4. Continuous Compounding:

$$FV = PVe^{rt} \Leftrightarrow PV = \frac{FV}{e^{rt}}$$

6.5.5. Simple Interest:

$$FV = PV(1 + rt) \Leftrightarrow PV = \frac{FV}{(1 + rt)}$$

6.6. Present Value of a Future Deposit Stream

Conditions: n compounding periods per year; total term t years with nt compounding periods; annual interest rate r ; nt identical deposits D made at beginning of each compounding period.

6.6.1. Periodic Deposit with no Final Deposit D_{nt+1} :

$$PV = \frac{Dn}{r}\left\{\left(1 + \tfrac{r}{n}\right)^{nt+1} - \left(1 + \tfrac{r}{n}\right)\right\}$$

6.6.2. Periodic Deposit with Final Deposit D_{nt+1}:

$$PV = \frac{Dn}{r}\left\{\left(1+\tfrac{r}{n}\right)^{nt+1} - 1\right\}$$

6.6.3. Annual Deposit with no Final Deposit D_{t+1}:

$$PV = \frac{D}{r_{eff}}\left\{\left(1+r_{eff}\right)^{t+1} - \left(1+r_{eff}\right)\right\}$$

6.6.4. Annual Deposit with Final Deposit D_{t+1}:

$$PV = \frac{D}{r_{eff}}\left\{\left(1+r_{eff}\right)^{t+1} - 1\right\}$$

6.7. Present Value of a Future Deposit Stream Coupled with Initial Lump Sum

Assume the initial lump sum $L > D$

6.7.1. Periodic Deposit with no Final Deposit D_{nt+1}:

$$PV = (L-D)\left(1+\tfrac{r}{n}\right)^{nt} + \frac{Dn}{r}\left\{\left(1+\tfrac{r}{n}\right)^{nt+1} - \left(1+\tfrac{r}{n}\right)\right\}$$

6.7.2. Periodic Deposit with Final Deposit D_{nt+1}:

$$PV = (L-D)\left(1+\tfrac{r}{n}\right)^{nt} + \frac{Dn}{r}\left\{\left(1+\tfrac{r}{n}\right)^{nt+1} - 1\right\}$$

6.7.3. Annual Deposit with no Final Deposit D_{t+1}:

$$PV = (L-D)\left(1+r_{eff}\right)^{t} + \frac{D}{r_{eff}}\left\{\left(1+r_{eff}\right)^{t+1} - \left(1+r_{eff}\right)\right\}$$

6.7.4. Annual Deposit with Final Deposit D_{t+1}:

$$PV = (L-D)\left(1+r_{eff}\right)^{t} + \frac{D}{r_{eff}}\left\{\left(1+r_{eff}\right)^{t+1} - 1\right\}$$

6.8. Present Value of a Continuous Future Deposit Stream

6.8.1. Annual Deposit Only: $PV = \frac{D}{r}(e^{rt} - 1)$

6.8.2. Annual Deposit plus Lump Sum: $PV = Le^{rt} + \frac{D}{r}(e^{rt} - 1)$

6.8.3. Increasing Annual Deposit $De^{\alpha t}$:

$$PV = \frac{D}{r-\alpha}(e^{rt} - e^{\alpha t})$$

6.8.4. **6.8.3** plus Lump Sum:

$$PV = Le^{rt} + \frac{D}{r-\alpha}(e^{rt} - e^{\alpha t})$$

6.9. Types of Retirement Savings Accounts

STANDARD IRA	**ROTH IRA**	**401 (K)**	**KEOGH PLAN**
Sponsored by Individual	Sponsored by Individual	Sponsored by Company	Plan for self employed
Taxes on contributions and interest are deferred until withdrawn	Taxes on contributions paid now. No taxes on any proceeds withdrawn	Taxes on contributions and interest are deferred until withdrawn	Taxes on contributions and interest are deferred until withdrawn
\$3000/year \$6000/year for jointly filing couples	\$3000/year \$6000/year for jointly filing couples	Increases every year. Currently \$15,000.00	Up to 25% of income
Withdrawals can begin at age 59.5, must begin at 70.5	Withdrawals can begin at age 59.5	Withdrawals can begin at age 59.5, must begin at 70.5	Withdrawals can begin at age 59.5, must begin at 70.5
Substantial penalty for early withdrawal	Lesser penalty for early withdrawal	Substantial penalty for early withdrawal	Substantial penalty for early withdrawal
Limited heir rights	Substantial heir rights	Limited heir rights	Limited heir rights

6.10. Loan Amortization

Assume monthly payments M

6.10.1. First Month's Interest: $I_{1st} = \frac{rP}{12}$

6.10.2. Amount of Payment: $M = \frac{rP}{12\left[1-\left(1+\frac{r}{12}\right)^{-12T}\right]}$

6.10.3. Total Loan Repayment: $SM = 12TM$

6.10.4. Total Interest Paid: $I_{total} = 12TM - P$

6.10.5. Payoff PO_j after the j^{th} Payment:

$$PO_j = P\left(1+\tfrac{r}{12}\right)^j - \frac{12M}{r}\left\{\left(1+\tfrac{r}{12}\right)^j - 1\right\}$$

6.10.6. Amount M_{Pj} of j^{th} Payment to Principle:

$$M_{Pj} = \left[\frac{12M - rP}{12}\right]\left(1+\tfrac{r}{12}\right)^{j-1}$$

6.10.7. Amount M_{Ij} of j^{th} Payment to Interest: $M_{Ij} = M - M_{Pj}$

6.10.8. Pros and Cons of Long-Term Mortgages:

PROS	CONS
Increased total mortgage costs are partially defrayed by tax breaks and inflation via payoff by cheaper dollars	Total mortgage costs are much more over time
Allows the borrower to buy more house sooner: with inflation, sooner means cheaper	Home equity buildup by mortgage reduction is much slower for long-term mortgages
Historically, inflation of home purchase prices contributes more to home equity buildup than home equity buildup by mortgage reduction	Mortgage is more vulnerable to personal misfortune such as sickness or job loss

6.11. Annuity Formulas

Note: Use the loan amortization formulas since annuities are nothing more than loans where the roles of the institution and the individual are reversed.

6.12. Markup and Markdown

C : Cost
OP : Old price
NP : New price
$P\%$; Given percent as a decimal equivalent

6.12.1. Markup Based on Original Cost: $NP = (1 + P\%)C$

6.12.2. Markup Based on Cost plus New Price: $C + P\% \cdot NP = NP$

6.12.3. Markup Based on Old Price: $NP = (1 + P\%)OP$

6.12.4. Markdown Based on Old Price: $NP = (1 - P\%)OP$

6.12.5. Percent given Old and New Price: $P\% = NP / OP$

6.13. Calculus of Finance

6.13.1. General Differential Equation of Elementary Finance: $\frac{dP}{dt} = r(t)P + D(t) : P(0) = P_0$

6.13.2. Differential Equation for Continuous Principle Growth or Continuous Loan Reduction Assuming a Constant Interest Rate and Fixed Annual Deposits/Payments

$$\frac{dP}{dt} = r_0 P \pm D_0 : P(0) = P_0 \Rightarrow$$

$$P(t) = P_0 e^{r_0 t} \pm \frac{D_0}{r}(e^{rt} - 1)$$

6.13.3. Present Value of Total Mortgage Repayment:

$$A_{PV} = \int_0^T \left[\frac{rP_0 e^{rT}}{e^{rT} - 1} \right] e^{-it} dt \Rightarrow A_{PV} = \frac{\left(\frac{r}{i}\right) P_0 (e^{rT} - e^{(r-i)T})}{(e^{rT} - 1)}$$

7. Probability and Statistics

7.1. Probability Formulas

Let U be a universal set consisting of all possible events.
Let Φ be the empty set consisting of no event.
Let $A, B \subset U$

7.1.1. Basic Formula:

$$P = \frac{favorable - number - of - ways}{total - number - of - ways}$$

7.1.2. Fundamental Properties:

$$P(U) = 1$$
$$P(\Phi) = 0$$

7.1.3. Order Relationship:

$$A \subset U \Rightarrow 0 \le P(A) \le 1$$

7.1.4. Complement Law:

$$P(A) = 1 - P(\sim A)$$

7.1.5. Addition Law:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

7.1.6. Conditional Probability Law:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

7.1.7. Multiplication Law:

$$P(A \cap B) = P(B) \cdot P(A \mid B)$$
$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

7.1.8. Definition of Independent Events (IE):

$$A \cap B = \Phi$$

7.1.9. IE Multiplication Law:

$$P(A \cap B) = P(A) \cdot P(B)$$

7.2. Basic Statistical Definitions

7.2.1. Set: an aggregate of individual items—animate or inanimate

7.2.2. Element: a particular item in the set

7.2.3. Observation: any attribute of interest associated with the element

7.2.4. Statistic: any measurement of interest associated with the element. Any statistic is an observation, but not all observations are statistics

7.2.5. Data set: a set whose elements are statistics

7.2.6. Statistics: the science of drawing conclusions from the totality of observations—both statistics and other attributes—generated from a set of interest

7.2.7. Population: the totality of elements that one wishes to study by making observations

7.2.8. Sample: that population subset that one has the resources to study

7.2.9. Sample Statistic: any statistic associated with a sample

7.2.10. Population Statistic: any statistic associated with a population

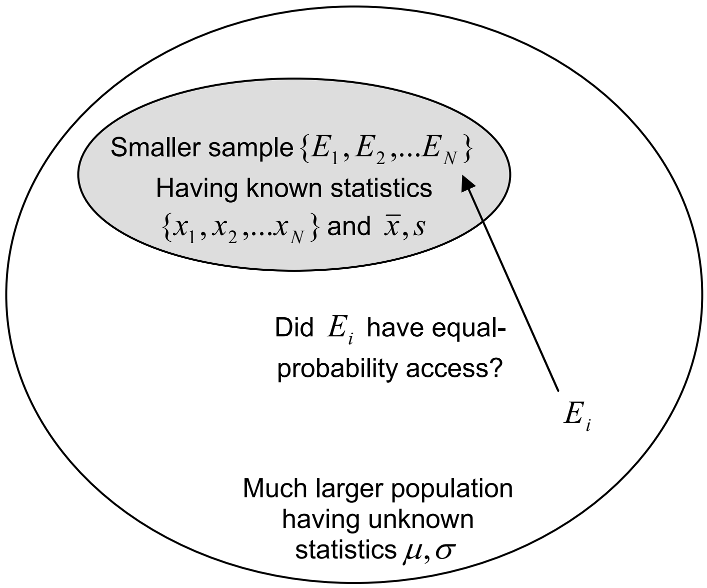
7.2.11. Random sample: a sample where all population elements have equal probability of access

7.2.12. Inference: the science of using sample statistics to predict population statistics

7.2.13. Brief Discussion Using the Above Definitions

Let a **set** consist of N **elements** $\{E_1, E_2, E_3, ..., E_N\}$ where there has been **observed** one **statistic** of a similar nature for each **element**. The **data set** of all **observed statistics** is denoted by $\{x_1, x_2, x_3, ..., x_N\}$. The corresponding rank-ordered **data set** is a re-listing of the individual **statistics** $\{x_1, x_2, x_3, ..., x_N\}$ in numerical order from smallest to largest. **Data sets** can come from either **populations** or from **samples**. Most **data sets** will be considered **samples**. As such, the **sample statistics** obtained from the **sample** will be utilized to make **inferential** predictions for corresponding **population statistics** characterizing a much larger **population. Inference** processes are valid if and only if one can be assured that the **sample** obtained is a **random sample**.

The diagram below supports sections **7.2** through **7.4** by illustrating some of the key concepts.



Example of Statistical Inference
Use $\bar{x}$ to predict μ.
Questions:

- ✓ Is my sample a random sample?
- ✓ How close is my prediction?
- ✓ How certain is my prediction?

7.3. Measures of Central Tendency

7.3.1. Sample Mean or Average $\bar{x}$: $\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$

7.3.2. Population Mean or Average μ : $\mu = \frac{1}{N} \sum_{i=1}^{N} x_i$

7.3.3. Median $\tilde{x}$: the middle value in a rank-ordered data set

7.3.4. Mode M : the data value or statistic that occurs most often.

7.3.5. Multi-Modal Data Set: a data set with two or more modes

7.3.6.Median Calculation Process:

Step 1: Rank order from smallest to largest all elements in the data set.

Step 2: The median $\tilde{x}$ is the actual middle statistic if there is an odd number of data points.

Step 3: The median $\tilde{x}$ is the average of the two middle statistics if there is an even number of data points.

7.4. Measures of Dispersion

7.4.1. Range $R : R = x_L - x_S$ where x_L is the largest data value in the data set and x_S is the smallest data value

7.4.2. Sample Standard Deviation s :

$$s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2} \; .$$

7.4.3. Population Standard Deviation σ :

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2} \; .$$

7.4.4. Sample Variance: s^2

7.4.5. Population Variance: σ^2

7.4.6. Sample Coefficient of Variation $C_{VS} : C_{VS} = \dfrac{s}{\bar{x}}$

7.4.7. Population Coefficient of Variation C_{VP} : $C_{VP} = \dfrac{\sigma}{\mu}$

7.4.8.Z-Score z_i for a Sample Value x_i : $z_i = \dfrac{x_i - \bar{x}}{s}$

7.5. Sampling Distribution of the Mean

The mean $\bar{x}$ is formed from a sample of individual data points randomly selected from either an infinite or finite population. The number of data points selected is given by n. The sample is considered a Large Sample if $n \geq 30$; a Small Sample if $n < 30$.

7.5.1. Expected Value of $\bar{x}$: $E(\bar{x}) = \mu$

7.5.2. Standard Deviation of $\bar{x}$:

Infinite Population	Finite Population of Count N
$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$	$\sigma_{\bar{x}} = \sqrt{\dfrac{N-n}{N-1}}\dfrac{\sigma}{\sqrt{n}}$

7.5.3. Large Sample Z-score for $\bar{x}_i$: $z_i = \dfrac{\bar{x}_i - \mu}{\sigma / \sqrt{n}}$

When σ is unknown, substitute s.

7.5.4. Interval Estimate of Population Mean:

Large-Sample Case	Small-Sample Case
$\bar{x} \pm z_{\frac{\alpha}{2}} \cdot \left[\dfrac{\sigma}{\sqrt{n}}\right]$	$\bar{x} \pm t_{\frac{\alpha}{2}} \cdot \left[\dfrac{s}{\sqrt{n}}\right]$

Note: No assumption about the underlying population needs to be made in the large-sample case. In the small-sample case, the underlying population is assumed to be normal or nearly so. When σ is unknown in the large-sample case, substitute s.

7.5.5. Sampling Error E_R : $E_R = z_{\frac{\alpha}{2}} \cdot \left[\dfrac{\sigma}{\sqrt{n}}\right]$

7.5.6. Sample Size Needed for a Given Error:

$$n = \left[\frac{z_{\frac{\alpha}{2}} \cdot \sigma}{E_R}\right]^2 .$$

7.6. Sampling Distribution of the Proportion

The proportion p is a quantity formed from a sample of individual data points randomly selected from either an infinite or finite population. The proportion can be thought of as a mean formulated from a sample where all the individual values are either zero (0) or one (1). The number of data points selected is given by n. The sample is considered a Large Sample if both $np \geq 5$ and $n(1-p) \geq 5$.

7.6.1. Expected Value E_X of $\overline{p}$: $E_X(\overline{p}) = \mu$

7.6.2. Standard Deviation of $\overline{p}$:

Infinite Population

$$\sigma_{\overline{p}} = \sqrt{\frac{p(1-p)}{n}}$$

Finite Population of Count N

$$\sigma_{\overline{p}} = \sqrt{\frac{N-n}{N-1}}\sqrt{\frac{p(1-p)}{n}}$$

7.6.3. Interval Estimate of Population Proportion:

$$\overline{p} \pm z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{\overline{p}(1-\overline{p})}{n}}$$

Note: Use $\overline{p} = .5$ *in* $\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}$ *if clueless on the initial size of* $\overline{p}$.

7.6.4. Sampling Error: $E_R = z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{p(1-p)}{n}}$

7.6.5. Sample Size Needed for Given Error:

$$n = \frac{z_{\frac{\alpha}{2}}^{2} \cdot p(1-p)}{ER^2}$$

7.6.6. Worse case for **7.6.5.**, proportion unknown:

$$n = \frac{z_{\frac{\alpha}{2}}^{2}}{4ER^2}$$

Section II

Tables

1. Numerical

1.1. Factors of Integers 1 through 192

The standard order-of-operations applies; ^ is used to denote the raising to a power; and * is used for multiplication.

INTEGER FOLLOWED BY FACTORIZATION					
1	1	29	Prime	57	3*19
2	Prime	30	2*3*5	58	2*29
3	Prime	31	Prime	59	Prime
4	2^2	32	2^5	60	2^2*3*5
5	Prime	33	3*11	61	Prime
6	2*3	34	2*17	62	2*31
7	Prime	35	5*7	63	3*3*7
8	2^3	36	2^2*3^2	64	2^6
9	3*3	37	Prime	65	5*13
10	2*5	38	2*19	66	2*3*11
11	Prime	39	3*13	67	Prime
12	2^2*3	40	2^3*5	68	2^2*17
13	Prime	41	Prime	69	3*23
14	2*7	42	2*3*7	70	2*5*7
15	3*5	43	Prime	71	Prime
16	2^4	44	2^2*11	72	2^3*3^2
17	Prime	45	3^3*5	73	Prime
18	2*3*3	46	2*23	74	2*37
19	Prime	47	Prime	75	3*5^2
20	2^2*5	48	2^4*3	76	2^2*19
21	3*7	49	7*7	77	7*11
22	2*11	50	2*5^2	78	2*3*13
23	Prime	51	3*17	79	Prime
24	2^3*3	52	2^2*13	80	2^4*5
25	5^5	53	Prime	81	3^4
26	2*13	54	2*3^3	82	2*41
27	3^3	55	5*11	83	Prime
28	2^2*7	56	2^3*7	84	2^2*3*7

Integer Followed By Factorization					
85	5*17	121	11^2	157	3*7^2
86	2*43	122	2*61	158	2*79
87	3*29	123	3*41	159	3*53
88	2^3*11	124	2^2*31	160	2^5*5
89	Prime	125	5^3	161	7*23
90	2*3^2*5	126	2*3^2*7	162	2*3^4
91	7*13	127	Prime	163	Prime
92	2^2*23	128	2^7	164	2^2*41
93	3*31	129	3*43	165	3*5*11
94	2*47	130	2*5*13	166	2*83
95	5*19	131	Prime	167	Prime
96	2^5*3	132	2*61	168	2^3*3*7
97	Prime	133	7*19	169	Prime
98	2*7^2	134	2*67	170	2*5*17
99	3^2*11	135	3^3*5	171	3^2*19
100	2^2*5^2	136	2^3*17	172	2^2*43
101	Prime	137	Prime	173	Prime
102	2*3*17	138	2*3*23	174	2*87
103	Prime	139	Prime	175	5^2*7
104	2^3*13	140	2^2*5*7	176	2^4*11
105	3*5*7	141	3*47	177	3*59
106	2*53	142	2*71	178	2*89
107	Prime	143	11*13	179	Prime
108	2^2*3^3	144	2^4*3^2	180	2^2*3^2*5
109	Prime	145	5*29	181	Prime
110	2*5*11	146	2*73	182	2*91
111	3*37	147	3*7^2	183	3*61
112	2^4*7	148	2^2*37	184	2^3*23
113	Prime	149	Prime	185	5*37
114	2*3*19	150	2*3*5^2	186	2*93
115	5*23	151	Prime	187	11*17
116	2^2*29	152	2^3*19	188	2^2*47
117	3*3*13	153	Prime	189	3^3*7
118	2*59	154	2*7*11	190	2*5*19
119	7*17	155	5*31	191	Prime
120	2^3*3*5	156	2^2*3*13	192	2^7*3

1.2. Prime Numbers less than 1000

2	*3*	*5*	*7*	*11*	*13*	*17*	*19*	23	*29*
31	37	*41*	*43*	47	53	*59*	*61*	67	*71*
73	79	83	89	97		*101*	*103*	*107*	*109*
113	127	131	*137*	*139*	*149*	*151*	157	163	167
173	*179*	*181*	*191*	*193*	*197*	*199*		211	223
227	*229*	233	*239*	*241*	251	257	263	269	271
277	*281*	*283*	293		307	*311*	*313*	317	331
337	*347*	*349*	353	359	367	373	379	383	389
397		401	409	*419*	*421*	*431*	*433*	439	443
449	457	*461*	*463*	467	479	487	491	499	
503	509	*521*	*523*	541	547	557	563	*569*	*571*
577	587	593	599		601	607	613	*617*	*619*
631	*641*	*643*	647	653	*659*	*661*	673	677	683
691		701	709	719	727	733	739	743	751
757	761	769	773	787	797		*809*	*811*	*821*
823	*827*	*829*	839	853	857	859	863	877	*881*
883	887		907	911	919	929	937	941	947
953	967	971	977	983	991	997			

1.3. Roman Numeral and Arabic Equivalents

ARABIC	ROMAN	ARABIC	ROMAN	ARABIC	ROMAN
1	I	10	X	101	CI
2	II	11	XI	200	CC
3	III	15	XV	500	D
4	IV	20	XX	600	DI
5	V	30	XXX	1000	M
6	VI	40	XL	5000	V bar
7	VII	50	L	10000	L bar
8	VIII	60	LX	100000	C bar
9	IX	100	C	1000000	M bar

1.4. Nine Elementary Memory Numbers

NUM	MEM	NUM	MEM	NUM	MEM
$\sqrt{2}$	1.4142	$\sqrt{7}$	2.6457	ϕ	0.6180
$\sqrt{3}$	1.7321	π	3.1416	$\ln(10)$	2.3026
$\sqrt{5}$	2.2361	e	2.7182	$Log(e)$	0.4343

1.5. American Names for Large Numbers

NUM	NAME	NUM	NAME	NUM	NAME
10^3	thousand	10^18	quintillion	10^33	decillion
10^6	million	10^21	sextillion	10^36	undecillion
10^9	billion	10^24	septillion	10^39	duodecillion
10^12	trillion	10^27	octillion	10^48	quidecillion
10^15	quadrillion	10^30	nontillion	10^63	vigintillion

1.6. Selected Magic Squares

1.6.1. 3X3 Magic Square with Magic Sum 15. The second square below is called a 3x3 Anti-Magic Square:

2	7	6
9	5	1
4	3	8

2	4	7
5	1	8
9	3	6

1.6.2. 4X4 Perfect Magic Square with Magic Sum 34:

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

1.6.3. 5X5 Perfect Magic Square with Magic Sum 65:

1	15	8	24	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

Note: For a Magic Square of size NXN, the Magic Sum is given by the formula

$$\frac{N(N^2+1)}{2}$$

1.6.4. Nested 5X5 Magic Square with Outer Magic Sum 65:

1	18	21	22	3
2	10	17	12	24
18	15	13	11	8
21	14	9	16	5
23	7	6	4	25

1.6.5. 6X6 Magic Square with Magic Sum 111:

1	32	3	34	35	6
12	29	9	10	26	25
13	14	22	21	23	18
24	20	16	15	17	19
30	11	28	27	8	7
31	5	33	4	2	36

1.6.6. 7X7 Magic Square: Magic Sum is 175.

