Solutions to Walter Rudin's Principles of Mathematical Analysis

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0 Notation

- "iff" means "if and only if."
- "WLOG" means "without loss of generality"
- $\mathbb{N}$ is the set of natural numbers, including 0.
- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are the sets of integers, rational, real, and complex numbers respectively.
- $\mathbb{Z}^+, \mathbb{Q}^+, \mathbb{R}^+$ are the sets of positive integers, rational, and real numbers respectively.
- $\mathbb{Z}[X], \mathbb{Q}[X], \mathbb{R}[X], \mathbb{C}[X]$ are the sets of polynomials in X with coefficients in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ respectively.
- $\mathrm{B}(x, r) = \{y \mid d(x, y) < r\}$ is the open ball centered at x with radius r.
- $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ is the sign function. It is given by

$$\operatorname{sgn}(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

- $\delta_{ij} : \mathbb{Z} \to \{0, 1\}$ is the Kronecker delta. It is given by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

- The coordinates of vectors will be indexed by superscripts.
- For $1 \leq j \leq k$, $\mathbf{e}_j \in \mathbb{R}^k$ are the standard basis vectors, each given by $\mathbf{e}_j^i = \delta_{ij}$.

1 The Real and Complex Number Systems

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution: Let $r \in \mathbb{Q}, r \neq 0$. If $r + x \in \mathbb{Q}$, then $x = -r + (r + x) \in \mathbb{Q}$. If $rx \in \mathbb{Q}$, then $x = r^{-1}(rx) \in \mathbb{Q}$. Take the contrapositive of both statements.

2. Prove that there is no rational number whose square is 12.

Solution: Suppose $r \in \mathbb{Q}$ and $r^2 = 12$. Let $n \in \mathbb{Z}^+$ be least such that $nr \in \mathbb{Z}$. Then

$$(nr)^2 = 12n^2. \tag{1}$$

Since 3 divides the right side of (1), it must divide the left side as well. If nr gives remainder 1 or 2 when divided by 3, then $(nr)^2$ gives remainder 1. Thus 3 divides nr. Cancel 3's from each side of (1) to get

$$3\left(\frac{nr}{3}\right)^2 = 4n^2. \tag{2}$$

Since 3 divides the left side of (2), it must divide the right side as well. Since 3 does not divide 4, 3 divides n^2, hence n. Thus $\frac{n}{3} \in \mathbb{Z}^+$ and $\frac{n}{3}r \in \mathbb{Z}$, which contradicts our choice of n.

3. Prove Proposition 1.15:
The axioms for multiplication imply the following statements.
   (a) If $x \neq 0$ and $xy = xz$, then $y = z$.
   (b) If $x \neq 0$ and $xy = x$, then $y = 1$.
   (c) If $x \neq 0$ and $xy = 1$, then $y = \frac{1}{x}$.
   (d) If $x \neq 0$ then $\frac{1}{\left(\frac{1}{x}\right)} = x$.

Solution: Let $x, y, z \in \mathbb{R}, x \neq 0$.

(a) If $xy = xz$, then

$$y = 1 \cdot y = \left(\left(\frac{1}{x}\right) x\right) y = \left(\frac{1}{x}\right)(xy) = \left(\frac{1}{x}\right)(xz) = \left(\left(\frac{1}{x}\right) x\right) z = 1 \cdot z = z$$

(b) If $xy = x$, then

$$y = 1 \cdot y = \left(\left(\frac{1}{x}\right) x\right) y = \left(\frac{1}{x}\right)(xy) = \left(\frac{1}{x}\right) x = 1$$

(c) If $xy = 1$, then

$$y = 1 \cdot y = \left(\left(\frac{1}{x}\right) x\right) y = \left(\frac{1}{x}\right)(xy) = \left(\frac{1}{x}\right) \cdot 1 = \frac{1}{x}$$

(d) Finally,

$$\frac{1}{\left(\frac{1}{x}\right)} = \left(\frac{1}{\left(\frac{1}{x}\right)}\right) \cdot 1 = \left(\frac{1}{\left(\frac{1}{x}\right)}\right)\left(\left(\frac{1}{x}\right) x\right) = \left(\left(\frac{1}{\left(\frac{1}{x}\right)}\right)\left(\frac{1}{x}\right)\right) x = 1 \cdot x = x$$

4. Let E be a nonempty subset of an ordered set; suppose that α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution: Let $\gamma \in E$. Then $\alpha \leq \gamma \leq \beta$ implies $\alpha \leq \beta$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Solution: Let $x \in A$, then $-x \leq -\inf A$ implies $\sup(-A) \leq -\inf A$ and $-\sup(-A) \leq x$ implies $-\sup(-A) \leq \inf A$.

6. Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = \frac{m}{n} = \frac{p}{q}$, prove that

$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}.$$

Hence is makes sense to define $b^r = (b^m)^{\frac{1}{n}}$.

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Solution:

(a) By hypothesis $mq = pn$, hence

$$\left((b^m)^{\frac{1}{n}}\right)^{nq} = \left(\left((b^m)^{\frac{1}{n}}\right)^n\right)^q = (b^m)^q = b^{mq} = b^{pn} = (b^p)^n = \left(\left((b^p)^{\frac{1}{q}}\right)^q\right)^n = \left((b^p)^{\frac{1}{q}}\right)^{nq}.$$

Thus $(b^m)^{\frac{1}{n}}, (b^p)^{\frac{1}{q}} \in (0, \infty)$ have the same nq-th power. Consequently, they are equal.

(b) Let $m, p \in \mathbb{Z}, n, q \in \mathbb{Z}^+, r = \frac{m}{n}, s = \frac{p}{q}$. Then

$$\begin{aligned}(b^{r+s})^{nq} &= \left(b^{\frac{m}{n}+\frac{p}{q}}\right)^{nq} = \left(b^{\frac{mq+pn}{nq}}\right)^{nq} = \left((b^{mq+pn})^{\frac{1}{nq}}\right)^{nq} = b^{mq+pn} \\ &= b^{mq}b^{pn} = \left((b^m)^{\frac{1}{n}}\right)^{nq}\left((b^p)^{\frac{1}{q}}\right)^{qn} = \left((b^m)^{\frac{1}{n}}(b^p)^{\frac{1}{q}}\right)^{nq} = (b^r b^s)^{nq}\end{aligned}$$

shows that $b^{r+s}, b^r b^s \in (0, \infty)$ have the same nq-th power. Thus they are equal.

(c) Let $m, p \in \mathbb{Z}, n, q \in \mathbb{Z}^+$ such that $\frac{m}{n} < \frac{p}{q}$. Then $mq < pn$, and

$$\left((b^m)^{\frac{1}{n}}\right)^{nq} = b^{mq} < b^{mq}b^{pn-mq} = b^{pn} = \left((b^p)^{\frac{1}{q}}\right)^{nq}.$$

If $(b^m)^{\frac{1}{n}} \geq (b^p)^{\frac{1}{q}}$, then $\left((b^m)^{\frac{1}{n}}\right)^{nq} \geq \left((b^p)^{\frac{1}{q}}\right)^{nq}$. Thus $(b^m)^{\frac{1}{n}} < (b^p)^{\frac{1}{q}}$.

Let $r \in \mathbb{Q}$. $b^r \in B(r)$ implies $b^r \leq \sup B(r)$. If $b^t \in B(r)$, then $b^t \leq b^r$. Hence, $\sup B(r) \leq b^r$.

(d) Let $x, y \in \mathbb{R}, b^s \in B(x), b^t \in B(y)$. Then $b^s b^t = b^{s+t} \in B(x+y)$ implies $b^s b^t \leq b^{x+y}$. Furthermore, $b^s b^y \leq b^{x+y}$, because $b^{-s}b^{x+y}$ is an upper bound of $B(y)$. Since $b^{x+y}(b^y)^{-1}$ is an upper bound of $B(x)$, we get

$$b^x b^y \leq b^{x+y}. \tag{3}$$

Now, $b^y = (b^{-y})^{-1}$ because $b^r \in B(y)$ iff $b^r = (b^{-r})^{-1} \leq (b^{-y})^{-1}$ for every $r \in \mathbb{Q}$. Substitute $x+y$ for x and $-y$ for y into (3) to get $b^{x+y}b^{-y} \leq b^x$. Thus

$$b^{x+y} = b^{x+y}b^{-y}(b^{-y})^{-1} \leq b^x(b^{-y})^{-1} = b^x b^y.$$

7. Fix $b > 1, y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b*.)

(a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

(b) Hence $b - 1 \geq n(b^{\frac{1}{n}} - 1)$.

(c) If $t > 1$ and $n > \frac{b-1}{t-1}$, then $b^{\frac{1}{n}} < t$.

(d) If w is such that $b^w < y$, then $b^{w+\frac{1}{n}} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$.

(e) If $b^w > y$, then $b^{w-\frac{1}{n}} > y$ for sufficiently large n.

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(g) Prove that this x is unique.

Solution:

(a) Since $b > 1$,

$$b^n - 1 = (b-1)(b^{n-1} + b^{n-2} + \ldots + b + 1) \geq n(b-1). \tag{4}$$

(b) $b^{\frac{1}{n}} > 1$ because $\frac{1}{n} > 0$. Thus

$$b - 1 = \left(b^{\frac{1}{n}}\right)^n - 1 \geq n(b^{\frac{1}{n}} - 1)$$

by (4).

(c) Using the hypothesis,

$$n(t-1) > \left(\frac{b-1}{t-1}\right)(t-1) = b - 1 \geq n(b^{\frac{1}{n}} - 1). \tag{5}$$

Divide both sides of (5) by n and add 1 to show $b^{\frac{1}{n}} < t$.

(d) If $n > \frac{b-1}{yb^{-w}-1}$, then $b^{\frac{1}{n}} < yb^{-w}$ by (c).

(e) If $n > \frac{b-1}{b^w y^{-1}-1}$, then $b^{\frac{1}{n}} < b^w y^{-1}$ by (c).

(f) If $b^x < y$, then for sufficiently large n, we have $b^{x+\frac{1}{n}} < y$ by part (d). In this case A has an element, $x + \frac{1}{n}$, larger than x. If $b^x > y$, then for sufficiently large n, we have $b^{x-\frac{1}{n}} > y$ by part (e). In this case A has an upper bound, $x - \frac{1}{n}$, that is less than x. Either case contradicts our choice of x. Thus we have $b^x = y$.

(g) Let $r \in \mathbb{R}$ such that $b^r = 1$. If $r > 0$, let $q \in \mathbb{Q}$ such that $0 < q < r$. Then $1 = b^0 < b^q < b^r$. If $r < 0$, let $q \in \mathbb{Q}$ such that $0 > q > r$. Then $1 = b^0 > b^q > b^r$. Thus $r = 0$.

Let $u \in \mathbb{R}$ such that $b^u = y$. Then $b^{u-x} = b^u b^{-x} = yy^{-1} = 1$. Thus $u - x = 0$, hence $u = x$.

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

Solution: Suppose that $\mathbb{C}$ can be made into an ordered field under the ordering $>$. Since $i \neq 0$, $-1 = i^2 > 0$ and $1 > 0$. Thus
$$0 < 1 = 1 + 0 < 1 + -1 = 0,$$
a contradiction.

9. Suppose $z = a + bi, w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

Solution: Let $x, y, z \in \mathbb{C}$. If $x \not< y$ and $x \not> y$, then $\Re(x) = \Re(y)$ and $\Im(x) = \Im(y)$, hence $x = y$.

Let $x < y < z$. Then $\Re(x) \leq \Re(y) \leq \Re(z)$. If $\Re(x) = \Re(y) = \Re(z)$, then $\Im(x) < \Im(y) < \Im(z)$, hence $x < z$. If $\Re(x) < \Re(y)$ or $\Re(y) < \Re(z)$, then $\Re(x) < \Re(z)$. Thus $x < z$.

Let $Y = i\mathbb{R}$. Observe that 1 is an upper bound of Y, $0 \in Y \neq \varnothing$, and Y does not contain an upper bound because $ir < i(r+1)$ for every $r \in \mathbb{R}$. Let $u + iv$ be an upper bound of Y. Then $u > 0$, and $\frac{u}{2} + iv$ is a strictly smaller upper bound of Y. Thus $\mathbb{C}$ does not have the least upper bound property when given the lexicographic order.

