SIAM REVIEW
Vol. 5, No. 3, July, 1963
Printed in U.S.A.

DIFFERENTIAL EQUATIONS: LINEARITY VS. NONLINEARITY*

JACK K. HALE[1] and JOSEPH P. LASALLE[1]

FOREMOST AMONG THE MATHEMATICAL CHALLENGES in modern science and technology is the field of nonlinear differential equations. They are becoming increasingly important in fields as diverse as economics and space flight, ichthyology and astronomy. Since many of the equations defy complete solution, a good starting point in understanding their value and significance comes with an examination of the geometric point of view, which can be used to study solutions qualitatively. A logical next step is to review some of the basic properties and peculiarities of linear systems, after which it is easy to see that almost none of these properties hold for nonlinear systems—which can approximate nature more closely. Some of Liapunov's simple geometric ideas are invaluable in discussing the problem of stable performance, and this in turn helps to illustrate the modern theory of automatic control. The contrast between linear and nonlinear systems is striking, and this will be illustrated by simple examples. Much has been learned recently about differential equations, but there are still many major unsolved problems.

1. INTRODUCTION

In trying to analyze some process or system that occurs in nature or is built by man, it is almost always necessary to approximate the system by a mathematical model, where the state of the system at each instant of time is given as the solution of a set of mathematical relations. Differential equations are particularly useful models, so that this field of mathematics has become increasingly important to mechanical systems, biological systems, economic systems, electronic circuits, and automatic controls. Differential equations are, as Lefschetz has said, "the cornerstone of applied mathematics."

The intricate history of differential equations began around 1690 with Newton and Leibniz, and since then the theory of differential equations has challenged most of the world's greatest mathematicians, providing a tremendous stimulus for much of the modern development of mathematics.

Up to the time when Cauchy proved in the early 1800's that differential equations do actually define functions, mathematicians were primarily concerned with finding explicit solutions to the differential equations which arose when they applied Newton's laws of motion to elementary problems in mechanics and physics. The few solutions which they found had profound scientific and technological implications, for they provided the basis for the development of mechanics and much of classical physics—and spurred the industrial revolution.

* An article entitled "Analyzing Nonlinearity," which appeared in the June 1963 issue of International Science and Technology, was based in part on this paper.

Received by the editors, April 10, 1963.

[1] Research Institute for Advanced Studies (RIAS) Baltimore, Md.

Ideally one would like to find all explicit solutions of every differential equation, but it soon became apparent that this was an impossible task. Of necessity the emphasis turned to qualitative properties of the family of functions which was defined by the solutions of each differential equation. Real progress in this direction began near the end of the last century with the work of two of the greatest men in differential equations—Henri Poincaré of France and Alexander Mikhailovich Liapunov of Russia. Most of the modern theory of differential equations can be traced to these two men. Liapunov's work founded the great school of differential equationists which still flourishes in the Soviet Union, while much of the American effort can be traced to some of Poincaré's disciples. Poincaré dealt primarily with the geometric properties of solutions, together with some techniques for the computation of special solutions. On the other hand, Liapunov generally sought as much information as possible about the stability properties of solutions without actually knowing explicit solutions. The work of Poincaré, besides contributing to the theory of differential equations, stimulated many of the abstract developments in today's mathematics. On the other hand, Liapunov's contributions have led to methods which yield quantitative as well as qualitative information about stability and automatic control.

At this point one might well ask why, with the advent of high speed computers, there are any problems left. The answer is quite simple. If the information that is desired concerns only one particular solution, then computers are adequate. Frequently, however, what is wanted is general but qualitative information about all of the solutions, as well as those properties of the solutions which remain unchanged even when the equations themselves are perturbed. Moreover, in problems of design the engineer has an infinity of systems to choose from and he wants to identify those which have some special property or properties. For example, he may be looking for a system in which all solutions tend in time to have the same behavior, regardless of the initial state of the system, or the question may be to classify all those systems which have the same qualitative behavior.

This is not to say that the differential equations of today can satisfy engineering requirements perfectly—or that their use eliminates all problems. The equations still represent an idealization, and in most cases the equations themselves cannot be known precisely. Furthermore, one is usually unable to predict the exact perturbations which will affect the actual system. It is necessary therefore to discuss the sensitivity of the model to slight changes. Any system which is to perform a specified task must maintain stable performance under a broad range of perturbations.

There are certain systems of differential equations for which it is possible to find general solutions, and these include the so-called linear systems. The general solution of a linear system can be obtained by finding only a finite number of specific solutions. This fact explains the simplicity and also the limitations of linear systems. Almost no systems are completely linear, and linearity is another approximation to reality. One purpose of this article is to point out some of the limitations of a linear approximation and also to point out some of the new and interesting phenomena that occur when the differential equations are nonlinear. It is not simply that nature imposes nonlinearity on us. The fact of the matter is that man can use nonlinearity to improve the performance of the instruments and machines that serve him.

2. GEOMETRIC ANALYSIS

A geometric point of view is helpful in obtaining qualitative information about the solutions of a differential equation. We can demonstrate this by starting with an old law—Newton's second law of motion, whose statement in mathematical terms is a differential equation. Take the simplest situation of a particle of mass m moving along a straight line and subject to a force (Fig. 1). We select a point 0 on this line from which to measure distance, so that a positive distance will mean the particle is to the right of 0 and negative distance will mean it is to the left. The distance of the particle from 0 at time t we will call $x(t)$ or simply x. The derivative dx/dt is the velocity of the particle and the derivative of velocity d^2x/dt^2 is its acceleration. For purposes of simplicity we can adopt the convenient notation of Newton, with $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$. The force may well depend on the position and velocity of the particle, and we denote this force and the dependence by $f(x, \dot{x})$. Newton's second law of motion then states that

$$m\ddot{x} = f(x, \dot{x});$$

the mass times the acceleration is equal to the force applied. Since we need not be concerned at the moment with units of measurement, let us take the mass to be unity ($m = 1$). The differential equation for the motion of the particle is then

$$\ddot{x} = f(x, \dot{x}). \tag{1}$$

Neglecting the effects of the atmosphere, Newton's law for a freely falling body of unit mass near the earth is

$$\ddot{x} = -g, \tag{2}$$

where g is the acceleration of gravity. Near the earth, the gravitational acceleration g may be assumed to be constant. If we know the initial state of the system, which means we know the initial distance x_0 of the particle from 0 and its initial velocity y_0, then at time t

$$\begin{aligned} \dot{x} &= -gt + y_0 \\ x &= -\tfrac{1}{2}gt^2 + y_0t + x_0, \end{aligned} \tag{3}$$

This is the solution of the differential equation and is the law of freely falling bodies partially verified by Galileo from the Tower of Pisa.