22	21	13	5	46	38	30
31	23	15	14	6	47	39
40	32	24	16	8	7	48
49	31	33	25	17	9	1
2	43	42	34	26	18	10
11	3	44	36	35	27	19
20	12	4	45	37	29	28

1.6.7. Quadruple-Nested 9X9 Magic Square with Outer Magic Sum 369:

16	81	79	78	77	13	12	11	2
76	28	65	62	61	26	27	18	6
75	23	36	53	51	35	30	59	7
74	24	50	40	45	38	32	58	8
9	25	33	39	41	43	49	57	73
10	60	34	44	37	42	48	22	72
14	63	52	29	31	47	46	19	68
15	64	17	20	21	56	55	54	67
80	1	3	4	5	69	70	71	66

1.7. Thirteen-by-Thirteen Multiplication Table

Different font sizes are used for, one, two, or three-digit entries.

×	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*
2	*2*	*4*	*6*	*8*	*10*	*12*	*14*	*16*	*18*	*20*	*22*	*24*	*26*
3	*3*	*6*	*9*	*12*	*15*	*18*	*21*	*24*	*27*	*30*	*33*	*36*	*39*
4	*4*	*8*	*12*	*16*	*20*	*24*	*28*	*32*	*36*	*40*	*44*	*48*	*42*
5	*5*	*10*	*15*	*20*	*25*	*30*	*35*	*40*	*45*	*50*	*55*	*60*	*65*
6	*6*	*12*	*18*	*24*	*30*	*36*	*42*	*48*	*54*	*60*	*66*	*72*	*78*
7	*7*	*14*	*21*	*28*	*35*	*42*	*49*	*56*	*63*	*70*	*77*	*84*	*91*
8	*8*	*16*	*24*	*32*	*40*	*48*	*56*	*64*	*72*	*80*	*88*	*96*	*104*
9	*9*	*18*	*27*	*36*	*45*	*54*	*63*	*72*	*81*	*90*	*99*	*108*	*117*
10	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*	*90*	*100*	*110*	*120*	*130*
11	*11*	*22*	*33*	*44*	*55*	*66*	*77*	*88*	*99*	*110*	*121*	*132*	*143*
12	*12*	*24*	*36*	*48*	*60*	*72*	*84*	*96*	*108*	*120*	*132*	*144*	*156*
13	*13*	*26*	*39*	*42*	*65*	*78*	*91*	*104*	*117*	*130*	*143*	*156*	*169*

Note: The shaded blocks on the main diagonal are the first thirteen squares

1.8. The Random Digits of PI

The digits of PI pass every randomness test. Hence, the first 900 digits of PI serve equally well as a random number table.

PI=3.-- READ LEFT TO RIGHT, TOP TO BOTTOM					
14159	26535	89793	23846	26433	83279
50288	41971	69399	37510	58209	74944
59230	78164	06286	20899	86280	34825
34211	70679	82148	08651	32823	06647
09384	46095	50582	23172	53594	08128
48111	74502	84102	70193	85211	05559
64462	29489	54930	38196	44288	10975
66593	34461	28475	64823	37867	83165
27120	19091	45648	56692	34603	48610
45432	66482	13393	60726	02491	41273
72458	70066	06315	58817	48815	20920
96282	92540	91715	36436	78925	90360
01133	05305	48820	46652	13841	46951
94151	16094	33057	27036	57595	91953
09218	61173	81932	61179	31051	18548
07446	23799	62749	56735	18857	52724
89122	79381	83011	94912	98336	73362
44065	66430	86021	39494	63952	24737
19070	21798	60943	70277	05392	17176
29317	67523	84674	81846	76694	05132
00056	81271	45263	56052	77857	71342
75778	96091	73637	17872	14684	40901
22495	34301	46549	58537	10507	92279
68925	89235	42019	95611	21290	21960
86403	44181	59813	62977	47713	09960
51870	72113	49999	99837	29784	49951
05973	17328	16096	31859	50244	59455
34690	83026	42522	30825	33446	85035
26193	11881	71010	00313	78387	52886
58753	32083	81420	61717	76691	47303

1.9. Standard Normal Distribution

THE STANDARD NORMAL DISTRIBUTION: TABLE VALUES ARE THE RIGHT TAIL AREA FOR A GIVEN Z										
Z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	.5000	.4960	.4920	.4880	.4840	.4800	.4761	.4761	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4051	.4013	.3974	.3936	.3897	.3858
0.3	.3821	.3783	.3744	.3707	.3669	.3631	.3594	.3556	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3263	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2980	.2946	.2911	.2877	.2843	.2809	.2776
0.6	.2742	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2482	.2451
0.7	.2420	.2389	.2358	.2327	.2297	.2266	.2236	.2206	.2176	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1761	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1250	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1074	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0837	.0822
1.4	.0807	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0642	.0630	.0618	.0606	.0594	.0582	.0570	.0559
1.6	.0548	.0536	.0526	.0515	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0445	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0280	.0274	.0268	.0262	.0255	.0250	.0244	.0238	.0232
2.0	.0228	.0222	.0217	.0212	.0206	.0202	.0197	.0192	.0187	.0183
2.1	.0178	.0174	.0170	.0165	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0128	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0101	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0058	.0057	.0055	.0054	.0052	.0050	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0020	.0020	.0019
2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
3.1	.0010	.0010	.0009	.0009	.0009	.0009	.0009	.0008	.0008	.0008
3.2	.0007	.0007	.0006	.0007	.0007	.0006	.0006	.0005	.0005	.0005
3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0004
3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
3.6	.0002	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001
3.7	.0001	.0001	.0001	**Right Tail Area starts to fall below 0.0001**						

1.10. Two-Sided Student's t Statistic

TABLE VALUES ARE T SCORES NEEDED TO GUARANTEE THE PERCENT CONFIDENCE			
Degrees of freedom: DF	**90%**	**95%**	**99%**
1	6.314	12.706	63.657
2	2.920	4.303	9.925
3	2.353	3.182	5.841
4	2.132	2.776	4.604
5	2.015	2.571	4.032
6	1.943	2.447	3.707
7	1.895	2.365	3.499
8	1.860	2.306	3.355
9	1.833	2.262	3.250
10	1.812	2.228	3.169
11	1.796	2.201	3.106
12	1.782	2.179	3.055
13	1.771	2.160	3.012
14	1.761	2.145	2.977
15	1.753	2.131	2.947
16	1.746	2.120	2.921
17	1.740	2.110	2.898
18	1.734	2.101	2.878
19	1.729	2.093	2.861
20	1.725	2.083	2.845
21	1.721	2.080	2.831
22	1.717	2.074	2.819
23	1.714	2.069	2.907
24	1.711	2.064	2.797
25	1.708	2.060	2.787
26	1.706	2.056	2.779
27	1.703	2.052	2.771
28	1.701	2.048	2.763
29	1.699	2.045	2.756
30	1.697	2.042	2.750
40	1.684	2.021	2.704
60	1.671	2.000	2.660
120	1.658	1.980	2.617
∞	1.645	1.960	2.576

1.11. Date and Day of Year

DATE	DAY	DATE	DAY	DATE	DAY
Jan 1	1	May 1	121	Sep 1	244
Jan 5	5	May 5	125	Sep 5	248
Jan 8	8	May 8	128	Sep 8	251
Jan 12	12	May 12	132	Sep 12	255
Jan 15	15	May 15	135	Sep 15	258
Jan 19	19	May 19	139	Sep 19	262
Jan 22	22	May 22	142	Sep 22	265
Jan 26	26	May 26	146	Sep 26	269
Feb 1	32	Jun 1	152	Oct 1	274
Feb 5	36	Jun 5	156	Oct 6	278
Feb 8	39	Jun 8	159	Oct 8	281
Feb 12	43	Jun 12	163	Oct 12	285
Feb 15	46	Jun 15	166	Oct 15	288
Feb 19	50	Jun 19	170	Oct 19	292
Feb 22	53	Jun 22	173	Oct 22	295
Feb 26	57	Jun 26	177	Oct 26	299
Mar 1	60**	Jul 1	182	Nov 1	305
Mar 5	64	Jul 5	186	Nov 5	309
Mar 8	67	Jul 8	189	Nov 8	312
Mar 12	71	Jul 12	193	Nov 12	316
Mar 15	74	Jul 15	196	Nov 15	319
Mar 19	78	Jul 19	200	Nov 19	323
Mar 22	81	Jul 22	203	Nov 22	326
Mar 26	85	Jul 26	207	Nov 26	330
Apr 1	91	Aug 1	213	Dec1	335
Apr 5	96	Aug 5	218	Dec 5	339
Apr 8	98	Aug 8	220	Dec 8	342
Apr 12	102	Aug 12	224	Dec 12	346
Apr 15	105	Aug 15	227	Dec 15	349
Apr 19	109	Aug 19	331	Dec 19	353
Apr 22	112	Aug 22	234	Dec 22	356
Apr 26	116	Aug 26	238	Dec 26	360
** Add one day starting here if a leap year					

2. Physical Sciences

2.1. Conversion Factors in Allied Health

2.1.1. Volume Conversion Table

Apothecary		Household		Metric
1minim		1drop	1gtt	
16minims				1mL (cc)
60minims	1fluidram	60gtts	1tsp	5mL (cc) *or 4mL*
4fluidrams	0.5fluidounce	3tsp	1tbsp	15mL (cc)
8fluidrams	1fluidounce	2tbsp		30mL (cc)
	8fluidounces	1cup		240mL (cc)
	16fluidounces	2cups	1pint	500mL (cc) *or 480mL*
	32fluidounces	2pints	1quart	1000mL (cc) *or 960mL*

2.1.2. Weight Conversion Table

Apothecary		Metric
1grain		60mg *or 64mg*
15grains		1g
60grains	1dram	4g
8drams	1ounce	32g
12ounces	1pound	384g

2.1.3. General Comments

- ✓ All three systems—apothecary, household and metric systems—have rough volume equivalents.
- ✓ Since the household system is a volume-only system, the Weight Conversion Table in **2.1.2** does not include household equivalents.
- ✓ Common discrepancies that are still considered correct are shown in *italics* in both tables **2.1.1** and **2.1.2**.

2.2. Medical Abbreviations in Allied Health

ABBREVIATION	MEANING
b.i.d.	Twice a day
b.i.w.	Twice a week
c	With
cap, caps	Capsule
dil.	Dilute
DS	Double strength
gtt	Drop
h, hr	Hour
h.s.	Hour of sleep, at bedtime
I.M.	Intramuscular
I.V.	Intravenous
n.p.o., NPO	Nothing by mouth
NS, N/S	Normal saline
o.d.	Once a day, every day
p.o	By or through mouth
p.r.n.	As needed, as necessary
q.	Every, each
q.a.m.	Every morning
q.d.	Every day
q.h.	Every hour
q2h	Every two hours
q4h	Every four hours
q.i.d.	Four times a day
ss	One half
s.c., S.C., s.q.	Subcutaneous
stat, STAT	Immediately, at once
susp	Suspension
tab	Tablet
t.i.d.	Three times a day
P% strength	P grams per 100 mL
A:B strength	A grams per B mL

2.3. Wind Chill Table

Grey area is the danger zone where exposed human flesh will begin to freeze within one minute.

	WIND SPEED (mph)								
		5	**10**	**15**	**20**	**25**	**30**	**35**	**40**
	35	31	27	25	24	23	22	21	20
	30	25	21	19	17	16	15	14	13
	25	19	15	13	11	9	8	7	6
T	**20**	13	9	6	4	3	1	0	-1
E	**15**	7	3	0	-2	-4	-5	-7	-8
M	**10**	1	-4	-7	-9	-11	-12	-14	-15
P	**5**	-5	-10	-13	-15	-17	-19	-21	-22
	0	-11	-16	-19	-22	-24	-26	-27	-29
^{0}F	**-5**	-16	-22	-26	-29	-31	-33	-34	-36
	-10	-22	-28	-32	-35	-37	-39	-41	-43
	-15	-28	-35	-39	-42	-44	-46	-48	-50
	-20	-34	-41	-45	-48	-51	-53	-55	-57
	-25	-40	-47	-51	-55	-58	-60	-62	-64

2.4. Heat Index Table

The number in the body of the table is the equivalent heating temperature at 0% humidity

	RELATIVE HUMIDITY (%)								
		30	**40**	**50**	**60**	**70**	**80**	**85**	**90**
T	**105**	114	123	135	148	163	180	190	199
E	**104**	112	121	131	144	158	175	184	193
M	**103**	110	118	128	140	154	169	178	186
P	**102**	108	116	125	136	149	164	172	180
	101	106	113	122	133	145	159	166	174
^{0}F	**100**	104	111	119	129	141	154	161	168
	97	99	105	112	120	129	140	145	152
	95	96	101	107	114	122	131	136	141
	90	89	92	96	100	106	112	115	119

2.5. Temperature Conversion Formulas

2.5.1. Fahrenheit to Celsius: $C = \frac{F - 32}{1.8}$

2.5.2. Celsius to Fahrenheit: $F = 1.8C + 32$

2.6. Unit Conversion Table

Arranged in alphabetical order

TO CONVERT	TO	MULTIPLY BY
acres	ft^2	43560
acres	m^2	4046.9
acres	rods	160
acres	hectares	0.4047
acre feet	barrels	7758
acre feet	m^3	1233.5
Angstrom (å)	cm	10E-8
Angstrom	nm	0.1
astronomical unit (AU)	cm	1.496E13
astronomical unit	km	1.496E8
atmospheres (atm)	feet H2O	33.94
atmospheres	in of Hg	29.92
atmospheres	mm of Hg	760
atmospheres	psi	14.7
bar	atm	.98692
bar	dyne/cm^2	10E6
bar	psi (lb/in^2)	14.5038
bar	mm Hg	750.06
bar	MPa	10E-1
barrels (bbl)	ft^3	5.6146
barrels	m^3	0.15898
barrels	gal (US)	42
barrels	liter	158.9

TO CONVERT	TO	MULTIPLY BY
BTU	Canadian BTU	1.000418022
BTU	cal	251.996
BTU	erg	1.055055853 E-10
BTU	joule	1054.35
calorie (cal)	joule	4.184
centimeter (cm)	inch	0.39370
cm	m	1E-2
darcy	m^2	9.8697E-13
dyne	g cm /s^2	1
dyne	Newton	10E-5
erg	cal	2.39006E-8
erg	dyne cm	1
erg	joule	10E-7
fathom	ft	6
feet (ft)	in	12
feet	m	0.3048
furlong	yd	220
gallon (US gal)	in^3	231
gallon	liter	3.78541
(Imperial) gal	in^3	277.419
gallon	liter	4.54608
gamma	Gauss	10E-5
gamma	Tesla	10E-9
gauss	Tesla	10E-4
gram (g)	pound	0.0022046
gram	kg	10E-3
hectare	acre	2.47105
hectare	cm^2	10E-8
horsepower	Watt (W)	745.700

TO CONVERT	TO	MULTIPLY BY
inch (in)	cm	2.54
inch (in)	mm	25.4
joule (J)	erg	10E7
joule	cal	0.239006
kilogram (kg)	g	10E3
kilogram	pound	2.20462
kilometer (km)	m	10E3
kilometer	ft	3280.84
kilometer	mile	0.621371
Kilometer/hr (kph)	mile/hr (mph)	0.621371
kilowatt	hp	1.34102
knot	mph	1.150779
liter	cm^3	10E3
liter	gal (US)	0.26417
liter	in^3	61.0237
meter	angstrom	10E10
meter	ft	3.28084
micron	cm	10E-4
mile	ft	5280
mile	km	1.60934
mm Hg	$dyne/cm^2$	1333.22
Newton	dyne	10E5
Newton	pound force	0.224809
Newton-meter (torque)	foot-pound-force	0.737562
ounce	lb	0.0625
Pascal	atmospheres	9.86923 x10E-6
Pascal	psi	1.45 x10E-4
Pascal	torr	7.501 x10E-3
pint	gallon	0.125
poise	g /cm/s	1
poise	kg /m/s	0.1

TO CONVERT	TO	MULTIPLY BY
pound mass	kg	0.453592
pound force	Newton	4.4475
rod	feet	16.5
quart	gallon	0.25
stoke	cm^2 /s	1
slug	kg	14.594
Tesla	Gauss	10E4
Torr	millibar	1.333224
Torr	millimeter hg	1
ton (long)	lb	2240
ton (metric)	lb	2205
ton (metric)	kg	1000
ton (short or net)	lb	2000
ton (short or net)	kg	907.185
ton (short or net)	ton (metric)	0.907
watt	J /s	1
yard	in	36
yard	m	0.9144
year (calendar)	days	365.242198781
year (calendar)	s	3.15576 x 10E7

2.7. Properties of Earth and Moon

PROPERTY	VALUE	PROPERTY	VALUE
Distance from sun	9.2.9x10^6 miles	Earth Surface g	32.2 ft/s^2
Equatorial diameter	7926 miles	Moon distance from earth	238,393 miles
Length of day	24 hours	Moon diameter	2160 miles
Length of year	365.26 days	Moon revolution	27 days, 7 hours

2.8. Metric System

2.8.1. Basic and Derived Units

QUANTITY	NAME	SYMBOL	UNITS
Length	meter	m	*basic unit*
Time	second	s	*basic unit*
Mass	kilogram	kg	*basic unit*
Temperature	Kelvin	K	*basic unit*
Electrical Current	ampere	A	*basic unit*
Force	Newton	N	$kg\ m\ s^{-2}$
Volume	Liter	L	m^3
Energy	joule	J	$kg\ m^2\ s^{-2}$
Power	watt	W	$kg\ m^2\ s^{-3}$
Frequency	hertz	Hz	s^{-1}
Charge	coulomb	C	A s
Capacitance	farad	F	$C^2\ s^2\ kg^{-1}\ m^{-2}$
Magnetic Induction	Tesla	T	$kg\ A^{-1}\ s^{-2}$

2.8.2. Metric Prefixes

PREFIX	FACTOR	SYMBOL	METER EXAMPLE
peta	10^15	E	Em
tera	10^12	P	Pm
giga	10^9	G	Gm
mega	10^6	M	Mm
kilo	10^3	k	km
hecto	10^2	h	hm
deca	10^1	da	dam
deci	10^(-1)	d	dm
centi	10^(-2)	c	cm
milli	10^(-3)	m	mm
micro	10^(-6)	μ	μm
nano	10^(-9)	n	nm
pica	10^(-12)	p	pm

2.9. British System

2.9.1. Basic and Derived Units

QUANTITY	NAME	SYMBOL	UNITS
Length	foot	ft	*basic unit*
Time	second	s	*basic unit*
Mass	slug		*basic unit*
Temperature	Fahrenheit	^{0}F	*basic unit*
Electrical Current	ampere	A	*basic unit*
Force	pound	lb	*derived unit*
Volume	gallon	gal	*derived unit*
Work	foot-pound	ft-lb	*derived unit*
Power	horsepower	hp	*derived unit*
Charge	coulomb	C	*derived unit*
Capacitance	farad	F	*derived unit*
Heat	British thermal unit	Btu	*basic unit*

2.9.2. Uncommon British Measures of Weight and Length

WEIGHT	LINEAR
Grain=Basic Unit	**Inch=Basic Unit**
1 scruple=20 grains	1 hand=4 inches
1 dram=3 scruples	1 link=7.92 inches
1 ounce=16 drams	1 span=9 inches
1 pound=16 ounces	1 foot=12 inches
1 hundredweight=100 pounds	1 yard=3 feet
1 ton=2000 pounds	1 fathom=2 yards
1 long ton=2240 pounds	1 rod=5.5 yards
	1 chain=100 links=22 yards
	1 furlong=220 yards
	1 mile=1760 yards
	1 knot mile=6076.1155 feet
	1 league=3 miles

2.9.3. Uncommon British Measures of Liquid and Dry Volume

LIQUID	DRY
Gill=Basic Unit	**Pint=Basic Unit**
1 pint=4 gills	1 quart=2 pints
1 quart= 2 pints	1 gallon=4 quarts
1 gallon=4 quarts	1 peck=2 gallons
1 hogshead=63 gallons	1 bushel=4 pecks
1 pipe (or butt)=2 hogsheads	
1 tun=2 pipes	

2.9.4. Miscellaneous British Measures

AREA	ASTRONOMY
1 square chain=16 square rods	1 astronomical unit (AU) = 93,000,000 miles
1 acre=43,560 square feet	1 light second = 186,000 miles =0.002 AU
1 acre=160 square rods	1 light year = 5.88x10^12 miles =6.3226x10^4 AU
1 square mile = 640 square acres	1 parsec (pc) = 3.26 light years
1 square mile = 1 section	1 kpc=1000pc
1 township = 36 sections	1 mpc = 1000000pc

VOLUME
1 U.S. liquid gallon= 231 cubic inches
I Imperial gallon=1.2 U.S. gallons=0.16 cubic feet
1 cord=128 cubic feet

This Page is Blank

Section III

Applications in Personal Finance

1. The Algebra of Interest

1.1. What is Interest?

Interest affects just about every adult in America. If you are independent, own a car or a home or both, or have a credit card or two, you probably pay or have paid interest. So, what exactly is interest? *Interest is a rent charge for the use of money. As a rent charge for the use of housing accumulates over time, likewise, an interest change for the use of money also accumulates over time*. Just as people sometimes borrow housing when shelter is needed, people sometimes borrow money when we want or need the items that money can buy.

Interest is normally stated in terms of a *percentage interest rate* such as $8\frac{\%}{year}$. Just as velocity ($60\frac{miles}{hour}$) is a rate of distance accumulation, percentage interest rate is a 'velocity' of percent accumulation. When driving in America, the customary units of velocity are *miles per hour*. Likewise, the customary units for interest rate are *percent per year*. The reader should be aware that other than customary units may be used in certain situations. For example, in space travel $7\frac{miles}{\sec}$ is used to describe escape velocity from planet earth; and, when computing a credit-card charge, a monthly interest rate of $1.5\frac{\%}{month}$ may be used. Both velocity and percentage interest rate need to be multiplied by time—specified in matching units—in order to obtain the total amount accumulated, either miles or percent, as illustrated below.

On the road: $D = 75\frac{miles}{hour} \cdot 2\frac{1}{3}hours = 175miles$

In the bank: $\% = 2\frac{percent}{month} \cdot 3\frac{1}{2}months = 7\,percent$ $

Once the total accumulated interest is computed, it is then multiplied by the amount borrowed, called the principal P , in order to obtain the total accumulated interest charge I .

The total accumulated interest charge I, the principal P, the percentage interest rate r (hereafter, to be simply called the interest rate), and the accumulated time t (called the term) during which a fixed principle is borrowed are related by the **Fundamental Interest Charge Formula** $I = \Pr t$ (also called the **Simple Interest Formula**). This formula applies as long as the principal P and the interest rate r remain constant throughout the time t.

