Selected Solutions To Problems in Complex Analysis

E. Chernysh

November 13, 2016

Contents

Page 28 2
- Problem 1 . . . 2
- Problem 4 . . . 2
- Problem 5 . . . 2

Page 78 3
- Problem 1 . . . 3
- Problem 2 . . . 4
- Problem 3 . . . 4
- Problem 4 . . . 5

Page 80 5
- Problem 1 . . . 5

Page 108 6
- Problem 3 . . . 6
- Problem 5 . . . 6

Page 123 7
- Problem 1. . . . 7
- Problem 2. . . . 8
- Problem 3. . . . 8
- Problem 5. . . . 8

Page 161 8
- Problem 1 . . . 8
- Problem 3. . . . 10

Page 154 11
- Problem 1. . . . 11
- Problem 2. . . . 11

Page 133 12
- Problem 1. . . . 12
- Problem 2. . . . 12
- Problem 3. . . . 12
- Problem 4. . . . 13

The Logarithm and a Useful Construction 13

Some Additional Problems **17**

These are mostly problems from Ahlfors' Complex Analysis.

PAGE 28

PROBLEM 1

Let $f : \Omega \subseteq \mathbb{C}$ be holomorphic in Ω and $g : \Lambda \to \mathbb{C}$ be holomorphic where $f(\Omega) \subseteq \Lambda$. Then the map

$$g \circ f : \Omega \to \mathbb{C}$$

is holomorphic. Indeed, it suffices to show that $g \circ f$ is (complex) differentiable at every point of Ω, and hence as Ω is open it will follow that the composition is a holomorphism. The chain rule easily yields this.

PROBLEM 4

If $f : \Omega \to \mathbb{C}$ is a holomorphism in a domain $\Omega \subseteq \mathbb{C}$ with $|f| \equiv M$, then f is constant. Note that the case where $M = 0$ is instant as this implies

$$|f|^2 = \Re(f)^2 + \Im(f)^2 \equiv 0$$

and hence $\Re(f)^2 \equiv \Im(f)^2 \equiv 0$. Now consider the case where $|f|^2 \equiv \Re(f)^2 + \Im(f)^2 =: u^2 + v^2 \equiv M > 0$ for any $z \cong (x, y) \in \mathbb{R} \times i\mathbb{R}$. Then differentiation with respect to x and y will yield

$$2\frac{\partial u}{\partial x}u + 2\frac{\partial v}{\partial x}v \equiv 0 \tag{1}$$
$$2\frac{\partial u}{\partial y}u + 2\frac{\partial v}{\partial y}v \equiv 0 \tag{2}$$

We now employ the Cauchy-Riemann equations to write this only in terms of $\dfrac{\partial}{\partial x}$ as follows:

$$\begin{aligned} 2\frac{\partial u}{\partial x}u + 2\frac{\partial v}{\partial x}v &\equiv 0 \\ -2\frac{\partial v}{\partial x}u + 2\frac{\partial u}{\partial x}v \equiv 2\frac{\partial u}{\partial x}v - 2\frac{\partial v}{\partial x}u &\equiv 0 \end{aligned}$$

Or, equivalently

$$\begin{pmatrix} u & v \\ v & -u \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note that the coefficient matrix is invertible as it has determinant $\det \begin{pmatrix} u & v \\ v & -u \end{pmatrix} = -(u^2 + v^2) = -M \neq 0$ by hypothesis. This yields that $\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \mathbf{0}$. A final application of the Cauchy Riemann equations yields $\nabla f \equiv \mathbf{0}$ over $\mathbb{C}$.