To understand the geometric point of view of Poincaré and Liapunov we introduce a new symbol y for the velocity $\dot{x}$. Then $\dot{y} = \ddot{x}$, and the equation (1) involving a second derivative (a second order equation) can be expressed as two equations involving first derivatives (two first order equations):

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(x, y); \end{aligned} \tag{4}$$

or, to consider the more general situation:

$$\begin{aligned} \dot{x} &= p(x, y) \\ \dot{y} &= q(x, y). \end{aligned} \tag{5}$$

In the physical problem the numbers x, y describe the "state" of the system—the position and velocity of the particle. Thus if we introduce coordinates (Fig. 2) each point (x, y) in the plane represents a state of the system. This plane is sometimes called the "phase plane" or "state space" of the system. We can now look upon the differential equations (5) as defining a flow in the plane. When we are given the differential equations, we are given the velocity of the flow at each point of the plane (Fig. 2). The horizontal or x-component of the velocity of the flow is $p(x, y)$ and the vertical or y-component of this velocity is $q(x, y)$. This velocity of the flow, which is now something quite different from the physical velocity of the system, can be represented by a vector (bold-face type) $\mathbf{v} = (p(x, y), q(x, y))$. This vector describes at each point (x, y) how—in magnitude and direction—the state of system is changing. Starting at a given initial point the flow of this point defines a curved path, which is the curve defined by a solution of the differential equation.

Since this picture is important to further understanding, let us look at several

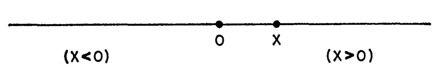

Fig. 1

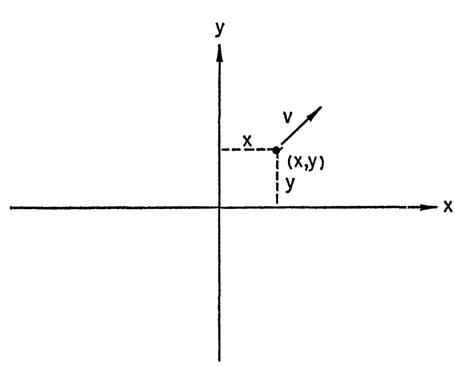

Fig. 2

examples. With $\dot{y} = x$, the equations for the freely falling body (eq. 2) are

$$\dot{x} = y$$
$$\dot{y} = -g.$$

Here the horizontal component of the flow is the height y of the point above the horizontal axis, and the vertical component is always negative and is constant. The solution (3) describes the flow precisely. Starting initially at the point (x_0, y_0), the solution is

$$y = -gt + y_0$$
$$x = -\tfrac{1}{2}gt^2 + y_0t + x_0.$$

The path defined by this solution is a parabola (Fig. 3). The arrows in Figure 4 show the direction of the flow with increasing time. Now remember that this flow describes the changing state of the system. Let the initial state be the point A of Figure 3. The particle—let us assume it is a ball—starts on the earth ($x = 0$) and is given an initial velocity upwards (y_0 is positive). The flow in the phase plane describing this motion is then along the parabola from A towards the point B. Until it reaches B the ball is rising (x is increasing) and its speed y is decreasing. At B the particle has reached its maximum height, it stops instantaneously ($y = 0$), and then starts to drop. Its state now follows

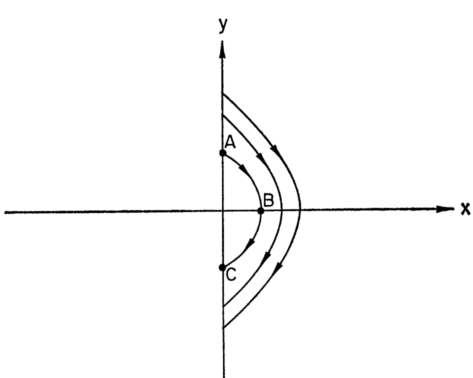

Fig. 3

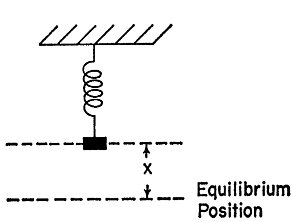

Fig. 4

the path from B to C. The velocity y of the ball becomes negative (it is falling) and its distance x from the ground is decreasing. The point C corresponds to the ball hitting the ground.

Let us also look quickly at a second example that will be of importance to us later on. It has to do with the motion of a mass attached to an elastic spring (Fig. 4), the so-called simple harmonic oscillator. The equations of motion are

$$\ddot{x} + x = 0, \tag{6}$$

where again since we need not be concerned with units of measurement we may take the mass and the coefficient of elasticity of the spring to be one. Here x measures the distance of the mass from its equilibrium position. The equivalent pair of first order equations is

$$\dot{x} = y$$

$$\dot{y} = -x.$$

At each point the flow is perpendicular (Fig. 5) to the radial line from the origin, and so we know that the flow is circular in a clockwise direction about the origin. This we may see directly from the differential equation, since $\frac{d}{dt}(x^2 + y^2) = 2x\dot{x} + 2y\dot{y} = 2xy - 2xy = 0$. The distance of a point in the flow from the origin does not change. For this particular system this is also the statement of the law of conservation of energy. Thus, if the initial state of the system corresponds to the point A, the circular flow describes the oscillation of the mass. At A it has its maximum displacement and the distance x_0 is the amplitude of the oscillation. Since the speed of the flow is $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{x^2 + y^2} = r$, the radius of the circle, the period of the oscillation (the time to make one revolution of the circle) is 2π. The period does not depend on the amplitude of the oscillation. The solutions of this differential equation give a good approximation of the small oscillations of a pendulum; and the fact that the period does not change with the amplitude of the oscillation was what Galileo observed as a young boy while watching a lamp swing in the cathedral of Pisa.

This geometric picture of the flow in state space is basic to the geometric or qualitative theory of differential equations. In the geometric theory one gives up the futile attempt of finding general solutions of all differential equations and attempts instead to obtain as much information as possible about the flow and the nature of the solutions defined by the flow without explicitly solving the equations. This was illustrated for the simple harmonic oscillator. Although the differential equations of motion could in this case have easily been solved (it is a linear problem), we obtained all of our information about the oscillations without exhibiting the solutions.

3. PROPERTIES OF LINEAR SYSTEMS

To understand some of the differences between linear and nonlinear differential equations, it is first necessary to discuss in some detail equation (5) when $p(x, y)$,

$q(x, y)$ are linear in x and y; that is, the equations

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned} \tag{7}$$

where a, b, c, d are real numbers.

The general solution of this linear system is obtained by knowledge of only two special solutions which are easy to compute. The other main implication of linearity is that the behavior of the solutions near the solution $x = y = 0$ yields the behavior of the solutions in the entire phase plane. Two further properties of linear systems are that all solutions are defined for all values of time and that there can never be isolated periodic solutions (see the example of the simple harmonic oscillator in the previous section). A few remarks will also be made about solutions which become unbounded in time (phenomena of resonance) when the simple harmonic oscillator is subjected to external forces. Even though the qualitative properties above are discussed only for the second order system (7), they are also true for n^{th} order systems.

