Solution Manual
for
Adaptive Filter Theory 5e

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Acknowledgments

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3. Erkan Baser for permitting us to reproduce his graduate student project in adaptive filter theory, 2013; the reproduction is verbatim, presented as an appendix at the end of the solution manual: The project entailed a revisit to the Adaptive Equalization Experiment in Chapter 6 on the LMS algorithm. This time, however, the projected involved using the IDBD algorithm and the Autostep Method, as well as the LMS and RLS algorithms as basis for comparative evaluation.

4. Ashkan Amir, Ph.D. student at McMaster University for helping the second co-author, Kelvin Hall during early stages of the work done on this Solutions Manual for the book on Adaptive Filter Theory.

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Table of Contents

Corrections for Question-Descriptions in the Textbook

Through the creation of the solution manual, several minor errors were noticed in the question descriptions in the textbook. Below is a collected list of which questions are affected and what changes are needed to be made. The information of what corrections are to be made is also present in the solutions manual at the question affected.

1.10 The expression $x(n) = \nu(n) + 0.75\nu(n-1) + 0.75\nu(n-2)$ should read $x(n) = \nu(n) + 0.75\nu(n-1) + 0.25\nu(n-2)$

5.6 The cost function should be $J_{\mathbf{s}}(\mathbf{w}) = |e(n)|^4$ not $J_{\mathbf{s}}(w) = |e(n)|^4$

5.6 a) The update formula should be given as:
$\hat{\mathbf{w}}(n+1) = \hat{\mathbf{w}}(n) + 2\mu\mathbf{u}(n)e^*(n)\,|e(n)|^2$
not $\hat{\mathbf{w}}(n+1) = \hat{\mathbf{w}}(n) + \mu\mathbf{u}(n-i)e^*(n)\,|e(n)|^2 \quad i = 0, 1, \ldots, M-1$

6.16 The problem is overdefined by providing both a noise variance and an AR process variance. The noise variance as such can be ignored. However if the solution is found with the prescribed noise variance the resulting graphs will be nearly identical to those included in the solution manual.

11.1 a) The question was meant to ask to show that $\mu^{-\frac{1}{2}}\tilde{\mathbf{w}}(n+1)$ equals
$\mu^{-\frac{1}{2}}\tilde{\mathbf{w}}(n) - \mu^{\frac{1}{2}}\mathbf{u}(n)\left(d(n) - \hat{\mathbf{w}}^T(n)\mathbf{u}(n)\right)$
not $\mu^{-\frac{1}{2}}\tilde{\mathbf{w}}(n) - \mu^{-\frac{1}{2}}\mathbf{u}(n)\left(d(n) - \hat{\mathbf{w}}^T(n)\mathbf{u}(n)\right)$

11.1 c) The last bracketed expression on the right hand side of the question should read $\left(d(n) - \hat{\mathbf{w}}^T(n)\mathbf{u}(n)\right)^2$ not $\left(d(n) - \tilde{\mathbf{w}}^T(n)\mathbf{u}(n)\right)^2$

11.1 d) The last bracketed expression on the right hand side of the question should read $\left(d(n) - \hat{\mathbf{w}}^T(n)\mathbf{u}(n)\right)^2$ not $\left(d(n) - \tilde{\mathbf{w}}^T(n)\mathbf{u}(n)\right)^2$

11.2 The denominator of the inequality should read

$\mu^{-1}\mathbf{w}^T\mathbf{w} + \sum_{n=0}^{i-1}\left(d(n) - \mathbf{w}^T\mathbf{u}(n)\right)^2$ not $\mu^{-1}\mathbf{w}^T\mathbf{w} + \sum_{n=0}^{i-1}\nu^2(n)$.

11.4 The question should be asking to find the optimizing $\mathbf{w}$ as shown by

$$\mathbf{w} = \left[\mu^{-1}\mathbf{I} - \mathbf{u}(i)\mathbf{u}^T(i)\right]^{-1}\left(\sum_{n=0}^{i-1} e(n)\mathbf{u}(n) - \hat{d}(i)\mathbf{u}(i)\right),$$

not $\mathbf{w} = \left[\mu\mathbf{I} - \mathbf{u}(i)\mathbf{u}^T(i)\right]^{-1}\left(\sum_{n=0}^{i-1} e(n)\mathbf{u}(n) - \hat{d}(i)\mathbf{u}(i)\right)$.

11.5 b) The experiment is described in Section 6.7 not 6.8.

11.7 d) Tildes are missing from above the step size parameters of the Normalized LMS algorithm entries in table P 11.1

Notes on the Computer Simulations and Provided Programs

The computer experiments completed for this solutions manual were completed almost exclusively using Matlab®, for ease of readability. To improve the accessibility of the solutions, to the users of this manual, specialized signal processing toolkits were not used in programs included. Graphical solutions are provided along with the .m files in case the user of the textbook is interested in completing the exercises in a different programming language, in which case a graphical solution is available for comparison.

The solutions of the computer problems in Chapter 13 were completed by Ashique Rupam Mahmood, Computer Science, University of Alberta, the creator of the Autostep algorithm. The solutions being completed prior to the rest of the manual are only available in the programming language python, which is similar to Matlab® and therefore quite readable.

