Solving Systems of 3x3 Linear Equations - Elimination

We will solve systems of 3x3 linear equations using the same strategies we have used before. That is, we will take something we don't recognize and change it into something we know how to do. With a 3x3 system ,we will convert the system into a single equation in $ax + b = c$ format.

When we solved a 2x2 system of linear equations, we had a choice of solving those by graphing, substitution, or linear combination (often called the addition method or the elimination method).

To determine which method to use, we often decided that if one of the variables in one of the equations was already solved for, we would use substitution, then plug that value into the other equation. If that did not occur, we would generally use linear combination/elimination. That is, we made one of the coefficients on one of the variables the same, but opposite in sign, then added the equations together to eliminate that variable. We then solved the resulting equation and substituted that value into another equation to find the value of the other variable. We wrote the answer as an ordered pair.

Solving three-variable, three-equation linear systems is not more difficult than solving the two-variable systems, it does take longer. What we do is change the 3x3 system to a 2x2 system by eliminating one of the variables using the elimination, then we solve the 2x2 system as we have done before.

Though the method of solution is based on addition/elimination, being organized and very neat will make the work a whole lot easier.

Let's start simple example.

Example 1. Solve the following system of equations.
$2x + y + 2z = 1$
$4x + 3z = -5$
$5y + 4z = 13$

To solve this system, I want to get rid of one the variables to get a 2x2 system. To do this efficiently, we need to <u>observe and think</u> first. I notice the third row has no x's.

If we multiply the first (top) row by (–2), that will result in a –4x in the first row that, if I added the first and second rows together would result in the elimination of the x's. So that is what we will do to get an equation with just two variables.

$2x + y + 2z = 1 \Rightarrow$ X(–2) results in

$$\begin{array}{rcrcrcl} -4x & - & 2y & - & 4z & = & -2 \\ \underline{4x} & & & \underline{+} & \underline{3z} & \underline{=} & \underline{-5} \\ & & \mathbf{-2y} & \mathbf{-} & \mathbf{z} & \mathbf{=} & \mathbf{-7} \end{array}$$

That gives me one equation in two variables, y and z.

The equation in the third row just happens to contain only the variables y and z. That equation, along with the equation I just obtained, gives me two equations in two variables – just what I wanted.

Copying those two equations, we have

$$\begin{aligned} -2y - z &= -7 \\ 5y + 4z &= 13 \end{aligned}$$

Now, we know how to solve the 2x2 system. If I multiply the first equation by 4, that will make the coefficients of the z's the same, but opposite in sign. By doing that, I can combine the two equations and have the z's eliminated.

$$-2y - z = -7 \rightarrow X(4) \text{ results in } \quad \begin{array}{r} -8y - 4z = -28 \\ \underline{5y + 4z = 13} \\ -3y \qquad = -15, \end{array} \quad \text{so } y = 5$$

Now substitute y = 5 into one of those equations, 5y + 4z = 13, we find

$$\begin{aligned} 5(5) + 4z &= 13, \\ 4z &= -12, \\ \text{so } z &= -3. \end{aligned}$$

Now we substitute y = 5 and z = –3 into one of the original equations to find the value of x.

$2x + y + 2z = 1$
$2x + 5 + 2(-3) = 1$
$2x - 1 = 1$
$2x = 2$
$x = 1$. Our solution is an ordered triple (1, 5, –3).

Those are values of the variables that make all three equations true.

Let's take a look at another 3x3 system of linear equations.

Example 2 Determine the solution set of the following system.

$2x + y + z = 13$
$x + 2y + z = 11$
$x + 3y + 3z = 19$

I could get rid of the x's easily by multiplying the second row by (–2) and adding the first and second equations together. We will leave the third equation alone for right now.

$$\begin{array}{ll} 2x + y + z = 13 & \quad 2x + y + z = 13 \\ x + 2y + z = 11 \rightarrow X(-2) & \underline{-2x - 4y - 2z = -22} \\ & \qquad \mathbf{-3y - z = -9} \end{array}$$

That results in an equation containing only y and z.

Now, notice the coefficients for the x's are the same in the second and third rows, I could multiply the third row by (–1) and add those two equations together, that would eliminate the x's leaving another equation in terms of y and z.