In the case of second order systems we notice first of all that if (x_1, y_1) and (x_2, y_2) are any two solutions of (7), then for any constants e_1, e_2, the pair $(e_1x_1 + e_2x_2, e_1y_1 + e_2y_2)$ is also a solution. This property is generally referred to as the principle of superposition. One can verify rather easily that *the principle of superposition is the distinguishing characteristic of linear systems*; that is, if the principle of superposition holds for equation (5) for any functions $p(x, y)$, $q(x, y)$ which are continuous, then $p(x, y)$, $q(x, y)$ must be linear.

This principle makes it possible to find explicit solutions for linear systems. In fact, suppose $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$ are any two solutions of (7) which have the property that, for any constants u, v, there are constants e_1, e_2 such that the system of equations

$$\begin{aligned}x_1(0)e_1 + x_2(0)e_2 &= u \\ y_1(0)e_1 + y_2(0)e_2 &= v\end{aligned}$$

is satisfied. Then if $(x(t), y(t))$ is a solution of (7) with $x(0) = u$, $y(0) = v$ then $x(t) = e_1x_1(t) + e_2x_2(t), y(t) = e_1y_1(t) + e_2y_2(t)$ for all values of t, since this is true at $t = 0$ and there is only one solution of the differential equation passing through any given point. Consequently, to solve the linear equation (7) it is only necessary to exhibit two solutions with the above property. If λ is a root of the equation

$$\lambda^2 - \lambda(a + d) + ad - bc = 0, \tag{8}$$

then $x = e^{\lambda t}u$, $y = e^{\lambda t}v$ is a solution of (7) for some constants u, v. Furthermore, if the two roots are λ_1, λ_2 and $\lambda_1 \neq \lambda_2$, then the two solutions needed are of this form. If $\lambda_1 = \lambda_2$, one may need to consider functions $te^{\lambda t}$. In any case, the solutions are defined for all values of t. We continue the discussion for the case in which system (7) is of the form

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x - 2\rho y, \qquad \rho^2 - 1 \neq 0\end{aligned} \tag{9}$$

Under these hypotheses equation (8) has two distinct roots whose real parts are the same sign as ρ and for $\rho^2 < 1$ an imaginary part equal to $\sqrt{1 - \rho^2}$.

For this situation, we reach the following conclusions:

I) *If* $\rho < 0$, *then* ***all*** *solutions of* (9) *approach zero as t increases indefinitely;*

II) *If* $\rho > 0$, *then* ***all*** *solutions of* (9) *become unbounded as time increases indefinitely;*

III) (*the simple harmonic oscillator*) *If* $\rho = 0$, *then* ***all*** *solutions of* (9) *are periodic with the same period*, 2π.

The paths of the solutions for these three cases are shown in Fig. 6 for $\rho^2 < 1$. The word *all* should be emphasized, since this is a specific characteristic of linear systems. To be precise, the behavior of the paths of a linear system in a local neighborhood of the solution $x = 0$, $y = 0$ yields the behavior of the paths in the entire phase plane. *Linear systems are by nature provincial—global behavior can be predicted from local behavior.*

If an equation (7) is an idealization of some physicial system, then the constants a, b, c, d are determined by the special characteristics of the physical system itself. What happens if the system is subjected to external forces? The

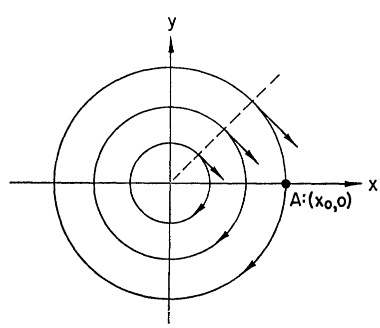

Fig. 5

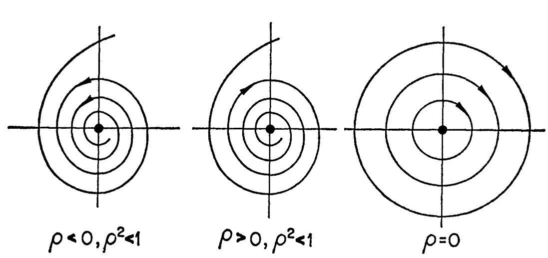

Fig. 6

new differential equations then have the form

$$\dot{x} = ax + by + f(t)$$
$$\dot{y} = cx + dy + g(t) \tag{10}$$

where $f(t)$, $g(t)$ are some given functions of time. We have already encountered one such equation in the discussion of a freely falling body near the earth; namely, equation (10) with $a = 0$, $b = 1$, $c = 0$, $d = 0$, $f(t) = 0$, $g(t) = g$, the acceleration due to gravity.

As another example consider the forced harmonic oscillator

$$\ddot{x} + x = \alpha \cos \beta t, \qquad \alpha \neq 0, \qquad \beta > 0,$$

or

$$\dot{x} = y$$
$$\dot{y} = -x + \alpha \cos \beta t \tag{11}$$

which has a particular solution

$$x_0 = \frac{\alpha}{1 - \beta^2} \cos \beta t, \qquad y_0 = -\frac{\alpha\beta}{1 - \beta^2} \sin \beta t$$

provided that $\beta^2 \neq 1$. Since $x_0^2 + y_0^2/\beta^2 = \alpha^2/(1 - \beta^2)$ the path in the phase plane is shown in Fig. 7 and is an ellipse whose semiaxes are $\alpha/(1 - \beta^2)$, $\beta\alpha/(1 - \beta^2)$. For any value of α, no matter how small, the amplitude of this oscillation can be made as large as desired by choosing β close to 1. For $\beta = 1$, the above solution is meaningless and (11) actually has an unbounded solution. This phenomenon is known as *resonance.*

In the case of an idealized pendulum, one can understand resonance intuitively. Suppose the pendulum is oscillating without the influence of an external force and suddenly one begins to apply a very small force at the point of maximum positive displacement from the origin in such a way as to assist the motion in its return to the origin (Fig. 8). Of course, if the model of the system is taken as linear, then the time the pendulum takes to return to the point of maximum positive displacement does not depend on the amplitude of the oscillation, and the applied force will be periodic of period 2π. If the model is truly linear, the amplitude of the motion will continually increase in time no matter how small the applied force; and eventually the pendulum will rotate around its point of support.

In the case of no damping in (9), that is, $\rho = 0$, we have seen that some types of bounded inputs or forcing functions to the system may yield unbounded outputs (resonance). On the other hand, a linear system (9) with damping ($\rho < 0$) always yields bounded outputs for all bounded inputs.