Chapter 1

Problem 1.1

Let

$$r_u(k) = \mathbb{E}[u(n)u^*(n-k)] \tag{1}$$

$$r_y(k) = \mathbb{E}[y(n)y^*(n-k)] \tag{2}$$

we are given that

$$y(n) = u(n+a) - u(n-a) \tag{3}$$

Hence, substituting Equation (3) into Equation (2), and then using Equation (1), we get

$$\begin{aligned} r_y(k) =& \mathbb{E}[(u(n+a) - u(n-a))(u^*(n+a-k) - u^*(n-a-k))] \\ =& 2r_u(k) - r_u(2a+k) - r_u(-2a+k) \end{aligned}$$

Problem 1.2

We know that the correlation matrix $\mathbf{R}$ is Hermitian; that is to say that

$$\mathbf{R}^H = \mathbf{R}$$

Given that the inverse matrix $\mathbf{R}^{-1}$ exists, we may write

$$\mathbf{R}^{-1}\mathbf{R}^H = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. Taking the Hermitian transpose of both sides:

$$\mathbf{R}\mathbf{R}^{-H} = \mathbf{I}$$

Hence,

$$\mathbf{R}^{-H} = \mathbf{R}^{-1}$$

That is, the inverse matrix $\mathbf{R}^{-1}$ is Hermitian.

Problem 1.3

For the case of a two-by-two matrix, it may be stated as

$$\begin{aligned}\mathbf{R}_u &= \mathbf{R}_s + \mathbf{R}_\nu \\ &= \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} + \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \\ &= \begin{bmatrix} r_{11}+\sigma^2 & r_{12} \\ r_{21} & r_{22}+\sigma^2 \end{bmatrix}\end{aligned}$$

For $\mathbf{R}_u$ to be nonsingular, we require

$$\det(\mathbf{R}_u) \neq 0$$

$$(r_{11}+\sigma^2)(r_22+\sigma^2) - r_{12}r_{22} \neq 0$$

With $r_{12} = r_{21}$ for real data, this condition reduces to

$$(r_{11}+\sigma^2)(r_{22}+\sigma^2) - r_{12}^2 \neq 0$$

Since this is a quadratic in σ^2, we may impose the following conditions on σ^2 for nonsingularity of $\mathbf{R}_u$:

$$\sigma^2 \neq \frac{1}{2}(r_{11}+r_{22})\left(\sqrt{1 - \frac{4\Delta_r}{(r_{11}+r_{22})^2 - 1}}\right)$$

where $\Delta_r = r_{11}r_{22} - r_{12}^2$

Problem 1.4

We are given

$$\mathbf{R} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

This matrix is positive definite because it satisfies the condition:

$$\begin{aligned} a^T\mathbf{R}\,a =& \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \\ =& a_1^2 + 2a_1a_2 + a_2^2 \\ =& (a_1+a_2)^2 > 0 \text{ for all nonzero values of } a_1 \text{ and } a_2 \end{aligned}$$

But the matrix $\mathbf{R}$ is singular because:

$$\det(\mathbf{R}) = (1)^2 - (1)^2 = 0$$

Hence, it is possible for a matrix to be both positive definite and singular at the same time.

Problem 1.5

a)

$$\mathbf{R}_{M+1} = \begin{bmatrix} r(0) & \mathbf{r}^H \\ \mathbf{r} & \mathbf{R}_M \end{bmatrix} \tag{1}$$

Let

$$\mathbf{R}_{M+1}^{-1} = \begin{bmatrix} a & \mathbf{b}^H \\ \mathbf{b} & \mathbf{C}_M \end{bmatrix} \tag{2}$$

where$a$, $\mathbf{b}$ and $\mathbf{C}$ are to be determined. Multiply Equation (1) by Equation (2):

$$\mathbf{I}_{M+1} = \begin{bmatrix} r(0) & \mathbf{r}^H \\ \mathbf{r} & \mathbf{R}_M \end{bmatrix} \begin{bmatrix} a & \mathbf{b}^H \\ \mathbf{b} & \mathbf{C} \end{bmatrix}$$

Where $\mathbf{I}_{M+1}$ is the identity matrix. Therefore,

$$r(0)a + \mathbf{r}^H\mathbf{b} = 1 \tag{3}$$

$$\mathbf{r}a + \mathbf{R}_M\mathbf{b} = \mathbf{0} \tag{4}$$

$$\mathbf{r}\mathbf{b}^H + \mathbf{R}_M\mathbf{C} = \mathbf{I}_M \tag{5}$$

$$r(0)\mathbf{b}^H + \mathbf{r}^H\mathbf{C} = \mathbf{0} \tag{6}$$

Equation (4) can be rearranged to solve for b as:

$$b = -\mathbf{R}_M^{-1}\mathbf{r}a \tag{7}$$

Hence, from equations (3) and (7):

$$a = \frac{1}{r(0) - \mathbf{r}^H \mathbf{R}_M^{-1} \mathbf{r}} \tag{8}$$

Correspondingly,

$$b = -\frac{\mathbf{R}_M^{-1} \mathbf{r} \mathbf{r}^H \mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^H \mathbf{R}_M^{-1} \mathbf{r}} \tag{9}$$