PROBLEM 5

Here we show that f is holomorphic in $\mathbb{C}$ if and only if $\overline{f(\overline{z})}$ also is. Note that by symmetry of conjugation, we need only show one implication. One can show this via algebraic manipulation of the limit definition. I, however, prefer to show that $\overline{f(\overline{z})}$ is also a solution to the Cauchy Riemann system of PDEs. Assuming that f is a holomorphism, we must have

$$u_x(x,y) = v_y(x,y), \quad u_y(x,y) = -v_x(x,y) \tag{3}$$

where $f(z) = u(x,y) + iv(x,y)$. Then, we also have $\overline{f(\overline{z})} = u(x,-y) + i(-v(x,-y))$. Thus, we may express

$$\overline{f(\overline{z})} = \alpha(x,y) + i\beta(x,y)$$

for $\alpha = u(x,-y)$ and $\beta = -v(x,-y)$. We now verify the Cauchy Riemann equations:

$$\alpha_x(x,y) = u_x(x,-y) = v_y(x,-y)$$
$$\beta_y(x,y) = \frac{\partial}{\partial y}\left[-v(x,-y)\right] = v_y(x,-y)$$

Moreover,

$$\alpha_y(x,y) = \frac{\partial}{\partial y}\left[u(x,-y)\right] = -u_y(x,-y) = v_x(x,-y)$$
$$-\beta_x(x,y) = -\frac{\partial}{\partial x}\left[-v(x,-y)\right] = v_x(x,-y)$$

Which concludes the proof, as being holomorphic is equivalent to solving the Cauchy Riemann equations.

Page 78

Problem 1

(Page 78). Suppose by way of contradiction that $\varphi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ given as above is a linear fractional transformation, and thus must be of the form

$$\overline{z} = \varphi(z) = \frac{az+b}{cz+d}, \quad \forall z \in \mathbb{C}$$

Note that if $\Im z = 0$ then $\overline{z} = z$. In particular,

$$0 \mapsto 0 = \frac{b}{d} \implies b = 0 \tag{4}$$

$$\tag{5}$$

Plugging in different values yields

$$1 \mapsto \frac{a}{c+d}$$
$$-1 \mapsto \frac{-a}{-c+d}$$
$$2 \mapsto \frac{2a}{2c+d}$$

Or,

$$c + d = a$$
$$d - c = a$$
$$2c + d = a$$

Thus, $a = d$ and hence $c = 0$. But then we have

$$\overline{z} = \frac{az}{d} = z$$

Which is absurd (for instance, $\overline{i} \neq i$).

Problem 2

(Page 78) Here we compute several compositions:

$$T_1T_2z = \frac{\frac{z}{z+1}+2}{\frac{z}{z+1}+3} = \frac{\frac{3z+2}{z+1}}{\frac{4z+3}{z+1}} = \frac{3z+2}{4z+3}$$
$$T_2T_2z = \frac{\frac{z+2}{z+3}}{\frac{z+2}{z+3}+1} = \frac{\frac{z+2}{z+3}}{\frac{2z+5}{z+3}} = \frac{z+2}{2z+5}$$

Now note that
$$T_1^{-1}(w) = \frac{3w-2}{1-w}$$

Thus,

$$T_1^{-1}T_2z = \frac{3\frac{z}{z+1}-2}{1-\frac{z}{z+1}} = \frac{z-2}{1} = z-2$$

Problem 3

(Page 78).

We argue in a similar manner. Let φ be a fractional linear transformation as given. Then, we have that $\varphi(0)=0$ and for any pair $(z,w) \in \mathbb{C}\times\mathbb{C}$:

$$|z-w| = |\varphi(z)-\varphi(w)| \tag{6}$$

since we assume that φ preserved distance under transformation. Of course, we know that then $\varphi(z) = \dfrac{az+b}{cz+d}$ for complex numbers a,b,c,d. The assumption that $0 \mapsto 0$ yields immediately
$$0 = \frac{b}{d}$$
and thus we see $b=0$. Now, as $\varphi(0)=0$ we see from (6) that

$$|z| = |\varphi(z)-\varphi(0)| = |\varphi(z)-0| = \left|\frac{az}{cz+d}\right| = |a|\cdot\frac{|z|}{|cz+d|}, \quad \forall z \in \mathbb{C}$$