10. Suppose $z = a + ib, w = u + iv$, and
$$a = \left(\frac{|w| + u}{2}\right)^{\frac{1}{2}}, b = \left(\frac{|w| - u}{2}\right)^{\frac{1}{2}}.$$

Prove that $z^2 = w$ if $v \geq 0$ and $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number(with one exception!) has two complex square roots.

Solution:

$$
\begin{aligned}
(a+\operatorname{sgn}(v)ib)^2 &= (a^2-b^2)+\operatorname{sgn}(v)2iab \\
&= \left(\left(\left(\frac{|w|+u}{2}\right)^{\frac{1}{2}}\right)^2 - \left(\left(\frac{|w|-u}{2}\right)^{\frac{1}{2}}\right)^2\right) + \operatorname{sgn}(v)2i\left(\frac{|w|+u}{2}\right)^{\frac{1}{2}}\left(\frac{|w|-u}{2}\right)^{\frac{1}{2}} \\
&= \left(\frac{|w|+u}{2}-\frac{|w|-u}{2}\right)+\operatorname{sgn}(v)2i\left(\frac{(|w|+u)(|w|-u)}{4}\right)^{\frac{1}{2}} \\
&= u+\operatorname{sgn}(v)2i\left(\frac{|w|^2-u^2}{4}\right)^{\frac{1}{2}} = u+\operatorname{sgn}(v)2i\left(\frac{v^2}{4}\right)^{\frac{1}{2}} = u+\operatorname{sgn}(v)i(v^2)^{\frac{1}{2}} = u+iv.
\end{aligned}
$$

Let $\alpha,\beta\in\mathbb{C}$ such that $\alpha^2=\beta^2=w$. Then $(\alpha-\beta)(\alpha+\beta)=0$. Thus $\beta=\pm\alpha$. If $w\neq 0$, then $\alpha-(-\alpha)=2\alpha\neq 0$ and w has exactly two square roots. If $w=0$, then $\alpha=0=-\alpha$ is the only square root.

11. If z is a complex number, prove that there exists an $r\geq 0$ and a complex number w with $|w|=1$ such that $z=rw$. Are w and r always uniquely determined by z?

Solution: Let $r=|z|\geq 0$. If $z=0$, let $w=1$ so that $rw=0\cdot 1=0=z$. If $z\neq 0$, let $w=\frac{z}{|z|}$. Then

$$|w|=\left|\frac{z}{|z|}\right|=\frac{|z|}{|z|}=1 \text{ and } rw=|z|\frac{z}{|z|}=z.$$

Suppose $0\neq z=st$ with $s\geq 0$ and $|t|=1$. Then $\frac{r}{s}=\left|\frac{r}{s}\right|=\left|\frac{t}{w}\right|=1$ shows that $r=s$. Thus $w=\frac{z}{r}=\frac{z}{s}=t$. If $z=0$ then w may be any complex number with $|w|=1$.

12. If $z_1,\ldots,z_n$ are complex, prove that

$$|z_1+z_2+\cdots+z_n|\leq|z_1|+|z_2|+\cdots+|z_n|.$$

Solution: Theorem 1.33(e) yields $|z_1+z_2|\leq|z_1|+|z_2|$. Suppose the inequality holds for $n-1$ complex numbers. Then

$$\left|\sum_{i=1}^{n}z_i\right|=\left|\sum_{i=1}^{n-1}z_i+z_n\right|\leq\left|\sum_{i=1}^{n-1}z_i\right|+|z_n|\leq\sum_{i=1}^{n-1}|z_i|+|z_n|=\sum_{i=1}^{n}|z_i|.$$

Thus the inequality holds for all $n\in\mathbb{Z}^+$.

13. If x,y are complex, prove that

$$||x|-|y||\leq|x-y|.$$

Solution: From

$$|x|=|(x-y)+y|\leq|x-y|+|y|$$

we have

$$|x|-|y|\leq|x-y|.$$

By symmetry,

$$-(|x| - |y|) = |y| - |x| \leq |y - x| = |x - y|.$$

Having established

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

we have $||x| - |y|| \leq |x - y|$.

14. If z is a complex number such that $|z| = 1$, that is, such that $z\overline{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

Solution:

$$\begin{aligned} |1 + z|^2 + |1 - z|^2 &= (1 + z)(1 + \overline{z}) + (1 - z)(1 - \overline{z}) \\ &= 1 + 2\Re(z) + |z|^2 + 1 - 2\Re(z) + |z|^2 \\ &= 4 \end{aligned}$$

15. Under what conditions does equality hold in the Schwarz inequality?

Solution: The proof of theorem 1.35 shows that equality holds in the Schwarz inequality when $\sum |Ba_j - Cb_j|^2 = 0$. Thus $a_j = \frac{C}{B} b_j$ for every j. Conversely, if $a_j = cb_j$ for every j and a fixed $c \in \mathbb{C}$, then

$$\left| \sum_{j=1}^{n} a_j \overline{b}_j \right|^2 = \left| \sum_{j=1}^{n} cb_j \overline{b}_j \right|^2 = |c|^2 \left| \sum_{j=1}^{n} |b_j|^2 \right|^2 = \left| \sum_{j=1}^{n} |cb_j|^2 \right| \cdot \left| \sum_{j=1}^{n} |b_j|^2 \right| = \left| \sum_{j=1}^{n} |a_j|^2 \right| \cdot \left| \sum_{j=1}^{n} |b_j|^2 \right|.$$

16. Suppose $k \geq 3, \mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:
    (a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

    (b) If $2r = d$, there is exactly one such $\mathbf{z}$.
    (c) If $2r < d$, there is no such $\mathbf{z}$.
How must these statements be modified if k is 2 or 1?

Solution: Let $\mathbf{u} = \mathbf{x} - \frac{\mathbf{x}+\mathbf{y}}{2}$. Suppose $\mathbf{w} \in \mathbb{R}^k$ such that

$$|\mathbf{w} - \mathbf{u}| = |\mathbf{w} + \mathbf{u}| = r.$$

Then

$$\begin{aligned} |\mathbf{w} - \mathbf{u}| &= |\mathbf{w} + \mathbf{u}| \\ |\mathbf{w} - \mathbf{u}|^2 &= |\mathbf{w} + \mathbf{u}|^2 \\ |\mathbf{w}|^2 - 2\mathbf{w} \cdot \mathbf{u} + |\mathbf{u}|^2 &= |\mathbf{w}|^2 + 2\mathbf{w} \cdot \mathbf{u} + |\mathbf{u}|^2 \\ 0 &= \mathbf{w} \cdot \mathbf{u}. \end{aligned}$$

For the other condition, we take

$$r^2 = |\mathbf{w} - \mathbf{u}|^2 = |\mathbf{w}|^2 + |\mathbf{u}|^2$$

to get

$$|\mathbf{w}|^2 = r^2 - |\mathbf{u}|^2.$$

Since $2|\mathbf{u}| = d > 0$, we know that $\mathbf{u} \neq 0$.

- If there is $i \in \{1, \ldots, k\}$ such that $\mathbf{u}^i = 0$, let $\mathbf{p} = \mathbf{e}_i$.
- If there is also $i \neq j \in \{1, \ldots, k\}$ such that $\mathbf{u}^j = 0$, then let $\mathbf{q} = \mathbf{e}_j$.
- If $\mathbf{u}^j \neq 0$ for all $j \neq i$, let $j, l \in \{1, \ldots, k\} \setminus \{i\}, j \neq l$, and let $\mathbf{q} = \mathbf{u}^l \mathbf{e}_j - \mathbf{u}^j \mathbf{e}_l$.
- If $\mathbf{u}^j \neq 0$ for all $1 \leq j \leq k$, then let $\mathbf{p} = \mathbf{u}^2 \mathbf{e}_1 - \mathbf{u}^1 \mathbf{e}_2$ and $\mathbf{q} = \mathbf{u}^3 \mathbf{e}_1 - \mathbf{u}^1 \mathbf{e}_3$.

In each case, $\mathbf{a}_n = \mathbf{p} + n\mathbf{q} \neq 0$ satisfies $\mathbf{a}_n \cdot \mathbf{u} = 0$ for all $n \in \mathbb{Z}^+$. Furthermore, if $m, n \in \mathbb{Z}^+, m \neq n$, then $\mathbf{a}_n \neq s\mathbf{a}_m$ for all $s \in \mathbb{R}$.

(a) If $2r > d$, then $r^2 - |\mathbf{u}|^2 > 0$. For each $n \in \mathbb{Z}^+$, define

$$\mathbf{w}_n = \frac{\mathbf{a}_n}{|\mathbf{a}_n|} \sqrt{r^2 - |\mathbf{u}|^2}.$$

By construction, $\mathbf{w}_n \cdot \mathbf{u} = 0$ and $|\mathbf{w}_n|^2 = r^2 - |\mathbf{u}|^2$. Thus

$$|\mathbf{w}_n - \mathbf{u}| = |\mathbf{w}_n + \mathbf{u}| = r.$$

Now define $\mathbf{z}_n = \mathbf{w}_n + \frac{\mathbf{x}+\mathbf{y}}{2}$ for every $n \in \mathbb{Z}^+$. The $\mathbf{z}_n$ are distinct because the $\mathbf{w}_n$ are. Furthermore,

$$|\mathbf{z}_n - \mathbf{x}| = |\mathbf{z}_n - \mathbf{y}| = r \text{ for every } n \in \mathbb{Z}^+.$$

(b) If $2r = d$, then $|\mathbf{u}| = r$, and $|\mathbf{w}|^2 = 0$. Thus $\mathbf{z} = \frac{1}{2}(\mathbf{x} + \mathbf{y})$ is the only solution of

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

(c) If $2r < d$, then any solution of

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r$$

would provide $\mathbf{w} \in \mathbb{R}^k$ such that $0 \leq |\mathbf{w}|^2 < 0$.

If $k = 1$, then $0 = \mathbf{w} \cdot \mathbf{u}$ implies that $\mathbf{w} = 0$ and $\mathbf{z} = \frac{\mathbf{x}+\mathbf{y}}{2}$. There is a solution iff $2r = d$.

If $k = 2$, then $0 = \mathbf{w} \cdot \mathbf{u}$ implies that $\mathbf{w}$ is a multiple of $(-u_2, u_1)$. There are two multiples with the required norm if $2r > d$, and only one if $2r = d$.

17. Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

Solution:

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = \left(|\mathbf{x}|^2 + 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2\right) + \left(|\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2\right) = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

The sum of the squares on the diagonals of a parallelogram equals the sum of the squares on the sides.

18. If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq 0$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

Soluion: If $\mathbf{x}^1 = 0$, let $\mathbf{y} = \mathbf{e}_1$. If $\mathbf{x}^1 \neq 0$, then let $\mathbf{y} = \mathbf{x}^2\mathbf{e}_1 - \mathbf{x}^1\mathbf{e}_2$.

Let $k = 1$. If $\mathbf{x} = 0$ and $\mathbf{x} \cdot \mathbf{y} = 0$, then $\mathbf{y}$ can be any real number. If $\mathbf{x} \neq 0$ and $\mathbf{x} \cdot \mathbf{y} = 0$, then

$$\mathbf{y} = \mathbf{x}^{-1}\mathbf{x}\mathbf{y} = \mathbf{x}^{-1}0 = 0.$$

19. Suppose $\mathbf{a} \in \mathbb{R}^k, \mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

iff $|\mathbf{x} - \mathbf{c}| = r$.
(*Solution:* $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}, 3r = 2|\mathbf{b} - \mathbf{a}|$.)