Copying the second and third row;

$$\begin{array}{lll} x + 2y + z = 11 & & x + 2y + z = 11 \\ x + 3y + 3z = 19 & \rightarrow (-1) & \underline{-x - 3y - 3z = -19} \\ \mathbf{-y - 2z = -8} & & \end{array}$$

***** Rather than multiplying the third equation by (–1), I could have subtracted those two equations getting the same result.**

Either way, we now have two equations (bolded) and two variables y and z – just as we wanted. Writing those as a 2x2 system, multiplying the First equation by (–2), we have:

$$\begin{array}{lll} -3y - z = -9 & \rightarrow (-2) & 6y + 2z = 18 \\ -y - 2z = -8 & & \underline{-y - 2z = -8} \\ & & 5y \quad\quad = 10 \quad \text{or} \quad y = 2 \end{array}$$

If y = 2, substituting that back into one of those equations, $-y - 2z = -8$,

we have $-2 - 2z = -8$

$-2z = -6$

or $z = 3$

If y = 2 and z = 3, we substitute those back into one of the original equations to find the value of x.

$$\begin{aligned} 2x + y + z &= 13 \\ 2x + 2 + 3 &= 13 \\ 2x + 5 \quad &= 13 \\ 2x &= 8, \\ x &= 4. \end{aligned}$$

The solution is the ordered triple (4, 2, 3)

<u>To solve these 3x3 systems, we:</u>

1) examined the three equations and made a judgment on which variable might be the easiest to eliminate.
2) used two of the equations to get rid of that variable by making the coefficients the same but opposite in sign.
3)used two other equations to get rid of the same variable.
4) then used those two resulting equations in two variables to solve the 2x2 system
5) and finally used substitution to find the values of the other variables and wrote the answer as an ordered triple.

We will continue in our study to solve 3x3 systems of equations by introducing new notation and using a matrix. By doing so, with practice, we will be able to solve these systems with less work.

Elimination Using a Matrix Format

In this section, we will solve systems pretty much the same way we solved them in the last section using elimination. Two differences, we will write our equations in a matrix and introduce some notation so it will be easier for others to follow our work. We will be doing a lot of arithmetic in our head.

The notation will be simple enough, for instance, if I multiply row 3 by 2 and add that to row 1, I will write that like this; $2R_3 + R_1$

Example 3. Solve the following system of equations using Gaussian elimination.

$$-3x + 2y - 6z = 6$$

$$5x + 7y - 5z = 6$$

$$x + 4y - 2z = 8$$

Copying these equations into a matrix, we have the first matrix. The third row has a coefficient of 1 on the x and the first row has a coefficient of –3 on the x. If I multiply row 3 by 3 and add it in my head to the first row, the x's will fall out.

Changing the order of the equations in a matrix has no bearing on the outcome. The order in which the equations are written in any system just does not matter.

$$\begin{bmatrix} -3x + 2y - 6z = 6 \\ 5x + 7y - 5z = 6 \\ x + 4y - 2z = 8 \end{bmatrix} \xrightarrow{3R_3 + R_1} \begin{bmatrix} 14y - 12z = 30 \\ 5x + 7y - 5z = 6 \\ x + 4y - 2z = 8 \end{bmatrix}$$

Notice, nothing was changed in the second and third rows.

The first row's coefficients are fairly large even numbers, to make them smaller, I will divide both sides of the equation by 2 or multiply by ½.

$$\begin{bmatrix} 14y-12z=30 \\ 5x+7y-5z=6 \\ x+4y-2z=8 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 7y-6z=15 \\ 5x+7y-5z=6 \\ x+4y-2z=8 \end{bmatrix}$$

To get rid of another x, I will multiply the third row by –5 and add that to the second row – again in my head.

I write that like this:

$$\begin{bmatrix} 7y-6z=15 \\ 5x+7y-5z=6 \\ x+4y-2z=8 \end{bmatrix} \xrightarrow{-5R_3+R_2} \begin{bmatrix} 7y-6z=15 \\ -13y+5z=-34 \\ x+4y-2z=8 \end{bmatrix}$$

When we reached this point before, two equations and two unknowns, we solved the 2x2 system in y and z, then used substitution to find the values of the other variables.

