Stokes' and Gauss' Theorems

Math 240 — Calculus III

Summer 2013, Session II

Monday, July 8, 2013

Agenda

1. Stokes' theorem

2. Gauss' theorem
   - Calculating volume with Gauss' theorem

Stokes' theorem

Theorem (Green's theorem)

Let D be a closed, bounded region in $\mathbb{R}^2$ with boundary $C = \partial D$. If $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ is a C^1 vector field on D then

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$$

Notice that $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} = \nabla \times \mathbf{F}$.

Theorem (Stokes' theorem)

Let S be a smooth, bounded, oriented surface in $\mathbb{R}^3$ and suppose that ∂S consists of finitely many C^1 simple, closed curves. If $\mathbf{F}$ is a C^1 vector field whose domain includes S, then

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

Stokes' theorem and orientation

Definition

A smooth, connected surface, S is **orientable** if a nonzero normal vector can be chosen continuously at each point.

Examples

Orientable planes, spheres, cylinders, most familiar surfaces

Nonorientable Möbius band

To apply Stokes' theorem, ∂S must be correctly oriented.

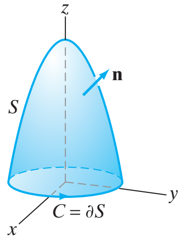
Right hand rule: thumb points in chosen normal direction, fingers curl in direction of orientation of ∂S.

Alternatively, when looking down from the normal direction, ∂S should be oriented so that S is on the *left*.

Stokes' theorem

Example

Let S be the paraboloid $z = 9 - x^2 - y^2$ defined over the disk in the xy-plane with radius 3 (i.e. for $z \geq 0$). Verify Stokes' theorem for the vector field



$$\mathbf{F} = (2z - y)\,\mathbf{i} + (x + z)\,\mathbf{j} + (3x - 2y)\,\mathbf{k}.$$

We calculate

$$\nabla \times \mathbf{F} = -3\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k} \text{ and } \mathbf{N} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}.$$

Therefore,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (-6x - 2y + 2)\,dx\,dy = 18\pi.$$

Stokes' theorem

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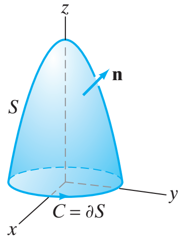

$$\mathbf{F} = (2z - y)\,\mathbf{i} + (x + z)\,\mathbf{j} + (3x - 2y)\,\mathbf{k}.$$

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (-6x - 2y + 2)\,dx\,dy = 18\pi.$$

Using Stokes' theorem, we can do instead

$$\begin{aligned} \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \oint_C -y\,dx + x\,dy \\ &= \int_0^{2\pi} (-3\sin t)^2 + (3\cos t)^2\,dt = 18\pi. \end{aligned}$$

Stokes' theorem

Example

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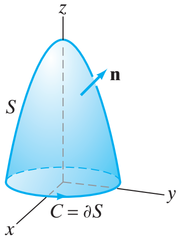


$$\mathbf{F} = (2z - y)\,\mathbf{i} + (x + z)\,\mathbf{j} + (3x - 2y)\,\mathbf{k}.$$

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (-6x - 2y + 2)\,dx\,dy = 18\pi.$$

Applying Stokes' theorem a second time yields

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S}$$
$$= \iint_D 2\,d\mathbf{S} = 2\,(\text{area of } D) = 18\pi.$$

Gauss' theorem

Theorem (Gauss' theorem, divergence theorem)

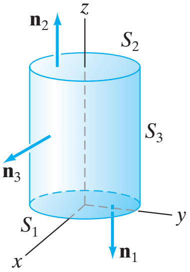
Let D be a solid region in $\mathbb{R}^3$ whose boundary ∂D consists of finitely many smooth, closed, orientable surfaces. Orient these surfaces with the normal pointing away from D. If $\mathbf{F}$ is a C^1 vector field whose domain includes D then

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

Gauss' theorem

Example

Let $\mathbf{F}$ be the radial vector field $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and let D the be solid cylinder of radius a and height b with axis on the z-axis and faces at $z = 0$ and $z = b$. Let's verify Gauss' theorem. Let S_1 and S_2 be the bottom and top faces, respectively, and let S_3 be the lateral face.



To orient ∂D for Gauss' theorem, choose normals

$$\mathbf{n}_1 = -\mathbf{k} \text{ for } S_1, \; \mathbf{n}_2 = \mathbf{k} \text{ for } S_2, \text{ and } \mathbf{n}_3 = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j}) \text{ for } S_3.$$

Now we integrate over the surface

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = b \iint_{S_2} dS + a \iint_{S_3} dS = 3\pi a^2 b.$$

Gauss' theorem

Example

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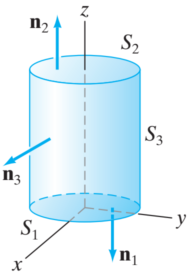

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = b \iint_{S_2} dS + a \iint_{S_3} dS = 3\pi a^2 b.$$

On the other hand, $\nabla \cdot \mathbf{F} = 3$.

Then

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV = 3 \iiint_D dV = 3\pi a^2 b.$$

Calculating volume

Recall how we used Green's theorem to calculate the area of a plane region via a line integral around its boundary.

Theorem

Suppose D is a solid region in $\mathbb{R}^3$ to which Gauss' theorem applies and $\mathbf{F}$ is a C^1 vector field such that $\nabla \cdot \mathbf{F}$ is identically 1 on D. Then the volume of D is given by

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S}$$

where ∂D is oriented as in Gauss' theorem.

Some examples are

$$\text{Volume of } D = \begin{cases} \oiint_{\partial D} (x\,\mathbf{i}) \cdot d\mathbf{S} \\ \oiint_{\partial D} (y\,\mathbf{j}) \cdot d\mathbf{S} \\ \oiint_{\partial D} (z\,\mathbf{k}) \cdot d\mathbf{S} \end{cases}.$$

Calculating volume

Example

Let's calculate the volume of a truncated cone via an integral over its surface. Let D be the solid bounded by the cone

$$x^2 + y^2 = (2 - z)^2$$

and the planes $z = 1$ and $z = 0$. Let's use the vector field $\mathbf{F} = x\,\mathbf{i}$, so that $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$ when S is the top or bottom face. Then we just need to calculate

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = x\,\mathbf{i} + y\,\mathbf{j} + r\,\mathbf{k}$$

and the volume of D is

$$\iint_S (x\,\mathbf{i}) \cdot d\mathbf{S} = \int_0^{2\pi} \int_1^2 (r\cos\theta)^2\,dr\,d\theta = \tfrac{7}{3}\pi.$$