Solution:

$$\begin{aligned}
|\mathbf{x} - \mathbf{a}| &= 2|\mathbf{x} - \mathbf{b}| \\
|\mathbf{x} - \mathbf{a}|^2 &= 4|\mathbf{x} - \mathbf{b}|^2 \\
|\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{a} + |\mathbf{a}|^2 &= 4|\mathbf{x}|^2 - 8\mathbf{x} \cdot \mathbf{b} + 4|\mathbf{b}|^2 \\
|\mathbf{a}|^2 - 4|\mathbf{b}|^2 &= 3|\mathbf{x}|^2 + 2\mathbf{x} \cdot (\mathbf{a} - 4\mathbf{b}) \\
\frac{|\mathbf{a}|^2 - 4|\mathbf{b}|^2}{3} + \frac{1}{9}|\mathbf{a} - 4\mathbf{b}|^2 &= |\mathbf{x}|^2 + \frac{2}{3}\mathbf{x} \cdot (\mathbf{a} - 4\mathbf{b}) + \frac{1}{9}|\mathbf{a} - 4\mathbf{b}|^2 \\
\frac{4}{9}|\mathbf{b} - \mathbf{a}|^2 &= \left|\mathbf{x} - \frac{1}{3}(4\mathbf{b} - \mathbf{a})\right|^2 \\
\frac{2}{3}|\mathbf{b} - \mathbf{a}| &= \left|\mathbf{x} - \frac{1}{3}(4\mathbf{b} - \mathbf{a})\right|
\end{aligned}$$

Let $\mathbf{c} = \frac{1}{3}(4\mathbf{b} - \mathbf{a})$ and $r = \frac{2}{3}|\mathbf{b} - \mathbf{a}|$.

20. With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.

Solution: Steps 1-3 work as in the appendix proof, omitting the statements related to (III). In step 4, define 0^* as the set of non-positive rational numbers. The proofs for (A1)-(A3) go through as stated.

Proof of (A4): If $r \in \alpha$ and $s \in 0^*$, then $r + s \leq r$, hence $r + s \in \alpha$. Thus $\alpha + 0^* \subseteq \alpha$. If $p \in \alpha + 0^*$, then $p = r + s$ with $r \in \alpha$ and $s \in 0^*$. Thus $p = r + s \leq r$, and we see that $p \in \alpha$. So $\alpha + 0^* \subseteq \alpha$.

Now we show that (A5) fails. Let β be the cut of negative rational numbers, and α any other cut. If α contains some $r > 0$, then $\frac{r}{2} = r + \left(-\frac{r}{2}\right) \in \alpha + \beta$ shows that $\alpha + \beta \neq 0^*$. If α has no positive elements, then $0 \notin \alpha + \beta$, and $\alpha + \beta \neq 0^*$.

2 Basic Topology

1. Prove that the empty set is a subset of every set.

   Solution: Every element of $\varnothing$ belongs to every set.

2. A complex number z is said to be *algebraic* if there are integers $a_0, \dots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

   Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots |a_n| = N.$$

   Solution: For every $p(X) = \sum_{i=0}^{n} a_i X^{n-i} \in \mathbb{C}[X]\,(a_0 \neq 0)$, let $\|p\| = n + \sum_{i=0}^{n} |a_i|$. For every algebraic number z, choose $q_z(X) \in \mathbb{Z}[X]$ such that $q_z(z) = 0$. The set of all algebraic numbers

$$\bigcup_{N \geq 0} \{z \mid \|q_z\| = N\}$$

   is a countable union of finite sets.

3. Prove that there exist real numbers which are not algebraic.

   Solution: Subsets of the algebraic numbers are countable. The real numbers are uncountable. Thus there are real numbers that are not algebraic.

4. Is the set of all irrational real numbers countable?

   Solution Since $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ is uncountable and $\mathbb{Q}$ is countable, $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.

5. Construct a bounded set of real numbers with exactly three limit points.

   Solution: Let $E_i = \{i + \frac{1}{n}\}_{n=2}^{\infty}$ for $i = -1, 0, 1$ and let $E = \bigcup_{i=-1,0,1} E_i$.

   Let $x \in E$. Then $-2 < x < 2$ shows that E is bounded.

   Let $i + \frac{1}{n} \in E, r \in \mathbb{R}$. If $i = r = -1, 0, 1$, then $\left|\left(i + \frac{1}{n}\right) - r\right| = \frac{1}{n} \to 0$ as $n \to \infty$. So $-1, 0, 1$ are limit points of E.

   Suppose that $r \neq -1, 0, 1$. Then

$$\left|\left(i + \frac{1}{n}\right) - r\right| \geq \left||i - r| - \frac{1}{n}\right| \geq \frac{|i - r|}{2}$$

for all $n > N = \lfloor \max\{\frac{2}{|i-r|} \mid i = -1, 0, 1\} \rfloor$. Thus

$$S_r = E \cap (r - N^{-1}, r + N^{-1}) \setminus \{r\}$$

is finite. Define $\rho = \min_{x \in S_r} d(x, r)$. Thus $E \cap (r-\rho, r+\rho) \setminus \{r\}$ is empty, and r is not a limit point of E.

6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?

Solution: Let $x \notin E'$. Then there is $r > 0$ such that $\mathrm{B}(x, r) \cap E = \varnothing$. Let $y \in \mathrm{B}(x, r), z \in E, \rho_y = r - d(x, y)$. Then

$$d(y, z) \geq d(x, z) - d(y, x) \geq \rho_y > 0$$

implies $\mathrm{B}(y, \rho) \cap E = \varnothing$. Thus $y \notin E'$. Since y was arbitrary, $\mathrm{B}(x, r) \cap E' = \varnothing$, and x is not a limit point of E'. Thus E' is closed.

If $x \in E'$, then for all $r > 0$, $\varnothing \neq \mathrm{B}(x, r) \cap E \subseteq \mathrm{B}(x, r) \cap \overline{E}$ implies $x \in \overline{E}'$. Suppose $x \in \overline{E}'$ and let $r > 0$. By hypothesis, there is $y \in B\left(x, \frac{r}{2}\right) \cap \overline{E}$. If $y \notin E$, then there is $z \in B\left(y, \frac{r}{2}\right) \cap E$. Thus

$$d(x, z) \leq d(x, y) + d(y, z) < r,$$

hence $z \in \mathrm{B}(x, r) \cap E$, and $x \in E'$.

Let E be the set $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$, then $E' = \{0\}$, and E'' is empty.

7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.
   (a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B}_n = \bigcup_{i=1}^n \overline{A}_i$, for $n = 1, 2, 3, \ldots$.
   (b) If $B = \bigcup_{i=1}^\infty A_i$, prove that $\overline{B}_n \supseteq \bigcup_{i=1}^\infty \overline{A}_i$.

   Show, by an example, that this inclusion can be proper.

Solution:

(a) $x \notin \overline{B}_n$ iff there is $r > 0$ such that

$$\varnothing = \mathrm{B}(x, r) \cap B_n = \mathrm{B}(x, r) \cap \left(\bigcup_{i=1}^n A_i \right) = \bigcup_{i=1}^n (\mathrm{B}(x, r) \cap A_i)$$

iff for each $1 \leq i \leq n$ there is an $r_i > 0$ such that $\mathrm{B}(x, r_i) \cap A_i = \varnothing$ (let $r = \min_{1 \leq i \leq n} r_i$) iff $x \notin \overline{A}_i$ for each $1 \leq i \leq n$ iff $x \notin \bigcup_{i=1}^n \overline{A}_i$.

(b) If $x \notin \overline{B}$, then there is $r > 0$ such that $\varnothing = \mathrm{B}(x, r) \cap B = \bigcup_{i=1}^\infty (\mathrm{B}(x, r) \cap A_i)$. Thus $\mathrm{B}(x, r) \cap A_i = \varnothing$ for all $i \geq 1$, and $x \notin \bigcup_{i=1}^\infty \overline{A}_i$. For a counterexample to the opposite inclusion, let the metric space be $[0, 1]$ and define $A_i = \left\{\frac{1}{i}\right\}$. Thus $0 \in \overline{B} \setminus \bigcup_{i=1}^\infty \overline{A}_i$.

8. Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Solution: Let $\mathbf{x} \in E^{\text{open}} \subseteq \mathbb{R}^2$ and let $r > 0$ be such that $\mathrm{B}(\mathbf{x}, r) \subseteq E$. For each $r > s > 0$, let $\mathbf{y}_s$ be a vector with $0 < |\mathbf{y}_s| < s$ (e.g. $\mathbf{y}_s = \frac{s}{2}\mathbf{e}_1$). Then $\mathbf{x} \neq \mathbf{x} + \mathbf{y}_s \in \mathrm{B}(\mathbf{x}, s) \cap E$ shows that $\mathbf{x} \in E'$. If $E = \varnothing$, then $E = E'$.

Let $E = \{\mathbf{0}\}$ and $\mathbf{x} \notin E$. Then $\mathrm{B}(\mathbf{x}, |\mathbf{x}|) \cap E = \varnothing$ demonstrates that E is closed. Furthermore, $\mathbf{0} \in E \setminus E'$.

9. Let E° denote the set of all interior points of a set E. [See Definition 2.18(e); E° is called the *interior* of E.]
   (a) Prove that E° is always open.
   (b) Prove that E is open iff $E^\circ = E$.
   (c) If $G \subseteq E$ and G is open, prove that $G \subseteq E^\circ$.
   (d) Prove that the complement of E° is the closure of the complement of E.
   (e) Do E and $\overline{E}$ always have the same interiors?
   (f) Do E and E° always have the same closures?

Solution:

(a) If $x \in E^\circ$, there is $r > 0$ such that $\mathrm{B}(x, r) \subseteq E$. If $y \in \mathrm{B}\left(x, \frac{r}{2}\right)$, then $\mathrm{B}\left(y, \frac{r}{2}\right) \subseteq \mathrm{B}(x, r) \subseteq E$ shows that $y \in E^\circ$. Thus $\mathrm{B}\left(x, \frac{r}{2}\right) \subseteq E^\circ$. Since x was arbitrary, E° is open.

(b) By (a), E is open iff $E \subseteq E^\circ$.

(c) Let $x \in G^{\text{open}} \subseteq E$. Then there is $r > 0$ such that $\mathrm{B}(x, r) \subseteq G \subseteq E$. Thus $x \in E^\circ$.

(d) $E^c \subseteq \overline{E^c}$ implies $E \supseteq \left(\overline{E^c}\right)^c$. Thus $\left(\overline{E^c}\right)^c \subseteq E^\circ$.

$E^\circ \subseteq E$ implies $(E^\circ)^c \supseteq E^c$. Thus $\overline{E^c} \subseteq (E^\circ)^c$.

(e) No. Let $E = \mathbb{Q}$ in $\mathbb{R}$. Then $E^\circ = \varnothing$ and $\left(\overline{E}\right)^\circ = \mathbb{R}$.

(f) No. Let $E = \mathbb{Q}$ in $\mathbb{R}$. Then $\overline{E} = \mathbb{R}$ and $\overline{E^\circ} = \varnothing$.

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Solution: The first two properties of a metric hold trivially. Let $p, q, r \in X$. If $d(p, r) + d(r, q) = 0$, then $p = r = q$ and $d(p, q) = 0$. If $d(p, r) + d(r, q) \geq 1$, then $d(p, q) \leq 1 \leq d(p, r) + d(r, q)$.

Singleton sets are open because $\mathrm{B}\left(p, \frac{1}{2}\right) = \{p\}$. Every subset of X is open because arbitrary unions of open sets are open. Hence every subset of X is closed.

Let $\varnothing \neq K^{compact} \subseteq X$. Then $U_q = \{q\}$ with $q \in K$ is an open cover of K. Since K is compact, there is a finite subcover $U_{q_1}, \ldots, U_{q_n}$. Thus $K \subseteq \bigcup_{i=1}^{n} U_{q_i} = \{q_1, q_2, \ldots, q_n\} \subseteq K$, and K is finite.

11. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \\ d_3(x,y) &= |x^2 - y^2|, \\ d_4(x,y) &= |x - 2y|, \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Solution:

$d_1(x,y)$ is not a metric because

$$d_1(1,-1) = 4 > 2 = d_1(1,0) + d_1(0,-1).$$

$d_2(x,y)$ is a metric. The first two properties of a metric hold trivially. By the triangle inequality,

$$\left(\sqrt{|x-y|}\right)^2 = |x-y| \leq |x-z|+|z-y| \leq |x-z|+2\sqrt{|x-z|}\sqrt{|z-y|}+|z-y| = \left(\sqrt{|x-z|}+\sqrt{|z-y|}\right)^2.$$

Taking square roots yields

$$\sqrt{|x-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|}.$$

$d_3(x,y)$ is not a metric because $d_3(1,-1) = 0$, but $1 \neq -1$.

$d_4(x,y)$ is not a metric because $d_4(1,1) = 1$, but $1 = 1$.