So next I have to work on the *y*-column. Be careful, since the third equation has an *x*-term, I cannot use it on either of the other two equations any more or I'll undo what I have done. I can work *on* the equation, but not *with* it.

If I add twice the first row to the second row, this will give me a leading 1 in the second row. I won't have gotten rid of the leading *y*-term in the second row, but I will have converted it (without getting involved in fractions) to a form that is simpler to do the arithmetic. (You should keep an eye out for this sort of simplification.)

$$\begin{bmatrix} 7y-6z=15 \\ -13y+5z=-34 \\ x+4y-2z=8 \end{bmatrix} \xrightarrow{2R_1+R_2} \begin{bmatrix} 7y-6z=15 \\ y-7z=-4 \\ x+4y-2z=8 \end{bmatrix}$$

Now I can use the second row to clear out the *y*-term in the first row. I'll multiply the second row by –7 and add.

$$\begin{bmatrix} 7y-6z=15 \\ y-7z=-4 \\ x+4y-2z=\ \ 8 \end{bmatrix} \xrightarrow{-7R_2+R_1} \begin{bmatrix} 43z=43 \\ y-7z=-4 \\ x+4y-2z=\ \ 8 \end{bmatrix}$$

I can tell what *z* is now, but, just to be thorough, I'll divide the first row by 43. Then I'll rearrange the rows to put them in a triangular pattern:

$$\begin{bmatrix} 43z=43 \\ y-7z=-4 \\ x+4y-2z=\ \ 8 \end{bmatrix} \xrightarrow{\frac{1}{43}R_1} \begin{bmatrix} z=\ \ 1 \\ y-7z=-4 \\ x+4y-2z=\ \ 8 \end{bmatrix} \longrightarrow \begin{bmatrix} x+4y-2z=\ \ 8 \\ y-7z=-4 \\ z=\ \ 1 \end{bmatrix}$$

Now I can start the process of finding the values of the other variables by substitution:

$y - 7(1) = -4$ $y - 7 = -4$ $y = 3$ $x + 4(3) - 2(1) = 8$ $x + 12 - 2 = 8$ $x + 10 = 8$ $x = -2$

Then the solution is **$(x, y, z) = (-2, 3, 1)$**.

There was nothing special about how the order in which this system was solved. You could work in a different order or simplify different rows, and still come up with the correct answer. It is unlikely to be one right way of computing the answer. So don't stress over "how would we know what to do that next?", because there is no rule. I did whatever seemed the easiest to get rid of a variable. Don't worry if you would have used completely different steps. As long as each step along the way is correct, you'll come up with the same answer.

In the above example, I could have gone further in my computations and been more thorough-going in my row operations, clearing out all the *y*-terms other than that in the second row and all the *z*-terms other than that in the first row. This is what the process would then have looked like:

$$\left[\begin{array}{r} -3x+2y-6z=6 \\ 5x+7y-5z=6 \\ x+4y-2z=8 \end{array}\right] \xrightarrow{3R_3+R_1} \left[\begin{array}{r} 14y-12z=30 \\ 5x+7y-5z=6 \\ x+4y-2z=8 \end{array}\right] \xrightarrow{\frac{1}{2}R_1}$$

$$\left[\begin{array}{r} 7y-6z=15 \\ 5x+7y-5z=6 \\ x+4y-2z=8 \end{array}\right] \xrightarrow{-5R_3+R_2} \left[\begin{array}{r} 7y-6z=15 \\ -13y+5z=-34 \\ x+4y-2z=8 \end{array}\right] \xrightarrow{2R_1+R_2}$$

$$\left[\begin{array}{r} 7y-6z=15 \\ y-7z=-4 \\ x+4y-2z=8 \end{array}\right] \xrightarrow{-7R_2+R_1} \left[\begin{array}{r} 43z=43 \\ y-7z=-4 \\ x+4y-2z=8 \end{array}\right] \xrightarrow{-4R_2+R_3}$$

$$\left[\begin{array}{r} 43z=43 \\ y-7z=-4 \\ x+26z=24 \end{array}\right] \xrightarrow{\frac{1}{43}R_1} \left[\begin{array}{r} z=1 \\ y-7z=-4 \\ x+26z=24 \end{array}\right] \begin{array}{c} \xrightarrow{7R_1+R_2} \\ \xrightarrow{-26R_1+R_3} \end{array}$$

$$\left[\begin{array}{r} z=1 \\ y=3 \\ x=-2 \end{array}\right] \longrightarrow \left[\begin{array}{r} x=-2 \\ y=3 \\ z=1 \end{array}\right]$$