From Equation (5):

$$\mathbf{C} = \mathbf{R}_M^{-1} - \mathbf{R}_M^{-1} \mathbf{r} \mathbf{b}^H$$

$$\mathbf{C} = \mathbf{R}_M^{-1} + \frac{\mathbf{R}_M^{-1} \mathbf{r} \mathbf{r}^H \mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^H \mathbf{R}_M^{-1} \mathbf{r}} \tag{10}$$

As a check, the results of Equations (9) and (10) should satisfy Equation (6)

$$\begin{aligned} r(0)\mathbf{b}^H + \mathbf{r}^H \mathbf{C} = & -\frac{r(0)\mathbf{r}^H \mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^H \mathbf{R}_M^{-1} \mathbf{r}} + \mathbf{r}^H \mathbf{R}_M^{-1} + \frac{\mathbf{r}^H \mathbf{R}_M^{-1} \mathbf{r} \mathbf{r}^H \mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^H \mathbf{R}_M^{-1} \mathbf{R}} \\ = & \ \mathbf{0} \end{aligned}$$

We have thus shown that

$$\begin{aligned} \mathbf{R}_{M+1}^{-1} = & \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_M^{-1} \end{bmatrix} + a \begin{bmatrix} 1 & -\mathbf{r}^H \mathbf{R}_M^{-1} \\ \mathbf{R}_M^{-1} \mathbf{r} & \mathbf{R}_M^{-1} \mathbf{r} \mathbf{r}^H \mathbf{R}_M^{-1} \end{bmatrix} \\ = & \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_M^{-1} \end{bmatrix} + a \begin{bmatrix} 1 \\ -\mathbf{R}_M^{-1} \mathbf{r} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{r}^{\mathbf{H}} \mathbf{R}_M^{-1} \end{bmatrix} \end{aligned}$$

where the scalar a is defined by Equation (8)

b)

$$\mathbf{R}_{M+1} = \begin{bmatrix} \mathbf{R}_M & \mathbf{r}^{B*} \\ \mathbf{r}^{BT} & r(0) \end{bmatrix} \tag{11}$$

Let

$$\mathbf{R}_{M+1}^{-1} = \begin{bmatrix} \mathbf{D} & \mathbf{e} \\ \mathbf{e}^H & f \end{bmatrix} \tag{12}$$

where $\mathbf{D}$, $\mathbf{e}$ and f are to be determined. Multiplying Equation (11) by Equation (12) you get:

$$\mathbf{I}_{M+1} = \begin{bmatrix} \mathbf{R}_M & \mathbf{r}^{B*} \\ \mathbf{r}^{BT} & r(0) \end{bmatrix} \begin{bmatrix} \mathbf{D} & \mathbf{e} \\ \mathbf{e}^H & f \end{bmatrix}$$

Therefore:

$$\mathbf{R}_M \mathbf{D} + \mathbf{r}^{B*}\mathbf{e}^H = \mathbf{I} \tag{13}$$

$$\mathbf{R}_M \mathbf{e} + \mathbf{r}^{B*} f = \mathbf{0} \tag{14}$$

$$\mathbf{r}^{BT}\mathbf{e} + r(0) f = 1 \tag{15}$$

$$\mathbf{r}^{BT}\mathbf{D} + r(0)\mathbf{e}^H = \mathbf{0} \tag{16}$$

From Equation (14):

$$\mathbf{e} = -\mathbf{R}_M^{-1}\mathbf{r}^{B*} \tag{17}$$

Hence, from Equation (15) and Equation (17):

$$f = \frac{1}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} \tag{18}$$

Correspondingly,

$$\mathbf{e} = -\frac{\mathbf{R}_M^{-1}\mathbf{r}^{B*}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} \tag{19}$$

From Equation (13):

$$\begin{aligned} \mathbf{D} &= \mathbf{R}_M^{-1} - \mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{e}^H \\ &= \mathbf{R}_M^{-1} + \frac{\mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{r}^{BT}\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} \end{aligned} \tag{20}$$

As a check, the results of Equation (19) and Equation (20) must satisfy Equation (16):

$$\mathbf{r}^{BT}\mathbf{D} + r(0)\mathbf{e}^H = \mathbf{0}$$

$$\mathbf{r}^{BT}\mathbf{R}_M^{-1} + \frac{\mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{r}^{BT}\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} - \frac{r(0)\mathbf{r}^{BT}\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} = \mathbf{0}$$

We have thus shown that

$$\begin{aligned} \mathbf{R}_{M+1}^{-1} &= \begin{bmatrix} \mathbf{R}_M^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} + f \begin{bmatrix} \mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{r}^{BT}\mathbf{R}_M^{-1} & \mathbf{R}_M^{-1}\mathbf{r}^{B*} \\ -\mathbf{r}^{BT}\mathbf{R}_M^{-1} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R}_M^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} + f \begin{bmatrix} -\mathbf{R}_M^{-1}\mathbf{r}^{B*} \\ 1 \end{bmatrix} \begin{bmatrix} -\mathbf{r}^{BT}\mathbf{R}_M^{-1} & 1 \end{bmatrix} \end{aligned}$$

where the scalar f is defined by Equation (18)

Problem 1.6

a)
We express the difference equation describing the first-order AR process $u(n)$ as

$$u(n) = \nu(n) + w_1 u(n-1)$$

where $w_1 = -a_1$. Solving the equation by repeated substitution, we get

$$\begin{aligned} u(n) =& \nu(n) + w_1\nu(n-1) + w_1 u(n-2) \\ =& \nu(n) + w_1\nu(n-1) + w_1^2\nu(n-2) + \ldots + w_1^{n-1}\nu(1) \end{aligned} \tag{1}$$