Ex 1.1.1: Suppose $\$10,000.00$ is borrowed at $7 \frac{\%}{year}$ over a 42 month period with no change in either principal or interest rate. How much are the total interest charges? Using $I = \Pr t$, we obtain (after converting percent to its fractional equivalent and months to their yearly equivalent)

$$I = (\$10,000.00)(\tfrac{8}{100} \tfrac{1}{years})(3\tfrac{1}{2}\, years) = \$2800.00\,.$$

Note: Notice how much the formula $I = \Pr t$ resembles the formula $D = Rt$, where D is distance, R is a constant velocity, and t is the time during which the constant velocity is in effect. The variable P in $I = \Pr t$ distinguishes the Fundamental Interest Charge Formula in that total interest charges are proportional to both the principal borrowed and the time during which the principal is borrowed.

There are two types of interest: *ordinary interest* and *banker's interest*. *Ordinary* interest is computed on the basis of a 365-day year, while bankers' interest is computed on the basis of a 360-day year. The distinction usually shows up in short duration loans of less than one year where the term is specified in days. **Given two identical interest rates, principals, and terms, the loan where interest is computed on the basis of bankers' interest will always cost more.**

Ex 1.1.2: Suppose $\$150,000.00$ is borrowed at $9 \frac{\%}{year}$ for 125 days. How much are the total interest charges using **A**) ordinary interest as the basis for computation, **B**) bankers' interest as the basis for computation?

Again, using $I = \Pr t$ as our fundamental starting point, we obtain

A) $I = (\$150{,}000.00)(9\,\tfrac{\%}{year})(\tfrac{125}{365}\,years) = \$4623.29$

B) $I = (\$150{,}000.00)(9\,\tfrac{\%}{year})(\tfrac{125}{360}\,years) = \$4687.50$.

Notice bankers' interest nets $\$64.21$ to the bank.

1.2. Simple Interest

Simple interest is interest charged according to the formula $I = \Pr t$. We normally find simple interest being used in loans where the term is relatively short or the principal is a few thousand dollars or less. At one time, simple interest was the interest method primarily used to compute changes in an automobile loan. Today, however, with some automobile prices approaching those of a small house—e.g. the Hummer—many automobile loans are set up just like shorter-term home mortgages.

When we borrow money via a simple interest contract, not only are we to pay the interest charges, but we also must pay back the principal borrowed in full. That is the meaning of the word borrowed: we are to return the item used in the same condition that it was originally loaned to us. When we borrow money, we are to return it in its original condition—i.e. all of it and with the *same purchasing power*. Since money invariably loses some of its purchasing power with the passage of time due to the effects of inflation, one can almost always be sure that the amount borrowed is worth less at the end of a specified term than at the beginning. Thus, any interest charge levied must, as a minimum, make up for the loss of purchasing power. In actuality, purchasing power is not only preserved but actually increased via the application of commercial interest charges. Remember, a bank is a business and should expect a profit (interest) on the sale of its particular business commodity (money).

Retiring a simple-interest loan requires the payment of both the principal borrowed and the simple interest charge incurred during its term. Thus we can easily write an algebraic formula for the total amount A to be returned, called the Simple Interest Formula, $A = P + I = P + \Pr t = P(1 + rt)$. We can easily use the simple interest formula to help calculate the monthly payment M for any loan issued on the basis of simple interest.

Ex 1.2.1: You borrow $\$38,000.00$ for an SUV at $3.5\frac{\%}{year}$ simple interest over a term of 7 years. What is your monthly payment? What is the total interest charge?

$$\overset{1}{\mapsto}: A = P + I = P(1 + rt) = \$38,000.00(1 + 0.035\{7\})$$

$$\Rightarrow A = \$47,310.00$$

$$\overset{2}{\mapsto}: M = \frac{A}{\# months} = \frac{\$47,310.00}{84} = \$563.22 \therefore$$

$$\overset{3}{\mapsto}: I = A - P = \$47,310.00 - \$38,000.00 = \$9,310.00 \therefore$$

Buyers should be aware that sometimes the actual interest rate is more than it is stated to be. A Simple Discount Note is a type of loan where this is indeed the case. Here, the borrower prepays all the interest up front from the principal requested. Thus, the funds F available for use during the term of the loan are in fact less, as given by the expression $F = P - I$. This leads to a hidden increase in interest rate if one considers the principal to be those funds F actually transferred to the borrower. This next example illustrates this common sleigh-of-hand scenario.

Ex 1.2.2: A Simple Discount Note for $\$100,000.00$ is issued for a term of 15 months at $10\frac{\%}{year}$. Find the 'hidden' interest rate.

$$\overset{1}{\mapsto}: I = \Pr t = \$100,000.00(10\tfrac{\%}{year})(\tfrac{15}{12}\, years) = \$12,500.00$$

$$\overset{2}{\mapsto}: F = P - I = \$100,000.00 - \$12,500.00 = \$87,500.00$$

$$\overset{3}{\mapsto}: I = Frt \Rightarrow$$
$$12{,}500.00 = 87{,}500.00(r)(\tfrac{15}{12}) \Rightarrow$$
$$109{,}375.00r = 12{,}500.00 \Rightarrow$$
$$r = \frac{12{,}500.00}{109{,}375.00} = 11.4 \tfrac{\%}{year} \therefore$$

Notice that the interest rate is increased by 1.4 percentage points by simply changing the type of loan, i.e. a Simple Discount Note. This will always be the case: not only does interest rate matter, but also the type of loan employing the interest rate. As shown in our last example, precise formulas allow one to easily calculate the various financial quantities without resorting to the use of extensive financial tables.

1.3. Compound Interest

The simple interest formula $A = P(1 + rt)$ is used in situations where the principal never changes during the term of the loan. But more often than not, the principal will change due to the fact that accrued interest is added to the original principal at regular intervals, where each interval is called a compounding period. This addition creates a new and enlarged principal from which future interest is calculated. Interest during any one compounding period is computed using the simple interest formula. To see how this works, let P be the initial principal and r_c be the interest rate during the compounding period (e.g. for an annual interest r applied via monthly compounding periods, $r_c = \frac{r}{12}$). Then after one compounding period, we have by the simple interest formula

$$A_1 = P + I = P + \Pr_c \cdot 1 = P(1 + r_c)^1 = P_1.$$

After the second compounding period, we have

$$A_2 = P_1 + I = P_1 + P_1 r_c = P_1(1+r_c)^1 \Rightarrow$$
$$A_2 = P(1+r_c)^1 \cdot (1+r_c)^1 = P(1+r_c)^2 = P_2.$$

After the third compounding period, the process cycles again with the result

$$A_3 = P_2 + I = P_2 + P_2 r_c = P_2(1+r_c)^1 \Rightarrow$$
$$A_3 = P(1+r_c)^2 \cdot (1+r_c)^1 = P(1+r_c)^3 = P_3.$$

Letting the process continue to the end of n compounding periods leads to the **Compound Interest Formula for Total Amount Returned** $A = A_n = P(1+r_c)^n$. If r is the annual interest rate and n is the number of compounding periods in one year, then the amount A after a term of t years is given by the familiar compound-interest formula $A = P(1+\frac{r}{n})^{nt}$.

In order to use either version of the compound interest formula, no addition to the initial principal P must occur (other than that generated by the compounding effect) during the totality of the compounding process (term). The amount A is the amount to be returned when the compounding process is complete (i.e. has cycled itself through a specified number of compounding periods). Both formulas are most commonly used in the case where an initial sum of money is deposited in a financial/investment institution and allowed to grow throughout a period of years under a specified set of compounding conditions.

Ex 1.3.1: A lump sum of $\$100{,}000.00$ is deposited at $3\frac{\%}{year}$ for 10 years compounded quarterly (four times per year). Find the amount A at the end of the term.

$\overset{1}{\mapsto}: A = P(1+\frac{r}{n})^{nt} \Rightarrow$

$A = \$100{,}000.00(1+\frac{0.03}{4})^{4\cdot 10} \Rightarrow$

$A = \$100{,}000.00(1.0075)^{40} = \$134{,}834.86 \therefore$

Ex 1.3.2: An amount of $\$25,000.00$ compounds at $1\frac{\%}{period}$ for 240 periods. Find the amount A at the end of the term.

$$\overset{1}{\mapsto}: A = P(1 + r_c)^n \Rightarrow$$

$$A = \$25,000.00(1 + 0.01)^{240} \Rightarrow$$

$$A = \$25,000.00(1.01)^{240} = \$272,313.84 \therefore$$

Ex 1.3.3: A grandfather invests \$5000.00 in a long-term growth fund for his newly-born granddaughter. The fund is legally inaccessible until the child reaches the age of 65. Assuming an effective interest rate of $9\frac{\%}{year}$ compounded annually, how much will the granddaughter have accumulated by age 65?

$$\overset{1}{\mapsto}: A = P(1 + \tfrac{r}{n})^{nt} \Rightarrow$$

$$A = \$5,000.00(1 + \tfrac{0.09}{1})^{1 \cdot 65} \Rightarrow$$

$$A = \$5,000.00(1.09)^{65} = \$1,354,229.81 \therefore$$

The last example shows the magic of compounding as it operates on an initial principal through a long period of time. A relatively small financial gain received when young can grow into a magnificent sum if left to accumulate over several decades. This simple but powerful fact leads to our first **Words of Wisdom**: *If properly managed, young windfalls become old fortunes.*

1.4. Continuous Interest

Consider the compound interest formula $A = P(1 + \frac{r}{n})^{nt}$. What would be the overall effect of increasing the number of compounding periods n in one year while holding both the annual interest rate r and the term t constant? One can immediately see that the exponent nt would grow in size, but the quantity inside the parentheses, $1 + \frac{r}{n}$, would become almost indistinguishable from the number 1 as n increases indefinitely.

Since $1^n = 1$ no matter how large n is, the diminishing of $1 + \frac{r}{n}$ to 1 may negate the effect of having a larger and larger exponent. Thus, we end up with a mathematical tug of war between the two affected quantities in $A = (1 + \frac{r}{n})^{nt}$. Our exponent is growing larger desperately trying to make A an indefinitely large number. By contrast, our base is nearing the number 1 trying to make $A = 1$. Which wins? Or, is there a compromise?

To explore this issue, we'll first look at a specific example where $r = 5\frac{\%}{year}$, $t = 10$ years, $P = \$1.00$, and, subsequently, $A = \$1.00(1 + \frac{0.05}{n})^{10n}$. The number of compounding periods n in a year will be allowed to increase through the sequence 1, 10, 12, 100, 365, 1000, $10{,}000$, $100{,}000$, and $1{,}000{,}000$. Modern calculators allow calculations such as these to be easily performed on a routine basis. The results are displayed in the table below with the corresponding amount generated by using the simple interest formula $A = P(1 + rt)$.

n	A
1	\$1.6288946
10	\$1.6466684
12	\$1.6470095
100	\$1.6485152
365	\$1.6486641
1000	\$1.6487006
10000	\$1.6487192
100000	\$1.6487210
1000000	\$1.6487212

Notice that as n progressively increases without bound, the amount A becomes more and more certain, stabilizing about one digit to the left of the decimal point for every power of ten. In conclusion, we can say that the battle ends in a tidy compromise with $1 < A < \infty$, in particular $A = 1.64872...$

The process of n progressively increasing without bound is called a limit process and is symbolized by the limit symbol $\lim\limits_{n\to\infty}$. Limit processes are extensively used to derive most of the mathematical tools and results associated with calculus. We now investigate A as $n \to \infty$ for the case of a fixed annual interest rate r and term t in years, $A = \lim\limits_{n\to\infty}[P(1+\frac{r}{n})^{nt}]$. To analyze this expression, we first move the limit process inside the parentheses and next to the part of the expression it directly affects to obtain $A = P\{\lim\limits_{n\to\infty}[(1+\frac{r}{n})^{n}]\}^{t}$. Again, we have set up our classic battle of opposing forces: the exponent grows without bound and the base gets ever closer to 1. What is the combined effect? To answer, first define $m = \frac{n}{r} \Rightarrow n = rm$. From this, we can establish the *towing relationship* $n \to \infty \Leftrightarrow m \to \infty$. Substituting, we obtain

$$A = \lim_{n\to\infty}[P(1+\tfrac{r}{n})^{nt}] \Rightarrow$$
$$A = P\{\lim_{n\to\infty}[(1+\tfrac{r}{n})^{n}]\}^{t} \Rightarrow .$$
$$A = P\{\lim_{m\to\infty}[(1+\tfrac{1}{m})^{m}]\}^{rt}$$

Now all we need to do is evaluate $\lim\limits_{m\to\infty}[(1+\frac{1}{m})^{m}]$, and we will do this evaluation the modern, easy way, via a scientific calculator.

m value	$(1+\frac{1}{m})^{m}$
1	2
10	2.5937
100	2.7048
1000	2.7169
10000	2.7181
100000	2.7183
1000000	2.7183

We will stop the evaluations at $m = 1{,}000{,}000$. Notice that each time m is increased by a factor of 10, one more digit in the expression $(1+\frac{1}{m})^m$ is stabilized. If more decimal places are needed, we can simply compute $(1+\frac{1}{m})^m$ to the accuracy desired. When m gets astronomically large, the expression $(1+\frac{1}{m})^m$ converges to the number $e = 2.7183...$. Correspondingly, our final limit becomes

$$A = P\{\lim_{m\to\infty}[(1+\tfrac{1}{m})^m]\}^{rt} \Rightarrow$$
$$A = P\{e\}^{rt} \Rightarrow \qquad .$$
$$A = Pe^{rt}$$

The last expression $A = Pe^{rt}$ is known as the **Continuous Interest Formula**. For a fixed annual interest rate r and initial deposit P, the formula gives the account balance A at the end of t years under the condition of continuously adding to the current balance the interest earned in a 'twinkling of an eye.' The continuous interest formula represents in itself an upper limit for the growth of an account balance given a fixed annual interest rate. Hence, it is a very important and easily used tool, which allows a person to quickly estimate account balances over a long period of time. The following example will illustrate this.

Ex 1.4.1: An initial deposit of $\$10{,}000.00$ is compounded monthly (typical turnover for a company 401K account, etc.) at $8\frac{\%}{year}$ for a period of 30 years. Compare the final amounts obtained by using both continuous and compound interest formulas.

$$\overset{1}{\mapsto} : A = Pe^{rt} \Rightarrow$$
$$A = \$10{,}000.00e^{(0.08\cdot 30)} \Rightarrow$$
$$A = \$110{,}231.76 \therefore$$

$\overset{2}{\mapsto}: A = P(1+\frac{r}{n})^{nt} \Rightarrow$

$A = \$10{,}000.00(1+\frac{0.08}{12})^{12\cdot 30} \Rightarrow$

$A = \$10{,}000.00(1.00667)^{360} \Rightarrow$

$A = \$109{,}487.73 \therefore$

Notice that there is less than $\$400.00$ difference between the two amounts, which shows the continuous interest formula a very valuable tool for making estimates when the number of compounding periods in a year exceeds twelve or more. By providing a quick upper bound for the total amount to be returned, the continuous interest formula can also be thought of as a fiscal 'gold standard' defining the limiting capabilities of the compounding process. In the next two examples, we explore the use of the continuous interest formula in providing rapid estimates for both interest rate and time needed to achieve a given amount A. In each example, the natural logarithm (denoted by '$\ln$') is first used to release the overall exponent in e^{rt}, which, in turn allows one to solve for either r or t.

Ex 1.4.2: A brokerage house claims that $\$10{,}000.00$ is 'guaranteed' to become $\$1{,}000{,}000.00$ in 40 years if left with them. What interest rate would make this so?

$\overset{1}{\mapsto}: A = Pe^{rt} \Rightarrow Pe^{rt} = A \Rightarrow$

$\$10{,}000e^{40r} = \$1{,}000{,}000.00 \Rightarrow$

$e^{40r} = 100$

$\overset{2}{\mapsto}: \ln(e^{40r}) = \ln(100) \Rightarrow$

$40r\ln(e) = \ln(100) \Rightarrow$

$40r = 4.605 \Rightarrow$

$r = 0.057 = 11.5\frac{\%}{year} \therefore$

The interest rate of $11.5\frac{\%}{year}$ may be obtainable, but represents an aggressive estimate since the average Dow-Jones-Industrial-Average annual rate of return has hovered around $9\frac{\%}{year}$ for the last 40 years. Hence the brochure is making a marketer's claim! Suppose we actively managed our account for 40 years where we were actually able to achieve $9\frac{\%}{year}$. Then $A = \$10{,}000.00e^{(0.09 \cdot 40)} = \$365{,}982.34$, which is a tidy sum, but no million. Let buyers beware, or, better yet, let buyers be able to figure for themselves.

Ex 1.4.3: How long does it take a starting principal P to quadruple at $5\frac{\%}{year}$ compounded monthly?

$$\overset{1}{\mapsto}: A = Pe^{rt} \Rightarrow$$
$$4P = Pe^{(0.05)t} \Rightarrow$$
$$Pe^{(0.05)t} = 4P \Rightarrow e^{(0.05)t} = 4$$
$$\overset{2}{\mapsto}: \ln(e^{(0.05)t}) = \ln(4) \Rightarrow$$
$$0.05t = 1.38629 \Rightarrow$$
$$t = 27.73 years \therefore$$

1.5. Effective Interest Rate

How do we compare one interest rate to another? The question arises since not only does actual interest rate matter, but also the way the rate interest is utilized (i.e. type of compounding mechanism). The effective annual interest rate, designated r_{eff}, provides a mathematical basis for comparing interest rates having different compounding mechanisms. r_{eff} is defined as *that annually-compounded interest rate that generates the same amount as the specified interest rate and associated compounding process at the end of t years*. In the case of the compound interest formula, we have

$$P(1+r_{eff})^t = P[(1+\tfrac{r}{n})^n]^t \Rightarrow$$
$$(1+r_{eff})^t = [(1+\tfrac{r}{n})^n]^t \Rightarrow$$
$$1+r_{eff} = (1+\tfrac{r}{n})^n \Rightarrow$$
$$r_{eff} = (1+\tfrac{r}{n})^n - 1$$

In the case of continuous interest, we have

$$P(1+r_{eff})^t = Pe^{rt} \Rightarrow$$
$$(1+r_{eff})^t = [e^r]^t \Rightarrow$$
$$1+r_{eff} = e^r \Rightarrow$$
$$r_{eff} = e^r - 1$$

In the case of simple interest, we have

$$P(1+r_{eff})^t = P(1+rt) \Rightarrow$$
$$(1+r_{eff})^t = (1+rt) \Rightarrow$$
$$1+r_{eff} = \sqrt[t]{1+rt}$$
$$r_{eff} = \sqrt[t]{1+rt} - 1$$

The effective interest rate, as defined above, is a simple and powerful consumer basis of comparison in that it combines both rate and process information into a single number. Banks and other lending institutes are legally required to state effective interest rate in their advertising and on their documents. Stock market returns over a long period of time are normally specified in terms of an average annual growth *or interest* rate. We definitely need to know the meaning of r_{eff} and its use if we are to survive the confusion of numbers tossed our way in modern society.

Ex 1.5.1: Which is the better deal, $7.25\frac{\%}{year}$ compounded continuously or $7.5\frac{\%}{year}$ compounded quarterly?

$$\stackrel{1}{\mapsto}: r_{eff} = e^{r} - 1 \Rightarrow$$

$$r_{eff} = e^{0.0725} - 1 = 0.07519 = 7.519\tfrac{\%}{year}$$

$$\stackrel{2}{\mapsto}: r_{eff} = (1 + \tfrac{r}{n})^{n} - 1 \Rightarrow$$

$$r_{eff} = (1 + \tfrac{0.075}{4})^{4} - 1 = 0.07713 = 7.713\tfrac{\%}{year} \therefore$$

The better deal is $7.5\frac{\%}{year}$ compounded quarterly where the effective interest rate is $R_{eff} = 7.713\frac{\%}{year}$.

When viewed as a general concept, the effective annual interest rate becomes a powerful economic and forecasting tool in that it can be easily adapted to determine the average annual growth rate for securities or *any phenomena where change occurs over a period of years*.

Ex 1.5.2: Securities valued at $\$5,000.00$ in 1980 have grown in value to $\$80,000.00$ in 2005. Assuming continuation of the average annual growth value as already displayed during the past 25 years, project the value of these same securities in 2045.

Diagramming the problem in two steps, we have

$$\stackrel{1}{\mapsto}: P = \$5,000.00 \underset{1980-2005}{\overset{25\,years}{\rightarrow\rightarrow}} A = \$80,000.00$$

$$\stackrel{2}{\mapsto}: P = \$80,000.00 \underset{2005-2045}{\overset{30\,years}{\rightarrow\rightarrow}} A?$$

Utilizing the general definition of r_{eff} as found in $A = P(1 + r_{eff})^{t}$ allows us to easily solve this problem for each step.

$$\overset{1}{\mapsto}: A = P(1+r_{eff})^t \Rightarrow$$

$$\$80{,}000.00 = \$5{,}000.00(1+r_{eff})^{25} \Rightarrow$$

$$(1+r_{eff})^{25} = 16 \Rightarrow 1+r_{eff} = \sqrt[25]{16} = 1.1172 \Rightarrow$$

$$r_{eff} = 0.1172 = 11.72\tfrac{\%}{year}$$

$$\overset{2}{\mapsto}: A = \$80{,}000.00(1+0.1172)^{30} \Rightarrow$$

$$A = \$80{,}000.00(1.1172)^{30} = \$2{,}223{,}401.00 \therefore$$

The average annual interest/growth rate of $11.72\tfrac{\%}{year}$ is very good and shows active management of the overall growth process. The final reward, $\$2{,}223{,}401.00$, is well worth it!

Ex 1.5.3: A professional's salary grows from \$9949.00 to $\$107{,}951.00$ over a period of 30 years. What is the average annual growth rate?

Diagramming: $P = \$9{,}949.00 \overset{30\,years}{\rightarrow\rightarrow} A = \$107{,}951.00$. Solving, we have

$$\overset{1}{\mapsto}: A = P(1+r_{eff})^t \Rightarrow$$

$$\$107{,}951.00 = \$9949.00(1+r_{eff})^{30} \Rightarrow$$

$$10.85 = (1+r_{eff})^{30} \Rightarrow$$

$$1+r_{eff} = \sqrt[30]{10.85} = 1.08271 \Rightarrow$$

$$r_{eff} = 0.08271 = 8.271\tfrac{\%}{year} \therefore$$

The final average growth rate of $8.271\tfrac{\%}{year}$ certainly exceeds the *average inflation annual rate* of $3\tfrac{\%}{year}$ and shows a steady increase in purchasing power over time.