$d_5(x,y)$ is a metric. The first two properties of a metric hold trivially. Let $0 < s < r$. Then $0 < \frac{1}{r} < \frac{1}{s}$. Hence $0 < \frac{1}{r} + 1 < \frac{1}{s} + 1$. Thus

$$0 < \frac{s}{1+s} = \frac{1}{\frac{1}{s}+1} < \frac{1}{\frac{1}{r}+1} = \frac{r}{1+r}. \tag{6}$$

By the triangle inequality and (6),

$$\begin{aligned} \frac{|x-y|}{1+|x-y|} &\leq \frac{|x-z|+|z-y|}{1+|x-z|+|z-y|} \\ &= \frac{|x-z|}{1+|x-z|+|z-y|} + \frac{|z-y|}{1+|x-z|+|z-y|} \\ &\leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|}. \end{aligned}$$

12. Let $K \subseteq \mathbb{R}^1$ consist of 0 and the numbers $\frac{1}{n}$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Solution: Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of K. Let α_0 be such that $0 \in U_{\alpha_0}$. There is $\varepsilon > 0$ such that $\mathrm{B}(0, \varepsilon) \subseteq U_{\alpha_0}$. Let $N = \lfloor \frac{1}{\varepsilon} \rfloor + 1$. Then $n > N$ implies $\frac{1}{n} \in U_{\alpha_0}$. For $1 \leq n \leq N$, there are α_n such that $\frac{1}{n} \in U_{\alpha_n}$. Thus $K \subseteq \bigcup_{n=0}^{N} U_{\alpha_n}$.

13. Construct a compact set of real numbers whose limit points form a countable set.

Solution: Let K be the set consisting of 0, numbers of the form $\frac{1}{n}$, and numbers of the form $\left(\frac{1}{mn} + 1\right)\frac{1}{n}$, where m, n are positive integers and $n \geq 2$. Notice that every element of K is non-negative and that

$$0 < \frac{1}{n} < \left(\frac{1}{mn} + 1\right)\frac{1}{n} \leq \left(\frac{1}{1 \cdot 2} + 1\right)\frac{1}{2} = \frac{3}{4} < 1.$$

Furthermore, $(n+1)(n-1) = n^2 - 1 < n^2$ shows that

$$\frac{1}{n} < \left(\frac{1}{mn} + 1\right)\frac{1}{n} \leq \left(\frac{1}{n} + 1\right)\frac{1}{n} = \frac{n+1}{n^2} < \frac{1}{n-1}.$$

Let r be a real number not in K. If $r < 0$, then r is not a limit point of K because $\mathrm{B}(r, |r|) \cap K = \varnothing$. If $r > \frac{3}{4}$, then r is not a limit point of K because $\mathrm{B}\left(r, \left|r - \frac{3}{4}\right|\right) \cap K = \varnothing$.

If $0 < r < \frac{3}{4}$, let $N = \lfloor \frac{1}{r} \rfloor + 1 \geq 2$. So we have $\frac{1}{N} < r < \frac{1}{N-1}$. Now define $M = \lfloor \frac{1}{N(Nr-1)} \rfloor + 1 \geq 1$. We get the following sequence of inequalities (disregard the division by zero if $M = 1$):

$$\begin{aligned} M - 1 \leq & \frac{1}{N(Nr-1)} < M \\ (M-1)N \leq & \frac{1}{Nr-1} < MN \\ \frac{1}{MN} < & Nr - 1 \leq \frac{1}{(M-1)N} \\ \left(\frac{1}{MN} + 1\right)\frac{1}{N} < & r < \left(\frac{1}{(M-1)N} + 1\right)\frac{1}{N}, \end{aligned}$$

where the weak inequality is strict in the last line because $r \notin K$. If $M = 1$, define

$$\varepsilon = \min\left\{\left|r - \left(\frac{1}{MN} + 1\right)\frac{1}{N}\right|, \left|r - \frac{1}{N-1}\right|\right\},$$

if $M > 1$, define

$$\varepsilon = \min\left\{\left|r - \left(\frac{1}{MN} + 1\right)\frac{1}{N}\right|, \left|r - \left(\frac{1}{(M-1)N} + 1\right)\frac{1}{N}\right|\right\}.$$

In either case, $\mathrm{B}(r, \varepsilon) \cap K = \varnothing$, and we conclude that no real numbers outside of K are limit points of K.

We have shown that K contains all of its limit points and that K is bounded. Thus K is compact. The following will show that K has countably many limit points.

Let $\varepsilon > 0$. If $n > \frac{1}{\varepsilon}$, then $\frac{1}{n} \in \mathrm{B}(0, \varepsilon)$. So 0 is a limit point of K. Let $n \geq 2$. If $m > \frac{1}{n\varepsilon}$, then $\left(\frac{1}{mn} + 1\right) \frac{1}{n} \in \mathrm{B}\left(\frac{1}{n}, \varepsilon\right)$. So $\frac{1}{n}$ is a limit point of K for all $n \geq 2$.

If $m \geq 1$, then the inequalities

$$\left(\frac{1}{(m+1)n} + 1\right) \frac{1}{n} < \left(\frac{1}{mn} + 1\right) \frac{1}{n} < \frac{1}{n-1}$$

demonstrate that there is a ball around $\left(\frac{1}{mn} + 1\right) \frac{1}{n}$ which contains no other points of K. Thus $\left(\frac{1}{mn} + 1\right) \frac{1}{n}$ is not a limit point of K.

Consequently, the limit points of K are 0 and the reciprocals of the positive integers greater than 1. We have given a compact set of real numbers with a countable number of limit points.

14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Solution: Let $x \in (0, 1), n > \frac{1}{x}$. Then $x \in \left(\frac{1}{n}, 1\right)$. Thus $\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1\right) \subseteq (0, 1) \subseteq \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1\right)$, and $\left\{\left(\frac{1}{n}, 1\right)\right\}_{n \geq 1}$ is an open cover of $(0, 1)$.

Given of a finite collection $\left(\frac{1}{n_k}, 1\right)$ with $k = 1, 2, \ldots, N$, let $n_0 = \max\{n_k \mid 1 \leq k \leq N\}$. Then $\frac{1}{n_0+1} \in (0, 1) \setminus \left(\frac{1}{n_0}, 1\right) = (0, 1) \setminus \left(\bigcup_{1 \leq k \leq N} \left(\frac{1}{n_k}, 1\right)\right)$.

15. Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}^1$, for example) if the word "compact" is replaced by "closed" or by "bounded."

Solution:

Closed: Let $K_\alpha = [\alpha, \infty)$ for all $\alpha \in \mathbb{R}$. Let $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ be any real numbers. Then $\alpha_n \in [\alpha_n, \infty) = \bigcap_{i=1}^{n} K_{\alpha_i}$. However, $\left(\bigcap_{\alpha \in \mathbb{R}} K_\alpha\right)^c = \bigcup_{\alpha \in \mathbb{R}} (-\infty, \alpha) = \mathbb{R}$ yields $\bigcap_{\alpha \in \mathbb{R}} K_\alpha = \varnothing$. Replace $\mathbb{R}$ by $\mathbb{Z}^+$ for a counterexample to the Corollary.

Bounded: Let $U_n = \left(0, \frac{1}{n}\right)$ for all $n \in \mathbb{Z}^+$. Given a finite subcollection $U_{n_1}, \ldots, U_{n_k}$, let $n_0 = \max\{n_i \mid 1 \leq i \leq k\}$. Then $\frac{1}{n_0+1} \in \left(0, \frac{1}{n_0}\right) = \bigcap_{i=1}^{k} U_{n_i}$. However, if $1 > r > 0$, let $N = \lfloor \frac{1}{r} \rfloor + 1$. Then $r \in (U_N)^c \subseteq \left(\bigcap_{n=1}^{\infty} U_n\right)^c$. Since r was an arbitrary, $\bigcap_{n=1}^{\infty} U_n = \varnothing$.

16. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

Solution: Let $p, q \in E$. Then

$$|p - q|^2 \leq (|p| + |q|) = |p|^2 + 2|pq| + |q|^2 \leq 4 \cdot 3 = 12.$$

Thus E is bounded.

E is both open and closed by theorem 2.30 and $E = \mathbb{Q} \cap [\sqrt{2}, \sqrt{3}] = \mathbb{Q} \cap (\sqrt{2}, \sqrt{3})$.

E is not closed in $\mathbb{R}$, hence not compact in $\mathbb{R}$. Thus E is not compact.

17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

Solution: The elements of E are in one to one correspondence with the sequences of 0's and 1's by matching 4's with 0's and 7's with 1's. Thus E is uncountable.

Let $r \in [0, 1] \setminus E$ have decimal expansion $0.d_1 d_2 d_3 \ldots$, and let $n \in \mathbb{Z}^+$ be least such that $d_n \neq 4, 7$. If there is $N \geq 1$ such that $d_k = 0$ for all $k \geq N$, then $\left(r - 10^{-(1+N)}, r + 10^{-(1+N)}\right) \cap E = \varnothing$. Otherwise, $(10^{-n} \lfloor 10^n r \rfloor, 10^{-n}(\lfloor 10^n r \rfloor + 1)) \cap E = \varnothing$. Thus E is closed and not dense.

Since E is closed and bounded, E is compact.

Let $x \in E$, $1 > \varepsilon > 0$, and $N = \lfloor \log_{10} \left(\frac{1}{\varepsilon}\right) \rfloor + 1$. Let y be the number obtained from x by swapping its $N + 1$th digit from 4 to 7, or visa versa. Then

$$|x - y| \leq \frac{3}{10^{-(N+1)}} < 10^{-N} < \varepsilon.$$

Hence $y \in \mathrm{B}(x, \varepsilon) \cap E$, and E is perfect.

18. Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?

Solution: Let $a, b \in \mathbb{R} \setminus \mathbb{Q}$, and let $\{q_n\}_{n \geq 1}$ be an enumeration of the rationals in (a, b). Let $a_1, b_1 \in (a, b) \setminus \mathbb{Q}$ such that $q_1 \in (a_1, b_1)$. For each $k > 1$, let n_k be least such that $q_{n_k} \notin \bigcup_{j=1}^{k-1} (a_j, b_j)$, and find $a_k, b_k \in (a, b) \setminus \mathbb{Q}$ such that $q_{n_k} \in (a_k, b_k)$, and $a_j, b_j \notin [a_k, b_k]$ for $j < k$. Let $E_0 = [a, b]$, $E_k = E_{k-1} \setminus (a_k, b_k)$ for $k \geq 1$, and $E = \bigcap_{k \geq 0} E_k$.

By construction, E is closed, $E \cap \mathbb{Q} = \varnothing$, and E is not empty because it contains a and b. Furthermore, each of the sets (a_k, b_k) are disjoint by construction, so $a_k, b_k \in E$ for all $k \geq 1$.

Let $p \in E, \varepsilon > 0$. Then $p \in E_k$ for all $k \geq 1$. Each E_k is a disjoint union of finitely many closed intervals, none of which are singletons. Let k be large enough that the interval $I_k(p)$ of E_k containing p is contained in $\mathrm{B}(p, \varepsilon)$. Let j be least such that $q_{n_j} \in I_k(p)$. By construction of E, $[a_j, b_j] \subsetneq I_k(p) \subseteq \mathrm{B}(p, \varepsilon)$. Thus, $a_j, b_j \in \mathrm{B}(p, \varepsilon) \cap E$ shows that $\mathrm{B}(p, \varepsilon) \cap E$ contains at least one point besides p. Since p was arbitrary, every element of E is a limit point and E is a non-empty perfect set containing no rationals.

19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

(b) Prove the same for disjoint open sets.

(c)Fix $p \in X, \delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

(d) Prove that every connected metric space with at least two points is uncountable. *Hint:* Use (c).

Solution:

(a) By hypothesis,

$$\overline{A} \cap B = A \cap \overline{B} = A \cap B = \varnothing.$$

(b) Let A, B be open with $A \cap B = \varnothing$. Then

$$\overline{A} \cap B \subseteq B^c \cap B = \varnothing \text{ and } A \cap \overline{B} \subseteq A \cap A^c = \varnothing.$$

(c) By construction, A is open. Let $q \in B$. If $r \in \mathrm{B}(q, d(p,q) - \delta)$, then

$$d(p,r) \geq d(p,q) - d(q,r) > \delta$$

implies $r \in B$. Thus B is open. Since A, B are open and disjoint, they are separated by (b).