Here we used the initial condition

$$u(0) = 0$$

Taking the expected value of both sides of Equation (1) and using

$$\mathbb{E}[\nu(n)] = \mu$$

we get the geometric series

$$\begin{aligned} \mathbb{E}[u(n)] =& \quad \mu + w_1\mu + w_1^2\mu + \ldots + w_1^{n-1}\mu \\ =& \quad \left\{ \begin{array}{ll} \mu(\frac{1-w_1^n}{1-w_1}), & w_1 \neq 1 \\ \mu n, & w_1 = 1 \end{array} \right\} \end{aligned}$$

This result shows that if $\mu \neq 0$, then $\mathbb{E}[u(n)]$ is a function of time n. Accordingly, the AR process $u(n)$ is not stationary. If, however, the AR parameter satisfies the condition:

$$|a_1| < 1 \text{ or } |w_1| < 1$$

then

$$\mathbb{E}[u(n)] \rightarrow \frac{\mu}{1-w_1} \text{ as } n \rightarrow \infty$$

Under this condition, we say that the AR process is asymptotically stationary to order one.

b)
When the white noise process $\nu(n)$ has zero mean, the AR process $u(n)$ will likewise have zero mean. Then

$$\text{var}[\nu(n)] = \sigma_\nu^2$$

$$\text{var}[u(n)] = \mathbb{E}[u^2(n)] \tag{2}$$

Substituting Equation (1) into Equation (2), and recognizing that for the white noise process

$$\mathbb{E}[\nu(n)\nu(k)] = \begin{cases} \sigma_\nu^2 & n = k \\ 0, & n \neq k \end{cases} \tag{3}$$

we get the geometric series

$$\begin{aligned} \text{var}[u(n)] =& \sigma_\nu^2(1 + w_1^2 + w_1^4 + \ldots + w_1^{2n-2}) \\ =& \begin{cases} \sigma_\nu^2(\dfrac{1 - w_1^{2n}}{1 - w_1^2}), & w_1 \neq 1 \\ \sigma_\nu^2 n, & w_1 = 1 \end{cases} \end{aligned}$$

When $|a_1| < 1$ or $|w_1| < 1$, then

$$\text{var}[u(n)] \approx \frac{\sigma_\nu^2}{1 - w_1^2} = \frac{\sigma_\nu^2}{1 - a_1^2} \text{ for large } n$$

c)

The autocorrelation function of the AR process $u(n)$ equals $\mathbb{E}[u(n)u(n-k)]$. Substituting Equation (1) into this formula, and using Equation (3), we get

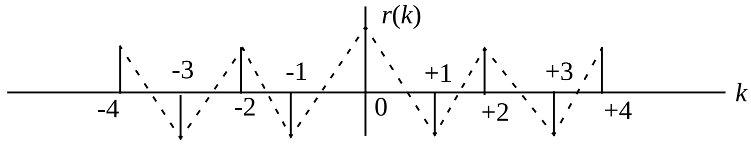
$$\begin{aligned} \mathbb{E}[u(n)u(n-k)] =& \quad \sigma_\nu^2(w_1^k + w_1^{k+2} + \ldots + w_1^{k+2n-2}) \\ =& \quad \begin{cases} \sigma_\nu^2 w_1^k(\frac{1-w_1^{2n}}{1-w_1^2}), & w_1 \neq 1 \\ \sigma_\nu^2 n, & w_1 = 1 \end{cases} \end{aligned}$$

For $|a_1| < 1$ or $|w_1| < 1$, we may therefore express this autocorrelation function as

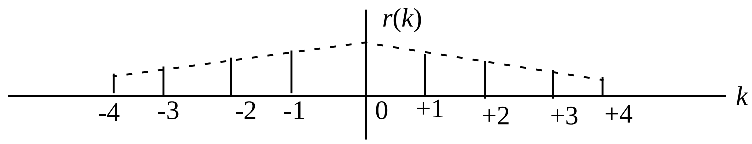
$$\begin{aligned} r(k) =& \mathbb{E}[u(n)u(n-k)] \\ \approx& \frac{\sigma_\nu^2 w_1^k}{1 - w_1^2} \text{ for large } n \end{aligned}$$

Case 1: $0 < a_1 < 1$

In this case, $w_1 = -a_1$ is negative, and $r(k)$ varies with k as follows:



Case 2: $-1 < a_1 < 0$
In this case, $w_1 = -a_1$ is positive, and $r(k)$ varies with k as follows:



Problem 1.7

a)

The second-order AR process $u(n)$ is described by the difference equation:

$$u(n) = u(n-1) - 0.5u(n-2) + \nu(n)$$

which, rewritten, states

$$w_1 = 1$$

$$w_2 = -0.5$$

as the AR parameters are equal to:

$$a_1 = -1$$

$$a_2 = 0.5$$

Accordingly, the Yule-Walker equation may be written as:

$$\begin{bmatrix} \mathbf{r}(0) & \mathbf{r}(1) \\ \mathbf{r}(1) & \mathbf{r}(0) \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} \mathbf{r}(1) \\ \mathbf{r}(2) \end{bmatrix}$$

b)