Especially $|a|>0$. As this is the case for all $z\in\mathbb{C}$ we must have

$$\frac{|a|}{|cz+d|} \equiv 1 \text{ in } \mathbb{C} \tag{7}$$

Of course, this can only happen if the denominator is also constant and hence we conclude that we require additionally that $c=0$. We are left with $\dfrac{|a|}{|d|}=1$, or $|a|=|d|$. Hence we may express the quotient $\dfrac{a}{d}$ in polar form:

$$\frac{a}{d} = \rho e^{i\theta}, \quad \theta\in[0,2\pi) \tag{8}$$

where $\rho=1$. This corresponds purely then to a rotation in the complex plane. Finally,

$$\varphi(z) = \frac{az}{d} = ze^{i\theta}, \quad 0\le\theta<2\pi$$

as was required.

Problem 4

(Page 78). Let $\varphi(z)$ be a linear transformation with the additional restriction that $\varphi(z) \in \mathbb{R}$ whenever $z \in \mathbb{R}$. Write now

$$\varphi(z) = \frac{az+b}{cz+d}$$

for $a,b,c,d \in \mathbb{C}$. We will now find $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ so that $\varphi(z) = \dfrac{\alpha z+\beta}{\gamma z+\delta}$. We need to distinguish separate cases:

1. If $a \neq 0$ do the following:

$$\varphi(z) = \frac{az+b}{cz+d} \cdot \frac{\overline{a}}{\overline{a}} = \frac{a'z+b'}{c'z+d'}, \quad a' \in \mathbb{R}$$

To alleviate notation, we will denote these constants again by a,b,c,d respectively. The important thing to observe is that we will then have $\Im a = 0$, i.e $a \in \mathbb{R}$. We now have a representation

$$\varphi(z) = \frac{az+b}{cz+d}, \quad a \in \mathbb{R}$$

In the above we see that $\infty \mapsto \dfrac{a}{c}$ implying that $c \in \mathbb{R}$ as well as a. We have seen in class that these transformations admit an inverse

$$\varphi^{-1}(z) = \frac{dz-b}{a-cz} \tag{9}$$

If $d = 0$ then taking $z = 0$ we see $\varphi(0) = b$ and hence achieve $b \in \mathbb{R}$. We now need to show that the same holds whenever $d \neq 0$. But we may repeat the above steps for this inverse φ^{-1} to see that whenever $d \neq 0$ we must be able to choose $d \in \mathbb{R}$. Returning to φ, taking again $z = 0$ we see

$$0 \mapsto \frac{b}{d} \in \mathbb{R} \implies b \in \mathbb{R} \because d \in \mathbb{R}$$

2. If $a = 0$ then we have $\varphi(z) = \dfrac{b}{cz+d}$. Here if $b = 0$ we have that $\varphi(z)$ is non-invertible which is impossible. So we may apply the same idea to this form to write $\varphi(z) = \dfrac{b}{cz+b} \cdot \dfrac{\overline{b}}{\overline{b}}$ thus obtaining

$$\varphi(z) = \frac{b}{cz+d}, \quad b \in \mathbb{R}$$

for suitable b. Taking $z = 0$ we see that $0 \mapsto \dfrac{b}{d}$. We then have $d \in \mathbb{R}$. From this we must also we able to take $c \in \mathbb{R}$.

Page 80

Problem 1

We seek a transformation φ which takes $0 \mapsto 1$, $i \mapsto -1$, $-i \mapsto 0$. We need to find coefficients $a,b,c,d \in \mathbb{C}$ so that

$$\varphi(z) = \frac{az+b}{cz+d} \tag{10}$$

satisfies the conditions. That is, we require

$$\begin{aligned}\frac{b}{d} &= 1\\ \frac{ai+b}{ci+d} &= -1\\ \frac{-ai+b}{-ci+d} &= 0\end{aligned}$$

From the above equations we may recover the following auxilary system

$$b-d=0 \tag{11}$$
$$(a+c)i+(b+d)=0 \tag{12}$$
$$-ai+b=0 \tag{13}$$

Hence $b=d=ai$. With this, we rewrite the second equation as

$$ai+ci+2d=0=3ai+ci=0 \tag{14}$$

Hence $3a=-c$. If we set $a=1$ it immediately follows that a particular solution is the 4-tuple:

$$(1,i,-3,i)$$

So,

$$\varphi(z):=\frac{z+i}{-3z+i} \tag{φ}$$

It is clear that this certainly satisfies the desired criteria.