(d) Let X be a connected metric space with at least two points p, q, and let A, B be as in (c). If $0 < \delta < d(p,q)$, then $A \neq \varnothing \neq B$, and there is $r_\delta \in X$ such that $d(p, r_\delta) = \delta$ by (c). Thus there is a family of distinct points in X indexed by $[0, d(p,q)]$, which is uncountable.

20. Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)

Solution: Let $E \subseteq X$ such that $\overline{E}$ is not connected. Then $\overline{E} = A \cup B$ with $A, B \neq \varnothing$, separated. Thus

$$\begin{gathered} E = (A \cap E) \cup (B \cap E), \\ \overline{A \cap E} \cap (B \cap E) \subseteq \overline{A} \cap B = \varnothing, \text{ and} \\ (A \cap E) \cap \overline{B \cap E} \subseteq A \cap \overline{B} = \varnothing. \end{gathered}$$

Hence E is not connected.

Define $C = \overline{\mathrm{B}(-1,1)} \cup \overline{\mathrm{B}(1,1)} \subseteq \mathbb{C}$. Exercise 2.21c demonstrates that $\overline{\mathrm{B}(-1,1)}$ and $\overline{\mathrm{B}(1,1)}$ are connected. Let $C = A \cup B$ with $A \cap B = \varnothing, A \neq \varnothing \neq B$. WLOG, $0 \in A$. Let $x \in B$. Then $0, x \in \overline{\mathrm{B}(\operatorname{sgn} \Re(x), 1)}$ implies $\overline{A} \cap B \neq \varnothing$ or $A \cap \overline{B} \neq \varnothing$. However, $C^\circ = \mathrm{B}(-1,1) \cup \mathrm{B}(1,1)$ is not connected.

21. Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\mathbf{a} \in A, \mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}^1$. Put $A_0 = \mathbf{p}^{-1}(A), B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ iff $\mathbf{p}(t) \in A$.]

(a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}^1$.

(b) Prove that there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

(c) Prove that every convex subset of $\mathbb{R}^k$ is connected.

Solution:

(a) WLOG, suppose $\overline{A_0} \cap B_0 \neq \varnothing$. Let $s \in \overline{A_0} \cap B_0$ and $\varepsilon > 0$. There is $r \in \mathrm{B}(s, \varepsilon) \cap A_0$. Thus

$$|\mathbf{p}(r) - \mathbf{p}(s)| = |(\mathbf{b} - \mathbf{a})(r - s)| < |\mathbf{b} - \mathbf{a}|\, \varepsilon$$

with $\mathbf{p}(r) \in A, \mathbf{p}(s) \in B$. Hence $\mathbf{p}(s) \in \overline{A} \cap B$, contrary to hypothesis.

(b) Since $(0,1)$ is connected, $(0,1) \nsubseteq A_0 \cup B_0$. Let $t_0 \in (0,1) \setminus (A_0 \cup B_0)$. Then $\mathbf{p}(t_0) \notin A \cup B$.

(c) Let $A, B \subseteq \mathbb{R}^k$ be separated and nonempty, $\mathbf{a} \in A, \mathbf{b} \in B$. Then (b) shows that there is $\lambda \in (0,1)$ such that

$$(1 - \lambda)\mathbf{a} + \lambda \mathbf{b} \notin A \cup B.$$

Thus $A \cup B$ is not convex. Take the contrapositive.

22. A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable. *Hint:* Consider the set of points which have only rational coordinates.

Solution: Let $\mathbf{r} \in \mathbb{R}^k, \varepsilon > 0$. For each $i = 1, 2, \ldots, k$, let $q_i \in \mathbb{Q}$ such that $|r_i - q_i| < \frac{\varepsilon}{\sqrt{k}}$, and let $\mathbf{q} = (q_1, q_2, \ldots, q_k) \in \mathbb{Q}^k$. Then

$$|\mathbf{r} - \mathbf{q}| = \sqrt{\sum_{i=1}^{k} (r_i - q_i)^2} < \sqrt{k \cdot \frac{\varepsilon^2}{k}} < \varepsilon.$$

Thus $\mathbf{q} \in \mathrm{B}(\mathbf{r}, \varepsilon)$.

23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have $x \in V_\alpha \subseteq G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.
Prove that every separable metric space has a *countable* base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.

Solution: Let D be a countable dense subset of the metric space X. The collection $\{\mathrm{B}(d, q) \mid d \in D, q \in \mathbb{Q}\}$ is countable. Let $x \in G^{open} \subseteq X$. Let $\varepsilon > 0$ such that $\mathrm{B}(x, \varepsilon) \subseteq G$, $\rho \in \mathbb{Q}$ such that $0 < \rho < \frac{\varepsilon}{2}$, and $\delta \in D \cap \mathrm{B}(x, \rho)$. If $y \in \mathrm{B}(\delta, \rho)$, then

$$d(x, y) \leq d(x, \delta) + d(\delta, y) < 2\rho < \varepsilon.$$

Thus $x \in \mathrm{B}(\delta, \rho) \subseteq \mathrm{B}(x, \varepsilon) \subseteq G$.

24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take

$\delta = \frac{1}{n}(n = 1, 2, 3, \ldots)$, and consider the centers of the corresponding neighborhoods.

Solution: Let $\delta > 0, x_1 \in X$. Suppose that for every $j \in \mathbb{Z}^+$ there is $x_{j+1} \in X$ such that $d(x_i, x_{j+1}) \geq \delta$ for $1 \leq i \leq j$. Let $x \in X$ be a limit point of $\{x_j\}_{j\in\mathbb{Z}^+}$. Then there is $N \in \mathbb{Z}^+$ such that $d(x, x_j) < \frac{\delta}{2}$ for all $j \geq N$. If $i, j \geq N$,

$$d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) < \delta,$$

a contradiction.

Consequently, there are $x_1, \ldots, x_N$ such that $y \in X$ implies $d(y, x_j) < \delta$ for some $j \in \{1, \ldots, N\}$. For $k \in \mathbb{Z}^+$, let $N_k \in \mathbb{Z}^+$ such that $\{\mathrm{B}\left(x_j^k, \frac{1}{k}\right)\}_{j=1}^{N_k}$ covers X. Let $\varepsilon > 0, y \in X, k > \frac{2}{\varepsilon}$. There is $j \in \{1, \ldots, N_k\}$ such that $y \in \mathrm{B}\left(x_j^k, \frac{1}{k}\right)$. Hence, $x_j^k \in \mathrm{B}(y, \varepsilon)$.

25. Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint:* For every positive integer n, there are finitely many neighborhoods of radius $\frac{1}{n}$ whose union covers K.

Solution: For each $n \in \mathbb{Z}^+$, let $x_1^n, x_2^n, \ldots, x_{N_n}^n \in K$ such that $K \subseteq \bigcup_{i=1}^{N_n} \mathrm{B}\left(x_i^n, \frac{1}{n}\right)$. Let

$$x \in G^{open} \subseteq K,\ \varepsilon > 0,\ \mathrm{B}(x, \varepsilon) \subseteq G.$$

If $n > \frac{2}{\varepsilon}$, then

$$x \in \mathrm{B}\left(x_i^n, \frac{1}{n}\right) \subseteq \mathrm{B}(x, \varepsilon) \subseteq G$$

for some $1 \leq i \leq N_n$. Thus $\left\{\mathrm{B}\left(x_i^n, \frac{1}{n}\right) \mid 1 \leq i \leq N_n, n \in \mathbb{Z}^+\right\}$ is a countable base. K is separable because $x_i^n \in G$.

26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint:* By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}, n = 1, 2, 3, \ldots$. If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \cup G \cdots \cup G_n$ is nonempty for each n, but $\cap F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E, and obtain a contradiction.

Solution: Let $\{U_k\}_{k\in\mathbb{N}}$ be a countable base and G_α be an open cover for X. For each $x \in X$, let $\beta(x)$ be an index such that $x \in G_{\beta(x)}$, define $\kappa(x) = \min\{k \in \mathbb{N} \mid x \in U_k \in G_{\alpha(x)}\}$, and let $I \subseteq \mathbb{N}$ be the image of κ. By construction, $\{U_k\}_{k\in I}$ is an open cover of X, and for each $k \in I$ there is $\gamma(k)$ such that $U_k \subseteq G_{\gamma(k)}$. Thus, $\{G_{\gamma(k)}\}_{k\in I}$ is a countable subcover of G_α.

Using the countability of I, reindex $\{G_{\gamma(k)}\}_{k\in I}$ as $\{G_k\}_{k\in\mathbb{N}}$, and define $F_n = \left(\bigcup_{k=1}^n G_k\right)^c$. Suppose that $F_n \neq \varnothing$ for all $n \in \mathbb{N}$. Let $x_n \in F_n$ be a sequence in X. It is non-constant because $\{G_k\}_{k\in\mathbb{N}}$ covers X. Let x be a limit point of $\{x_n\}_{n\in\mathbb{N}}$. Then for every $n \in \mathbb{N}$, $x \in F_n$ because F_n is closed. Thus $x \in \bigcap_{n\in\mathbb{N}} F_n = \left(\bigcup_{k\in\mathbb{N}} G_k\right)^c = \varnothing$, a contradiction. Consequently, there is $n \in \mathbb{N}$ such that $X = \varnothing^c = F_n^c = \bigcup_{k=1}^n G_k$.

27. Define a point p in a metric space X to be a *condensation point* of a set $E \subseteq X$ if every neighborhood of p contains uncountably many points of E.
Suppose $E \subseteq \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. *Hint:* Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$, let W be a union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Solution: Let E be an uncountable set in a separable metric space X, $\{V_n\}_{n\in\mathbb{N}}$ be a countable base for X, $I = \{n \in \mathbb{N} \mid E \cap V_n \text{ is at most countable}\}$, $W = \bigcup_{n\in I} V_n$, and P be the set of condensation points of E. First, $E \cap W = \bigcup_{n\in I} E \cap V_n$ is countable because it is a countable union of at most countable sets. Second, if $x \in P$, then $E \cap V_n$ is uncountable for every V_n containing x. Hence, $x \notin W$. Third, let $x \notin W$, $x \in G^{open}$, and $x \in V_n \subseteq G$. Then $E \cap V_n$ is uncountable. Thus $E \cap G \supseteq E \cap V_n$ is uncountable, and $x \in P$. Thus $P = W^c$ is closed. Finally, let $x \in P$. If G is an open neighborhood of x, then $G \cap E$ is uncountable and at most countably many elements of $G \cap E$ are in W. Thus P is perfect, $E \setminus P$ is at most countable, and $E \subseteq P \cup (E \setminus P) \subseteq \overline{E}$.

28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary:* Every countable closed set in $\mathbb{R}^k$ has isolated points.) *Hint:* Use Exercise 27.

Solution: Let E be closed. If E is at most countable, then $E = \varnothing \cup E$ is the desired union. If E is uncountable, the solution to exercise 27 yields $E = P \cup (E \setminus P)$.

If $E \subseteq \mathbb{R}^k$ is a closed, countable set, then E is not perfect. Hence $E \setminus E' \neq \varnothing$.

29. Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments. *Hint:* Use Exercise 22.

Solution: Let $\varnothing \neq U^{open} \subseteq \mathbb{R}^1$. For each $q \in U \cap \mathbb{Q}$, let $V_q = \bigcup_{q\in(a,b)\subseteq U}(a,b)$. By construction, each $V_q = (\inf V_q, \sup V_q)$. If $q \in V_{q_1} \cap V_{q_2}$, then

$$q \in (\min_{i=1,2} \inf V_{q_i}, \max_{i=1,2} \sup V_{q_i}) \subseteq U$$

demonstrates that $V_{q_1} = V_{q_2}$.

Let $\{q_n\}_{n\in\mathbb{N}}$ be an enumeration of the rationals and define $U_0 = V_{q_0}$. For each $k \in \mathbb{Z}^+$, let

$$n_k = \min\left\{n \in \mathbb{N} \mid q_n \notin \bigcup_{j=0}^{k-1} U_j\right\},$$

and define $U_k = V_{q_{n_k}}$. If $r \in U$, then $r \in V_q = U_k$ for some $q \in U \cap \mathbb{Q}, k \in \mathbb{N}$. Thus each U_k is an interval, $U = \bigcup_{k\in\mathbb{N}} U_k$, and the U_k are disjoint.