Writing the Yule-Walker equations in expanded form:

$$\mathbf{r}(0) - 0.5\mathbf{r}(1) = \mathbf{r}(1)$$

$$\mathbf{r}(1) - 0.5\mathbf{r}(0) = \mathbf{r}(2)$$

Solving the first relation for $\mathbf{r}(1)$:

$$\mathbf{r}(1) = \frac{2}{3}\mathbf{r}(0) \tag{1}$$

Solving the second relation for $\mathbf{r}(2)$:

$$\mathbf{r}(2) = \frac{1}{6}\mathbf{r}(0) \tag{2}$$

c)

Since the noise $\nu(n)$ has zero mean, the associated AR process $u(n)$ will also have zero mean. Hence,

$$\begin{aligned} \text{var}[u(n)] &= \mathbb{E}[(u^2)] \\ &= \mathbf{r}(0) \end{aligned}$$

It is known that

$$\begin{aligned} \sigma_\nu^2 =& \sum_{k=0}^{2} a_k \mathbf{r}(k) \\ =& \mathbf{r}(0) + a_1\mathbf{r}(1) + a_2\mathbf{r}(2) \end{aligned} \tag{3}$$

Substituting Equation (1) and Equation (2) into Equation (3), and solving for $\mathbf{r}(0)$, we get:

$$\mathbf{r}(0) = \frac{\sigma_\nu^2}{1 + \frac{2}{3}a_1 + \frac{1}{6}a_2} = 1.2$$

Problem 1.8

By Definition,

$$\begin{aligned} P_0 =& \text{Average power of the AR process } u(n) \\ =& \mathbb{E}[|u(n)|^2] \\ =& \mathbf{r}(0) \end{aligned} \tag{1}$$

where $\mathbf{r}(0)$ is the autocorrelation function of $u(n)$ with zero lag. We note that

$$\{a_1, a_2, \ldots, a_M\} \rightleftharpoons \left\{\frac{\mathbf{r}(1)}{\mathbf{r}(0)}, \frac{\mathbf{r}(2)}{\mathbf{r}(0)}, \ldots, \frac{\mathbf{r}(M)}{\mathbf{r}(0)}\right\}$$

Equivalently, except for the scaling factor $\mathbf{r}(0)$,

$$\{a_1, a_2, \ldots, a_M\} \rightleftharpoons \{\mathbf{r}(1), \mathbf{r}(2), \ldots, \mathbf{r}(M)\} \tag{2}$$

Combining Equation (1) and Equation (2):

$$\{P_0, a_1, a_2, \ldots, a_M\} \rightleftharpoons \{\mathbf{r}(0), \mathbf{r}(1), \mathbf{r}(2), \ldots, \mathbf{r}(M)\} \tag{3}$$

Problem 1.9

a)

The transfer function of the MA model of Fig. 1.3 is

$$H(z) = 1 + b_1^* z^{-1} + b_2^* z^{-2} + \ldots + b_K^* z^{-K}$$

b)

The transfer function of the ARMA model of Fig. 1.4 is

$$H(z) = \frac{b_0 + b_1^* z^{-1} + b_2^* z^{-2} + \ldots + b_K^* z^{-K}}{1 + a_1^* z^{-1} + a_2^* z^{-2} + \ldots + a_M^* z^{-M}}$$

c)

The ARMA model reduces to an AR model when

$$b_0 = b_1 = \ldots = b_K = 0$$

The ARMA model reduces to MA model when

$$a_1 = a_2 = \ldots = a_M = 0$$

Problem 1.10

[*] Taking the z-transform of both sides of the correct equation:

$$X(z) = (1 + 0.75z^{-1} + 0.25z^{-2})V(z)$$

Hence, the transfer function of the MA model is:

$$\begin{aligned}\frac{X(z)}{V(z)} =& 1 + 0.75z^{-1} + 0.75z^{-1} \\ =& \frac{1}{(1 + 0.75z^{-1} + 0.75z^{-1})^{-1}}\end{aligned} \quad (1)$$

Using long division we may perform the following expansion of the denominator in Equation (1):

$$\begin{aligned}&(1 + 0.75z^{-1} + 0.75z^{-1})^{-1} \\ &\quad = 1 - \frac{3}{4}z^{-1} + \frac{5}{16}z^{-2} - \frac{3}{64}z^{-3} - \frac{11}{256}z^{-4} - \frac{45}{1024}z^{-5} \\ &\quad\quad - \frac{91}{4096}z^{-6} + \frac{93}{16283}z^{-7} - \frac{85}{65536}z^{-8} - \frac{627}{262144}z^{-9} + \frac{1541}{1048576}z^{-10} + \dots \\ &\quad \approx 1 - 0.75z^{-1} + 0.3125z^{-2} - 0.0469z^{-3} - 0.043z^{-4} - 0.0439z^{-5} \\ &\quad\quad - 0.0222z^{-6} + 0.0057z^{-7} - 0.0013z^{-8} - 0.0024z^{-9} + 0.0015z^{-10}\end{aligned} \quad (2)$$

a)

$$M = 2$$

Retaining terms in Equation (2) up to z^{-2}, we may approximate the MA model with an AR model of order two as follows:

$$\frac{X(z)}{V(z)} \approx \frac{1}{1 - 0.75z^{-1} + 0.3125z^{-2}}$$

[*]Correction: the question was meant to ask the reader to consider an MA process $x(n)$ of order two described by the difference equation