Ex 1.5.4: $\$10,000.00$ is lent to a friend at $2\frac{\%}{year}$ simple interest for a period of 5 years. What is r_{eff} ?

$$\overset{1}{\mapsto}: r_{eff} = \sqrt[t]{1+rt} - 1 \Rightarrow$$

$$r_{eff} = \sqrt[5]{1+(0.02)5} - 1 = \sqrt[5]{1.1} - 1 = 0.0192 \Rightarrow$$

$$r_{eff} = 1.92\frac{\%}{year} \because$$

Ex 1.5.5: You have \$25,000 to invest for 10 years. Which of the following three deals is most advantageous to you, the investor: $12\frac{\%}{year}$ simple interest for the entire time period, $7\frac{\%}{year}$ interest compounded daily for the entire time period, or $8\frac{\%}{year}$ interest compounded quarterly for the entire time period?

We analyze problem in two stages. First, we will compute the r_{eff} for the three cases noting that daily interest (365 compounding periods a year) is for all effects and purposes indistinguishable from continuous interest. The highest r_{eff} will then provide our answer. Secondly, in the modern spirit of 'show me the money', we will compute the expected earnings in all three cases. Comparing r_{eff}

$$\overset{1}{\mapsto}: r_{eff} = \sqrt[10]{1+(0.12)10} - 1 \Rightarrow$$

$$r_{eff} = \sqrt[10]{1+1.2} - 1 = \sqrt[10]{2.2} - 1 = 8.204\frac{\%}{year} \therefore$$

$$\overset{2}{\mapsto}: r_{eff} = e^{0.07} - 1 = 7.251\frac{\%}{year} \therefore$$

$$\overset{3}{\mapsto}: r_{eff} = (1+\tfrac{0.08}{4})^4 - 1 = 8.243\frac{\%}{year} \therefore$$

Quarterly compounding at $8\frac{\%}{year}$ provides the best deal. Calculating the associated expected earnings gives

$$\overset{1}{\mapsto}: A = P(1+rt) \Rightarrow$$

$$A = \$25{,}000.00(1+[0.12]\cdot 10) = \$55{,}000.00 \therefore$$

$$\overset{1alt}{\mapsto}: A = P(1+r_{eff})^t \Rightarrow$$

$$A = \$25{,}000.00(1+0.08204)^{10} = \$55{,}001.32 \therefore$$

$$\overset{2}{\mapsto}: A = Pe^{rt} \Rightarrow$$

$$A = \$25{,}000.00e^{0.07\cdot 10} = \$50{,}343.82 \therefore$$

$$\overset{2alt}{\mapsto}: A = P(1+r_{eff})^t \Rightarrow$$

$$A = \$25{,}000.00(1+0.07251)^{10} = \$50{,}344.67 \therefore$$

$$\overset{3}{\mapsto}: A = P(1+\tfrac{r}{n})^{nt} \Rightarrow$$

$$A = \$25{,}000.00(1+\tfrac{0.08}{4})^{4\cdot 10} = \$55{,}200.99 \therefore$$

$$\overset{3alt}{\mapsto}: A = \$25{,}000.00(1+0.08243)^{10} = \$55{,}199.89 \therefore$$

Case three, quarterly compounding at $8\frac{\%}{year}$, has the highest expected earnings as predicted by the associated r_{eff}.

The three alternate calculations use the effective annual interest-rate construct formula to arrive at the exact same answers (within a dollar or two) as those produced by the associated compounding formulas. This would be expected; for this is how the three r_{eff} formulas were derived in the first place!

2. The Algebra of the Nest Egg

2.1. Present and Future Value

Money changes its value with time. This fact is as certain as the proverbial 'death and taxes'. Inflation is a force beyond an individual's control that lessens the value of money over time. Smart investing counters inflation in that it enhances the value of money over time. The value of money right now is called the present value PV. The time-changed equivalent value in the future is called the future value FV. This can be diagramed as

$$PV \underset{time}{\overset{process}{\to\to}} FV .$$

In order for a present value to become a future value, both time and a process need to be specified. This is exactly the case in the familiar compound interest formula $A = P(1+\frac{r}{n})^{nt}$. Using the above general diagrammatic pattern, we can diagram the compound-interest formula as follows

$$P \underset{t}{\overset{(1+\frac{r}{n})^{nt}}{\to\to}} A .$$

Replacing $P \,\&\, A$ with $PV \,\&\, FV$ respectively leads to

$$PV \underset{t}{\overset{(1+\frac{r}{n})^{nt}}{\to\to}} FV .$$

Note: The above formula is not completely correct until one takes in account the effects of inflation, an analysis option. To account for inflation, subtract the annual inflation rate from the given annual interest rate. Use the modified rate in present-to-future value formulas to project an inflation-adjusted future value.

With this last note in mind, we present the four *coupled* **Present-to-Future-Value Formulas**. All interest rates in the formulas below need to be inflation adjusted per $r_{adj} = r - i$ *if one wants to obtain an inflation-adjusted future value*.

Compound Interest: $FV = PV(1+\frac{r}{n})^{nt} \Leftrightarrow PV = \dfrac{FV}{(1+\frac{r}{n})^{nt}}$

Effective Interest: $FV = PV(1+r_{eff})^{t} \Leftrightarrow PV = \dfrac{FV}{(1+r_{eff})^{t}}$

Continuous Interest: $FV = PVe^{rt} \Leftrightarrow PV = \dfrac{FV}{e^{rt}}$

Simple Interest: $FV = PV(1+rt) \Leftrightarrow PV = \dfrac{FV}{(1+rt)}$

Notice that the coupled Present-to-Future-Value Formulas allow us to easily move from present value to future value (or visa versa) as long as the compounding process, time period, and one of the two values—present or future—is specified. Coupled present-to-future-value formulas allow us to *estimate* total change in monetary value as either investments or durable goods move forwards or backwards in time under a *given set of process conditions*.

Ex 2.1.1: Bill wishes to have $\$1{,}800{,}000.00$ in his Individual Retirement Account (IRA) when he retires in 35 years. What is the present value of this amount assuming an average annual compounding rate of $11.5\frac{\%}{year}$?

$$\overset{1}{\mapsto}: PV = \frac{FV}{(1+r_{eff})^{t}} \Rightarrow$$

$$PV = \frac{\$1{,}800{,}000.00}{(1+.115)^{35}} \Rightarrow$$

$$PV = \frac{\$1{,}800{,}000.00}{(1.115)^{35}} = \$39{,}870.54$$

Ex 2.1.2: Repeat the calculation in Ex. 2.1.1 if an average inflation rate $i = 3\frac{\%}{year}$ acts through the same 35 year time period.

Bill's wish can be restated in terms of buying power. What Bill really wants is $\$1{,}800{,}000.00$ in current buying power by the time he retires in 35 years. Thus

$$\overset{1}{\mapsto}: FV = PV(1 + r_{eff})^t \Rightarrow$$

$$FV = PV(1+i)^t$$

$$FV = \$1{,}800{,}000.00(1.03)^{35} \Rightarrow$$

$$FV = \$5{,}064{,}952.42$$

Interpreted, $\$5{,}064{,}952.42$ is the amount needed 35 years from now just to preserve the buying power inherent in $\$1{,}800{,}000.00$ today assuming a long-term steady inflation rate of $i = 3\frac{\%}{year}$. Turning to the present value of this new amount assuming the same $11.5\frac{\%}{year}$, we have

$$\overset{2}{\mapsto}: PV = \frac{FV}{(1 + r_{eff})^t} \Rightarrow$$

$$PV = \frac{\$5{,}064{,}952.42}{(1 + .115)^{35}} \Rightarrow$$

$$PV = \frac{\$5{,}064{,}952.42}{(1.115)^{35}} = \$96{,}147.83$$

When inflationary price increases for durable goods are stated in terms of an annually-compounded percentage jump, we typically use present-to-future-value formulas to estimate the future price. This is especially true for single 'big ticket' items such as houses, cars, boats, jewelry, etc. Our next example illustrates the use of a present-to-future value formula to estimate the future price of a newly-built house.

Ex 2.1.3: The price of a new house in a certain city increases at an average rate of $5\frac{\%}{year}$. If a particular 3-bedroom model in a certain subdivision is priced at $\$235{,}000.00$ in 2006, estimate the price of a similar model in the same subdivision in 2010.

$$\overset{1}{\mapsto}: FV = PV(1+r_{eff})^t \Rightarrow$$

$$FV = \$235{,}000.00(1+0.05)^4 \Rightarrow$$

$$FV = \$285{,}644.00$$

This is some disconcerting news in that the same house will sell for approximately $\$285{,}644.00$ four years from now. If you can afford it, you better buy now. Waiting costs money!

Ex 2.1.4: Calculate the present value of a $\$100{,}000.00$ corporate bond coming due in 15 years at $5\frac{\%}{year}$ compounded quarterly.

$$\overset{1}{\mapsto}: PV = \frac{FV}{(1+\frac{r}{n})^{nt}} \Rightarrow$$

$$PV = \frac{\$100{,}000.00}{(1+\frac{0.05}{4})^{60}} = \$47{,}456.76$$

If redeemed today, the bond would fetch $47,456.76.

2.2. Growth of an Initial lump Sum Deposit

If an initial lump-sum deposit is the only means by which monetary growth is achieved, then the **Present-to-Future-Value Formulas** are sufficient to perform the associated calculations. We need only to identify the process by which the growth is occurring: annual compounding via an effective interest rate, continuous compounding, or compounding for a finite number of compounding periods per year. Each compounding process has an associated formula to which a total time and interest rate must be supplied.

Ex 2.2.1: What is the future value (non-inflation adjusted) at age 65 of $\$13{,}000.00$ invested at age 25 assuming $r_{eff} = 8\frac{\%}{year}$ throughout the 40-year term?

Note: the making of a monetary-growth diagram is strongly recommended as a first step for all present-to-future-value problems since pictures engage the use of one's right brain and the associated spatial problem-solving capabilities. Hence, for Example 2.2.1, the associated monetary-growth diagram is

$$\overset{1}{\mapsto}: \underset{age25}{\$13{,}000.00} \overset{r_{eff}=8\frac{\%}{year}}{\longrightarrow\rightarrow} \underset{age65}{FV\,?}$$

Solving:

$$\overset{2}{\mapsto}: FV = PV(1 + r_{eff})^t \Rightarrow$$

$$FV = \$13{,}000.00(1 + 0.08)^{40} \Rightarrow$$

$$FV = \$282{,}417.77$$

Ex 2.2.2: Calculate the effective annual interest rate needed to turn $\$10{,}000.00$ into $\$1{,}000{,}000.00$ over a 25 year period.

$$\overset{1}{\mapsto}: \underset{t=0}{\$10{,}000.00} \overset{r_{eff}\,?}{\longrightarrow\rightarrow} \underset{t=25}{\$1{,}000{,}000.00}$$

Note that the process mechanism implicitly assumed is annual compounding via the referencing of an unknown r_{eff}. Solving:

$$\overset{2}{\mapsto}: \$1{,}000{,}000.00 = \$10{,}000.00(1 + r_{eff})^{25} \Rightarrow$$

$$100 = (1 + r_{eff})^{25} \Rightarrow$$

$$\sqrt[25]{100} = 1 + r_{eff} \Rightarrow$$

$$1.2022 = 1 + r_{eff} \Rightarrow$$

$$r_{eff} = 0.2022 = 20.22\frac{\%}{year}$$

The effective annual interest rate of $r_{eff} = 20.22 \frac{\%}{year}$ is probably impossible to sustain for an extended period of 25 years. Even in the go-go high-tech 90s, rates of this magnitude lasted for only six years or so.

Ex 2.2.3: What continuous interest rate is needed to quadruple a given present value in 15 years?

Asking for a continuous interest rate r_{cont} means that the continuous interest form of the present-to-future value formula $FV = PVe^{rt}$ should be used. Also, the problem states that the required future value is $FV = 4PV$. Annotating this information on the monetary-growth diagram and solving gives

$$\overset{1}{\mapsto}: \underset{t=0}{PV} \overset{r_{cont}\,?}{\rightarrow\rightarrow} \underset{t=15}{4PV}$$

$$\overset{2}{\mapsto}: 4PV = PVe^{15r} \Rightarrow$$

$$4 = e^{15r} \Rightarrow$$

$$\ln(4) = \ln(e^{15r}) \Rightarrow$$

$$1.38629 = 15r \Rightarrow$$

$$r = r_{cont} = 0.0924 = 9.24 \frac{\%}{year}$$

The stated continuous interest rate of $9.24 \frac{\%}{year}$ is certainly achievable in today's markets; however, it is not automatic and will require active management of one's investments.

Our last example illustrates what happens if more than one deposit is made during the overall investment period.

Ex 2.2.4: What is the projected future value (ignoring inflation) of a retirement fund where an initial deposit of $\$40,000.000$ is made at age 30 and a subsequent deposit of $\$60,000.00$ is made at age 40. Assume an effective annual interest rate of $r_{eff} = 10 \frac{\%}{year}$ and an anticipated retirement age of 68.

Understandably, the monetary-growth diagram increases in complexity as it is modified to show the $\$60,000.00$ deposit (or *insertion into the investment process*) at age 40. Again, by the stating of an effective annual interest rate r_{eff}, the monetary-growth process is understood to be annual compounding.

$$\overset{1}{\mapsto}: \underset{age30}{\$40,000} \xrightarrow{r_{eff}=10\frac{\%}{year}} \underset{\substack{\uparrow \\ \$60,000 \\ age\,40}}{\to} \underset{age68}{FV\,?}$$

Solving for the projected future value requires direct addition of two algebraic terms.

$$\overset{2}{\mapsto}: FV = \$40,000(1+r_{eff})^{38} + \$60,000(1+r_{eff})^{28} \Rightarrow$$

$$FV = \$40,000(1.1)^{38} + \$60,000(1.1)^{28} \Rightarrow$$

$$FV = \$1,496,173.73 + \$865,259.61 \Rightarrow$$

$$FV = \$2,361,433.35$$

To summarize **Ex 2.2.4**, $\$100,000.00$ invested by the age of 40 becomes $\$2,361,433.35$ by age 68 *if the stated conditions hold throughout the investment period*.

Suppose that in **Ex 4.2.4** a single deposit could be made at age 30 in order to create the same $\$2,361,433.35$ by age 68. How much would such a deposit be? By direct application of the coupled Present-to-Future Value Formulas

$$PV = \frac{\$2,361,433.35}{(1.1)^{38}} = \$63,132.59,$$

a net savings to the investor of \$36,867.40. Calculating the inflation-adjusted future value of $\$2,361,433.35$ over the same 38 years, we obtain

$$FV_{adj} = \frac{\$2,361,433.35}{(1.03)^{38}} = \$767,999.88.$$

2.3. Growth of a Deposit Stream

Most of us don't have an initial lump sum of $\$40{,}000.00$ (or $\$63{,}132.59$) by which to build a retirement fund. The more typical way we build our retirement funds is by means of a periodic deposit—either through payroll deduction or direct self-discipline—that accumulates in value year after year. And, after thirty years or so, we are talking about a sum jokingly referred to as 'real money'. But it is no joke on how the sum is obtained: *through discipline, sacrifice, and attentive money management*. In this section, we will develop and use the equations that determine the future value of a regular *deposit stream* over an extended period of time.

Let $D_i \equiv D : i = 1, nt$ be a deposit stream of identically-sized payments made over a period of t years where n is the number of compounding periods per year and r is the annual interest rate. Suppose that each deposit D_i is sequenced to coincide with the beginning of the corresponding compounding period and that the last deposit D_{nt} begins the last of the nt compounding periods. Under these conditions, what is the future value of the entire deposit stream? Diagramming,

$$D_1 \xrightarrow{\frac{r}{n}} \underset{D_2}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_3}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_4}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_5}{\uparrow} \cdots \xrightarrow{\frac{r}{n}} \underset{D_{nt-1}}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_{nt}}{\uparrow} \xrightarrow{\frac{r}{n}} FV\,?\,.$$

Now, each deposit D_i contributes a portion FV_i to the total future value FV where $FV_i = D_i\left(1+\frac{r}{n}\right)^{nt+1-i}$. Thus,

$$FV = \sum_{i=1}^{nt} FV_i \Rightarrow$$

$$FV = \sum_{i=1}^{nt} D_i\left(1+\tfrac{r}{n}\right)^{nt+1-i} \Rightarrow .$$

$$FV = D\sum_{i=1}^{nt}\left(1+\tfrac{r}{n}\right)^{nt+1-i}$$

The expression

$$D\sum_{i=1}^{nt}\left(1+\tfrac{r}{n}\right)^{nt+1-i} = D\left\{\left(1+\tfrac{r}{n}\right)^{1} + \left(1+\tfrac{r}{n}\right)^{2} + \ldots + \left(1+\tfrac{r}{n}\right)^{nt}\right\}$$

is a geometric series and can be summed accordingly as

$$D\sum_{i=1}^{nt}\left(1+\tfrac{r}{n}\right)^{nt-i} = \frac{Dn}{r}\left\{\left(1+\tfrac{r}{n}\right)^{nt+1} - \left(1+\tfrac{r}{n}\right)\right\},$$

leads to the following formula: $FV = \dfrac{Dn}{r}\left\{\left(1+\tfrac{r}{n}\right)^{nt+1} - \left(1+\tfrac{r}{n}\right)\right\}.$

Suppose we want to conclude our term of t years with one final deposit D_{nt+1} as shown in the modified deposit stream

$$D_1 \xrightarrow{\frac{r}{n}} \underset{D_2}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_3}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_4}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_5}{\uparrow} \cdots \xrightarrow{\frac{r}{n}} \underset{D_{nt-1}}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_{nt}}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_{nt+1}}{\uparrow} \; FV\,?$$

To do so, add one more D to obtain $FV = \dfrac{Dn}{r}\left\{\left(1+\tfrac{r}{n}\right)^{nt+1} - 1\right\}.$

In the case of annual compounding where $\frac{r}{n} = r_{eff}$ and D is a yearly total (or rate), the two formulas become

Without Final Deposit: $FV = \dfrac{D}{r_{eff}}\left\{\left(1+r_{eff}\right)^{t+1} - \left(1+r_{eff}\right)\right\}$

No Final Deposit: $FV = \dfrac{D}{r_{eff}}\left\{\left(1+r_{eff}\right)^{t+1} - 1\right\}$

Similar formulas are developed for the case of continuous compounding in Section III, Topic 5. As discussed previously, all future values must be adjusted for inflation in order to ascertain true buying power.

Ex 2.3.1: After a term of 30 years, what is the projected future value of a retirement fund where 30 annual deposits of $\$5000.00$ are faithfully made on 1 January of each succeeding each year. Assume $r_{eff} = 11\frac{\%}{year}$.

A modified monetary-growth diagram can be used to show the periodic annual deposits as follows:

$$\overset{1}{\mapsto}: \underset{t=0}{\$5,000.00} \rightarrow 29 \times (\overset{r_{eff}=11\frac{\%}{year}}{\underset{\$5000.00}{\uparrow}} \rightarrow) \underset{t=30}{FV}?$$

Here, the diagram starts with the first annual deposit of $\$5000.00$ at $t = 0$ and annotates via multiplication the subsequent 29 annual $\$5000.00$ deposits made at the start of each annual compounding period. Solving,

$$\overset{2}{\mapsto} FV = \frac{D}{r_{eff}} \left\{ \left(1 + r_{eff}\right)^{t+1} - \left(1 + r_{eff}\right) \right\} \Rightarrow$$

$$FV = \frac{\$5000.00}{.11} \left\{ (1.11)^{31} - (1.11) \right\} \Rightarrow \quad .$$

$$FV = \$1,104,565.87$$

To summarize, 30 annual deposits of $\$5000.00$ totaling $\$150,000.00$ have grown to a future value of $\$1,104,565.87$ over a 30-year term assuming $r_{eff} = 11\frac{\%}{year}$.

Ex 2.3.2: Suppose a single lump-sum deposit could be made at the start of the 30-year period in Example 2.3.1 in an amount sufficient to create the same future value of $\$1,104,565.87$. How much would be needed? Assume $r_{eff} = 11\frac{\%}{year}$.

$$\overset{1}{\mapsto}: \underset{t=0}{PV}? \overset{r_{eff}=11\frac{\%}{year}}{\rightarrow\rightarrow} \underset{t=30}{\$1,104,565.87}$$

$$\overset{2}{\mapsto}: PV(1+r_{eff})^t = FV \Rightarrow$$

$$PV(1.11)^{30} = \$1{,}104{,}565.87 \Rightarrow$$

$$PV = \frac{\$1{,}104{,}565.874}{(1.11)^{30}} = \$48{,}250.54$$

Ex 2.3.3: Sam contributes $\$200.00$ per month to a college savings account for his daughter Mary, who just turned 12. In addition, he makes 'bonus deposits' of $\$1000.00$ on Mary's birthday. Sam started this practice with a combined $\$1200.00$ deposit on the day of Mary's birth and will 'cash out' on Mary's 18th birthday with a final deposit of $\$1200.00$. How much will be in Mary's college savings account at that time assuming $r = 7\frac{\%}{year}$ and monthly compounding?

This problem can be thought of as two sub-problems: 1) a monthly deposit stream of 217 individual deposits over a term of 18 years and 2) a parallel yearly deposit stream of 19 individual deposits over a period of 18 years. The total future value will be the sum of both parallel deposit streams the day Mary turns 18.

For the monthly deposit stream, we slightly modify the monetary-growth diagram to show the inclusion of the final deposit.

$$\overset{1}{\mapsto}: \underset{t=0}{\$200.00} \rightarrow 215 \times (\underset{\$200.00}{\uparrow} \overset{r=7\frac{\%}{year}}{\longrightarrow}) + \underset{t=18}{\$200.00} = FV_{month}\ ?$$

Solving:

$$\overset{2}{\mapsto}: FV_{month} = \frac{Dn}{r}\left\{\left(1+\tfrac{r}{n}\right)^{nt+1} - 1\right\} \Rightarrow$$

$$FV_{month} = \frac{\$200.00(12)}{0.07}\left\{\left(1+\tfrac{0.07}{12}\right)^{217} - 1\right\} \Rightarrow$$

$$FV_{month} = \$86{,}846.71$$

For the yearly deposit stream, we will first need to compute the effective annual interest rate: $r_{eff} = (1+\frac{0.07}{12})^{12} - 1 = 7.229\frac{\%}{year}$.
Now, we have all the information needed to compute FV_{year} and, consequently, $FV_{total} = FV_{month} + FV_{year}$

$$\overset{1}{\mapsto}: \underset{t=0}{\$1000.00} \to 17 \times (\underset{\$1000.00}{\overset{r_{eff}=7.229\frac{\%}{year}}{\uparrow}} \to) + \$1000.00 = \underset{t=18}{FV_{year}} ?$$

$$\overset{2}{\mapsto}: FV_{year} = \frac{D}{r_{eff}}\left\{\left(1+r_{eff}\right)^{nt} - 1\right\} \Rightarrow$$

$$FV_{year} = \frac{\$1000.00}{0.07229}\left\{\left(1+0.07229\right)^{19} - 1\right\} \Rightarrow$$

$$FV_{year} = \$38{,}268.93 \Rightarrow$$

$$FV_{total} = FV_{month} + FV_{year} = \$125{,}115.54$$

Each of the four deposit-stream formulas can also be used to determine the periodic deposit D needed in order to accumulate a specified future value under a given set of conditions.