30. Imitate the proof of Theorem 2.43 to obtain the following result:

If $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.

Equivalent statement: If G_n is a dense open subset of $\mathbb{R}^k$, for $n = 1, 2, 3, \ldots$, then $\cap_{n=1}^{\infty} G_n$ is not empty (in fact, it is dense in $\mathbb{R}^k$).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

Solution: For each $n \in \mathbb{N}$, let $F_n \subseteq \mathbb{R}^k$ be closed such that $F_n^\circ = \varnothing$. Let $N \in \mathbb{N}$ and $U = \left(\bigcup_{n=1}^{N} F_n\right)^\circ$. Then

$$U \subseteq \bigcup_{n=1}^{N} U \cap F_n = \varnothing.$$

Let $f_0 \in F_0^c$ and $r_0 > 0$ such that $\mathrm{B}(f_0, 2r_0) \subseteq F_0^c$. For each $N \geq 1$, let $f_N \in \mathrm{B}(f_{N-1}, r_{N-1})$ and $r_{N-1} > 2r_N > 0$ such that $\mathrm{B}(f_N, 2r_N) \subseteq \mathrm{B}(f_{N-1}, r_{N-1}) \cap F_N^c$. Then

$$\overline{\mathrm{B}(f_N, r_N)} \subseteq \mathrm{B}(f_N, 2r_N) \subseteq \mathrm{B}(f_{N-1}, r_{N-1}) \subseteq \overline{\mathrm{B}(f_{N-1}, r_{N-1})}$$

and

$$\overline{\mathrm{B}(f_N, r_N)} \subseteq \left(\bigcup_{n=1}^{N} F_n\right)^c$$

for every $N \geq 1$. Since $\{\overline{\mathrm{B}(f_N, r_N)}\}_{N \in \mathbb{N}}$ is a family of compact sets with nonempty finite intersections,

$$\varnothing \neq \bigcap_{N \in \mathbb{N}} \overline{\mathrm{B}(f_N, r_N)} \subseteq \left(\bigcup_{n \in \mathbb{N}} F_n\right)^c.$$

Take the contrapositive.

3 Numerical Sequences and Series

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

   Solution: Let $\lim_{n\to\infty} s_n = s$, let $\varepsilon > 0$, and let $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \varepsilon$. Then $||s_n| - |s|| \leq |s_n - s| < \varepsilon$ for all $n \geq N$. Thus $\lim_{n\to\infty} |s_n| = |s|$.

   Let $s_n = (-1)^n$. Then $|s_n| = 1$ is a convergent sequence. On the other hand, $|s_{n+1} - s_n| = 2$ (for all $n \in \mathbb{N}$) shows that s_n is not convergent because it is not a Cauchy sequence.

2. Calculate $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

   Solution: Let $r \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $0 < r < \frac{1}{2}$ and $\frac{r^2}{1-2r} < n$. Then $2rn + r^2 < n$ yields

$$\begin{aligned} r &= \sqrt{(n+r)^2} - n \\ &= \sqrt{n^2 + 2rn + r^2} - n \\ &< \sqrt{n^2+n} - n \\ &< \sqrt{n^2 + n + \frac{1}{4}} - n \\ &= \frac{1}{2}. \end{aligned}$$

   Thus $\lim_{n\to\infty}(\sqrt{n^2+n} - n) = \frac{1}{2}$.

3. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \;\; (n = 1, 2, 3, \ldots),$$

   prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \ldots$.

   Solution: The identity

$$s_{n+1}^2 - s_n^2 = \sqrt{s_n} - \sqrt{s_{n-1}}$$

   demonstrates that s_n is monotone. Since

$$s_2^2 = 2 + \sqrt[4]{2} > 2 = s_1^2,$$

   s_n is increasing. If $s_n < 2$ then $s_{n+1} < 2$ is shown by

$$4 - s_{n+1}^2 = 2 - \sqrt{s_n} > 0.$$

   Thus s_n is a bounded increasing sequence, hence convergent.

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \; s_{2m} = \frac{s_{2m-1}}{2}; \; s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Solution: Suppose that $s_{2m} = \frac{1}{2} - 2^{-m}$. Then

$$s_{2(m+1)} = \frac{1}{2}\left(\frac{1}{2} + s_{2m}\right) = \frac{1}{2}\left(1 - 2^{-m}\right) = \frac{1}{2} - 2^{-(m+1)}.$$

Suppose that $s_{2m-1} = 1 - 2^{1-m}$. Then

$$s_{2(m+1)-1} = \frac{1}{2}(1 + s_{2m-1}) = \frac{1}{2}(2 - 2^{1-m}) = 1 - 2^{1-(m+1)}.$$

By induction $s_{2m} = \frac{1}{2} - 2^{-m}$ and $s_{2m-1} = 1 - 2^{1-m}$ for all $m \in \mathbb{Z}^+$. If $n > 2m$, then $\frac{1}{2} - 2^{-m} < s_n < 1$. Thus

$$\frac{1}{2} \leq \liminf s_n \leq \limsup s_n \leq 1$$

yields

$$\limsup_{n\to\infty} s_n = \lim_{m\to\infty} (1 - 2^{1-m}) = 1$$

and

$$\liminf_{n\to\infty} s_n = \lim_{m\to\infty} \left(\frac{1}{2} - 2^{-m}\right) = \frac{1}{2}.$$

5. For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Solution: Let $\{n_k\}$ be a sequence in $\mathbb{N}$ such that

$$\lim_{k\to\infty} (a_{n_k} + b_{n_k}) = \limsup (a_n + b_n).$$

Let A and B be the sets of subsequential limits of $\{a_{n_k}\}$ and $\{b_{n_k}\}$ respectively. Then

$$\limsup (a_n + b_n) = \lim_{k\to\infty} (a_{n_k} + b_{n_k}) \leq \sup A + \sup B \leq \limsup a_n + \limsup b_n.$$

6. Investigate the behavior (convergence or divergence) of $\sum a_n$ if
   (a) $a_n = \sqrt{n+1} - n$;
   (b) $a_n = \frac{\sqrt{n+1}-n}{n}$;
   (c) $a_n = (\sqrt[n]{n} - 1)^n$;
   (d) $a_n = \frac{1}{1+z^n}$, for complex values of z.

Solution:

(a) $\sum_{n=0}^{N} \sqrt{n+1} - n = \sqrt{N+1} \to \infty$ as $N \to \infty$. Thus the sum diverges.

(b) $\sum_{n=1}^{N} \frac{\sqrt{n+1}-n}{n} = \sum_{n=1}^{N} \frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)} \leq \frac{1}{2}\sum_{n=1}^{N} \frac{1}{n^{3/2}}$. The latter sum converges. Thus the former does as well.

(c) Let $\varepsilon > 0$ and $K \in \mathbb{N}$ such that $2^K > \frac{2}{\varepsilon}$ and $\sqrt[n]{n} - 1 < \frac{1}{2}$ for $n \geq K$. If $N \geq M \geq K$, then

$$\sum_{n=M}^{N} \left(\sqrt[n]{n} - 1\right)^n \leq 2^{-M} \sum_{n=0}^{N-M} 2^{-n} \leq 2^{1-K} < \varepsilon.$$

Thus the sum converges by the Cauchy criteriion.

(d) If $\frac{1}{1+z^n} \to 0$, as $n \to \infty$, then $|z| > 1$. Now,

$$\frac{1}{z} - \frac{1+z^n}{1+z^{n+1}} = \frac{z^{-(n+1)} - z^{-n}}{z^{-n} + z} \to 0$$

as $n \to \infty$, if $|z| > 1$. Finally, the ratio test yields

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1+z^n}{1+z^{n+1}}\right| = \left|\frac{1}{z}\right| < 1$$

iff $|z| > 1$. Thus the sum converges iff $|z| > 1$.

7. Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

Solution: Let $x, y \geq 0$. Then

$$4xy \leq (x-y)^2 + 4xy = (x+y)^2$$

yields the arithmetic-geometric mean inequality:

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

If $N \in \mathbb{Z}^+$, then

$$\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n} = \sum_{n=1}^{N} \sqrt{a_n \cdot \frac{1}{n^2}} \leq \frac{1}{2}\sum_{n=1}^{N} a_n + \frac{1}{2}\sum_{n=1}^{N} \frac{1}{n^2}.$$

The sums on the right converge, thus the sum on the left does as well.

8. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Solution: Let $A_n = \sum_{k=1}^{n} a_k, \lim_{n\to\infty} b_n = b, \beta_n = |b_n - b|$. Since $\{A_n\}$ converges, it is bounded. Furthermore, $\{\beta_n\}$ is a decreasing sequence converging to 0. By theorem 3.42, $\sum a_n \beta_n$ converges. Thus

$$\sum_{n=1}^{N} a_n b_n = \operatorname{sgn}(b_1 - b) \sum_{n=1}^{N} a_n \beta_n + b \sum_{n=1}^{N} a_n$$

converges as well.

9. Find the radius of convergence of each of the following power series:
   (a) $\sum n^3 z^n$,
   (b) $\sum \frac{2^n}{n!} z^n$,
   (c) $\sum \frac{2^n}{n^2} z^n$,
   (d) $\sum \frac{n^3}{3^n} z^n$.

Solution:

(a)

$$\lim_{n\to\infty} \sqrt[n]{n^3} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^3 = 1.$$

By the root test, $R = 1$.

(b)

$$\lim_{n\to\infty} \left(\frac{n!}{(n+1)!} \cdot \frac{2^{n+1}}{2^n} \right) = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

By the ratio test, $R = \infty$.

(c)

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty} 2 \left(\frac{1}{\sqrt[n]{n}} \right)^2 = 2.$$

By the root test, $R = \frac{1}{2}$.

(d)

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{1}{3} \left(\sqrt[n]{n}\right)^3 = \frac{1}{3}.$$

By the root test, $R = 3$.

10. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Solution: Since $1 \le \sqrt[n]{|a_n|}$ infinitely often, $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \ge 1$. Thus the radius of convergence is at most 1.

11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.
   (a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.
   (b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \ge 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.
   (c) Prove that

$$\frac{a_n}{s_n^2} \le \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \text{ and } \sum \frac{a_n}{1+n^2a_n}?$$

Solution: Since $\sum a_n$ is a series of positive terms that diverges, its partial sums are unbounded.

(a) If $a_n \geq 1$ for infinitely many n, let a_{n_k} be a subsequence such that $a_{n_k} \geq 1$ for all $k \in \mathbb{N}$. Then

$$\sum_{n=1}^{N} \frac{a_n}{1+a_n} \geq \sum_{1 \leq n_k \leq N} \frac{1}{2}$$

for all $N \in \mathbb{N}$. As $N \to \infty$, the sum on the right is unbounded, thus $\sum \frac{a_n}{1+a_n}$ diverges.

If there is $N \in \mathbb{N}$ such that $a_n < 1$ for $n \geq N$, then

$$\sum_{n=N}^{N+M} \frac{a_n}{1+a_n} \geq \frac{1}{2} \sum_{n=N}^{N+M} a_n$$

for all $M \in \mathbb{N}$. The sum on the right is unbounded as $M \to \infty$, thus $\sum \frac{a_n}{1+a_n}$ diverges.

(b)

$$\sum_{j=1}^{k} \frac{a_{N+j}}{s_{N+j}} \geq \sum_{j=1}^{k} \frac{a_{N+j}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.$$

Since the partial sums of $\sum a_n$ are increasing and unbounded, for every $N \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $1 - \frac{s_N}{s_{N+k}} > \frac{1}{2}$. Hence, $\sum_{j=1}^{k} \frac{a_{N+j}}{s_{N+j}} > \frac{1}{2}$, and $\sum \frac{a_n}{s_n}$ fails the Cauchy criterion.

(c)

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{a_n}{s_n \cdot s_{n-1}} \geq \frac{a_n}{s_n^2}.$$

If $N \in \mathbb{N}$, then

$$\sum_{n=2}^{N} \frac{a_n}{s_n^2} \leq \sum_{n=2}^{N} \left(\frac{1}{s_{n-1}} - \frac{1}{s_n} \right) = \frac{1}{s_1} - \frac{1}{s_N} \leq \frac{1}{s_1}.$$

Thus $\sum \frac{a_n}{s_n^2}$ converges.