$$x(n) = \nu(n) + 0.75\nu(n-1) + 0.25\nu(n-2)$$

not the equation

$$x(n) = \nu(n) + 0.75\nu(n-1) + 0.75\nu(n-2)$$

b)

$$M = 5$$

Retaining terms in Equation (2) up to z^{-5}, we may approximate the MA model with an AR model of order two as follows:

$$\frac{X(z)}{V(z)} \approx \frac{1}{1 - 0.75z^{-1} + 0.3125z^{-2} - 0.0469z^{-3} - 0.043z^{-4} + 0.0439z^{-5}}$$

c)

$$M = 10$$

Retaining terms in Equation (2) up to z^{-10}, we may approximate the MA model with an AR model of order two as follows:

$$\frac{X(z)}{V(z)} \approx \frac{1}{D(z)}$$

where $D(z)$ is given by the polynomial on the right-hand side of Equation (2).

Problem 1.11

a)

The filter output is

$$x(n) = \mathbf{w}^H \mathbf{u}(n)$$

where $\mathbf{u}(n)$ is the tap-input vector. The average power of the filter output is therefore

$$\begin{aligned} \mathbb{E}[|x(n)|^2] &= \mathbb{E}[\mathbf{w}^H \mathbf{u}(n)\mathbf{u}^H(n)\mathbf{w}] \\ &= \mathbf{w}^H \mathbb{E}[\mathbf{u}(n)\mathbf{u}^H(n)]\mathbf{w} \\ &= \mathbf{w}^H \mathbf{R}\mathbf{w} \end{aligned}$$

b)

If $\mathbf{u}(n)$ is extracted from a zero-mean white noise with variance σ^2, then

$$\mathbf{R} = \sigma^2 \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. Hence,

$$\mathbb{E}[|x(n)|^2] = \sigma^2 \mathbf{w}^H \mathbf{w}$$

Problem 1.12

a)
The process $u(n)$ is a linear combination of Gaussian samples. Hence, $u(n)$ is Gaussian.

b)
From inverse filtering, we recognize that $\nu(n)$ may also be expressed as a linear combination of samples relating to $u(n)$. Hence, if $u(n)$ is Gaussian, then $\nu(n)$ is also Gaussian.

Problem 1.13

a)
From the Gaussian moment factoring theorem:

$$\begin{aligned}\mathbb{E}[(u_1^* u_2)^k] =& \mathbb{E}[u_1^* \dots u_1^* u_2 \dots u_2] \\ =& k!\mathbb{E}[u_1^* u_2] \dots \mathbb{E}[u_1^* u_2] \\ =& k!(\mathbb{E}[u_1^* u_2])^k \end{aligned} \tag{1}$$

b)
By allowing $u_2 = u_1 = u$, Equation (1) reduces to:

$$\mathbb{E}[|u|^{2k}] = k!(\mathbb{E}[|u|^2])^k$$

Problem 1.14

It is not permissible to interchange the order of expectation and limiting operation in Equation (1.113). The reason is that the expectation is a linear operation, whereas the limiting operation with respect to the number of samples N is nonlinear.

Problem 1.15

The filter output is

$$y(n) = \sum_i h(i)u(n-i)$$

Similarly, we may write

$$y(m) = \sum_k h(k)u(m-k)$$

Hence,

$$\begin{aligned} r_y(n,m) &= \mathbb{E}[y(n)y^*(m)] \\ &= \mathbb{E}\left[\sum_i h(i)u(n-i)\sum_k h^*(k)u^*(m-k)\right] \\ &= \sum_i\sum_k h(i)h^*(k)\mathbb{E}\left[u(n-i)u^*(m-k)\right] \\ &= \sum_i\sum_k h(i)h^*(k)r_u(n-i,m-k) \end{aligned}$$

Problem 1.16

The mean-square value of the filter output response to white noise input is

$$P_0 = \frac{2\sigma^2\Delta\omega}{\pi}$$

The value P_0 is linearly proportional to the filter bandwidth $\Delta\omega$. This relation holds irrespective of how small $\Delta\omega$ is compared to the mid-band frequency of the filter.

Problem 1.17

a)

The variance of the filter output is

$$\sigma_y^2 = \frac{2\sigma^2\Delta\omega}{\pi}$$

It has been stated that

$$\sigma^2 = 0.1 \text{ volts}^2$$

$$\Delta\omega = 2\pi \times 1 \text{ radians/sec}$$

Hence,

$$\sigma_y^2 = \frac{2 \times 0.1 \times 2\pi}{\pi} = 0.4 \text{ volts}^2$$

b)

The pdf of the filter output y is

$$\begin{aligned} f(y) = & \quad \frac{1}{\sqrt{2\pi}\sigma_y} \exp(-y^2/0.8) \\ = & \quad \frac{3.1623}{\sqrt{2\pi}} \exp(-y^2/0.8) \end{aligned}$$

Problem 1.18

a)