Ex 2.3.4: Suppose Sam is not happy with the $125,115.64 accumulated by Mary's 18th birthday and, instead, would like to accumulate a future value of $160,000.00 via the single mechanism of monthly deposits. **A)** How much should this deposit be, again, assuming monthly compounding and $r = 7\frac{\%}{year}$? **B)** What single lump-sum deposit made on the day Mary was born would generate an equivalent future value?

A) $\overset{1}{\mapsto}: \underset{t=0}{D?} \to 215 \times (\underset{D?}{\overset{r=7\frac{\%}{year}}{\uparrow}} \to) + D? = \underset{t=18}{FV} ?$

$$\overset{2}{\mapsto}: FV = \frac{Dn}{r}\left\{\left(1+\tfrac{r}{n}\right)^{nt+1} - 1\right\} \Rightarrow$$

$$D = \frac{rFV}{n\left\{\left(1+\tfrac{r}{n}\right)^{nt+1} - 1\right\}} \Rightarrow$$

$$D = \frac{0.07(\$160{,}000.00)}{12\left\{\left(1+\tfrac{0.07}{12}\right)^{217} - 1\right\}} \Rightarrow$$

$$D = \$368.47$$

B) $$\overset{1}{\mapsto}: \underset{t=0}{PV}\,? \xrightarrow{r=7\frac{\%}{year}} \underset{t=18}{\$160{,}000.00}$$

$$\overset{2}{\mapsto}: PV\left(1+\tfrac{r}{n}\right)^{nt} = FV \Rightarrow$$

$$PV(1.005833)^{216} = \$160{,}000.00 \Rightarrow$$

$$PV = \frac{\$160{,}000.00}{(1.005833)^{216}} = \$45{,}551.09$$

This example suggests the old maxim of *pay me now or pay me later.* One could think of *now* as a single payment of $\$45{,}551.09$ and *later* as a deposit stream of 217 payments, each $\$368.47$, totaling $79,957.99.

2.4. The Two Growth Mechanisms in Concert

Sometimes, we may have the opportunity to open up a retirement or college investment account with a respectable lump-sum deposit (denote by L_S)—perhaps gained by winning a lottery or receiving an inheritance. From then on, we contribute to this deposit by means of a deposit stream as shown in the monetary-growth diagram

$$L_S \xrightarrow{\frac{r}{n}} \underset{D_2}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_3}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_4}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_5}{\uparrow} \cdots \xrightarrow{\frac{r}{n}} \underset{D_{nt-1}}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_{nt}}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_{nt+1}}{\uparrow} FV?.$$

If $L_S > D_i = D$ (which would surely be the case for 99% of the time), then we could redraw the monetary-growth diagram as follows

$$(L_S - D_1) + D_1 \xrightarrow{\frac{r}{n}} \underset{D_2}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_3}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_4}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_5}{\uparrow} \cdots \xrightarrow{\frac{r}{n}} \underset{D_{nt-1}}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_{nt}}{\uparrow} \xrightarrow{\frac{r}{n}} \underset{D_{nt+1}}{\uparrow} FV\,?\,.$$

Examining this last diagram, one algebraic expression can be easily written for the associated future value by summing the two embedded monetary-growth processes:

$$FV = (L_S - D)\left(1 + \tfrac{r}{n}\right)^{nt} + \frac{Dn}{r}\left\{\left(1 + \tfrac{r}{n}\right)^{nt+1} - 1\right\}.$$

Ex 2.4.1: Suppose Bill makes quarterly deposits of $\$2000.00$ to a retirement fund over a period of 35 years that is started with an initial deposit of $\$5000.00$ and concluded with a final deposit of $\$2000.00$. What is the future value assuming quarterly compounding and $r = 8\frac{\%}{year}$?

The monetary-growth diagram increases in complexity again.

$$\overset{1}{\mapsto}: \$3000.00 \underset{t=0}{+} \$2000.00 \rightarrow 138 \times \Big(\underset{\$2000.00}{\uparrow} \xrightarrow{r=8\frac{\%}{year}} \Big)$$
$$+ \$2000.\underset{t=35}{00} = FV\,?$$

Solving:

$$\overset{2}{\mapsto}: FV = (L_S - D)\left(1 + \tfrac{r}{n}\right)^{nt} + \frac{Dn}{r}\left\{\left(1 + \tfrac{r}{n}\right)^{nt+1} - 1\right\} \Rightarrow$$
$$FV = \$3000.00\left(1 + \tfrac{0.08}{4}\right)^{140}$$
$$+ \frac{\$2,000.00(4)}{0.08}\left\{\left(1 + \tfrac{0.08}{4}\right)^{141} - 1\right\} \Rightarrow$$
$$FV = \$47,989.39 + \$1,531,639.53 = \$1,579,628.92$$

Note: The reader may ask, "Is this the only way that a monetary-growth diagram can be drawn?" The answer is an emphatic no! These diagrams are offered as a suggested approach for two reasons: 1) they visually imply a flow of money and 2) they have been classroom tested. The important thing is to make a monetary-growth diagram that has meaning to you and upon which you can assemble all the relevant information.

Ex 2.4.2: Suppose in **Ex 2.4.1**, Bill starts his retirement account on his 25th birthday and stops contributing on his 60th birthday with plans not to withdraw from his account until the age of 70. Bill is becoming increasingly wary of higher-risk investments as he grows older. Hence, Bill rolls his retirement account over into a safe U.S. government-bond fund paying an effective annual interest rate of $r_{eff} = 4.5\frac{\%}{year}$ on his 60th birthday. What will be the future value of Bill's retirement account at age 70?

$$\overset{1}{\mapsto}: \underset{t=0}{\$1{,}579{,}628.92} \overset{r_{eff}=4.5\frac{\%}{year}}{\to\to} \underset{t=10}{FV}?$$

$$\overset{2}{\mapsto}: PV(1+r_{eff})^t = FV \Rightarrow$$

$$\$1{,}579{,}628.92(1.045)^{10} = FV \Rightarrow$$

$$FV = \$2{,}453{,}115.42$$

Ex 2.4.2 illustrates the importance of being able to choose the right formula for the right scenario. In many investment scenarios, several formulas may have to be used in order to obtain the sought-after answer. Understanding of the underlying concepts and facility with algebra are the two keys to success. We will now list all four future-value formulas with initial lump sum deposit L_S corresponding to the four deposit-stream formulas.

Final Deposit & Other-than-Annual Compounding:

$$FV = (L_S - D)\left(1+\tfrac{r}{n}\right)^{nt} + \frac{Dn}{r}\left\{\left(1+\tfrac{r}{n}\right)^{nt+1} - 1\right\}$$

No Final Deposit & Other-than-Annual Compounding:

$$FV = (L_S - D)\left(1+\tfrac{r}{n}\right)^{nt} + \frac{Dn}{r}\left\{\left(1+\tfrac{r}{n}\right)^{nt+1} - \left(1+\tfrac{r}{n}\right)\right\}$$

Final Deposit and Yearly Compounding:

$$FV = (L_S - D)\left(1+r_{eff}\right)^{t} + \frac{D}{r_{eff}}\left\{\left(1+r_{eff}\right)^{t+1} - 1\right\}$$

No Final Deposit and Yearly Compounding:

$$FV = (L_S - D)\left(1+r_{eff}\right)^{t} + \frac{D}{r_{eff}}\left\{\left(1+r_{eff}\right)^{t+1} - \left(1+r_{eff}\right)\right\}$$

Our next example illustrates the integration of a mid-life windfall into one's retirement program.

Ex 2.4.3: George graduates from nursing school at age 22 and accepts a sign-on bonus of $\$7000.00$ to go to work at a local hospital. At the time, George used $\$2000.00$ of the money to open up a *Roth IRA* (see **Section I: 6.9**). He contributes $\$1000.00$ per year making the final deposit at age 60.

George is a fairly astute investor and was able to achieve an effective annual interest rate of $r_{eff} = 12.5\frac{\%}{year}$ over the course of 38 years. Additionally, at age 45, George received a small inheritance of $\$15{,}000.00$ that he promptly invested in tax-free municipals paying an effective annual interest rate of $r_{eff} = 4.5\frac{\%}{year}$. What are George's total holdings at age 60?

For the Roth portion

$$\overset{1}{\mapsto}: \$1000.00 + \underset{age22}{\$1000.00} \rightarrow 38 \times (\overset{r_{eff}=12.5\frac{\%}{year}}{\underset{\$1000.00}{\uparrow}} \rightarrow)$$

$$+ \underset{age60}{\$1000.00} = FV_{Roth}\ ?$$

$$\overset{2}{\mapsto}: FV_{Roth} = (L_S - D)\left(1 + r_{eff}\right)^t + \frac{D}{r_{eff}}\left\{\left(1 + r_{eff}\right)^{t+1} - 1\right\} \Rightarrow$$

$$FV_{Roth} = \$1000.00\left(1.125\right)^{38} + \frac{\$1000.00}{0.125}\left\{\left(1.125\right)^{39} - 1\right\} \Rightarrow$$

$$FV_{Roth} = \$1000.00\left(1.125\right)^{38} + \frac{\$1000.00}{0.125}\left\{\left(1.125\right)^{39} - 1\right\} \Rightarrow$$

$$FV_{Roth} = \$87{,}860.94 + \$782{,}748.47 = \$870{,}609.41$$

For the tax-free-municipals portion

$$\overset{1}{\mapsto}: \underset{age\,45}{\$15{,}000.00} \xrightarrow{r_{eff} = 4.5 \frac{\%}{year}} \rightarrow\rightarrow \underset{age\,60}{FV_{taxfree}}\ ?$$

$$\overset{2}{\mapsto}: FV_{taxfree} = PV\left(1 + r_{eff}\right)^t \Rightarrow$$

$$FV_{taxfree} = \$15{,}000.00\left(1.045\right)^{15} \Rightarrow$$

$$FV_{taxfree} = \$29{,}029.23$$

Finally: $FV = FV_{Roth} + FV_{taxfree} = \$899{,}638.64$

To recap, through smart investing, George was able to turn contributions totaling $\$55{,}000.00$ into $\$899{,}638.64$ over a 38-year period.

2.5. Summary

This article is not intended to be a treatise on retirement planning. All serious retirement planning should start with a licensed financial consultant in order to devise detailed long-term action plans that meet individual goals. The important thing in this day of age is to 'just do it!' This leads to a second **Words of Wisdom**: *You must first plan smart! Then, you must do smart in order to achieve that coveted economic security!*

We close this article with the table below, a powerful motivational aid that shows the future value of a $\$4000.00$ yearly deposit for various terms and effective annual interest rates. Notice that the shaded million-dollar levels can be reached in four of the five columns. Reaching a net worth of one million dollars or more is a matter of both time and average annual interest rate. The formula used to construct the table is

$$FV = \frac{D}{r_{eff}}\left\{\left(1 + r_{eff}\right)^{t+1} - 1\right\}.$$

GROWTH OF $4000.00 YEARLY DEPOSIT					
	EFFECTIVE ANNUAL INTEREST RATE				
TERM	**5%**	**7%**	**9%**	**11%**	**13%**
5 yr	$27,207	$28,613	$30,093	$31,651	$33,290
10 yr	$56,827	$63,134	$70,241	$78,245	$87,257
15 yr	$94,629	$111,552	$132,013	$156,759	$186,686
20 yr	$142,877	$179,460	$227,058	$289,060	$369,879
25 yr	$204,453	$274,705	$373,295	$511,995	$707,400
30 yr	$283,043	$408,292	$598,300	$887,652	$1,329,260
35 yr	$383,345	$595,653	$944,498	$1,520,657	$2,474,997
40 yr	$511,359	$858,438	$1,477,167	$2,587,307	$4,585,943
45 yr	$674,740	$1,227,007	$2,296,744	$4,384,675	$8,475,224
50 yr	$883,261	$1,743,943	$3,557,764	$7,413,343	$15,640,972

1. The Algebra of Consumer Debt

3.1 Loan Amortization

Very few people buy a house with cash. For most of us, the mortgage is the time-honored way to home ownership. A mortgage is a long-term collateralized loan, usually with a financial institution, where the title-deed to the house itself is the collateral. Once a mortgage is secured, mortgage payments are then made month-by-month and year-by-year until the amount originally borrowed is fully paid, usually within a pre-specified time in years. We call this process of methodically paying back—payment by payment—the amount originally borrowed *amortizing a loan*. The word 'amortize' means to liquidate, extinguish, or put to death. So, to *amortize a loan* means to put the loan to death. In generations past (especially those in the 'Greatest Generation'), the final payment in 'putting a loan to death' was celebrated with the ceremonial burning of some of the mortgage paperwork. This symbolized the death of the mortgage and the associated transference of the title deed to the proud and debt-free homeowners. Nowadays, we Baby Boomers or Generation Xers don't usually hang on to a mortgage long enough to have the satisfaction of burning it.

Suppose we borrow a mortgage amount A, which is scheduled to be compounded monthly for a term of T years at an annual interest rate r. If no payments are to be made during the term, and a single balloon payment is to be made at the end of the term, then the future value FV_A of this single balloon payment is

$$FV_A = A\left(1 + \tfrac{r}{12}\right)^{12T} .$$

Now, let $D = D_i : i = 1,12T$ be a stream of identically-sized mortgage payments made over the same term of T years where the first payment is made exactly one-month after mortgage inception and the last payment coincides with the end of the term.

Then, the total future value FV_D associated with the payment stream is

$$FV_D = \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^{12T} - 1\right\}$$

For the mortgage to be paid, the future value of the mortgage-amount borrowed must be equal to the total future value of the mortgage-payments made. Hence,

$$\begin{aligned}
&FV_A = FV_D \Rightarrow FV_D = FV_A \Rightarrow \\
&\frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^{12T} - 1\right\} = A\left(1+\tfrac{r}{12}\right)^{12T} \Rightarrow \\
&D = \frac{rA\left(1+\tfrac{r}{12}\right)^{12T}}{12\left\{\left(1+\tfrac{r}{12}\right)^{12T} - 1\right\}} \Rightarrow \\
&D = \frac{rA}{12\left\{1-\left(1+\tfrac{r}{12}\right)^{-12T}\right\}}
\end{aligned}.$$

The last expression is the monthly payment D needed to amortize a mortgage amount A at the end of T years given a *fixed annual interest rate*. Once D is determined, we can compute the present dollar value of the entire payment stream

$$PV_{PS} = 12TD$$

and the present dollar value of all the interest paid via the entire payment stream

$$PV_{IPS} = 12TD - A.$$

Another fundamental quantity associated with a loan, particularly a mortgage loan, undergoing the process of amortization is the actual dollar value of the original loan still unpaid—called the payoff or payout value—after a given number j of monthly payments D have been made. We will denote this payoff value by the algebraic symbol PO_j.

Recall that the j^{th} payment is made at the end of the j^{th} compounding period. By that time, the amount borrowed will have grown via the compounding mechanism to $A\left(1+\frac{r}{12}\right)^{j}$. In like fashion, the future value of the first monthly payment D will have grown to $D\left(1+\frac{r}{12}\right)^{j-1}$, and the total future value of the first j monthly payments D will have grown to $\frac{12D}{r}\left\{\left(1+\frac{r}{12}\right)^{j}-1\right\}$. Hence, the amount of the payoff PO_j that corresponds to exactly the first j monthly payments D is

$$PO_j = A\left(1+\tfrac{r}{12}\right)^{j} - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^{j}-1\right\}.$$

For any fixed amortization term T, the payoff amount undergoes a negative change from the $j-1^{st}$ payment to the j^{th} payment as it is incrementally reduced throughout the life of the loan. The negative of this change is the actual dollar amount D_{Aj} of the j^{th} payment actually applied to loan reduction (or to *principal*, see next note). Thus,

$$D_{Aj} = -\left(PO_j - PO_{j-1}\right) = PO_{j-1} - PO_j \Rightarrow$$

$$D_{Aj} = A\left(1+\tfrac{r}{12}\right)^{j-1} - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^{j-1}-1\right\}$$
$$-\left[A\left(1+\tfrac{r}{12}\right)^{j} - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^{j}-1\right\}\right] \Rightarrow$$

$$D_{Aj} = A\left[\left(1+\tfrac{r}{12}\right)^{j-1} - \left(1+\tfrac{r}{12}\right)^{j}\right] - \frac{12D}{r}\left[\left(1+\tfrac{r}{12}\right)^{j-1} - \left(1+\tfrac{r}{12}\right)^{j}\right] \Rightarrow$$

$$D_{Aj} = \left[A - \frac{12D}{r}\right]\left[1-\left(1+\tfrac{r}{12}\right)\right]\left(1+\tfrac{r}{12}\right)^{j-1} \Rightarrow$$

$$D_{Aj} = \left[\frac{12D-rA}{12}\right]\left(1+\tfrac{r}{12}\right)^{j-1}$$

Finally, the dollar amount of the j^{th} payment D going towards the payment of interest I is

$$D_{Ij} = D - D_{Aj}\,.$$

Note: In this hand book, we have deliberately shied away from the term 'principal' in favor of more user-friendly terms that allow the construction of non-overlapping and pneumonic algebraic symbols. Traditionally, the principal P is a capital sum initially borrowed or initially deposited to which a compounding mechanism is applied.

The six loan-amortization formulas presented thus far can be split into two groups: **Global Amortization Formulas** and **Payment Specific Formulas**. One must first compute the monthly payment D in order calculate all remaining quantities in either group.

Global Amortization Formulas

Monthly Payment: $D = \dfrac{rA}{12\left\{1-\left(1+\frac{r}{12}\right)^{-12T}\right\}}$

Sum of Payments in Payment Stream: $PV_{PS} = 12TD$

Total Interest Paid in Payment Stream: $PV_{IPS} = 12TD - A$

Payment Specific Formulas

Payoff after the j^{th} Monthly Payment:

$$PO_j = A\left(1+\tfrac{r}{12}\right)^j - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^j - 1\right\}$$

Amount of j^{th} Monthly Payment to Principal:

$$D_{Aj} = \left[\frac{12D - rA}{12}\right]\left(1+\tfrac{r}{12}\right)^{j-1}$$

Amount of j^{th} Monthly Payment to Interest: $D_{Ij} = D - D_{Aj}$

Ex 3.1.1: A $\$400,000.00$ business-improvement loan is negotiated with a local bank for an interest rate of $r = 7 \frac{\%}{year}$ and an amortization term of 17 years. Find the quantities D, PV_{PS}, PV_{IPS}, PO_{180}, D_{A100}, and D_{I100}.

Since these six quantities are a direct single-step application of the associated formulas, a process diagram is not needed.

$$\overset{1}{\mapsto}: D = \frac{rA}{12\left\{1 - \left(1 + \frac{r}{12}\right)^{-12T}\right\}} \Rightarrow$$

$$D = \frac{0.07 \cdot (\$400,000.00)}{12\left\{1 - \left(1 + \frac{0.07}{12}\right)^{-204}\right\}} \Rightarrow$$

$$D = \$3358.64 mo \therefore$$

$$\overset{2}{\mapsto}: PV_{PS} = 12TD \Rightarrow$$

$$PV_{PS} = 12 \cdot 17 \cdot (\$3358.64) \Rightarrow$$

$$PV_{PS} = \$685,163.09 \therefore$$

$$\overset{3}{\mapsto}: PV_{IPS} = 12TD - A \Rightarrow$$

$$PV_{IPS} = \$685,163.09 - \$400,000.00 \Rightarrow$$

$$PV_{IPS} = \$285,163.09 \therefore$$

The last three quantities are payment specific.

$$\overset{4}{\mapsto}: PO_j = A\left(1 + \frac{r}{12}\right)^j - \frac{12D}{r}\left\{\left(1 + \frac{r}{12}\right)^j - 1\right\} \Rightarrow$$

$$PO_{180} = \$400,000.00\left(1 + \frac{0.07}{12}\right)^{180}$$

$$- \frac{12(\$3358.64)}{0.07}\left\{\left(1 + \frac{0.07}{12}\right)^{180} - 1\right\} \Rightarrow$$

$$PO_{180} = \$1,139,578.69 - \$1,064,562.24 \Rightarrow$$

$$PO_{180} = \$75,015.61$$

PO_{180} is also the 'balloon' payment needed in order to amortize the loan 2 years ahead of schedule at the end of 15 years.

$$\overset{5}{\mapsto}: D_{Aj} = \left[\frac{12D - rA}{12}\right]\left(1 + \tfrac{r}{12}\right)^{j-1} \Rightarrow$$

$$D_{A100} = \left[\frac{12(\$3358.64) - (0.07)(\$400{,}000.00)}{12}\right]\left(1 + \tfrac{0.07}{12}\right)^{99} \Rightarrow$$

$$D_{A100} = \$1834.24 \therefore$$

$$\overset{6}{\mapsto}: D_{I100} = D - D_{A100} \Rightarrow$$

$$D_{I100} = \$3358.64 - \$1834.24 \Rightarrow$$

$$D_{I100} = \$1524.39 \therefore$$

Ex 5.1.2: Bill borrows $\$38{,}000.00$ in order to buy a new SUV. The $5\frac{\%}{year}$ declining-balance loan (*another name for a loan that is being reduced via an amortization schedule*) has a term of 7 years. **A**) Calculate the monthly payment D, the sum of all monthly payments PV_{PS}, and the sum of all interest payments PV_{IPS}. **B**) Calculate D_{A1} and D_{I1}. **C**) Find the payment number J where the amount being applied to principal starts to exceed 90% of the payment.

$$\overset{1}{\mapsto}: D = \frac{rA}{12\left\{1 - \left(1 + \frac{r}{12}\right)^{-12T}\right\}} \Rightarrow$$

$$D = \frac{0.05 \cdot (\$38{,}000.00)}{12\left\{1 - \left(1 + \frac{0.05}{12}\right)^{-84}\right\}} \Rightarrow$$

A) $$D = \$537.09mo \therefore$$

$$\overset{2}{\mapsto}: PV_{PS} = 12TD \Rightarrow$$

$$PV_{PS} = 12 \cdot 7 \cdot (\$539.09) \Rightarrow$$

$$PV_{PS} = \$45{,}115.43 \therefore$$

$$\overset{3}{\mapsto}: PV_{IPS} = 12TD - A \Rightarrow$$

$$PV_{IPS} = \$45{,}115.43 - \$38{,}000.00 \Rightarrow$$

$$PV_{IPS} = \$7{,}115.43 \therefore$$

…

$$\overset{1}{\mapsto}: D_{A1} = \left[\frac{12D - rA}{12}\right]\left(1 + \tfrac{r}{12}\right)^{1-1} = \left[\frac{12D - rA}{12}\right] \Rightarrow$$

B) $$D_{A1} = \left[\frac{12(\$539.64) - (0.05)(\$38{,}000.00)}{12}\right] \Rightarrow$$

$$D_{A1} = \$381.30 \therefore$$

$$\overset{2}{\mapsto}: D_{I1} = D - D_{A1} \Rightarrow D_{I1} = \$157.99 \therefore$$

…

$$\overset{1}{\mapsto}: D_{Aj} = \left[\frac{12D - rA}{12}\right]\left(1 + \tfrac{r}{12}\right)^{j-1}$$

$$\overset{2}{\mapsto}: D_{AJ} = 0.9D \Rightarrow$$

$$\left[\frac{12D - rA}{12}\right]\left(1 + \tfrac{r}{12}\right)^{J-1} = 0.9D \Rightarrow$$

C) $$\$381.30(1.004167)^{J-1} = \$485.67 \Rightarrow$$

$$(1.004167)^{J-1} = 1.2737 \Rightarrow$$

$$(J-1)\ln(1.004167) = \ln(1.2737) \Rightarrow$$

$$J - 1 = \frac{\ln(1.2737)}{\ln(1.004167)} \Rightarrow J - 1 = 58.17 \Rightarrow$$

$$J = 60 \therefore$$

Note: Notice the use of the natural logarithm $\ln$ *when solving for* $J-1$*. Taking the logarithm of both sides is the standard technique when solving algebraic equations where the variable appears as an exponent. In theory, one can use any base, but* $\ln$ *is a standard key available on most scientific calculators.*

An interesting question associated with loan amortization asks, what percent of the first payment is applied towards principal and what percent pays interest charges? We already have the algebraic machinery in place to answer this question. To start,

$$D_{Aj} = \left[\frac{12D - rA}{12}\right]\left(1 + \tfrac{r}{12}\right)^{j-1} \Rightarrow$$
$$D_{A1} = \left[\frac{12D - rA}{12}\right].$$

Recall that

$$D = \frac{rA}{12\left\{1 - \left(1 + \tfrac{r}{12}\right)^{-12T}\right\}}.$$

Substituting the expression for D into that for D_{A1} gives

$$D_{A1} = \frac{\left(12\left[\frac{rA}{12\left\{1 - \left(1 + \tfrac{r}{12}\right)^{-12T}\right\}}\right] - rA\right)}{12} \Rightarrow$$
$$D_{A1} = \frac{rA}{12}\left[\frac{1}{1 - \left(1 + \tfrac{r}{12}\right)^{-12T}} - 1\right] \Rightarrow$$
$$D_{A1} = \frac{rA}{12}\left[\frac{1}{\left(1 + \tfrac{r}{12}\right)^{12T} - 1}\right]$$

Next, we form the ratio

$$\frac{D_{A1}}{D} = \frac{\frac{rA}{12}\left[\frac{1}{\left(1 + \tfrac{r}{12}\right)^{12T} - 1}\right]}{\frac{rA}{12\left\{1 - \left(1 + \tfrac{r}{12}\right)^{-12T}\right\}}}$$

Finally, we obtain after algebraic simplification

$$\frac{D_{A1}}{D} = \left(1+\tfrac{r}{12}\right)^{-12T} \therefore$$

Ex 3.1.3: Calculate $\frac{D_{A1}}{D}$ for $r = 8.25 \frac{\%}{year}$ and the following values for T : 15, 20, and 30 years.

$$15 years \Rightarrow \frac{D_{A1}}{D} = (1.006875)^{-180} = .291 = 29.1\% :$$

$$20 years \Rightarrow \frac{D_{A1}}{D} = 19.3\% : 30 years \Rightarrow \frac{D_{A1}}{D} = 8.48\%$$

The expression $\frac{D_{A1}}{D} = \left(1+\tfrac{r}{12}\right)^{-12T}$ can be used to build a lookup table for various annual interest rates and typical loan amortization terms where the entries in the body of the table will be the corresponding *principal-to-overall-payment* ratios $\frac{D_{A1}}{D}$ for the very first mortgage payment.