4. SOME ASPECTS OF NONLINEAR SYSTEMS.

The principle of superposition cannot hold for nonlinear systems. This implies that in general the solutions of a nonlinear system cannot be represented in

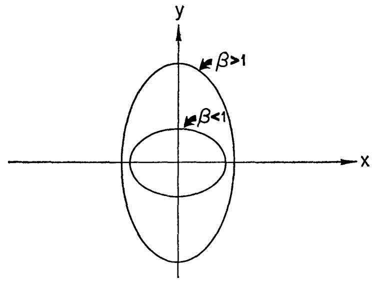

Fig. 7

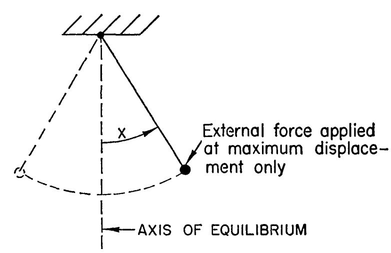

Fig. 8

a simple manner in terms of a special set of solutions, and thus the qualitative discussion of the solutions becomes even more important. In particular, it can be shown that the solutions of nonlinear systems generally have completely different local and global properties. Furthermore, all solutions need not be defined for all values of time, and there may be isolated periodic solutions (closed curves in phase space). Also, with a more realistic approximation to the harmonic oscillator, the phenomenon of resonance cannot occur. Even though none of the properties of linear systems are preserved, *some of the local properties of nonlinear systems can be determined by a linear analysis.*

An *equilibrium* or *rest point* of a differential system is a constant solution. For system (5), all equilibrium points are the constant pairs (c, d) such that $p(c, d) = 0$, $q(c, d) = 0$. At an equilibrium point, there is no flow in the phase space.

To make a very simple comparison between linear and nonlinear systems, consider the linear first order equation

$$\dot{y} = -y \tag{12}$$

and the nonlinear first order equation

$$\dot{y} = -y(1 - y) \tag{13}$$

The general solution of the linear equation (12) is given by $y = ae^{-t}$ where a

is an arbitrary constant and is equal to the initial state of the system at time zero. On the other hand, if the initial state of system (13) at time zero is a, then by direct calculation one observes that the solution of equation (13) is

$$y = \frac{a}{1 - a + ae^{-t}} e^{-t}$$

The curves in the phase space y and the phase-time space (y, t) for the two equations for various values of a are shown in Fig. 9. As is observed from this figure, new phenomena occur even for the simplest monlinear equations. For nonlinear systems there can be more than one equilibrium point, each of which is isolated; and some solutions of a nonlinear system may become unbounded in a finite interval of time. Also, the behavior of the solutions of (13) with initial values greater than one is completely different from those with initial values less than one. None of these situations occurs in linear systems.

It is possible for linear analysis to yield some information about (13). We first study the solutions near the equilibrium point $y = 0$. Since the equation is $\dot{y} = -y + y^2$ it would seem reasonable (and it was proved by Liapunov) that the equation $\dot{y} = -y$ is a good approximation near $y = 0$. An inspection of the curves in Figures 9a and 9b shows this is actually the case. However, an analysis of $\dot{y} = -y(1 - y)$ which is based on $\dot{y} = -y$ for all initial values is erroneous, since all of its solutions do not approach zero and some are actually unbounded. To analyze the behavior of (13) near $y = 1$, let $y = 1 + z$ and study the behavior near $z = 0$. The new equation for z is

$$\dot{z} = z + z^2.$$

Near $z = 0$, it again seems reasonable to take the linear approximation $\dot{z} = z$ whose solutions are $z = be^t$, where b is the initial state at zero. The linear analysis then yields the result that solutions of (13) near $y = 1$ tend to diverge from $y = 1$, which is actually the case. On the other hand, the linear analysis would also say that all solutions are defined for all values of t, whereas we have seen this is not the case. Thus we see that *linear approximation is often useful but has its limitations.*

As indicated above, it is impossible for linear homogeneous equations (7) to have isolated periodic solutions; but this is clearly not the case for nonlinear equations. Consider the second order equation

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + 2\rho(1 - x^2)y \end{aligned} \tag{14}$$

where ρ is a positive constant.

Equation (14) is called van der Pol's equation. He studied it in connection with his work on the triode oscillator; and for large values of ρ, he also suggested that this equation explains some irregularities in the heart beat. The same equation in a different form was studied by Lord Rayleigh in his investigations of the theory of sound.

To understand (14), we first observe that $x = 0$, $y = 0$, is an equilibrium point and analyze the behavior of nearby solutions by linearizing the equations. These linear equations are (9) with $\rho > 0$. Thus, all solutions of the linear equations leave the equilibrium point as t increases. Liapunov has shown that the same result is true for the solutions of (14) when the initial values are sufficiently close to the equilibrium point but different from it. Near the rest point and for $\rho^2 < 1$, the solution curves of (14) in the phase plane are shown in Fig. 6a.

On the other hand, the term $2\rho(1 - x^2)y$ in (14) represents a frictional force and for large values of x, this frictional force actually has a damping effect. As a result of this, one can show that the solutions of (14) are bounded and there actually is a curve C in the (x, y)-plane across which the solution curves move from the outside inward. By this linear analysis, one can also construct a curve C_1 which lies inside C such that the solution curves move from the inside of C_1 to the outside. This is depicted in Fig. 10 where the arrows designate the direction of the motion along solution curves. Since there are no equilibrium points in the region between C_1 and C, intuition leads to the conjecture that there must be a closed solution curve (yielding a periodic motion) in this region. This is precisely the case and is a consequence of a nontrivial result proved around the turn of the century by Bendixson. By a more detailed argument, one can actually

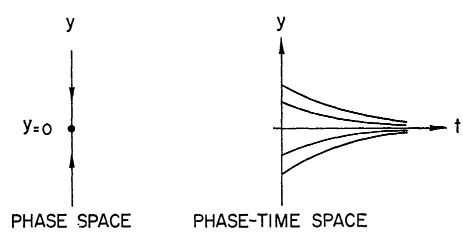

Fig. 9 (a) MOTION OF LINEAR SYSTEM (12)

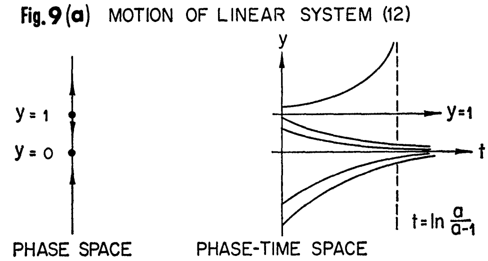

Fig. 9 (b) MOTION OF NONLINEAR SYSTEM (13)

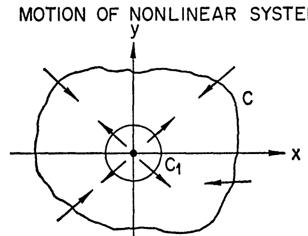

Fig. 10

show there is only one closed solution curve of (14), and all other solution curves except $x = 0$, $y = 0$, approach this curve as t increases indefinitely (see Fig. 11).