We are given

$$U_k = \sum_{0}^{N-1} u(n) \exp(-\mathrm{j}\, n\omega_k), \quad k = 0, 1, ..., N-1$$

where $u(n)$ is real valued and

$$\omega_k = \frac{2\pi}{N} k$$

Hence,

$$\begin{aligned} \mathbb{E}[U_k U_l^*] =& \mathbb{E}\left[\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} u(n)u(m) \exp(-\mathrm{j}\, n\omega_k + \mathrm{j}\, m\omega_l)\right] \\ =& \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \exp(-\mathrm{j}\, n\omega_k + \mathrm{j}\, m\omega_l)\mathbb{E}[u(n)u(m)] \\ =& \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \exp(-\mathrm{j}\, n\omega_k + \mathrm{j}\, m\omega_l) r(n-m) \\ =& \sum_{n=0}^{N-1} \exp(\mathrm{j}\, m\omega_k) \sum_{m=0}^{N-1} r(n-m) \exp(-\mathrm{j}\, n\omega_k) \end{aligned} \quad (1)$$

By definition, we also have

$$\sum_{n=0}^{N-1} r(n)\exp(-\mathrm{j}\, n\omega_k) = S_k$$

Moreover, since $r(n)$ is periodic with period N, we may invoke the time-shifting property of the discrete Fourier transform to write

$$\sum_{n=0}^{N-1} r(n-m)\exp(-\mathrm{j}\, n\omega_k) = \exp(-\mathrm{j}\, m\omega_k)S_k$$

Recognizing that $\omega_k = (2\pi/N)k$, Equation (1) reduces to

$$\begin{aligned}\mathbb{E}[U_k U_l^*] = & \; S_k \textstyle\sum_{m=0}^{N-1} \exp(\mathrm{j}\, m(\omega_l - \omega_k)) \\ = & \begin{cases} S_k & l = k \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

b)

Part **A**) shows that the complex spectral samples U_k are uncorrelated. If they are Gaussian, then they will also be statistically independent. Hence,

$$f_{\mathbf{U}}(U_0, U_1, \ldots, U_{N-1}) = \frac{1}{(2\pi)^N \det(\Lambda)} \exp\left(-\frac{1}{2}\mathbf{U}^H \Lambda \mathbf{U}\right)$$

where

$$\mathbf{U} = [U_0, U_1, \ldots, U_{N-1}]^T$$

$$\begin{aligned}\Lambda = & \; \frac{1}{2}\mathbb{E}[\mathbf{U}\mathbf{U}^H] \\ = & \; \frac{1}{2}\mathrm{diag}(S_0, S_1, ..., S_{N-1})\end{aligned}$$

$$\det(\Lambda) = \frac{1}{2^N}\prod_{k=0}^{N-1} S_k$$

Therefore,

$$\begin{aligned} f_{\mathbf{U}}(U_0, U_1, \ldots, U_{N-1}) = & \frac{1}{(2\pi)^N 2^{-N} \prod_{k=0}^{N-1} S_k} \exp\left(-\frac{1}{2}\sum_{k=0}^{N-1} \frac{|U_k|^2}{\frac{1}{2}S_k}\right) \\ = & \pi^{-N} \exp\left(\sum_{k=0}^{N-1}\left(-\frac{|U_k|^2}{S_k}\right) - \ln S_k\right) \end{aligned}$$

Problem 1.19

The mean-square value of the increment process $\mathrm{d}\, z(\omega)$ is

$$\mathbb{E}[|\,\mathrm{d}\, z(\omega)|^2] = S(\omega)\,\mathrm{d}\,\omega$$

Hence, $\mathbb{E}[|\,\mathrm{d}\, z(\omega)|^2]$ is measured in watts.

Problem 1.20

The third-order cumulant of a process $u(n)$ is

$$\begin{aligned} c_3(\tau_1, \tau_2) = & \mathbb{E}[u(n)u(n+\tau_1)u(n+\tau_2)] \\ = & \text{third-order moment.} \end{aligned}$$

All odd-order moments of a Gaussian process are known to be zero; hence,

$$c_3(\tau_1, \tau_2) = 0$$

The fourth-order cumulant is

$$\begin{aligned} c_r(\tau_1, \tau_2, \tau_3) = & \mathbb{E}[u(n)u(n+\tau_1)u(n+\tau_2)u(n+\tau_3)] \\ & -\mathbb{E}[u(n)u(n+\tau_1)]\mathbb{E}[u(n+\tau_2)u(n+\tau_3)] \\ & -\mathbb{E}[u(n)u(n+\tau_2)]\mathbb{E}[u(n+\tau_1)u(n+\tau_3)] \\ & -\mathbb{E}[u(n)u(n+\tau_3)]\mathbb{E}[u(n+\tau_1)u(n+\tau_2)] \end{aligned}$$

For the special case of τ=τ_1=τ_2=τ_3, the fourth-order moment of a zero-mean Gaussian process of variance σ^2 is $3\sigma^4$, and its second-order moments of σ^2. Hence, the fourth-order cumulant is zero. Indeed, all cumulants higher than order two are zero

Problem 1.21

The trispectrum is

$$C_4(\omega_1, \omega_2, \omega_3) = \sum_{\tau_1=-\infty}^{\infty} \sum_{\tau_2=-\infty}^{\infty} \sum_{\tau_3=-\infty}^{\infty} c_4(\tau_1, \tau_2, \tau_3) \exp(-\,\mathrm{j}(\omega_1\tau_1 + \omega_2\tau_2 + \omega_3\tau_3))$$

Let the process be passed through a three-dimensional band-pass filter centered on ω_1, ω_2, and ω_3. We assume that the bandwidth (along each dimension) is small compared to the respective center frequency. The average power of the filter output is therefore proportional to the trispectrum, $C_4(\omega_1, \omega_2, \omega_3)$.