(d) The divergence of $\sum a_n$ does not solely determine the divergence of $\sum \frac{a_n}{1+na_n}$:

- If $a_n = \frac{1}{n}$, then $\sum \frac{a_n}{1+na_n} = \frac{1}{2} \sum \frac{1}{n}$ is divergent.
- If

  $$a_n = \begin{cases} 1 & \text{if } n = 2^k \\ 2^{-n} & \text{if } n \neq 2^k \end{cases},$$

  then

  $$\sum \frac{a_n}{1+na_n} \leq 2 \cdot \sum 2^{-n} = 4.$$

If $N \in \mathbb{N}$, then

$$\sum_{n=1}^{N} \frac{a_n}{1+n^2 a_n} = \sum_{n=1}^{N} \frac{1}{\frac{1}{a_n}+n^2} \leq \sum_{n=1}^{N} \frac{1}{n^2}.$$

Thus $\sum \frac{a_n}{1+n^2 a_n}$ converges.

12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Solution:

(a) For fixed m and sufficiently large n, $\frac{r_n}{r_m} < \frac{1}{2}$. Thus

$$\sum_{k=m}^{n} \frac{a_k}{r_k} > \sum_{k=m}^{n} \frac{a_k}{r_m} > 1 - \frac{r_n}{r_m}$$

shows that $\sum \frac{a_n}{r_n}$ fails the Cauchy criterion.

(b)

$$\frac{a_n}{\sqrt{r_n}} \cdot \left(\sqrt{r_n} + \sqrt{r_{n+1}}\right) = a_n\left(1 + \sqrt{\frac{r_{n+1}}{r_n}}\right) < 2a_n = 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right) \cdot \left(\sqrt{r_n} + \sqrt{r_{n+1}}\right).$$

Thus

$$\sum_{n=1}^{N} \frac{a_n}{\sqrt{r_n}} < 2\sum_{n=1}^{N} \left(\sqrt{r_n} - \sqrt{r_{n+1}}\right) = \sqrt{r_1} - \sqrt{r_{N+1}} < \sqrt{r_1},$$

shows that $\sum \frac{a_n}{\sqrt{r_n}}$ is a bounded series of positive terms. Hence, it converges.

13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Solution: Let $A = \sum |a_n|, B = \sum |b_n|$. Let $C = \sum c_n$ be the Cauchy product of A and B. Since A and B converge, so does C. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $0 \leq \sum_{k=n}^{m} c_k < \varepsilon$. Then

$$\sum_{k=n}^{m} \left| \sum_{j=0}^{k} a_j b_{k-j} \right| \leq \sum_{k=n}^{m} \sum_{j=0}^{k} |a_j| \, |b_{k-j}| = \sum_{k=n}^{m} c_k < \varepsilon.$$

Thus the Cauchy product of $\sum a_n$ and $\sum b_n$ converges absolutely.

14. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \; (n = 0, 1, 2, \ldots).$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?

(d) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

Assume that $\lim(na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $na_n \to 0$.]

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:

If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^{n} (s_n - s_i).$$

For these i,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n - \varepsilon}{1 + \varepsilon} < m + 1.$$

Then $(m+1)/(n-m) \leq 1/\varepsilon$ and $|s_n - s_i| < M\varepsilon$. Hence

$$\limsup |s_n - \sigma| \leq M\varepsilon.$$

Since ε was arbitrary, $\lim s_n = \sigma$.

Solution:

(a) Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \varepsilon$. Then

$$|\sigma_n - s| = \left| \frac{1}{n+1} \sum_{k=0}^{N} s_k + \frac{1}{n+1} \sum_{k=N+1}^{n} s_k - s \right| \leq \frac{1}{n+1} \sum_{k=0}^{N} |s_k - s| + \frac{n-N}{n+1}\varepsilon.$$

Thus $\limsup |\sigma_n - s| \leq \varepsilon$ for every $\varepsilon > 0$.

(b) Let $s_{2m} = 1, s_{2m-1} = -1$ for $m \in \mathbb{N}$. Then $\sigma_{2m} = \frac{1}{2m+1}$ and $\sigma_{2m-1} = 0$ for all $m \in \mathbb{N}$. Thus $\lim \sigma_n = 0$.

(c) Let $s_n = \begin{cases} k & \text{if } n = 2^k, \\ \frac{1}{n!} & \text{otherwise.} \end{cases}$ Since s_n is unbounded, $\limsup s_n = \infty$. On the other hand, let $k = \lfloor \log_2 n \rfloor$. Then

$$\sigma_n < \frac{e}{n+1} + \frac{1}{n+1}(1 + \ldots + k) < e \cdot 2^{-k} + \frac{k(k+1)}{2^{1+k}}$$

As $n \to \infty$, $k \to \infty$. Thus $\lim \sigma_n = 0$.

(d)

$$\begin{aligned} s_n - \sigma_n &= \frac{(n+1)s_n - \sum_{j=0}^n s_j}{n+1} \\ &= \frac{1}{n+1}\sum_{j=0}^n (s_n - s_j) \\ &= \frac{1}{n+1}\sum_{j=0}^n \sum_{k=j+1}^n (s_k - s_{k-1}) \\ &= \frac{1}{n+1}\sum_{0\le j<k\le n} a_k \\ &= \frac{1}{n+1}\sum_{k=1}^n k a_k \end{aligned}$$

Let $\lim \sigma_n = \sigma, \varepsilon > 0$, and $N \in \mathbb{N}$ such that $n > N$ implies $|\sigma_n - \sigma| < \frac{\varepsilon}{2}$ and $na_n < \frac{\varepsilon}{2}$. Then

$$|s_n - \sigma_n| = \left| \frac{1}{n+1}\sum_{k=1}^N k a_k + \frac{1}{n+1}\sum_{k=N+1}^n k a_k \right| \le \frac{\sum_{k=1}^N k\,|a_k|}{n+1} + \frac{n-N}{n+1}\frac{\varepsilon}{2}.$$

Thus

$$\limsup |s_n - \sigma| \le \limsup |s_n - \sigma_n| + \limsup |\sigma_n - \sigma| \le \varepsilon.$$

Since ε was arbitrary, $\lim s_n = \sigma$.

(e) If $m < n$, then

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{1}{n+1}\sum_{i=0}^m s_i - \frac{1}{n+1}\sum_{i=m+1}^n s_i \\ &= s_n - \frac{1}{n+1}\sum_{i=0}^m s_i - \left(\frac{1}{n-m}\right)\left(1 - \frac{m+1}{n+1}\right)\sum_{i=m+1}^n s_i \\ &= \frac{m+1}{n-m}\left[\frac{-(n-m)\sum_{i=0}^m s_i + (m+1)\sum_{i=m+1}^n}{(n+1)(m+1)}\right] + s_n - \frac{1}{n-m}\sum_{i=m+1}^n s_i \\ &= \frac{m+1}{n-m}\left[\frac{(m+1)\sum_{i=0}^n s_i - (n+1)\sum_{i=0}^m s_i}{(n+1)(m+1)}\right] + \frac{1}{n-m}\left((n-m)s_n - \sum_{i=m+1}^n s_i\right) \\ &= \frac{m+1}{n-m}\left[\frac{1}{n+1}\sum_{i=0}^n s_i - \frac{1}{m+1}\sum_{i=0}^m s_i\right] + \frac{1}{n-m}\sum_{i=m+1}^n (s_n - s_i) \\ &= \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{i=m+1}^n (s_n - s_i). \end{aligned}$$

If $i > m$, then

$$|s_n - s_i| \leq \left| \sum_{j=i+1}^{n} s_j - s_{j-1} \right| \leq \sum_{j=i+1}^{n} |a_j| \leq M \sum_{j=i+1}^{n} \frac{1}{j} \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Let $1 > \varepsilon > 0$ and $m = \lfloor \frac{n-\varepsilon}{1+\varepsilon} \rfloor$. Then

$$\begin{aligned} m \leq & \frac{n-\varepsilon}{1+\varepsilon} < m+1 \\ m + \varepsilon m = (1+\varepsilon)m \leq & n - \varepsilon < (m+1)(1+\varepsilon) = m + 1 + \varepsilon m + \varepsilon \\ \varepsilon(m+1) \leq & n - m < \varepsilon(m+2) + 1. \end{aligned}$$

Thus

$$\frac{m+1}{n-m} \leq \frac{1}{\varepsilon} \text{ and } \frac{(n-m-1)M}{m+2} < M\varepsilon.$$

Putting the above together yields

$$|s_n - \sigma_n| \leq \left| \frac{m+1}{n-m} \right| \cdot |\sigma_n - \sigma_m| + \frac{1}{n-m} \sum_{i=m+1}^{n} |s_n - s_i| \leq \frac{1}{\varepsilon} |\sigma_n - \sigma_m| + M\varepsilon.$$

Let $N \in \mathbb{N}$ such that $i, j \geq N$ implies that $|\sigma_i - \sigma_j| < \varepsilon^2$. If $n \geq 2N + 3$, then

$$m > \frac{n-\varepsilon}{1+\varepsilon} - 1 > \frac{n-3}{2} \geq N$$

and $\frac{1}{\varepsilon} |\sigma_n - \sigma_m| < \varepsilon$. Hence,

$$|s_n - \sigma| \leq |s_n - \sigma_n| + |\sigma_n - \sigma| \leq (2 + M)\varepsilon,$$

if $n \geq 2N + 3$. Since ε was arbitrary, $\lim s_n = \sigma$.

15. Definition 3.21 can be extended to the case in which the a_n lie in some fixed $\mathbb{R}^k$. Absolute convergence is defined as convergence of $\sum |\mathbf{a}_n|$. Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)

Solution:

- The statement and proof of 3.22, 3.23, 3.25(a), 3.33, 3.45, 3.47, and 3.55 are identical to the text.
- In the statement and proof of 3.34, replace $\left| \frac{a_{n+1}}{a_n} \right|$ with $\frac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|}$.
- In the statement and proof of 3.42, $\mathbf{a}_n \in \mathbb{R}^k, b_n \in \mathbb{R}^1$.

16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots,$ by the recursion formula

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{\alpha}{x_n} \right).$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

(b) Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \quad (n = 1, 2, 3, \ldots).$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < \frac{1}{10}$ and that therefore

$$\varepsilon_5 < 4 \cdot 10^{-16}, \ \varepsilon_6 < 4 \cdot 10^{-32}.$$

Solution:

(a) By the arithmetic-geometric mean inequality,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) > \sqrt{x_n \cdot \frac{\alpha}{x_n}} = \sqrt{\alpha}.$$

Consequently,

$$x_{n+1} = \frac{1}{2}\left(\frac{x_n^2 + \alpha}{x_n}\right) < \frac{1}{2} \cdot \frac{2x_n^2}{x_n} = x_n.$$

Since $\{x_n\}$ is decreasing and bounded, it converges.

Let $\lim x_n = x$. Then $x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right)$ implies $x^2 = \alpha$. Thus $x = \sqrt{\alpha}$, because $x \geq \sqrt{\alpha} > 0$.

(b)

$$\varepsilon_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} = \frac{x_n^2 - 2x_n\sqrt{\alpha} + (\sqrt{\alpha})^2}{2x_n} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}.$$

If $\varepsilon_n < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^{n-1}}$, then

$$\varepsilon_{n+1} < \frac{\varepsilon_n^2}{\beta} = \beta\left(\frac{\varepsilon_n}{\beta}\right)^2 < \beta\left(\left(\frac{\varepsilon_1}{\beta}\right)^{2^{n-1}}\right)^2 = \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}.$$

(c) Let $\alpha = 3, x_1 = 2$. Then

$$\begin{aligned} 25 &< 27 \\ 5 &< 3\sqrt{3} \\ 10 &< 6\sqrt{3} \\ \frac{10}{\sqrt{3}} - 5 &< 1. \end{aligned}$$

Thus $10 \cdot \frac{\varepsilon_1}{\beta} = 10 \cdot \left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right) = \frac{10}{\sqrt{3}} - 5 < 1$. This gives $\frac{\varepsilon_1}{\beta} < \frac{1}{10}$.