Oscillations of the above type are called self-sustained, since they occur without the influence of any external forces but arise simply from the internal structure of the system and the manner in which energy is transferred from one state to another. The point to be made is that such *self-sustained oscillations can only be explained by a nonlinear theory.*

The motion of a clock or watch can be explained by an investigation of self-sustained oscillations, and many attempts have even been made to apply the concept in explaining the interaction of various biological species. For example, suppose there are two isolated species of fishes, of which one species feeds on the other—which in turn feeds on vegetation which is available in unlimited quantity. For this situation, it is possible to construct two first-order differential equations (whose solution gives the quantities of each species at time t) in such a way as to give very realistic information. The differential equations have a self-sustained oscillation (that is, the quantity of each species varies periodically in time); and for every initial state (except the equilibrium states) the number of each species approaches this particular periodic behavior.

A further contrast between linearity and nonlinearity is that nonlinearity can prevent resonance in the sense described in the section on linear systems. In our discussion of the pendulum, the linear equation (6) was said to be a good approximation for small amplitudes. A more realistic approximation is the equation

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \tfrac{1}{6}x^3 \end{aligned} \tag{15}$$

Observing that $\frac{d}{dt}(x^2 + x^4/12 + y^2) = 0$, we see that the solution curves lie on the curves $x^2 + x^4/12 + y^2 = \text{constant}$. In a neighborhood of $x = y = 0$, the paths are closed and thus the solutions are periodic, but now the period varies with the path, in contrast to linear systems. In our intuitive discussion of resonance on p. 257, the constancy of the period was the main element for the plausibility of resonance. But, if the period varies with amplitude, then a periodic disturbance will become out of phase with the free motion and the forcing function should be a hindrance to increasing amplitude. This can be shown to be the case for systems without damping in which the period of the oscillations varies with amplitude (in which case the system must be nonlinear), small periodic disturbances do not cause instability. Here is a situation where stability is a consequence of nonlinearity, even though no frictional forces are present. It is typical of the advantages which may sometimes be obtained by deliberately introducing nonlinearities into systems.

5. STABILITY

Stability is one of those concepts that we all understand even though we might not be able to define it precisely. We speak of a stable personality or a stable economy, and we recognize the relative stability in both voltage and frequency of the electric power supplied to our homes. Basically we have a system of some type which operates in some specified way under certain conditions. If these conditions change slightly, how does this affect the operation of the system? If the effect is small, the system is stable. If not, the system is unstable. Take the simple example of a ball rolling along a curve in a plane. The points A, B, C and D of Figure 12 are the rest positions (equilibrium states) of the ball. Positions A and C are clearly unstable, since the slightest perturbation will send the ball rolling down the hill. The positions B and D are stable, since a slight perturbation causes the ball to oscillate about the rest position but remain close to it. This is a particular instance of a quite general physical principle. In mechanics, potential energy at a point is the amount of work expended in moving a unit mass to that point from some arbitrary but fixed point. The principle then states that *a minimum of potential energy corresponds to a stable rest position. If the rest position is not a minimum it is unstable.* This is so easily comprehended that physics teachers seldom bother to prove it.

This law was enunciated by Lagrange around 1800, but it was Liapunov some 90 years later who was the first to appreciate fully this principle and its extensions. Liapunov's extension is called his "direct" or "second" method for the study of stability. The method is said to be direct because his criteria for stability

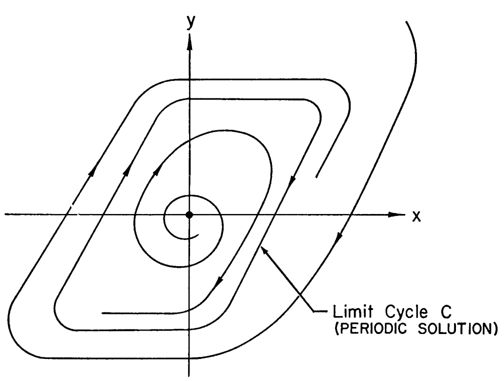

Fig. 11

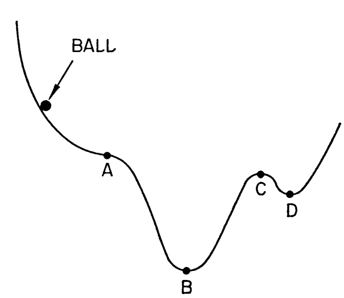

Fig. 12

and instability can be applied as soon as the differential equations of the system are known and requires no specific knowledge of their solutions.

But let us now examine these concepts more precisely in terms of our fundamental system

$$\begin{aligned} \dot{x} &= p(x, y) \\ \dot{y} &= q(x, y). \end{aligned} \tag{16}$$

Here (x, y) represents the deviation from some desired state or performance of the system. The origin 0 then corresponds to the desired behavior and is an equilibrium state of the system: $p(0, 0) = 0$ and $q(0, 0) = 0$. There is no flow at this point, and if the system is initially at 0 it remains for all time at that point. However, such a statement concerns the mathematical model and neglects reality, and this is precisely why we are interested in stability. Stability is concerned with the question of what happens when the system is perturbed from this equilibrium state.

Let C_1 be an arbitrary circle about the origin. The if for each such C_1 we can find a concentric circle C_2, which is so located that solutions which start inside C_2 remain inside C_1 (Fig. 13), we say that the origin is *stable*. If this is not the case we say that it is *unstable*. If there is also another circle C_0 about a stable origin which is such that solutions starting inside C_0 tend to *return* to the origin (Fig. 14), we say that the origin is *asymptotically stable*; and if *all* solutions tend to return to the origin, the origin is *asymptotically stable in the large*. For example,

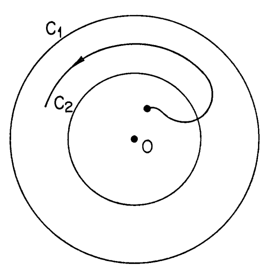

Fig. 13

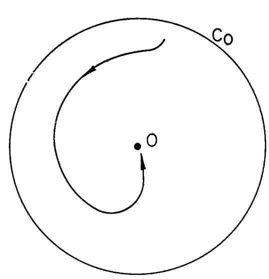

Fig. 14

in Figure 6c the origin is stable but not asymptotically stable. In Figure 6a the origin is asymptotically stable and in Figure 6b the origin is unstable.

Consider once again van der Pol's equation (14) which can be written as

$$\ddot{x} - 2\rho(1 - x^2)\dot{x} + x = 0.$$

Letting $y = \dot{x} - 2\rho(x - x^3/3)$, $\epsilon = -2\rho$, we obtain the two first-order equations

$$\begin{aligned} \dot{x} &= y + \epsilon(\tfrac{1}{3}x^3 - x) \\ \dot{y} &= -x. \end{aligned} \tag{17}$$

In contrast to the previous discussion, we will assume $\epsilon = -2\rho$ is positive ($\epsilon > 0$). This is equivalent to reversing the flow in the phase plane, and the picture of the flow is now that of Figure 15. The origin is asymptotically stable, since every solution inside the limit cycle tends toward the origin. However, if the system were to be perturbed to a state outside the limit cycle this is no longer the case, and it can actually be shown that there are solutions outside the limit cycle that go to infinity in finite time. The region inside the limit cycle is called the *region of asymptotic stability*, and it is the size and shape of this region that determines just how stable the system is.