This way, we can just read off the values of *x*, *y*, and *z*, and we don't have to bother with the back-substitution. This more-complete method of solving is called "Gauss-Jordan elimination" (with the equations ending up in what is called "reduced-row-echelon form").

Let’s look at the second example we did in the first section and solve it using the Gauss-Jordan Elimination Method with a slight modification. We will not write the variables, the first column will be the x, the second column is the y, and the third column is the z variable. The rows represent the three equations. I must include the signs of the coefficients.

When finished working with the row operations, I should have the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$$

Using the elimination method, you might recall we got rid of the x's by multiplying and adding equations together. In matrix format, that means the coefficients of the x's in row 2 and row 3 will be zero.

Example 2 Determine the solution set of the following system.

$$
\begin{aligned}
2x + y + z &= 13 \\
x + 2y + z &= 11 \\
x + 3y + 3z &= 19
\end{aligned}
$$

To get started, we will copy the coefficients into the matrix.

$$\begin{bmatrix} 2 & 1 & 1 & 13 \\ 1 & 2 & 1 & 11 \\ 1 & 3 & 3 & 19 \end{bmatrix}$$

Now I want to get rid of two x's in the first column. So I will multiply rows (equations) making the coefficients the same, but opposite in sign and add that to another row (equation).

$$\begin{bmatrix} 2 & 1 & 1 & 13 \\ 1 & 2 & 1 & 11 \\ 1 & 3 & 3 & 19 \end{bmatrix} \xrightarrow{-2R3+R1} \begin{bmatrix} 0 & -5 & -5 & -25 \\ 1 & 2 & 1 & 11 \\ 1 & 3 & 3 & 19 \end{bmatrix} \xrightarrow{R1\div(-5)}$$

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 1 & 2 & 1 & 11 \\ 1 & 3 & 3 & 19 \end{bmatrix} \xrightarrow{-1R3+R2} \begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & -1 & -2 & -8 \\ 1 & 3 & 3 & 19 \end{bmatrix}$$

Notice, that gets rid of the x's in the first and second equations. Since there was a common factor in the first rwo of the second matrix, I divided by (–5) to get smaller numbers. Now I will continue with these row operations to get zeros in positions that will lead me to the final matrix solving for x, y and z. That is get zeros in the y column.

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & -1 & -2 & -8 \\ 1 & 3 & 3 & 19 \end{bmatrix} \xrightarrow{R1+R2} \begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & -1 & -3 \\ 1 & 3 & 3 & 19 \end{bmatrix} \xrightarrow{R2\div(-1)}$$

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \\ 1 & 3 & 3 & 19 \end{bmatrix} \xrightarrow{-1R2+R1} \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 1 & 3 & 3 & 19 \end{bmatrix}$$

From here we can see that y = 2 and z = 3. And that we could substitute those values into to equation in row 3 and find the value for x. For now, we will just pretend not to know that and continue to work with this matrix.

The good news is we have the ones in each row representing an x, y, and z. We have to fill in the other numbers with zeros.

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 1 & 3 & 3 & 19 \end{bmatrix} \xrightarrow{-3R1+R3} \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 3 & 13 \end{bmatrix} \xrightarrow{-3R2+R3}$$

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 4 \end{bmatrix}$$

Rearranging the rows, we see that x = 4, y = 2 and z = 3

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The answer is written as an ordered triple - (4, 2, 3).

Anytime you do something differently, it always seems more cumbersome. However, overtime, if you solved 3x3 systems using this matrix reduction (Gauss-Jordan Elimination Method), you would be able to do more than one operation at time, you would save workspace and time – with practice.