Page 108

Problem 3

Here we compute $\oint_{|z|=2}\frac{1}{z^2-1}\mathrm{d}z$. Note that $(z^2-1)^{-1}=(z-1)^{-1}(z+1)^{-1}$ has two poles of order 1 inside the circle $|z|=2$, at $z=\pm1$. The residues are easy to compute:

$$\begin{aligned}\oint_{|z|=2}\frac{1}{z^2-1}\mathrm{d}z=2\pi i\sum_w \mathrm{Res}_{z=w}f(z) &= 2\pi i\left(\frac{1}{-2}+\frac{1}{2}\right)\\ &= 2\pi i\cdot 0=0\end{aligned}$$

Problem 5

Let f be holomorphic on a closed curve $\Gamma\subset\mathbb{C}$, we claim that $\oint_\gamma \overline{f(z)}f'(z)\mathrm{d}z$ is **purely imaginary**. Note that as f is holomorphic it is a regular function that if infinitely many times differentiable and and hence all partials (of any order) of it's real and imaginary parts exist and are continuous. We may derive from the limit definition that

$$\begin{aligned}f'(z)=\lim_{\substack{h\to0\\h\in\mathbb{C}}}\frac{f(z+h)-f(z)}{h} &= \lim_{h\to0}\frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h}\\ &= \lim_{h\to0}\frac{u(x+h,y)-u(x,y)}{h}+i\lim_{h\to0}\frac{v(x+h,y)-v(x,y)}{h}\\ &= \frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y)\end{aligned}$$

Now,

$$\oint_\gamma \overline{f(z)} f'(z)\mathrm{d}z = \oint_\gamma \left(u(x,y) - iv(x,y)\right)\left(u_x(x,y) + iv_x(x,y)\right)(dx + idy)$$
$$= \oint_\gamma \left(u(x,y) - iv(x,y)\right)\left(u_x(x,y) + iv_x(x,y)\right)(dx + idy)$$
$$= \oint_\gamma (uu_x + vv_x - ivu_x + iuv_x)(dx + idy)$$
$$= \oint_\gamma (uu_x dx + vv_x dx - ivu_x dx + iuv_x dx) + (iuu_x dy + ivv_x dy + vu_x dy - uv_x dy)$$
$$= \oint_\gamma uu_x dx + vv_x dx + vu_x dy - uv_x dy + \Im(I)$$
$$= \oint_\gamma (uu_x + vv_x)\mathrm{d}x + (vu_x - uv_x)\mathrm{d}y + \Im(I)$$

where I is the integral being considered. Now, if $\mathcal{D}$ denotes the region enclosed by the closed curve γ we apply Green's theorem to see the real part is given by

$$\iint_{\mathcal{D}} \left(v_x u_x + vu_{xx} - u_x v_x - uv_{xx} - u_y u_x - uu_{xy} - v_y v_x - vv_{xy}\right)\mathrm{d}A$$
$$= \iint_{\mathcal{D}} \left(vu_{xx} - uv_{xx} - uu_{xy} - vv_{xy}\right)\mathrm{d}A$$
$$= \iint_{\mathcal{D}} \left(vv_{yx} + uu_{yx} - uu_{xy} - vv_{xy}\right)\mathrm{d}A$$
$$= \iint_{\mathcal{D}} 0\mathrm{d}A = 0$$

Page 123

Some of these computations are **very** tedious and combinatorial and do not (in my humble opinion) teach one much. It is more instructive to do one nice example and not get bogged down on the computations that follow. The only crucial step is to notice the applicability of Cauchy's theorem. The computations that may follows are not unique to complex analysis and are not deep.

Problem 1.