Since $12 < 16$, we know that $\beta = 2\sqrt{3} < 4$. Therefore the bounds on ε_5 and ε_6 follow.

17. Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(a) Prove that $x_1 > x_3 > x_5 > \cdots$.

(b) Prove that $x_2 < x_4 < x_6 < \cdots$.

(c) Prove that $\lim x_n = \sqrt{\alpha}$.

(d) Compare the rapidity of convergence of this process with the one described in Exercise 16.

Solution:

- If $x_n > \sqrt{\alpha}$, then

$$x_{n+2} = \frac{(1+\alpha)x_n + 2(\sqrt{\alpha})^2}{2x_n + (1+\alpha)} < \frac{(1+\sqrt{\alpha})^2 x_n}{(1+\sqrt{\alpha})^2} = x_n. \tag{7}$$

- If $x_n < \sqrt{\alpha}$, then

$$x_{n+2} = \frac{(1+\alpha)x_n + 2(\sqrt{\alpha})^2}{2x_n + (1+\alpha)} > \frac{((1+\alpha) + 2x_n)x_n}{2x_n + (1+\alpha)} = x_n. \tag{8}$$

-

$$x_{n+1} - \sqrt{\alpha} = x_n - \sqrt{\alpha} + \frac{\alpha - x_n^2}{1 + x_n} = \frac{(x_n - \sqrt{\alpha})(1 - \sqrt{\alpha})}{x_n + 1}. \tag{9}$$

Thus $x_n \neq \sqrt{\alpha}$ for all n and

$$\frac{x_{n+1} - \sqrt{\alpha}}{x_n - \sqrt{\alpha}} < 0. \tag{10}$$

(a) Since $x_1 > \sqrt{\alpha}$, (7) and (10) show that

$$s_{2n+1} > s_{2(n+1)+1} > \sqrt{\alpha}$$

for all $n \in \mathbb{N}$.

(b) By (10), $x_2 < \sqrt{\alpha}$. By (8),

$$s_{2n} < s_{2(n+1)} < \sqrt{\alpha}$$

for all $n \in \mathbb{N}$.

(c) By (7) and (8), $\lim x_{2n+1} = \sqrt{\alpha} = \lim x_{2n}$. (If either equality failed to hold then (7) or (8) would provide a contradiction.)

(d) Let $N \in \mathbb{N}$ such that $n > N$ implies $|x_n - \sqrt{\alpha}| < 1$. By (9),

$$\begin{aligned} |\varepsilon_{n+1}| &= \frac{\sqrt{\alpha} - 1}{x_n + 1} \cdot |\varepsilon_n| \\ &= (x_1 - \sqrt{\alpha}) \prod_{k=1}^{n} \frac{\sqrt{\alpha} - 1}{x_n + 1} \\ &< (x_1 - \sqrt{\alpha}) \left(\prod_{k=1}^{N} \frac{\sqrt{\alpha} - 1}{x_n + 1} \right) \left(1 - \frac{1}{\sqrt{\alpha}} \right)^{n-N}. \end{aligned}$$

This process converges slowly if α is close to 1.

18. Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1}$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.

Solution: Let $\alpha > 0, x_1 > \sqrt[p]{\alpha}$. By the arithmetic-geometric mean inequality,

$$x_{n+1} = \frac{(p-1)x_n + \alpha x_n^{-p+1}}{p} \geq \sqrt[p]{x_n^{p-1} \alpha x_n^{-p+1}} = \sqrt[p]{\alpha}.$$

If $x_n > \sqrt[p]{\alpha}$, then

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1} < \frac{p-1}{p} x_n + \frac{x_n^p}{p} x_n^{-p+1} = x_n.$$

Thus $\{x_n\}$ is a convergent sequence with limit $x \geq \sqrt[p]{\alpha}$.

The recursion formula yields

$$\begin{aligned} x &= \frac{p-1}{p} x + \frac{\alpha}{p} x^{1-p} \\ 1 &= 1 - \frac{1}{p} + \frac{\alpha}{p} x^{-p} \\ x^p &= \alpha \\ x &= \sqrt[p]{\alpha}. \end{aligned}$$

Furthermore,

19. Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.

Solution: Using the notation of 2.44, suppose that left endpoint of each interval of E_n has only 0's and 2 in its ternary expansion. Let $[a, b]$ be one of the intervals in E_{n+1}. Then there is an interval $[c, d]$ in E_n such that $[a, b] \subseteq [c, d]$ and $c = a$ or $b = d$. If $c = a$, then the left endpoint of $[a, b]$ has only 0's and 2's in its ternary expansion. If $b = d$, then

$$a = b - 3^{-(n+1)} = d - 3^{-(n+1)} = c + 3^{-n} - 3^{-(n+1)} = c + 2 \cdot 3^{-(n+1)}.$$

By induction, every interval in the construction of the Cantor set has only 0's and 2's in its ternary expansion. Since there are 2^n intervals in E_n,

$$E_n = \bigcup [r, r + 3^{-n}]$$

where the union is over all $r \in [0, 1]$ such that $3^n r \in \mathbb{N}$ and r's ternary expansion contains no 1's.

Given a sequence a,

$$x(a) \in \left[\sum_{k=1}^{n} \frac{\alpha_k}{3^k}, \sum_{k=1}^{n} \frac{\alpha_k}{3^k} + 3^{-n}\right] \subseteq E_n$$

for every $n \geq 1$. Thus $x(a)$ is in the Cantor set.

Conversely, let x be an element of the cantor set. Since $x \in E_1$, there is $\alpha_1 \in \{0,2\}$ such that $x \in \left[\frac{\alpha_1}{3}, \frac{\alpha_1}{3} + 3^{-1}\right]$. Given $x \in \left[\sum_{k=1}^{n} \frac{\alpha_k}{3^k}, \sum_{k=1}^{n} \frac{\alpha_k}{3^k} + 3^{-n}\right]$, there is $\alpha_{n+1} \in \{0,2\}$ such that

$$x \in \left[\sum_{k=1}^{n+1} \frac{\alpha_k}{3^k}, \sum_{k=1}^{n} \frac{\alpha_k}{3^k} + 3^{-(n+1)}\right] \subseteq E_{n+1}.$$

Thus

$$x(a) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\alpha_k}{3^k} \leq x \leq \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\alpha_k}{3^k} + 3^{-n}\right) = x(a).$$

20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Solution: Let $\varepsilon > 0, N \in \mathbb{N}$ such that $m, n, i \geq N$ implies that $d(p_n, p_m) < \frac{\varepsilon}{2}$ and $d(p_{n_i}, p) < \frac{\varepsilon}{2}$. Note that $n_i \geq i$ for all $i \in \mathbb{N}$. If $k \geq N$, then

$$d(p_k, p) \leq d(p_k, p_{n_k}) + d(p_{n_k}, p) < \varepsilon.$$

21. Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space X, if $E_n \supseteq E_{n+1}$, and if

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0,$$

then $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.

Solution: Choose $x_n \in E_n$ for each $n \in \mathbb{Z}^+$. Let $\varepsilon > 0, N \in \mathbb{Z}^+$ such that $\operatorname{diam} E_N < \varepsilon$. If $m, n \geq N$, then $d(x_m, x_n) < \varepsilon$. Thus $\{x_n\}$ is a Cauchy sequence with limit x. For each $n \in \mathbb{Z}^+$, $\{x_k\}_{k=n}^{\infty} \subseteq E_n$ and E_n is closed. Thus $x \in E_n$ for all n, and $x \in \bigcap_{n=1}^{\infty} E_n$. Furthermore, if $y \in \bigcap_{n=1}^{\infty} E_n$, then $d(x,y) \leq \operatorname{diam} E_n$ for all n. Thus $d(x,y) = 0$, and $y = x$.

22. Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_{n=1}^{\infty} G_n$ is not empty. (In fact, it is dense in X.) *Hint:* Find a shrinking sequence of neighborhoods E_n such that $\overline{E}_n \subseteq G_n$, and apply Exercise 21.

Solution: Let $x_1 \in G_1, 0 < r_1$ such that $\mathrm{B}(x_1, r_1) \subseteq G_1$. Given $x_1, \ldots, x_n$, let

$$x_{n+1} \in \mathrm{B}\left(x_n, \frac{r_n}{2}\right) \cap G_{n+1}, 0 < r_{n+1} < \frac{r_n}{2}$$

such that

$$\mathrm{B}(x_{n+1}, r_{n+1}) \subseteq \mathrm{B}\left(x_n, \frac{r_n}{2}\right) \cap G_{n+1}.$$

Then $\overline{\mathrm{B}\left(x_n, \frac{r_n}{2}\right)} \subseteq G_n$ for all n, and $\{\overline{\mathrm{B}\left(x_n, \frac{r_n}{2}\right)}\}$ satisfy the hypothesis of exercise 21. Thus

$$\varnothing \neq \bigcap_{n=1}^{\infty} \overline{\mathrm{B}\left(x_n, \frac{r_n}{2}\right)} \subseteq \bigcap_{n=1}^{\infty} G_n.$$

Given an open set U, restrict x_1, r_1 such that $\mathrm{B}(x_1, r_1) \subset U \cap G_1$. In this case, $\lim x_n \in U \cap \bigcap_{n=1}^{\infty} G_n$.

23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. Hint: For any m, n,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

Solution: For any $m, n \in \mathbb{N}$,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n).$$

Thus

$$d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n).$$

Swapping the roles of m, n gives

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_m, p_n) + d(q_m, q_n).$$

Let $N \in \mathbb{N}$ be such that $m, n \geq N$ implies $d(p_m, p_n) < \frac{\varepsilon}{2}$ and $d(q_m, q_n) < \frac{\varepsilon}{2}$. Then

$$|d(p_n, q_n) - d(p_m, q_m)| < \varepsilon.$$

Consequently, $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{d(p_n, q_n)\}$ converges.

24. Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}$,$\{q_n\}$ in X *equivalent* if

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*, Q \in X^*, \{p_n\} \in P, \{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

(c) Prove that the resulting metric space X^* is complete.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping ϕ defined by $\phi(p) = P_p$ is an isometry (i.e. distance-preserving mapping) of X into X^*.

(e) Prove that $\phi(X)$ is dense in X^*, and that $\phi(X) = X^*$ if X is complete. By (d), we may identify X and $\phi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X.

Solution:

(a) The following bullets show that this is an equivalence relation:

- $\lim d(p_n, p_n) = 0$,
- $\lim d(p_n, q_n) = \lim d(q_n, p_n)$, and
- $\lim d(p_n, q_n) \leq \lim [d(p_n, r_n) + d(r_n, q_n)] = \lim d(p_n, r_n) + \lim d(r_n, q_n)$.

(b) Let $\{p_n\}$ and $\{q_n\}$ be equivalent to $\{p'_n\}$ and $\{q'_n\}$, respectively. Then

$$\lim d(p_n, q_n) \leq \lim [d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)] = \lim d(p'_n, q'_n).$$

By symmetry, $\lim d(p'_n, q'_n) = \lim d(p_n, q_n)$.

Let $P, Q, R \in X^*$, and $\{p_n\} \in P, \{q_n\} \in Q, \{r_n\} \in R$. The following bullets show that Δ is a metric:

- $\Delta(P, Q) = \lim d(p_n, q_n) \geq 0$. In the case of equality, $\{p_n\}$ is equivalent to $\{q_n\}$. Hence $P = Q$.
- $\Delta(P, Q) = \lim d(p_n, q_n) = \lim d(q_n, p_n) = \Delta(Q, P)$.
- $\Delta(P, Q) = \lim d(p_n, q_n) \leq \lim d(p_n, r_n) + \lim d(r_n, q_n) = \Delta(P, R) + \Delta(R, Q)$.

(c) Let $\{P_n\}$ be a Cauchy sequence in X^*, and $\{p^n_k\} \in P_n$ for each n. Let $\varepsilon > 0, N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies

$$\Delta(P_m, P_n) < \frac{\varepsilon}{3}.$$

(d)

(e)

25. Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space? (Compare Exercise 24.)

4 Continuity

5 Differentiation

6 The Riemann-Stieltjes Integral

7 Sequences and Series of Functions

8 Some Special Functions

9 Functions of Several Variables

10 Integration of Differential Forms

11 The Lebesgue Theory