It so happens in this case it is possible to decide the asymptotic stability of the origin by omitting the nonlinear term $x^3/3$ and examining only the linear approximation. Since the linear approximation is asymptotically stable it is always asymptotically stable in the large. The region of asymptotic stability for the linear system is always the whole space, and the linear approximation does not and cannot give any information as to how stable the system is. *The extent of the stability of a system is determined by its nonlinearities.* A decided advantage of Liapunov's method is that it takes into account the nonlinearities of the system and permits an estimation of the region of asymptotic stability without calculating any solutions. Even in van der Pol's equation, the calculation of the limit cycle C (which determines the region of asymptotic stability) is extremely complicated, but fortunately the existence of a rather large region of asymptotic stability can be established without knowing C.

Liapunov's method, like Lagrange's principle, is extremely simple. In essence Liapunov's method consists of selecting a suitable function $V(x, y)$ in phase space such that (a) it has a minimum at the equilibrium point being investigated, and (b) the contours (surfaces along which the function is a constant) of the function surround the equilibrium point. Then if the flow of solutions in phase space can be shown to cross these contours from the outside towards the inside the equilibrium point is stable, at least for perturbations that keep the state of the system within the largest contour for which the flow is always inward.

In carrying out Liapunov's method we actually check whether the Liapunov function, $V(x, y)$, decreases along solution-flow lines. Since we required that $V(x, y)$ have a minimum at the equilibrium point, the fact that it decreases along solutions means these solutions are crossing the contours of $V(x, y)$ in the desired direction. Remember, we do this without solving the differential equation.

Of course, nothing comes for free in this life—the trick is to be able to find a suitable Liapunov function. We chose the coordinate system in the example of van der Pol's equation that follows so that $V(x, y)$ would have a particularly simple form; don't be deceived thereby, it is not always that easy.

For van der Pol's equation we can take

$$V(x, y) = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2.$$

At the origin $V(0, 0) = 0$ and everywhere else $V(x, y)$ is greater than zero. Computing the rate of change $dV/dt = \dot{V}$ of V along solutions we obtain from equation (17)

$$\dot{V} = x\dot{x} + y\dot{y} = -\tfrac{1}{3}\epsilon x^2(3 - x^2).$$

Within the circle of radius $x^2 + y^2 < 3$, x^2 is less than 3 and $\dot{V} \leqq 0$. Actually within this circle $\dot{V} < 0$ if $x \neq 0$. This tells us that within any circle $x^2 + y^2 = c^2 < 3$ the solutions cross the circle as shown in Figure 16. This means that every solution starting inside $x^2 + y^2 = 3$ tends to the origin as t tends to infinity. The origin is asymptotically stable, and we have the additional information that independently of ϵ (all $\epsilon > 0$) the region of asymptotic stability in this coordinate system is always at least as large as this circle. This also gives us information about the limit cycle C. In fact, for all non-zero ϵ, the limit cycle lies outside this circle of radius $\sqrt{3}$.

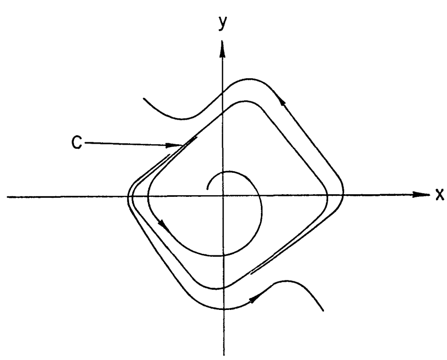

Fig. 15

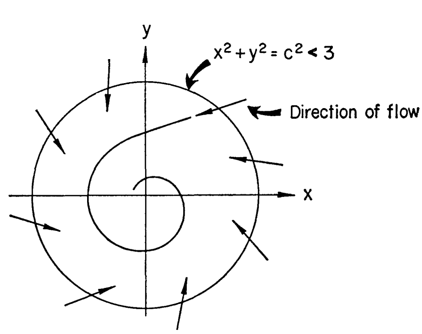

Fig. 16

The Liapunov method which we have discussed is the only general method available at present for the study of stability and has been widely used, particularly in the Soviet Union, to solve practical problems of stability. Success has been achieved in studying the stability of motion of rigid bodies, in investigating the stability of a large class of nonlinear control systems, and in determining the stability of nuclear reactors. The method and its extensions also play an important theoretical role in the study of differential equations.

In theory it should always be possible to decide stability by this method, but there remains the practical difficulty of constructing suitable Liapunov functions —functions of the type V used above. There are some methods which guide one in the construction of Liapunov functions but effective use of the method requires ingenuity and experience. There are, as yet, no general schemes for using computers to decide the stability of nonlinear systems.

6. THEORY OF AUTOMATIC CONTROL

Man has been aware of the feedback principle of automatic control through most of recorded history, and man himself embodies the principle. The Babylonians, some 2000 years before the birth of Christ, recognized a primitive form of feedback in their opening and closing of ditches to control the moisture content of the soil. A more sophisticated feedback was used in highly accurate Arabian water clocks developed near the beginning of the Christian era. It was a float-operated valve, similar to that still used today in our bathrooms, to control the water level in a tank. A classic example of a conscious application of feedback to achieve automatic control is the flyball-governor which Watt used in 1788 to regulate the speed of a steam engine under a varying load. As the engine's speed deviated from the desired value, the change in speed itself was used to regulate the throttle and correct the error. Through the development of electronics and more sophisticated mechanical devices, we now have automatic control systems of great complexity, yet the basic principle of feedback control remains the same—whether it be used for the automatic control of an industrial process, a phase of our economy, a nuclear reactor, or the attitude of a space vehicle.

Our interest is in the fact that this principle and its application can be subjected to mathematical analysis. This was first demonstrated by a British physicist, Maxwell, in 1868 and independently by a Russian engineer, Vyshnegradskii, around 1876. They concerned themselves mainly with small errors, that is, small deviations from the performance desired, and they approximated the actual system by a linear one. They observed that an improper use of feedback could enhance the error instead of diminishing it and that the proper use of feedback involves an analysis of stability. Does the error tend to zero or not?

This same procedure of linear approximation and analysis of stability was the mathematical foundation for Minorsky's design of an automatic steering device for the battleship New Mexico in the early 1920's. The behavior he predicted on the basis of his simple mathematical model was then confirmed by observed fact. In one of those strange quirks of history, however, the U. S. Navy did not adopt automatic steering at that time for the simple reason that it might deprive helmsman of practice. The logic is similar to that of the British Admiralty when they first rejected the idea of using steam engines on warships on the grounds that sparks from the engines would burn holes in the sails.