	ANNUAL INTEREST RATE				
TERM	**5%**	**6%**	**7%**	**8%**	**9%**
15 yr	.473	.407	.351	.302	.260
20 yr	.368	.302	.247	.202	.166
30 yr	.223	.166	.123	.091	.067
40 yr	.135	.091	.061	.041	.027

The above table helps answer questions such as, 'by what percentage would I have to increase my monthly payment in order to reduce my amortization term from 30 years to 20 years?' If your mortgage interest rate is $7 \frac{\%}{year}$, the answer from table lookup is roughly

$$\begin{gathered}\Delta\% = .247 - .123 \Rightarrow \\ \Delta\% = .124 = 12.4\%\end{gathered}.$$

3.2 Your Home Mortgage

In his pop hit "Philadelphia Freedom", Elton John sings about the 'good old family home.' The vast majority of all Americans purchase that 'good old family home' via a collateralized declining-balance loan where the collateral is the title deed to the house being purchased. This is the traditional *home mortgage* as we Americans know it.

Two terms associated with the word mortgage are: *mortgager*, the lending institution granting the mortgage; and *mortgagee*, the individual obtaining the mortgage. The responsibility of the mortgagee is to make monthly payments on time until that time when the loan is amortized. In return, the mortgagee is guaranteed a place to live—i.e. the house cannot be legally resold or the mortgagee legally evicted. However, if the mortgagee fails to make payments, then the mortgager can start the *legal process of eviction* as a means of recovering the unpaid balance associated with the home mortgage. After eviction occurs, the lending institution will 1) sell the house, 2) recover the unpaid balance, 3) recover expenses associated with the sale, and 4) return any proceeds left to the mortgagee. The aforementioned scenario is not a happy one and should be avoided at all 'costs'. Remember, as long as there is an unpaid mortgage balance, the lending institution holds the title deed to the home that you and your family occupy. Always make sure that the payment you sign up for is a payment that you can continually meet month after month and year after year!

The many examples in this article address various aspects of making mortgage payments and the total lifetime costs associated with the mortgage process. Let's begin with the most frequently asked question, *how much is my payment?*

Ex 3.2.1: The Bennett family is in the process of buying a new home for a purchase price of $\$300,000.00$. They plan to put 20% down and finance the remainder of the purchase price via a conventional fixed-interest-rate home mortgage with a local lending institution.

The amortization options are as follows: 1) $T = 15 yrs @ r = 6.25 \frac{\%}{year}$, 2) $T = 20 yrs @ r = 6.90 \frac{\%}{year}$, and 3) $T = 30 yrs @ r = 7.25 \frac{\%}{year}$. Compute the monthly payment for each of the three options.

The interest-rate range of $1.00 \frac{\%}{year}$ is fairly typical for a term range of 15 years. The amount borrowed will be $\$240,000.00$ after the 20% down payment is made. Proceeding with the calculations, we have

$$\overset{1}{\mapsto}: T = 15 yrs @ r = 6.25 \frac{\%}{year}$$

$$D = \frac{0.0625(\$260,000.00)}{12\left\{1 - \left(1 + \frac{0.0625}{12}\right)^{-12(15)}\right\}} \Rightarrow$$

$$D = \$2,229.30 mo \therefore$$

$$\overset{2}{\mapsto}: T = 20 yrs @ r = 6.90 \frac{\%}{year}$$

$$D = \frac{0.0690(\$260,000.00)}{12\left\{1 - \left(1 + \frac{0.0625}{12}\right)^{-12(20)}\right\}} \Rightarrow$$

$$D = \$2,098.30 mo \therefore$$

$$\overset{3}{\mapsto}: T = 30 yrs @ r = 7.25 \frac{\%}{year}$$

$$D = \frac{0.0725(\$260,000.00)}{12\left\{1 - \left(1 + \frac{0.0725}{12}\right)^{-12(30)}\right\}} \Rightarrow$$

$$D = \$1773.66 mo \therefore$$

Of interest would be the present value $PV_{PS} = 12TD$ of all mortgage payments comprising the payment stream for each of the three options. Once PV_{PS} is determined, we can determine PV_{IPS} by the formula $PV_{IPS} = 12TD - A$. The results from **Ex 3.2.1** are shown in the next table

PRESENT VALUE FOR THREE PAYMENT STREAMS			
TERM	PV_{PS}	A	PV_{IPS}
15 yr	\$401,274.00	\$260,000.00	\$141,274.00
20 yr	\$503,592.00	\$260,000.00	\$243,592.00
30 yr	\$638,517.60	\$260,000.00	\$378,517.00

The facts displayed in the above table are a real eye-opener for most of us when first exposed. The bottom line is that longer-term mortgages with lower monthly payments cost more money—much more money—in the long run. These considerations have to be factored in when buying a home. **Section I: 6.10.8** lists some of the pros and cons associated with long-term mortgages.

The next example answers the question, *how much house can I afford?*

Ex 3.2.2: Based on income, Bill Johnson has been approved for a monthly mortgage payment not to exceed $\$3000.00$ including real-estate taxes and homeowners insurance. If, on the average, real-estate taxes are $\$4000.00$ per year and homeowners insurance is $\$1600.00$ for homes in the subdivision where Bill wants to move, how much house can he afford assuming 30-year mortgage rates are $r = 6.5 \frac{\%}{year}$?

We are only quoting the 30-year rate since the associated mortgage payment will most likely be the lowest payment available. The mortgage payment that includes principal, interest, taxes, and insurance is traditionally known as the PITI payment, whereas the payment that just includes principal and interest is known as the PI payment. The first step will be the subtracting out of the monthly portion of the $\$3000.00$ mortgage payment that must be allocated to taxes and insurance.

$$\stackrel{1}{\mapsto}: D = \$3000 - \left[\frac{\$4000.00 + \$1600.00}{12}\right] \Rightarrow$$

$$D = \$2533.00 mo \therefore$$

In the second step, we set $\$2533.00$ equal to the monthly payment formula and solve for the associated mortgage amount A.

$$\overset{2}{\mapsto}: \$2533.00 = \frac{0.0650(A)}{12\left\{1-\left(1+\frac{0.065}{12}\right)^{-12(30)}\right\}} \Rightarrow$$

$$A = \frac{12\left\{1-\left(1+\frac{0.065}{12}\right)^{-12(30)}\right\}\{\$2533.00\}}{0.065} \Rightarrow$$

$$A = \$400{,}800.74 \therefore$$

In summary, Bill qualifies for a $\$400{,}000.00$ mortgage. If one assumes that Bill has enough money to make a 20% down payment, then Bill would be qualified to buy a house having a purchase price P_P of $\$500{,}000.00$ as shown in the algebraic calculation below.

$$P_P - 0.20P_P = \$400{,}000.00 \Rightarrow 0.80P_P = \$400{,}000.00 \Rightarrow$$

$$P_P = \frac{\$400{,}00.00}{0.80} \Rightarrow$$

$$P_P = \$500{,}000.00$$

Notice that the down payment needed under the above scenario is a hefty $\$100{,}000.00$.

The next example answers the question, *if I increase my payment by so many dollars per month, how much sooner will I be able to pay off my mortgage?*

Ex 3.2.3: Nathan and his wife Nancy purchased a house seven years ago, financing $\$175{,}000.00$ for 30 years at $r = 7\frac{\%}{year}$. The couple's monthly income has recently increased by $\$500.00$. Nathan and Nancy decide to use $\$250.00$ of this increase for an additional monthly principle payment. **A**) If the couple follows this plan, how many years will they be able to save from the current 23 years remaining on the mortgage? **B**) How much money will they save in interest charges?

In Step 1, we calculate the existing monthly payment by the usual method.

$$\overset{1}{\mapsto}: T = 30\,yrs\,@\,r = 7.00\,\tfrac{\%}{year}$$

$$D = \frac{0.070(\$175{,}000.00)}{12\left\{1-\left(1+\frac{0.070}{12}\right)^{-12(30)}\right\}} \Rightarrow D = \$1164.28mo \therefore$$

In Step 2, we calculate the balance (payoff) remaining on the mortgage at the end of seven years.

$$\overset{2}{\mapsto}: PO_j = A\left(1+\tfrac{r}{12}\right)^j - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^j - 1\right\} \Rightarrow$$

$$PO_{84} = \$175{,}000\left(1+\tfrac{0.070}{12}\right)^{84}$$

$$-\frac{12\{\$1164.28\}}{0.070}\left\{\left(1+\tfrac{0.070}{12}\right)^{84} - 1\right\} \Rightarrow PO_{84} = \$159{,}507.97 \therefore$$

Keeping the same payment of $D = \$1164.28mo$ allows the remaining principle of $\$159{,}507.97$ to be paid off in 23 years—right on schedule. Increasing the payment to $D = \$1414.28mo$ will logically result in a compression of the remaining term.

Our approach for the remainder of the problem is to use the existing monthly payment formula

$$D = \frac{rA}{12\left\{1-\left(1+\frac{r}{12}\right)^{-12T}\right\}}$$

in reverse in order to solve for T when D, A, and r is known. First notice that

$$D = \frac{rA}{12\left\{1-\left(1+\frac{r}{12}\right)^{-12T}\right\}} \Rightarrow$$

$$D = \frac{0.070(\$159{,}507.97)}{12\left\{1-\left(1+\frac{0.070}{12}\right)^{-12(23)}\right\}} \Rightarrow .$$

$$D = \$1164.28mo \therefore$$

The previous result confirms the power of the existing monthly payment formula in that this formula retains the algebraic linkage amongst principal, payment, interest rate and term at any stage in the amortization process. It also allows one to solve for any one of the four variables provided the other three variables are known. With this in mind, we finally proceed to Step 3 where D is increased to $D = \$1414.28mo$.

$$\overset{3}{\mapsto}: D = \frac{rA}{12\left\{1-\left(1+\frac{r}{12}\right)^{-12T}\right\}}$$

$$\$1414.28 = \frac{0.070(\$159{,}507.97)}{12\left\{1-\left(1+\frac{0.070}{12}\right)^{-12T}\right\}} \Rightarrow$$

$$\$16{,}971.36\left\{1-\left(1+\tfrac{0.070}{12}\right)^{-12T}\right\} = \$11{,}165.56 \Rightarrow$$

$$\left\{1-\left(1+\tfrac{0.070}{12}\right)^{-12T}\right\} = .6579 \Rightarrow$$

$$\left(1+\tfrac{0.070}{12}\right)^{-12T} = 0.34209 \Rightarrow$$

$$-12T \cdot \ln\left(1+\tfrac{0.070}{12}\right) = \ln(0.34209) \Rightarrow$$

$$-12T \cdot (0.005816) = -1.07268$$

$$T = 15.36 years$$

The answer $T = 15.36 years$ represents 185 payments where the final payment is a small fractional payment that would ceremoniously pay off the mortgage. Going back to the original question, Nathan and Nancy would compress the original mortgage by

A) $\overset{4}{\mapsto}: 23.00 years - 15.36 years = 7.64 years$

by increasing the payment to $D = \$1414.28mo$.

To answer part **B**), we calculate the original amount programmed to interest (assuming the full thirty-year schedule) and then recalculate it for the amount actually paid. The difference is the savings.

$$\overset{5}{\mapsto}: PV_{IPS}(original) = 12TD - A \Rightarrow$$
$$PV_{IPS} = \$419{,}140.8 - \$175{,}000.00 \Rightarrow$$
$$PV_{IPS} = \$244{,}140.80$$

B) $$\overset{6}{\mapsto}: PV_{IPS}(reclaculated) = 12(7)(\$1164.28)$$
$$+12(15.36)(\$1414.28) - \$175{,}000.00 \Rightarrow$$
$$PV_{IPS}(recalculated) = \$183{,}479.61 \Rightarrow$$
$$Savings = \$60{,}661.19$$

Thus Nathan and Nancy will be able to save $\$60{,}661.19$ in interest charges if they faithfully follow their original plan.

In the next example, the mortgage initially has a term of 30 years and the mortgagee wishes to amortize it on an accelerated 20 year schedule after five years have elapsed in the original term.

Ex 3.2.4: Brian Smith purchased a house five years ago and financed $\$215{,}000.00$ for 30 years at $r = 7.2\frac{\%}{year}$. He would like to pay off his house in 15 years. **A**) By how much should he increase his monthly payment in order to make this happen? **B**) How much does he save in the long run by following the compressed repayment schedule?

Step 1 is the calculation of the existing monthly payment.

$$\overset{1}{\mapsto}: T = 30yrs\,@\,r = 7.20\frac{\%}{year}$$
$$D = \frac{0.072(\$215{,}000.00)}{12\left\{1 - \left(1 + \frac{0.072}{12}\right)^{-12(30)}\right\}} \Rightarrow$$
$$D = \$1459.39mo \therefore$$

In Step 2, we calculate the payoff at the end of five years.

$$\overset{2}{\mapsto}: PO_j = A\left(1+\tfrac{r}{12}\right)^j - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^j - 1\right\} \Rightarrow$$

$$PO_{60} = \$215{,}000\left(1+\tfrac{0.072}{12}\right)^{60}$$

$$-\frac{12\{\$1459.39\}}{0.072}\left\{\left(1+\tfrac{0.072}{12}\right)^{60} - 1\right\} \Rightarrow$$

$$PO_{60} = \$202{,}809.89 \therefore$$

Brian wants to accelerate the mortgage repayment schedule so that the remaining $\$202{,}809.89$ is paid off in 15 years. This, in effect, creates a brand new 15-year mortgage having the same annual interest rate. Step 3 is the calculation for Brian's new payment.

$$\overset{3}{\mapsto}: T = 15yrs\,@\,r = 7.20\tfrac{\%}{year}$$

$$D = \frac{0.072(\$202{,}809.89)}{12\left\{1-\left(1+\tfrac{0.072}{12}\right)^{-12(15)}\right\}} \Rightarrow D = \$1845.66mo \therefore$$

Once the old and revised payments are known, Part **A**) is easily answered.

A) $\overset{4}{\mapsto}: increase = \$1845.66 - \$1459.89 = \$385.77mo$

Part **B**): Follow the exact process as presented in Example 5.2.3, Steps 5) and 6), to obtain Brian's overall projected savings of $\$105{,}748.20$.

In our next example, a mortgage is initially taken out for a term of 20 years. Three years into the term, the mortgage is refinanced in order to obtain a lower interest rate.

Ex 3.2.5: In buying a new home, the Pickles financed $\$159{,}000.00$ for 20 years at $r = 6.2\tfrac{\%}{year}$. Three years later, 15-year rates dropped to $4.875\tfrac{\%}{year}$. The Pickles decide to refinance the remaining balance and the associated $\$1500.00$ refinancing closing costs at the lower rate. How much do they save overall by completing this transaction?

$\overset{1}{\mapsto}: T = 20\,yrs\,@\,r = 6.20\,\frac{\%}{year}$

$$D = \frac{0.062(\$159{,}000.00)}{12\left\{1-\left(1+\frac{0.062}{12}\right)^{-12(20)}\right\}} \Rightarrow$$

$$D = \$1157.55\,mo \therefore$$

$$\overset{2}{\mapsto}: PO_j = A\left(1+\tfrac{r}{12}\right)^j - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^j - 1\right\} \Rightarrow$$

$$PO_{36} = \$159{,}000\left(1+\tfrac{0.062}{12}\right)^{36}$$
$$-\frac{12\{\$1157.55\}}{0.062}\left\{\left(1+\tfrac{0.062}{12}\right)^{36} - 1\right\} \Rightarrow$$

$$PO_{36} = \$145{,}741.48 \therefore$$

$$\overset{3}{\mapsto}: T = 15\,yrs\,@\,r = 4.875\,\tfrac{\%}{year}$$

$$D = \frac{0.04875(\$147{,}241.48)}{12\left\{1-\left(1+\frac{0.04875}{12}\right)^{-12(15)}\right\}} \Rightarrow$$

$$D = \$1154.81\,mo \therefore$$

Notice that the monthly payment actually drops a little bit, and we have compressed the overall term by two years! Using our standard methodology, the overall savings is

$$\overset{4}{\mapsto}: 240(\$1157.55) - \{36(\$1157.55) + 180(\$1154.81)\} = \$28{,}274.19.$$

Our last example illustrates the devastating cumulative effects of making partial mortgage payments over a period of time. Hopefully, this is a situation that most of us will strive to avoid.

Ex 5.2.6: Teresa bought a new home for a purchase price of $\$450{,}000.00$. She made a $\$90{,}000.00$ down payment and financed the remainder at $7\frac{\%}{year}$ for a term of 30 years. Three years into the loan, Teresa was cut to half-time work for a period of 24 months.

Teresa was able to negotiate with her lending institution a partial mortgage payment (half the normal amount) for the same period. At the end of the 24 months, Teresa was able to go back to full-time employment and make full house payments. **A**) Calculate her mortgage balance at the end of five years. **B**) Calculate the revised remaining term if the original payment is maintained. **C**) Calculate the revised payment needed in order to amortize the loan via the original schedule.

First, we need to calculate Teresa's original payment:

$$\overset{1}{\mapsto}: T = 30\,yrs\,@\,r = 7.00\tfrac{\%}{year}$$

$$D = \frac{0.07(\$360{,}000.00)}{12\left\{1-\left(1+\frac{0.07}{12}\right)^{-12(30)}\right\}} \Rightarrow .$$

$$D = \$2395.09mo \therefore$$

At the end of three years, the mortgage balance is

$$\overset{2}{\mapsto}: PO_j = A\left(1+\tfrac{r}{12}\right)^j - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^j - 1\right\} \Rightarrow$$

$$PO_{36} = \$360{,}000\left(1+\tfrac{0.07}{12}\right)^{36}$$

$$-\frac{12\{\$2395.09\}}{0.07}\left\{\left(1+\tfrac{0.07}{12}\right)^{36} - 1\right\} \Rightarrow$$

$$PO_{36} = \$348{,}217.03 \therefore$$

We use the same formula the second time in order to calculate the effects of making a monthly half payment of $\$1197.54$ for a period of two years on a partially-amortized loan having a starting balance $\$348{,}217.03$.

$$\overset{3}{\mapsto}: PO_j = A\left(1+\tfrac{r}{12}\right)^j - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^j - 1\right\} \Rightarrow$$

$$PO_{24} = \$348{,}217.03\left(1+\tfrac{0.07}{12}\right)^{24}$$

$$-\frac{12\{\$1197.54\}}{0.07}\left\{\left(1+\tfrac{0.07}{12}\right)^{24} - 1\right\} \Rightarrow$$

$$PO_{24} = \$369{,}627.84 \therefore$$

A) Teresa's revised mortgage balance at the end of five years is $\$369{,}627.84$, a sum which is $\$9627.84$ more than she originally borrowed.

At the end of five years, the original payment of $\$2395.08$ comes back into play, a payment that must pay off a balance of $\$369{,}627.84$ over a yet-to-be-calculated number of years.

$$\overset{4}{\mapsto}: D = \frac{rA}{12\left\{1 - \left(1 + \frac{r}{12}\right)^{-12T}\right\}}$$

$$\$2395.08 = \frac{0.070(\$369{,}627.84)}{12\left\{1 - \left(1 + \frac{0.070}{12}\right)^{-12T}\right\}} \Rightarrow$$

$$\$28{,}740.96\left\{1 - \left(1 + \tfrac{0.070}{12}\right)^{-12T}\right\} = \$25{,}873.95 \Rightarrow$$

$$\left\{1 - \left(1 + \tfrac{0.070}{12}\right)^{-12T}\right\} = .90025 \Rightarrow \left(1 + \tfrac{0.070}{12}\right)^{-12T} = 0.09975 \Rightarrow$$

$$-12T \cdot \ln\left(1 + \tfrac{0.070}{12}\right) = \ln\left(0.09975\right) \Rightarrow$$

$$-12T \cdot \left(0.005816\right) = -2.30505$$

$$T = 33.02748\, years$$

B) With the original payment, Teresa will not pay off her mortgage until another 33 years have passed. When added to the five years that have already transpired, this mortgage will require 38 years to amortize assuming no other changes occur.