Problem 1.22

a)
Starting with the formula

$$c_k(\tau_1, \tau_2, \ldots, \tau_{k-1}) = \gamma_k \sum_{i=-\infty}^{\infty} h_i h_{i+\tau_1} \ldots h_{i+\tau_{k-1}}$$

The third-order cumulant of the filter output is

$$c_3(\tau_1, \tau_2) = \gamma_3 \sum_{i=-\infty}^{\infty} h_i h_{i+\tau_1} h_{i+\tau_2}$$

where γ_3 is the third-order cumulant of the filter input. The bispectrum is

$$\begin{aligned} c_3(\tau_1, \tau_2) =& \gamma_3 \sum_{\tau_1=-\infty}^{\infty} \sum_{\tau_2=-\infty}^{\infty} c_3(\tau_1, \tau_2) \exp(-\,\mathrm{j}(\omega_1\tau_1 + \omega_2\tau_2)) \\ =& \gamma_3 \sum_{i=-\infty}^{\infty} \sum_{\tau_1=-\infty}^{\infty} \sum_{\tau_2=-\infty}^{\infty} h_i h_{i+\tau_1} h_{i+\tau_2} \exp(-\,\mathrm{j}(\omega_1\tau_1 + \omega_2\tau_2)) \end{aligned}$$

Hence,

$$C_3(\omega_1, \omega_2) = \gamma_3 H\left(e^{\mathrm{j}\,\omega_1}\right) H\left(e^{\mathrm{j}\,\omega_2}\right) H^*\left(e^{\mathrm{j}(\omega_1+\omega_2)}\right) \quad (1)$$

b)
From the formula found in part **a)**, Equation (1), can be clearly deduced that

$$\arg[C_3(\omega_1, \omega_2)] = \arg\left[H\left(e^{\mathrm{j}\,\omega_1}\right)\right] + \arg\left[H\left(e^{\mathrm{j}\,\omega_2}\right)\right] - \arg\left[H\left(e^{\mathrm{j}(\omega_1+\omega_2)}\right)\right]$$

Problem 1.23

The output of a filter, which is defined by the impulse response h_i due to an input $u(i)$, is given by the convolution sum

$$y(n) = \sum_i h_i u(n-i)$$

The third-order cumulant of the filter output is, for example,

$$\begin{aligned} C_3(\tau_1, \tau_2) =& \mathbb{E}[y(n)y(n+\tau_1)y(n+\tau_2)] \\ =& \mathbb{E}\left[\sum_i h_i u(n-i) \sum_k h_k u(n+\tau_1-k) \sum_l h_l u(n+\tau_2-l)\right] \\ =& \mathbb{E}\left[\sum_i h_i u(n-i) \sum_k h_{k+\tau_1} u(n-k) \sum_l h_{l+\tau_2} u(n-l)\right] \\ =& \sum_i \sum_k \sum_l h_i h_{k+\tau_1} h_{l+\tau_2} \mathbb{E}[u(n-i)u(n-k)u(n-l)] \end{aligned}$$

For an input sequence of independent and identically distributed random variables, we note that

$$\mathbb{E}[u(n-i)u(n-k)u(n-l)] = \begin{cases} \gamma_3, & i=k=l \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$C_3(\tau_1, \tau_2) = \gamma_3 \sum_{i=-\infty}^{\infty} h_i h_{i+\tau_1} h_{i+\tau_2}$$

In general, we may thus write

$$C_k(\tau_1, \tau_2, \ldots, \tau_{k-1}) = \gamma_k \sum_{i=-\infty}^{\infty} h_i h_{i+\tau_1} \ldots h_{i+\tau_{k-1}}$$

Problem 1.24

By definition:

$$r^{(\alpha)}(k) = \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}[u(n)u^*(n-k)e^{-\mathrm{j}\,2\pi\alpha n}]e^{\mathrm{j}\,\pi\alpha k}$$

Hence,

$$r^{(\alpha)}(-k) = \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}[u(n)u^*(n+k)e^{-\mathrm{j}\,2\pi\alpha n}]e^{-\mathrm{j}\,\pi\alpha k}$$

$$r^{(\alpha)*}(k) = \frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}[u^*(n)u(n-k)e^{\mathrm{j}\,2\pi\alpha n}]e^{-\mathrm{j}\,\pi\alpha k}$$

We are told that the process $u(n)$ is cyclostationary, which means that

$$\mathbb{E}[u(n)u^*(n+k)e^{-\mathrm{j}\,2\pi\alpha n}] = \mathbb{E}[u^*(n)u(n-k)e^{\mathrm{j}\,2\pi\alpha n}]$$

It follows therefore that

$$r^{(\alpha)}(-k) = r^{(\alpha)*}(k)$$

Problem 1.25

For $\alpha = 0$, the input to the time-average cross-correlator reduces to the squared amplitude of a narrow-band filter with mid-band frequency ω. Correspondingly, the time-average cross-correlator reduces to an average power meter. Thus, for $\alpha = 0$, the instrumentation of Fig. 1.16 reduces to that of Fig. 1.13 in the book.