Even during World War II, in spite of all the sophisticated developments in control, the theory was restricted for the most part to linear approximations, linear feedback, and a linear analysis of stability. Even though the analysis was made in this limited fashion, optimization of certain performance criteria was quite successful. However, in the past decade there has been an explosive growth in the development of a theory of control in the United States and the Soviet Union. A complete bibliography on almost any one aspect of the subject may amount to 300 or 400 papers. Fortunately, it is possible to point out a few basic aspects of the subject and to illustrate the theory simply by comparing different solutions of a single not-too-complex problem.

The problem is that of moving a body through a viscous fluid with a force (thrust) F which can drive or impede the motion of the body. The equations of motion are

$$\ddot{x} + \beta\dot{x} = F \tag{18}$$

where β is positive and is the coefficient of the viscosity of the fluid. Let us suppose that what we want to control is the speed $\dot{x}$ of the body; that is, we want $\dot{x} = a$ where a is a given constant. Our ability to control the motion of the body lies in our ability to select F. We are interested only in velocity, so with $\dot{x} = y$, we have the first order equation

$$\dot{y} + \beta y = F.$$

First let us look at what the engineer calls "open loop" control as opposed to "closed loop" or feedback control. In open loop control no information concerning the deviation from desired performance is used to adjust the control force F. There is no connection between the output y and the input F. The control loop is open. Thus, since we are not using information on whether the body is going too fast or too slow, the best that we can do is to make F a constant. In this case the solution for the velocity is

$$y = y_0 e^{-\beta t} + \frac{F}{\beta}(1 - e^{-\beta t}),$$

where y_0 is the initial velocity of the body. Thus, independently of the initial velocity, y approaches F/β exponentially as t approaches infinity. We want this terminal velocity to be a and select $F = \beta a$. This is the desired result but the approach to a will be quite slow if β is small. As we shall see, this open loop control also has a more serious disadvantage.

Now let us compare this with the feedback control $F = -\delta(y - a) + \beta a$, where δ is some positive constant. The control force F now depends linearly upon the difference between the actual velocity of the body and that desired. This is linear feedback control. The differential equation of the controlled system is

$$\dot{y} + \beta t = -\delta(y - a) + \beta a$$

or

$$\dot{y} + (\beta + \delta)y = (\beta + \delta)a.$$

The feedback has the effect of changing the coefficient of viscosity and now

$$y = y_0 e^{-(\beta+\delta)t} + a(1 - e^{-(\beta+\delta)t}).$$

Again, independently of the initial velocity, y approaches a but at a faster rate. Note also that for a small error in control ($(y - a)$ small) the control force being used is about the same as before.

Although this in itself illustrates some gain in using feedback control, a more important advantage is the increased stability of operation achieved by feedback. Suppose that we do not know the value of β exactly or that we want a system which will operate over a range of values of β. To make matters simple, suppose that when we originally designed the system we thought that the coefficient of viscosity was β but now the body is moving in a fluid whose coefficient of viscosity is $\frac{1}{2}\beta$ (it has become warmer). Under open loop control the equation of motion is in this circumstance

$$\dot{y} + \tfrac{1}{2}\beta y = \beta a$$

and with feedback control

$$\dot{y} + \tfrac{1}{2}\beta y = -\delta(y - a) + \beta a.$$

Under open loop control the velocity y now approaches $2a$, an error of 100 per cent. Under feedback control y approaches

$$\frac{\beta + \delta}{\frac{1}{2}\beta + \delta} a,$$

and by making δ large we can make the control insensitive to changes in viscosity. For example, if $\delta = 10\beta$, y approaches $\frac{22}{21}a$, which is an error of less than five per cent. *Feedback control makes the system relatively insensitive to changes in environment.*

But there is still a more important feature to feedback control. In a sense feedback control enables a system to adapt to its environment. The linear feedback of the above example will now be adjusted by means of a nonlinear feedback in such a way that the system adapts to changes in viscosity. This suggests that *nonlinearity is essential to adaptation.*

Assume now that we have no information about β except that it is constant. We also no longer assume that β is positive (if β is negative, then the system we are controlling is unstable). The controlled system will now be of the following form:

$$\dot{y} + \beta y = (b - \beta_0)(y - a) + ba$$
$$\dot{b} = g(y - a).$$

The linear feedback control, which is the right-hand side of the first equation, was selected by asking ourselves what would work if we knew β and afterwards replacing β by an adjustable parameter b. The second differential equation determines the adjustment of b, and this adjustment is to depend only upon the observation of the error $y - a$. The function g remains to be selected, and β_0 is any positive number which we are also free to select. Thus the control assumes no knowledge of the constant β. What we want is to have y approach the desired velocity a for all initial velocities y_0 and any initial value of b. The function g, which determines the design of the mechanism for adjusting b, can be selected using an extension of Liapunov's method. We want y to approach a and we may suspect that b will approach the unknown constant β. This then suggests that we take as a Liapunov function

$$V = \tfrac{1}{2}(y - a)^2 + \tfrac{1}{2}(b - \beta)^2.$$

Using the differential equations of our system, we obtain after a simple computation the rate of change of V along solutions to be

$$\dot{V} = (y - a)\dot{y} + (b - \beta)\dot{b} = -\beta_0(y - a)^2 + (b - \beta)(g + y(y - a)).$$

Thus, if we take $g = -y(y - a)$,

$$\dot{V} = -\beta_0(y - a)^2.$$

This means that V is always decreasing as long as there is an error in control, and it can be shown that y approaches a and that b does in fact approach β. The choice of β_0 is still open, but the appearance of β_0 in the equation for $\dot{V}$ tells us that the larger the value of β_0 the faster y approaches a. Thus, regardless of the value of β, this control system always reduces the error to zero and the feedback adjusts itself to adapt to the environment.

This leads to the most recent advancement in control theory—the study of optimal processes of control. The problem is, relative to some performance criterion, to select within a given class of available controls the best possible control. The limitations of space make it impossible to carry out the solution in detail in this article, but it is feasible to illustrate the nature of the problem and the nature of the solution that can be derived from the modern theory of optimal control. We are still dealing with a body moving in a viscous fluid, but now let us assume that we want to bring it to rest at the point (0, 0) in the phase plane in minimum time. Realistically, assume that our power source is limited, placing constraints on the force F available for control. We might assume, for example, that F must lie between -1 and 1 ($-1 \leqq F \leqq 1$). The equations of motion are as before

$$\ddot{x} + \dot{x} = F$$

and the equivalent system is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -y + F, \qquad -1 \leqq F \leqq 1. \end{aligned}$$

The desired state of the system is $x = 0$, $y = 0$, and we want to bring the system to this state in the shortest possible time. There is available, as we have said, a theory for solving this problem; and the optimal control law is shown in Figure 17. The phase plane is divided into two parts by a curve C whose equation is

$$x = -y + \log_e (1 + y), \qquad y \geqq 0$$
$$x = -y - \log_e (1 - y), \qquad y \leqq 0.$$

This curve is called the "switching curve." At a state (x, y) above this switching curve optimal control is obtained by taking $F = -1$ and below the curve $F = 1$. In other words, F changes sign at the switching curve. This optimal control law is unique. Any other control law satisfying the constraint $-1 \leqq F \leqq 1$ takes longer to bring the system to rest at $x = 0$, $y = 0$.