To bring Teresa back on schedule, we will need to calculate a revised mortgage payment that allows her to amortize the balance of $\$369{,}627.84$ in 25 years.

$$\overset{5}{\mapsto}: T = 25\, yrs\ @\ r = 7.00 \tfrac{\%}{year}$$

$$D = \frac{0.07(\$369{,}627.84)}{12\left\{1 - \left(1 + \frac{0.07}{12}\right)^{-12(25)}\right\}} \Rightarrow D = \$2612.45 mo \therefore$$

C) Teresa's revised mortgage payment is $\$2612.45 mo$, $\$317.36 mo$ more than her original payment of $\$2395.08$. Playing catch up is costly!

3.3 Car Loans and Leases

Nowadays, most car loans are set up on declining-balance amortization schedules. The mathematics associated with car loans set up on a declining-balance amortization schedule is identical to the mathematics associated with home mortgages. Two major differences are that the term is much shorter for a car loan and that the annual interest rate is often less. Let's start off by computing a car payment.

Ex 3.3.1: Bob bought a 2004 SUV having a sticker price of $\$45{,}000.00$. The salesperson knocked 12% off, a 'deal' that Bob gladly agreed too. After factoring in a 7% state sales tax on the agreed-to sales price, Rob put $\$2000.00$ down and financed the balance for 66 months at $4\frac{\%}{year}$. The lending institution happens to be a subsidiary of the car manufacturer. **A**) Calculate Bob's car payment. **B**) Calculate the interest paid to the lending institution assuming the loan goes full term.

$$\overset{1}{\mapsto}: Sales\,\Pr ice =$$

$$(0.88)\cdot(\$45{,}000.00) = \$39{,}600.00 \therefore$$

$$Sales\,\Pr ice + Tax =$$

$$(1.07)\cdot(\$39{,}600.00) = \$42{,}372.00 \therefore$$

A) $AmountFinanced = \$40{,}372.00 \therefore$

$$\overset{2}{\mapsto}: T = 5.5yrs\,@\,r = 4.00\tfrac{\%}{year}$$

$$D = \frac{0.04(\$40{,}372.00)}{12\left\{1-\left(1+\frac{0.04}{12}\right)^{-66}\right\}} \Rightarrow$$

$$D = \$682.46mo \therefore$$

B) $\overset{3}{\mapsto}: PV_{IPS} = 66(\$682.46) - \$40{,}372.00 \Rightarrow$

$$PV_{IPS} = \$4670.60 \therefore$$

The fascinating thing about Example 5.3.1 is that total interest $\$4670.60$ to be paid to the lending institution (part of the car conglomerate) just about equals $\$5400.00$, the dollar amount 'knocked off' the original sales price. Could this be a classic case of pay me now or pay me later?

A real danger in financing large amounts for expensive vehicles is that vehicles—unlike houses—depreciate over time. This means that there may be a period of time within the term of the loan where the actual balance remaining on the loan exceeds the current value of the vehicle itself. Such a period of time is properly characterized as a financial 'danger zone' since insurance proceeds paid via the 'totaling' of a fully-insured vehicle in the danger zone will not be enough to retire the associated loan. Thus, the once proud owner is not only stuck with a trashed vehicle, but also a partially unpaid debt and, most assuredly, significantly higher insurance premiums in the future. Motorized vehicles, as much as Americans love 'em, are definitely a major family money drain.

So, by how much does a vehicle typically depreciate? The standing rule of thumb is between 15% and 20% per year where the starting value is the manufacturers suggested retail price. The 15% figure is a good number for higher-priced vehicles equipped with desirable standard options such as air conditioning and an automatic transmission. The 20% figure is usually reserved for cheaper stripped-down models having few customer-enticing features. Either percentage figure leads to a simple mathematical model describing car depreciation. Let SRP be the suggested retail price of a particular car model, P be the assumed annual depreciation rate (as a decimal fraction), and t be the number of years that have elapsed since purchase. Then the current vehicle value $V = V(t)$ can be estimated by $V(t) = SRP \cdot (1-P)^t$ where SRP is the manufacturers suggested retail price; P is the annual depreciation rate; t be the number of years since purchase.

Note: Some estimators say that one must immediately reduce a vehicle's value from resale value to wholesale value as soon as it leaves the showroom. That amount is roughly equivalent to a normal year's depreciation, which increases the exponent up by one in the previous model $V(t) = SRP \cdot (1-P)^{t+1}$.

Ex 3.3.2: Project the value of Bob's SUV over the life of the corresponding loan with and without immediate 'Showroom Depreciation'. Use an annual depreciation rate of $P = .15$ and calculate the two values at six-month intervals.

Looking back at the previous example, we see that $SRP = \$45,000.00$. The results obtained via the two vehicle-depreciation models are shown in the table below.

DEPRECIATION OF BOB'S SUV		
Time in months	**With Showroom Depreciation**	**Without Showroom Depreciation**
0	$38,250.00	$45,000.00
6	$35,264.00	$41,487.00
12	$32,512.00	$38,250.00
18	$29,975.00	$35,264.00
24	$27,635.00	$32,512.00
30	$25,478.00	$29,975.00
36	$23,490.00	$27,635.00
42	$21,656.00	$25,478.00
48	$19,966.00	$23,490.00
54	$18,408.00	$21,656.00
60	$16,971.00	$19,966.00
66	$15,647.00	$18,408.00

One can argue about 'with' or 'without' showroom depreciated, but even with no depreciation, Bob's SUV drops about $\$3500.00$ of its sticker price in the first six months. The important thing to note is that the table values are the insurance value of the vehicle—i.e. the cash that an insurance company will pay you if the vehicle is totally destroyed. Yes, you may be able to sell it for more; but what if it is involved in an accident? The table value will be your legal compensation.

Let's see how Bob's SUV loan progresses towards payout during the same 66-month term. We will compute the remaining balance at six-month intervals using the now-familiar formula

$$PO_j = A\left(1+\tfrac{r}{12}\right)^j - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^j - 1\right\}.$$

The results are:

AMORTIZATION OF BOB'S SUV LOAN	
Time in months	**Remaining Loan Balance**
0	$40,372.00
6	$37,057.16
12	$33,675.47
18	$30,225.58
24	$26,706.12
30	$23,115.68
36	$19,452.83
42	$15,716.11
48	$11,904.27
54	$8,015.06
60	$4,047.67
66	$0.26

Note the few cents remaining on the loan balance. Increasing the monthly loan payment to an even $683.00 will easily eliminate that problem (caused by rounding errors)—an approach most lending institutions would take.

Now for the moment of truth! We will merge the last two tables into a new table in order to compare depreciated value to current loan balance line-by-line.

BOB'S SUV LOAN, A LOAN ON THE EDGE!			
Time in months	**With Showroom Depreciation**	**With No Showroom Depreciation**	**Remaining Loan Balance**
0	$38,250.00	$45,000.00	$40,372.00
6	$35,264.00	$41,487.00	$37,057.16
12	$32,512.00	$38,250.00	$33,675.47
18	$29,975.00	$35,264.00	$30,225.58
24	$27,635.00	$32,512.00	$26,706.12
30	$25,478.00	$29,975.00	$23,115.68
36	$23,490.00	$27,635.00	$19,452.83
42	$21,656.00	$25,478.00	$15,716.11
48	$19,966.00	$23,490.00	$11,904.27
54	$18,408.00	$21,656.00	$8,015.06
60	$16,971.00	$19,966.00	$4,047.67
66	$15,647.00	$18,408.00	$0.26

The above table shows a loan on the edge! If we factor in showroom depreciation, the insurance value of the vehicle is actually less than the balance remaining on the loan for about the first two years. We could term that period of time a financial danger zone since the insurance proceeds from a totaled vehicle will not be enough to pay off the loan in full. If we don't factor in showroom depreciation, we are in reasonably good shape throughout the same two years—a big if. So, we might conclude that Bob is not in too great of danger. But, how about Mr. Harvey, whose story is in our next example.

Ex 3.3.3: Mr. Robert Harvey bought a new Camry for his son John, who planned to use it while attending college. The original Camry sticker price of $\$24{,}995.00$ was discounted by $\$1500.00$ due to a Toyota advertised sale. State and county sales taxes then added 6% to the remaining purchase price. Mr. Harvey made a $\$1000.00$ down payment and financed the balance for five years at $3.5\frac{\%}{year}$, figuring the car would be paid off when John graduated. Alas, fate had a different plan because poor John totaled it seventeen months later. Project the unpaid loan balance, if any, after insurance proceeds are received.

$$\overset{1}{\mapsto}: Sales\,\Pr ice =$$

$$\$24{,}995.00 - \$1500.00 = \$23{,}495.00 \therefore$$

$$Sales\,\Pr ice + Tax =$$

$$(1.06)\cdot(\$23{,}495.00) = \$24{,}904.70 \therefore$$

$$AmountFinanced = \$24{,}904.70 - \$1{,}000.00 = \$23{,}904.70 \therefore$$

$$\overset{2}{\mapsto}: T = 5\,yrs\,@\,r = 3.50\tfrac{\%}{year}$$

$$D = \frac{0.035(\$23{,}904.00)}{12\left\{1-\left(1+\frac{0.035}{12}\right)^{-60}\right\}} \Rightarrow$$

$$D = \$434.85 mo \therefore$$

At the seventeen-month point, we need to calculate both the remaining wholesale value of the Camry (which hopefully equals the insurance proceeds) and the remaining balance on the loan. Also, as a rule, the Toyota Camry holds its resale value rather well. Thus, we will be optimistic and use $P = 0.13$ in conjunction with showroom depreciation. Notice the rescaling of the time t to months.

$$\overset{3}{\mapsto}: V(t) = SRP\cdot(1-P)^{\frac{t+12}{12}} \Rightarrow$$

$$V(t) = \$24{,}995.00\cdot(0.87)^{\frac{29}{12}} \Rightarrow$$

$$V(t) = \$17{,}852.18$$

$$\overset{4}{\mapsto}: PO_j = A\left(1+\tfrac{r}{12}\right)^j - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^j - 1\right\} \Rightarrow$$

$$PO_{17} = \$23{,}904.00\left(1+\tfrac{0.035}{12}\right)^{17}$$

$$-\frac{12(\$434.85)}{0.035}\left\{\left(1+\tfrac{0.035}{12}\right)^{17} - 1\right\} \Rightarrow$$

$$PO_{17} = \$17{,}549.82$$

$$\overset{5}{\mapsto}: SettlementBalance = \$17{,}852.18 - \$17{,}549.82 \Rightarrow$$

$$SettlementBalance = \$302.36$$

Mr. Harvey escaped by the skin of his teeth. After the loan balance is paid off, he will have pocketed $\$302.36$. But wait, Mr. Harvey will have to come up with an additional down payment because John now needs another car. Life on the edge!

The last story might have been significantly different if another model of automobile was involved. Let's assume that the purchase price, discount, taxes, and loan conditions remain identical but the make and model of car is one for which $P = 0.20$. Then, starting again at Step 3, we have

$$\overset{3}{\mapsto}: V(t) = SRP \cdot (1-P)^{\frac{t+12}{12}} \Rightarrow$$
$$V(t) = \$24{,}995.00 \cdot (0.80)^{\frac{29}{12}} \Rightarrow$$
$$V(t) = \$14{,}576.52$$
$$\overset{4}{\mapsto}: PO_{17} = \$17{,}549.82$$
$$\overset{5}{\mapsto}: SettlementBalance = \$14{,}576.52 - \$17{,}549.82 \Rightarrow$$
$$SettlementBalance = -2973.30$$

In this scenario, Mr. Harvey still owes $\$2973.30$ to the lending institution once insurance proceeds are received. Plus, he'll need some additional cash for a new down payment on a replacement vehicle. Hence, by signing on to this 'deal', Mr. Harvey rolled on the edge and eventually fell off.

Our next example is taken from an advertisement in a local newspaper.

Ex 3.3.4: A Ford dealership is advertising a brand new 2004 Freestar for a sales price of $\$17{,}483.00$, which is $\$5000.00$ less than the manufacturers suggested retail price. Ford will finance the whole amount—with nothing down for qualified buyers—for 84 months at $5.89\frac{\%}{year}$. The advertised payment is $\$269.00mo$. Analyze this deal for correctness, true cost and "edginess'.

We first need to add in the 7% State-of-Ohio sales tax to get the true amount financed; then, we compute the monthly payment.

$$\overset{1}{\mapsto}: Sales\,\mathrm{Pr}\,ice = \$17{,}483.00 \therefore$$

$$Sales\,\mathrm{Pr}\,ice + Tax =$$

$$(1.07)\cdot(\$17{,}483.00) = \$18{,}706.81 \therefore$$

$$AmountFinanced = \$18{,}706.81 \therefore$$

$$\overset{2}{\mapsto}: T = 7\,yrs\,@\,r = 5.89\tfrac{\%}{year}$$

$$D = \frac{0.0589(\$18{,}706.81)}{12\left\{1-\left(1+\frac{0.0589}{12}\right)^{-84}\right\}} \Rightarrow$$

$$D = \$272.29mo \therefore$$

Notice that we are only about $\$3.00$ away from the advertised payment; hence we will accept the dealership's calculations as valid. *Note: the small difference is probably due on how we interpreted the stated rate of* $5.89\frac{\%}{year}$ *—as either an effective annual rate* r_{eff} *or an actual annual rate* r *.*

Next, let's compute the sum of all interest payments during the life of the loan.

$$\overset{3}{\mapsto}: PV_{IPS} = 12TD - A \Rightarrow$$

$$PV_{IPS} = (84)\cdot(\$272.29) - \$18{,}706.81 \Rightarrow$$

$$PV_{IPS} = \$4165.55$$

An important thing to note here is that the dealership is gaining back 80% of the advertised rebate $\$5000.00$ in interest charges. The hook is the lure of no money down.

Lastly, let's examine loan 'edginess' in terms of *remaining loan balance versus the depreciated value of the Freestar*. Considering the size of the initial rebate, assume that the initial showroom discount has already occurred.

Hence, the appropriate depreciation model is $V(t) = SRP \cdot (1-P)^t$; and, since the Freestar has desirable features, we will use $P = 0.15$. Table 5.7 shows the frightful results—a Freestar on the edge for nearly four years!

A FREESTAR ON THE EDGE		
Time in months	**With no Showroom Depreciation**	**Remaining Loan Balance**
0	$17,483.00	$18,706.81
6	$16,118.52	$17,610.61
12	$14,860.55	$16,487.42
18	$13,700.75	$15,319.19
24	$12,631.47	$14,121.99
30	$11,645.64	$12,889.11
36	$10,736.75	$11,619.46
42	$9,898.79	$10,311.96
48	$9,126.23	$8,965.48
54	$8,413.97	$7,557.85

Our last example in this section examines a vehicle lease. A lease is a loan that finances the corresponding amount of vehicle depreciation that transpires during the term of the loan. At the end of the period, the vehicle is returned to the dealership. All leases have stipulations where the amount of miles aggregated on the vehicle must remain below (usually 12,000 miles) per year.

Ex 3.3.5: A Grand Cherokee is advertised for a 'red tag' sales price of $\$21{,}888.00$ after rebates. The corresponding red-tag lease payment is $\$248.00 mo$ plus tax for a term of 39 months with $\$999.00$ due at signing. From the information just given, analyze this transaction.

The sales price of $\$21{,}888.00$ represents about 20% off and may actually be a little bit below wholesale. But, what does it matter, for the vehicle is going to eventually be returned to the dealership and resold as a 'premium' used car!

Predicting the original manufacturers suggested retail price (SRP), we have

$$\overset{1}{\mapsto} : (0.80) \cdot SRP = \$21{,}888.00 \Rightarrow SRP = \$27{,}360.00 \therefore$$

Next, we predict the depreciation during the 39 month term of the lease using the showroom depreciation model with $P = 0.15$.

$$\overset{2}{\mapsto} : V(t) = SRP \cdot (1 - P)^{t+1} \Rightarrow$$

$$V(39) = \$27{,}360.00 \cdot (0.85)^{\frac{51}{12}} \Rightarrow V(39) = \$13{,}713.44 \therefore$$

Once the depreciation is calculated, we can determine the actual amount financed and the interest charged.

$$\overset{3}{\mapsto} : AF = \$21{,}888.00 - \$13{,}713.44 - (\$999.00) \Rightarrow$$

$$AF = \$8174.55 - \$999.00 = \$7175.55$$

$$\overset{4}{\mapsto} : PV_{PS} = 39 \cdot (\$248.00) = \$9672.00$$

$$\overset{5}{\mapsto} : PV_{IPS} = \$9672.00 - \$7175.55 = \$2496.45$$

The difference PV_{IPS} is due to the applied interest rate over the term of 39 months, which we will now determine by:

$$\overset{6}{\mapsto} : FV = PV \cdot (1 + \tfrac{r}{12})^{T} \Rightarrow PV \cdot (1 + \tfrac{r}{12})^{T} = FV \Rightarrow$$

$$\$7175.55 \cdot (1 + \tfrac{r}{12})^{39} = \$9692.00 \Rightarrow (1 + \tfrac{r}{12})^{39} = 1.3479 \Rightarrow$$

$$39 \ln(1 + \tfrac{r}{12}) = \ln(1.3479) = 0.298556 \Rightarrow$$

$$\ln(1 + \tfrac{r}{12}) = 0.00765 \Rightarrow (1 + \tfrac{r}{12}) = 1.007679 \Rightarrow r = 9.2\% \therefore$$

Notice the sky-high interest rate of $r = 9.2\%$, a rate that is approaching low-end credit-card rates!

In closing **Article 3.3**, we will leave it to the reader to verify the following statement: **To avoid living on the edge when signing up for an automobile loan, make a down payment equivalent to the first year's depreciation, including showroom depreciation**.

3.4 The Annuity as a Mortgage in Reverse

An annuity can be thought of as a *mortgage in reverse* where the annuitant (the one receiving the payment) becomes the lender and the institution from which the annuity 'is purchased' becomes the borrower. Thus, monthly annuity payments are computed via the same methods used for computing monthly mortgage payments.

With the last statement in mind, we proceed with just one comprehensive example that addresses both annuity creation and annuity usage.

Ex 3.4.1: Mike, age 25, receives $\$10{,}000.00$ as an inheritance. Using his inheritance money as an initial deposit, Mike wisely decides to open a company-sponsored 401K account. For 42 years, he makes an annual payroll deposit of $\$2000.00$ which the company matches. **A**) Project the value of Mike's 401K account at age 67 assuming an average effective annual rate of return of $r_{eff} = 9\frac{\%}{year}$. **B**) If the total value in Mike's 401K account is used to buy a thirty-year-fixed-payment annuity paying $r = 5\frac{\%}{year}$ at age 67, calculate Mike's monthly retirement payment. **C**) If Mike dies at age 87, how much is left in his 401K account?

A) Annuity Creation Phase

Step 1 is the construction of a monetary-growth diagram.

$$\overset{1}{\mapsto}: \underset{t=0}{\$10{,}000.00} \rightarrow 41 \times (\underset{\$4000.00}{\overset{r_{eff}=9\frac{\%}{year}}{\uparrow}} \rightarrow) \underset{t=42}{FV}?$$

Step 2 is projecting the Future Value of Mike's 401K

$$\overset{2}{\mapsto}: FV_{401K} = (L_S - D)(1 + r_{eff})^t + \frac{D}{r_{eff}}\{(1 + r_{eff})^{t+1} - 1\} \Rightarrow$$

$$FV_{401K} = \$6000.00(1.09)^{42} + \frac{\$4000.00}{0.09}\{(1.09)^{43} - 1\} \Rightarrow .$$

$$FV_{401K} = \$223{,}905.19 + \$1{,}763{,}382.65 = \$1{,}987{,}287.84$$

B) Annuity Payment Phase

Using the formula $D = \frac{rA}{12\left\{1-\left(1+\frac{r}{12}\right)^{-12T}\right\}}$ for monthly payments needed to amortize a mortgage, we obtain

$$D = \frac{0.05(\$1{,}987{,}287.84)}{12\left\{1-\left(1+\frac{0.05}{12}\right)^{-12(30)}\right\}} \Rightarrow$$
$$D = \frac{\$8{,}280.36}{0.77617} \Rightarrow$$
$$D = \$10{,}668.18 mo$$

C) Balance Left in Annuity at the End of 20 Years

$$PO_j = A\left(1+\tfrac{r}{12}\right)^j - \frac{12D}{r}\left\{\left(1+\tfrac{r}{12}\right)^j - 1\right\} \Rightarrow$$
$$PO_{240} = \$1{,}987{,}287.84\left(1+\tfrac{0.05}{12}\right)^{240}$$
$$-\frac{12\{\$10{,}668.18\}}{0.05}\left\{\left(1+\tfrac{0.05}{12}\right)^{240} - 1\right\} \Rightarrow$$
$$PO_{240} = \$1{,}005{,}815.89 \therefore$$

When Mike dies at age 87, he leaves $\$1{,}005{,}815.89$ in non-liquidated funds. Hopefully his annuity is such that any unused amount reverts to Mike's estates and heirs as specified in a will.

4. The Calculus of Finance

4.1 Jacob Bernoulli's Differential Equation

A question commonly asked by those students struggling with a required mathematics course is, "What is this stuff good for?" Though asked in every mathematics course that I have taught, I think business calculus is the one course where this question requires the strongest response. For in my other classes—pre-algebra, algebra, etc.—I can argue that one is learning a universal language of quantification. Subsequently, to essentially ask 'of what good is this algebraic language?' is to miss the whole point of having available a new, powerful, and exact means of communication. To not have this communication means at my disposal could be likened to not being able to speak English in a primarily English-speaking country. To say that this would be a handicap definitely is an understatement! Yet this is precisely what happens when one doesn't speak mathematics in a technological world bubbling over with mathematical language: e.g. numbers, data, charts, and formulas. I have found through experience that the previous argument makes a good case for pre-algebra and algebra; however, making a similar case for business calculus may require more specifics in a day when Microsoft EXCEL rules. In this article, we will explore one very essential specific in the modern world of finance, namely the growth and decay of money by the use of **differential equations**, one of the last topics encountered in a standard business calculus course.

Jacob Bernoulli (1654-1705) was nestled in between the lifetimes of Leibniz and Newton, the two co-founders of calculus. Jacob was about 10 years younger than either of these men and continued the tradition of 'standing on the shoulders of giants'.

One of Jacob's greatest contributions to mathematics *and physics* was made in the year 1696 when he found a solution to the differential equation below, which bears his name.

$$\frac{dy}{dx} = f(x)y + g(x)y^n$$

Of particular interest in this article is the case for $n = 0$:

$$\frac{dy}{dx} = f(x)y + g(x).$$

The solution is obtained via Bernoulli's 300-year-old methodology as follows.