The control law here is said to be "bang-bang". It jumps rapidly from -1 to 1 and at all times uses the maximum control available, which is intuitively what one expects. The control, although made up of linear pieces, is highly nonlinear on the whole and does what no linear control could do—it reduces the error to zero in finite time. Under linear control the error can only tend to zero exponentially. *Nonlinear control can achieve performance far beyond that possible by linear control.*

7. FUTURE TRENDS

Most of the important questions in second order systems have now been answered. This is not surprising since the geometry of integral curves in two

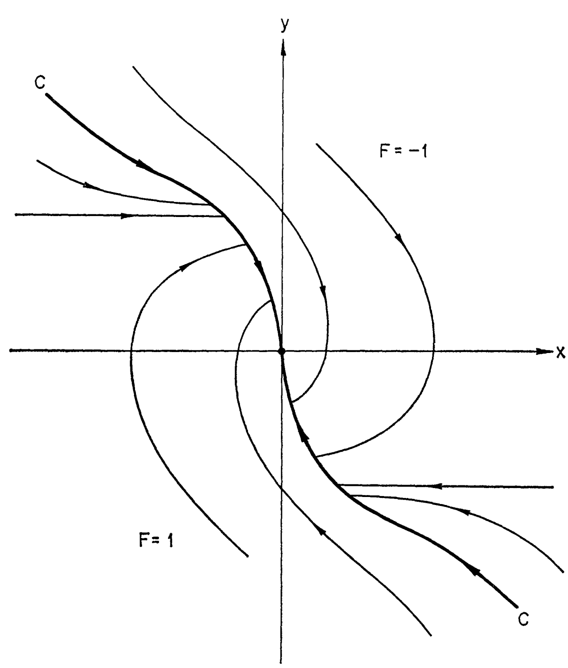

Fig. 17

dimensions (in a plane) is not too complicated. More precisely, the only possible solution curves which can be limits of other solution curves (that is, approached as time increases indefinitely or decreases indefinitely) are points of equilibrium or closed curves. As soon as third order systems are considered, however, the geometry becomes vastly more complicated, and it is possible to have limit curves which "fill up" a torus (the surface of a doughnut). Such limit curves still possess oscillatory properties, and one of the most important problems today, as it has also been for the last 75 years, is to characterize the possible solution curves for an n^{th} order system which can be limits of other solution curves. Much intelligent experimentation by engineers is still needed to discover the type of oscillatory phenomena that can occur in practice for higher order systems of differential equations, and this should lead to interesting new classes of nonlinear differential equations which will have to be investigated. For cases in which the nonlinearities are small, specialized techniques are already available for systems of any order.

One of the most important remaining problems which concerns stability of an equilibrium point is that of obtaining general methods or algorithms for the construction of Liapunov functions. A solution of this problem could lead to an efficient use of computers in deciding questions of stability. Another important area involves characterising those properties of the geometric structure in the phase space of all of the solution curves of an n^{th} order system which remain unchangedeven when the differential equations themselves are subject to arbitrary but small perturbations. This problem is completely solved for second order equations, but very little is known for higher order systems.

Among astronomical problems, it has still not been proved that a system involving the interaction of n planets is stable. Even if it is, however, it is important to know whether it would still be stable under relatively small perturbations. For example, what would happen if a planet exploded? This problem has recently been solved for periodic disturbances, and is attracting the attention of some of the best young mathematicians in the world.

Since the current views of automatic control are somewhat different from those of the past, there are of course many interesting and unsolved problems in this area. Given a performance criterion and a specified law of control, it is not always clear that an optimum control law exists. Pontryagin's maximum principle gives necessary conditions for the existence of an optimum, but sufficient conditions have only been given in very particular situations. After the existence has been established, efficient methods must be devised for the computation of the optimal control laws. Computational methods exist for systems where the control enters linearly, but the computation for nonlinear systems still must proceed by local linearization. The solution of many more specific examples of automatic control will certainly lead to great strides in the entire area.

Finally, it is difficult to see how a system can be made to be truly adaptive without having the control law depend upon a certain portion of the past history of the system. Insertion of such hereditary dependence in the problem takes one beyond the realm of differential equations. Much research is now being conducted on this extended type of equation with the initial aim being to understand the essential differences between these and ordinary differential equations. It is quite clear that the applications of equations with hereditary dependence will reach beyond automatic control, and, historically, they were being discussed long before the conception of adaptive control.

In a way, this is typical of the entire history of differential equations—particularly those involving nonlinearity. Specific practical application has usually stimulated the initial interest in classes of differential equations. As soon, however, as significant progress is made in any given area, a whole host of applications becomes apparent.

RECOMMENDED READING

Good elementary introductions to the modern theory of differential equations are a book by Birkhoff and Rota, *Ordinary Differential Equations*, Ginn, Boston, 1962, and a book by Walter Leighton, *Ordinary Differential Equations*, Wadsworth, Belmont Caliifornia, 1963. For a discussion of the theory of nonlinear oscillations which is not too sophisticated mathematically and contains many physical applications, the reader should consult Minorsky, *Nonlinear Oscillations*, Van Nostrand, Princeton, N. J., 1962. A book by LaSalle and Lefschetz, *Stability by Liapunov's Direct Method*, Academic Press, New York, 1961, gives a self-contained and rather elementary introduction to the subject of stability as well as some applications to control problems.

A well-written treatment of oscillations of all sorts is contained in J. J. Stoker's *Nonlinear Vibrations*, Interscience, New York, N. Y., 1950. Russian contributions to the field are summarized succinctly and critically in a chapter by one of the co-authors, LaSalle and Solomon Lefschetz, that forms Part 6 of, *Recent Soviet Contributions to Mathematics*, Macmillan, New York, N. Y., 1962, which LaSalle and Lefschetz also edited.

A unified treatment of oscillations is given in *Oscillations in Nonlinear Systems* by Hale, McGraw-Hill, New York, N. Y., 1963.

The only book presently available in English on modern control theory is Pontryagin, Boltjanskii, Gamkrelidze, and Mishchenko, *The Mathematical Theory of Optimal Processes*, Interscience, New York, N. Y., 1962. More elementary books on this subject will probably appear in the near future.

More advanced books on the theory of differential equations are S. Lefschetz, *Differential Equations: Geometric Theory*, 2nd Edition, Interscience, New York, N. Y., 1963 and E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, N. Y., 1955.