Step1: Let $F(x)$ be such that $F'(x) = -f(x)$

Step 2: Formulate the integrating factor $e^{F(x)}$

Step 3: Multiply both sides of $\frac{dy}{dx} = f(x)y + g(x)$ by $e^{F(x)}$ to obtain

$$e^{F(x)}\left[\frac{dy}{dx}\right] = e^{F(x)}\left[f(x)y + g(x)\right] \Rightarrow$$

$$e^{F(x)}\left[\frac{dy}{dy}\right] + e^{F(x)}[-f(x)]y = e^{F(x)} \cdot g(x)$$

Where the left-hand side of the last equality is the derivative of a product

$$e^{F(x)}\left[\frac{dy}{dy}\right] + e^{F(x)}[-f(x)]y = e^{F(x)} \cdot g(x) = \frac{d}{dx}\left[e^{F(x)} \cdot y\right]$$

.

Step 4: To complete the solution, perform the indefinite integration.

$$\frac{d}{dx}\left[e^{F(x)} \cdot y\right] = e^{F(x)} \cdot g(x) \Rightarrow$$

$$e^{F(x)} \cdot y = \int e^{F(x)} \cdot g(x)dx + C \Rightarrow$$

$$y = y(x) = e^{-F(x)} \cdot \left[\int e^{F(x)} \cdot g(x)dx\right] + Ce^{-F(x)} \therefore$$

4.2 Differentials and Interest Rate

Everyone will agree that a fixed amount of money p will change with time. Even though $p = \$10{,}000.00$ is stuffed under a mattress for twenty years in the hopes of preserving its value, the passage of twenty years will change p into something less due to the ever-present action of inflation (denoted by i in this article), which can be thought of as a negative interest rate. So properly, $p = p(t)$ where t is the independent variable and p is the dependent variable.

Let dt be a differential increment of time. Since $p = p(t)$, dt will induce a corresponding differential change dp in p via a first-order linear expression linking dp to dt:

$$dp = Kdt \Rightarrow dp = K(t)dt\,.$$

The exact form of the proportionality expression $K(t)$ will depend on whether principle is growing, decaying, or whether there is a number of complementary and/or competing monetary-change mechanisms at work. Any one of these mechanisms may be time dependent in and of itself necessitating the writing of K as $K = K(t)$. The simplest case is the monetary growth mechanism where $K = rp_0$, the product of a constant interest rate r and the initial principle p_0. This implies a constant rate of dollar increase with time for a given p_0, which is the traditional simple-interest growth mechanism. Thus

$$dp = rp_0 dt : p(0) = p_0\,.$$

The preceding is a first-order linear differential equation written in separated form with stated initial condition. It can be easily solved in three steps:

$$\overset{1}{\mapsto}: p(t) = p_0 rt + C$$
$$\overset{2}{\mapsto}: p(0) = p_0 \Rightarrow C = p_0 \qquad .$$
$$\overset{3}{\mapsto}: p(t) = p_0 rt + p_0 = p_0(1 + rt)$$

One might recognize the last expression as the functional form of the *simple interest formula*. The same differential equation can be written as

$$\frac{dp}{dt} = rp_0 : p(0) = p_0 \text{ after division by } dt \text{ .}$$

This form highlights the differential-based definition of the first derivative. In words it states that *the ratio of an induced differential change of principle with respect to a corresponding, intrinsic differential change in time is constant, being equal to the applied constant interest rate times the initial principal, also constant.* Simple examination of both sides of the above differential equation reveals common and consistent units for both sides with

$$\frac{dp}{dt} \equiv \frac{dollars}{year} \,\&\, rp_0 \equiv \frac{dollars}{year} \text{ .}$$

The expression $\frac{dp}{dt} \equiv p'(t)$ is known as the Leibniz form of the first derivative, equal to the instantaneous change of principle with respect to time—which one could immediately liken to an instantaneous "velocity" of money growth.

4.3 Bernoulli and Money

Returning to $dp = K(t)dt$, we have for the general case that $K(t) = r(t) \cdot p(t) + d(t)$ where $r(t)$ is a time-varying (variable) interest rate, $p(t)$ is the principal *currently present*, and $d(t)$ is an independent variable deposit rate.

Substituting into $dp = K(t)dt$ gives

$$dp = [r(t) \cdot p(t) + d(t)]dt : p(0) = p_0$$

or

$$\frac{dp}{dt} = r(t) \cdot p(t) + d(t) : p(0) = p_0$$

where $p(0) = p_0$ is the amount of principal present at the onset of the process.

Translating the differential equation into words, *the instantaneous rate of change of principal with respect to time equals the sum of two independently acting quantities: 1) the product of the variable interest rate with the principal concurrently present and 2) a variable direct-addition rate.* The preceding differential equation is applicable in the business world if the principal p is continuously growing (or declining) with time. When the interest rate is fixed $r(t) \equiv r_0$ and the independent direct-addition rate is zero $d(t) \equiv 0$, the differential equation reduces to

$$\frac{dp}{dt} = r_0 p : p(0) = p_0 .$$

Solving using separation of variables gives

$$\overset{1}{\mapsto} : \frac{dp}{p} = r_0 dt$$

$$\overset{2}{\mapsto} : \ln(p) = r_0 t + C \Rightarrow p(t) = e^C e^{r_0 t} .$$

$$\overset{3}{\mapsto} : p(0) = p_0 \Rightarrow p(t) = p_0 e^{r_0 t}$$

The final expression $p(t) = p_0 e^{r_0 t}$ is the familiar Continuous-Interest Formula for principle growth given a starting principal p_0 and constant interest rate r_0.

Returning to the general differential equation

$$\frac{dp}{dt} = r(t) \cdot p(t) + d(t) : p(0) = p_0 ,$$

we see that it is Bernoulli in form with the solution given again by an atrocious expression

$$F(t) = -\int r(t)dt$$
$$p(t) = e^{-F(t)} \cdot \left[\int e^{F(t)} \cdot d(t)dt \right] + Ce^{-F(t)}$$

Upon comparison with the general solution developed in detail earlier. The initial condition $p(0) = p_0$ will be applied on a case-by-case basis as we explore the various and powerful uses of the above solution in the world of finance. Depending on the complexity of $r(t)$ and $d(t)$, the coupled solution

$$F(t) = -\int r(t)dt$$
$$p(t) = e^{-F(t)} \cdot \left[\int e^{F(t)} \cdot d(t)dt \right] + Ce^{-F(t)} : p(0) = p_0$$

may or may not be expressible in terms of a simple algebraic expression.. Thus, since interest rates are unpredictable and out of any one individual's control (I have seen double-digit swings in both savings-account rates and mortgage rates in my lifetime), we will assume for the purpose of predictive analysis that the interest rate is constant throughout the time interval of interest $r(t) \equiv r_0$. This immediately leads to

$$p(t) = e^{r_0 t} \cdot \left\lfloor \int e^{-r_0 t} \cdot d(t)dt \right\rfloor + Ce^{r_0 t} : p(0) = p_0 ,$$

a considerable simplification.

The last result is our starting point for concrete applications in investment planning, mortgage analysis, and annuity planning.

4.4 Applications

4.4.1 Growing a Nest Egg

<u>Case 1</u>: If $d(t) \equiv d_0$, a constant annual deposit rate, then the last expression for $p(t)$ further simplifies to

$$p(t) = d_0 e^{r_0 t}\left\lfloor \int e^{-r_0 t} dt \right\rfloor + Ce^{r_0 t} : p(0) = p_0 .$$

This can be easily solved to give

$$p(t) = p_0 e^{r_0 t} + \frac{d_0}{r_0}\left[e^{r_0 t} - 1\right]$$

after applying the boundary condition $p(0) = p_0$.

Notice that the above expression consists of two distinct terms. The term $p_0 e^{r_0 t}$ corresponds to the principal accrued in a continuous interest-bearing account over a time period t at a constant interest rate r_0 given an initial lump-sum investment p_0. Likewise, the term $\frac{d_0}{r_0}\left[e^{r_0 t} - 1\right]$ results from direct principal addition via annual metered contributions into the same interest-bearing account. If either of the constants p_0 or d_0 is zero, then the corresponding term drops away from the overall expression. The following two-stage investment problem illustrates the use of

$$p(t) = p_0 e^{r_0 t} + \frac{d_0}{r_0}\left[e^{r_0 t} - 1\right].$$

Ex 4.4.1: You inherit $\$12{,}000.00$ at age 25 and immediately invest $\$10{,}000.00$ in a corporate-bond fund paying $6 \frac{\%}{year}$. Five years later, you roll this account over into a solid stock fund (whose fifty-year average is $8 \frac{\%}{year}$) and start contributing $\$3000.00$ annually. **A)** Assuming continuous and steady interest, how much is this investment worth at age 68? **B)** What percent of the final total was generated by the initial $\$10{,}000.00$?

A) In the first five years, the only growth mechanism in play is that induced by the initial investment of $\$10{,}000.00$. Thus, the amount at the end of the first five years is given by

$$p(5) = \$10{,}000.00e^{0.06(5)} = \$13{,}498.58.$$

The output from Stage 1 is now input to Stage 2 where both growth mechanisms act for an additional 38 years.

$$p(38) = 13{,}498.58e^{0.08(38)} + \frac{3000}{0.08}(e^{0.08(38)} - 1) \Rightarrow$$

$$p(38) = \$148{,}797.22 + \$375{,}869.11 \Rightarrow$$

$$p(38) = \$528{,}666.34$$

B) The % of the final total accrued by the initial $\$10{,}000.00$ is

$$\frac{\$148{,}792.22}{\$528{,}666.34} = .281 = 28.1\%$$

Note: The initial investment of $\$10{,}000.00$ *is generating* 28.1% *of the final value even though it represents only* 8% *of the overall investment of* $\$124{,}000.00$. *The earlier a large sum of money is inherited or received by an individual, the wiser it needs to be invested; and the more it counts later in life.*

Holding the annual contribution rate to $\$3000.00$ over a period of 38 years is not a realistic thing to do. As income grows, the corresponding annual retirement contribution should also grow. One mathematical model for this is

$$\frac{dp}{dt} = r_0 p + d_0 e^{\alpha t} : p(0) = p_0$$

where the constant annual contribution rate d_0 in the previous model d_0 has been replaced with the expression $d_0 e^{\alpha_0 t}$, allowing the annual contribution rate to be continuously compounded over a time period t at an average annual growth rate α_0 .

The above equation is yet another example of a solvable Bernoulli-in-form differential equation per the sequence

$$p(t) = e^{r_0 t} \cdot \left[\int e^{-r_0 t} \cdot d_0 e^{\alpha_0 t} dt \right] + Ce^{r_0 t} : p(0) = p_0 \Rightarrow$$
$$p(t) = d_0 e^{r_0 t} \cdot \left[\int e^{(\alpha_0 - r_0)t} \cdot dt \right] + Ce^{r_0 t} : p(0) = p_0 \Rightarrow .$$
$$p(t) = p_o e^{r_0 t} + \frac{d_0}{r_0 - \alpha_0} \left[e^{r_0 t} - e^{\alpha_0 t} \right] \therefore$$

Ex 4.4.2: Repeat **Ex 4.4.1** using the annual contribution model $d(t) = 3000e^{0.03t}$.

A) Stage 1 remains the same with $p(5) = \$13{,}498.58$. The Stage 2 calculation now becomes

$$p(38) = 13{,}498.58e^{0.08(38)} + \frac{3000}{0.08 - .03}(e^{0.08(38)} - e^{0.03(38)}) \Rightarrow$$
$$p(38) = \$148{,}797.22 + \$1{,}066{,}708.49 \Rightarrow$$
$$P(38) = \$1{,}215{,}500.71$$

The final annual contribution is $\$3000.00e^{0.03(38)} = \$9380.31$ with the total contribution throughout the 38 years is given by the definite integral

$$\int_0^{38} \$3000.00e^{0.03t} dt =$$
$$\$100{,}000.00e^{0.03t} \Big|_0^{38} = \$212{,}676.83 .$$

B) The % of the final total accrued by the initial $\$10{,}000.00$ is

$$\frac{\$148{,}792.22}{\$1{,}215{,}500.71} = .122 = 12.2\%$$

Most of us don't receive a large amount of money early in our lives. That is the reason we are a nation primarily made up of middle-class individuals. So with this in mind, we will forgo the early inheritance in our next example.

Ex 4.4.3: Assume we start our investment program at age 25 with an annual contribution of $\$3000.00$ grown at a rate of $\alpha_0 = 5\%$ per year. Also assume an aggressive annual interest rate of $r_0 = 10\%$ (experts tell us that this is still doable in the long term through smart investing). **A**) How much is our nest egg worth at age 68? **B**) How does an assumed average annual inflation rate of 3% throughout the same time period alter the final result?

A) Direct substitution gives

$$p(43) = \frac{3000}{0.10 - 0.05}(e^{0.10(43)} - e^{0.05(43)}) \Rightarrow$$
$$p(43) = \$3,906,896.11$$

B) Inflation is nothing more than a negative growth rate (or interest rate) that debits the given rate. For a 3% average annual inflation rate, the true interest r_{T0} and income growth rates α_{T0} are given by the two expressions

$$r_{T0} = r_0 - i_0 = 10\% - 3\% = 7\% = 0.07$$
$$\alpha_{T0} = \alpha_0 - i_0 = 5\% - 3\% = 2\% = 0.02$$

Sadly, our true value after 43 years in terms of today's buying power is

$$p(43) = \frac{3000}{0.07 - 0.02}(e^{0.07(43)} - e^{0.02(43)}) \Rightarrow$$
$$p(43) = \$1,075,454.35$$

4.4.2 Paying for the Nest

Both mortgage loans and annuities are, in actuality, investment plans in reverse where one starts with a given amount of principle $p(0) = p_0$ and chips away at this initial amount until that point in time T when $p(T) = 0$. The governing equation for the case where the interest rate r_0 is fixed throughout the amortization period T is

$$p(t) = p_0 e^{r_0 t} + \frac{d_0}{r_0}\left[e^{r_0 t} - 1\right]$$

where d_0 now becomes the required annual payment.

Applying the condition $p(T) = 0$ leads to

$$d_0 = \frac{r_0 p_0 e^{r_0 t}}{e^{r_0 t} - 1}.$$

The fixed monthly payment m_0 is given by

$$m_0 = \frac{d_0}{12} = \frac{r_0 p_0 e^{r_0 t}}{12\left\{e^{r_0 t} - 1\right\}}$$

The continuous-interest-principal-reduction model does an excellent job of calculating nearly-correct payments when the number of compounding or principal recalculation periods exceeds four per year. Below are three other mortgage-payment formulas based on the continuous-interest model.

First Month's Interest: $\frac{r_0 p_0}{12}$

Total Interest I Payment : $I = p_0\left[\frac{r_0 T e^{r_0 T}}{e^{r_0 T} - 1} - 1\right]$

Total Amount Paid $A = p_0 + I$: $A = \frac{r_0 T p_0 e^{r_0 T}}{e^{r_0 T} - 1}$

Ex 4.4.4: $\$250{,}000.00$ is borrowed for 30 years at 5.75%. Calculate the monthly payment, total repayment , and total interest repayment assuming no early payout.

$$m_0 = \frac{0.0575(\$250{,}000.00)e^{0.0575(30)}}{12(e^{0.0575(30)} - 1)} = \$1457.62$$

$$A = \frac{0.0575(30)(\$250{,}000.00)e^{0.0575(30)}}{(e^{0.0575(30)} - 1)} = \$524{,}745.50$$

$$I = A - p_0 = \$524{,}745.50 - \$250{,}000.00 = \$274{,}745.51$$

Many people justify an initially-high mortgage payment due to the fact that 'the mortgage is being paid off in cheaper dollars." This statement refers to the effects of inflation on future mortgage payments. Future mortgage payments are simply not worth as much in today's terms as current mortgage payments. In fact, if we project t years into the loan and the continuous annual inflation rate has been i_0 throughout that time period, then the present value of our future payment m_{PV} is

$$m_{PV} = \frac{r_0 P_0 e^{r_0 T}}{12(e^{r_0 T} - 1)} e^{-i_0 t}.$$

To illustrate using **Ex 4.4.4**, the present value of a payment made 21 years from now, assuming $i_0 = 3 \frac{\%}{year}$ is $m_{PV} = \$1457.62e^{-0.03(21)} = \$776.31$. Thus, under stable economic conditions, our ability to comfortably *afford* the mortgage should increase over time. This is a case where inflation actually works in our favor. Continuing with this discussion, if we are paying off our mortgage with cheaper dollars, then what is the present value of the total amount paid A_{PV}? A simple definite integral—interpreted as continuous summing—provides the answer

$$A_{PV} = \int_0^T \left[\frac{r_0 P_0 e^{r_0 T}}{e^{r_0 T} - 1} \right] e^{-i_0 t} dt = \frac{r_0 P_0 (e^{r_0 T} - e^{(r_0 - i_0)T})}{i_0 (e^{r_0 T} - 1)}$$

Returning again to **Ex 4.4.4**, the present value of the total 30-year repayment stream is $A_{PV} = \$345{,}999.90$.

Ex 4.4.5: Compare m_0, A, and A_{PV} for a mortgage where $p_0 = \$300{,}000.00$ if the fixed interest rates are: $r_{30\,years} = 6\%$, $r_{20\,years} = 5.75\%$, and $r_{15\,years} = 5.0\%$. Assume a steady annual inflation rate of $i_0 = 3\%$ and no early payout. In this example, we dispense with the calculations and present the results in the table below.

FIXED RATE MORTGAGE COMPARISON FOR A PRINCIPAL OF $P_0 = \$300{,}000.00$				
Terms	r	M	A	A_{PV}
$T = 30$	6.00%	$1797.05	$646,938.00	$426,569.60
$T = 20$	5.75%	$2103.57	$504,856.80	$379.642.52
$T = 15$	5.00%	$2369.09	$426,436.20	$343,396.61

The table definitely shows the mixed advantages/disadvantages of choosing a short-term or long-term mortgage. For a fixed principal, long-term mortgages have lower monthly payments. They also have a much higher overall repayment, although the total repayment is dramatically reduced by the inflation factor. The mortgage decision is very much an individual one and should be done considering all the facts within the scope of the broader economic picture.

Ex 4.4.6: Our last example is an annuity problem. Annuities are simply mortgages in reverse where monthly payouts are made, instead of monthly payments, until the principal is reduced to zero. You retire at age 68 and invest money earned via **Ex 4.4. 3** in an annuity paying $4.5\frac{\%}{year}$ to be amortized by age 92. What is the monthly payout to you in today's terms? The phrase, 'in today's terms", means we let $p_0 = p_{PV} = \$1{,}075{,}454.35$. Thus,

$$m_0 = \frac{(0.045)(\$1{,}075{,}454.35)e^{(0.045)24}}{12(e^{(0.045)24} - 1)} = \$6{,}106.79\,.$$

The monthly income provided by the annuity looks very reasonable referencing to the year 2005. But, unfortunately, it is a fixed-income annuity that will continue as fixed for 24 years. And, what happens during that time? Inflation! To calculate the present value of that monthly payment, say at age 84, our now well-known inflation factor $i = 3\frac{\%}{year}$ is used to obtain

$$m_0 = \$6{,}106.79e^{-.03(16)} = \$3778.80\,.$$

In conclusion, the power provided by the techniques in this short section on finance is nothing short of miraculous. We have used Bernoulli-in-form differential equations to model and solve problems in inflation, investment planning, and installment payment determination (whether loans or annuities). We have also revised the interpretation of the definite integral as a continuous sum in order to obtain the present value of a total repayment stream many years into the future. These economic and personal issues are very much today's issues, and calculus still very much remains a worthwhile tool-of-choice (even for mundane earthbound problems) some 300 years after its inception.

Appendices

A. Greek Alphabet

GREEK LETTER		ENGLISH NAME
Upper Case	Lower Case	
Α	α	Alpha
Β	β	Beta
Γ	γ	Gamma
Δ	δ	Delta
Ε	ε	Epsilon
Ζ	ζ	Zeta
Η	η	Eta
Θ	θ	Theta
Ι	ι	Iota
Κ	κ	Kappa
Λ	λ	Lambda
Μ	μ	Mu
Ν	ν	Nu
Ξ	ξ	Xi
Ο	ο	Omicron
Π	π	Pi
Ρ	ρ	Rho
Σ	σ	Sigma
Τ	τ	Tau
Υ	υ	Upsilon
Φ	φ	Phi
Χ	χ	Chi
Ψ	ψ	Psi
Ω	ω	Omega

B. Mathematical Symbols

SYMBOL	MEANING
$+$	Plus or Add
-	Minus or Subtract or Take Away
$\pm$	Plus or Minus (do both for two results)
$\div$	Divide
/	Divide
$\cdot$	Multiply or Times
^	Power raising
•	Scalar product of vectors
$\{\ \}$ or $[\ ]$ or $(\ )$	Parentheses
$=$	Is equal to
$\equiv$	Is defined as
$\neq$	Does not equal
$\cong$	Is approximately equal to
$\approx$	Is similar too
$>$	Is greater than
$\geq$	Is greater than or equal to
$<$	Is less than
$\leq$	Is less than or equal to
$x, t, etc.$	Variables or 'pronumbers'
$f(x)$ or y	Function of an independent variable x
$\rightarrow$	Approaches a limit
$dx, dt, dy, etc.$	differentials
$f'(x)$ or y'	First derivative of a function
$f''(x)$ or y''	Second derivative of a function

SYMBOL	MEANING
$\overset{1}{\mapsto}$, $\overset{2}{\mapsto}$	Step 1, Step 2, etc.
$A \Rightarrow B$	A implies B
$A \Leftarrow B$	B implies A
$A \Leftrightarrow B$	A implies B implies A
!	Factorial
$\sum_{i=1}^{n}$	Summation sign summing n terms
$\int$	Sign for indefinite integration or anti-differentiation
$\int_{a}^{b}$	Sign for definite integration
$\prod_{i=1}^{n}$	Product sign multiplying n terms
$\sqrt{\ }$	Sign for square root
$\sqrt[n]{\ }$	Sign for n^{th} root
∞	Infinity symbol or the process of continuing indefinitely in like fashion
$\parallel$	Parallel
$\perp$	Perpendicular
$\angle$	Angle
$\neg$	Right angle
Δ	Triangle
$\bigcup$	Set union
$\bigcap$	Set intersection

SYMBOL	MEANING
$x \in A$	Membership in a set A
$x \notin A$	Non-membership in a set A
$A \subset B$	Set A is contained in set B
$A \not\subset B$	Set A is not contained in set B
ϕ	The empty set
$\therefore$	QED: thus it is shown
$\forall$	For every
$\ni$	There exists
π	The number Pi such as in 3.1…
e	The number e such as in 2.7…
φ	The Golden Ratio such as in 1.6…

C. My Most Used Formulas

	Formula	Page Ref
1.		
2.		
3.		
4.		
5.		
6.		
7.		
8.		
9.		
10.		
11.		
12.		
13.		
14.		
15.		
16.		
17.		
18.		