We compute $\oint_{|z|=1} e^z z^{-n}\mathrm{d}z$. The first thing to observe is that if $n \leq 0$ the integrand is holomorphic (even **entire**) and hence this integral vanishes. We now consider powers $n \in \mathbb{N}$. Recall *Cauchy's Integral Formula*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta \tag{15}$$

Here we have $z = 0$, and $n \mapsto n - 1$. Thus,

$$\oint_{|\zeta|=1} \frac{e^\zeta}{\zeta^n}\mathrm{d}\zeta = \frac{2\pi i}{(n-1)!}\cdot\frac{\mathrm{d}^{n-1}}{\mathrm{d}\zeta^{n-1}}\left[e^\zeta\right]_{\zeta=0} = \frac{2\pi i}{(n-1)!}$$

Problem 2.

We claim here that if f is entire and for all $f(z) \in \mathcal{O}(z^n)$ for some n, then f is a polynomial of at most degree n. Here we will develop of modified form of Cauchy's estimate. Fix $z \in \mathbb{C}$ and take $R \gg 0$ and consider the circle $|\zeta - z| = R$. We may let R be so large that for ζ on this circle we have $|f(\zeta)| \leq C |\zeta|^n$. Then,

$$\begin{aligned} f^{(n+1)}(z) = \frac{(n+1)!}{2\pi} \left| \oint_{|\zeta - z| = R} \frac{f(\zeta)}{(\zeta - z)^{n+2}} \mathrm{d}\zeta \right| &\leq \frac{(n+1)!}{2\pi} \oint_{|\zeta - z| = R} \frac{|f(\zeta)|}{|\zeta - z|^{n+2}} |\mathrm{d}\zeta| \\ &\leq \frac{(n+1)!}{2\pi} \oint_{|\zeta - z| = R} \frac{C|\zeta|^n}{R^{n+2}} \cdot |\mathrm{d}z| \\ &= \frac{C(n+1)!}{2\pi R^2} \oint_{|\zeta - z| = R} |\mathrm{d}z| \\ &= \frac{K(n+1)!}{2\pi R} \xrightarrow{R \to \infty} 0 \end{aligned}$$

Hence, $f^{(n+1)}(z) = 0$, and since $z \in \mathbb{C}$ was arbitrary we see $f^{(n+1)} \equiv 0$ in $\mathbb{C}$ and hence f is a polynomial of at most degree n.

Problem 3.

It is implied that we need to find an upper-bound that is uniform for all ρ and z inside the domain. Note for any ρ the largest disc we can take for all z with $|z| \leq \rho < R$ is simply the open disc with radius $R - \rho$. Thence, as this disc is contained in the larger domain where f is bounded:

$$\left| f^{(n)}(z) \right| \leq \frac{n!}{2\pi} \oint_{|\zeta - z| = R - \rho} \frac{M}{(R - \rho)^{n+1}} \mathrm{d}\zeta = \frac{n! M}{(R - \rho)^n}$$

Problem 5.

We show that at a point where $f(z)$ is holomorphic we cannot have for successive n: $\left| f^{(n)}(z) > n! n^n \right|$. We may pick a small disc γ centred at z of radius r contained in a domain in which f is holomorphic and consequently bounded. Thus, express

$$\left| f^{(n)}(z) \right| \leq \frac{n!}{2\pi} \oint_{|\zeta - z| = r} \frac{M}{|\zeta - z|^{n+1}} |\mathrm{d}\zeta| = \frac{n! \cdot M}{r^n}$$

Shrink r so that if $\mu^n = M$ then $\left(\dfrac{M}{r} \right) \leq n$. Since the integral takes the same value regardless of the size of the ball, this can be done for all z and all subsequent n.

Page 161

Problem 1

Here we calculate poles and residues. Let's do this!

a) $f(z) = \dfrac{1}{z^2 + 5z + 6}$. Note that we can factor the denominator as $(z+3)(z+2)$ which has zeros of order 1 at $z = -2, -3$. The residues are then

$$\mathrm{Res}(f, -2) = 1, \quad \mathrm{Res}(f, -3) = -1$$

b) Here we consider $\dfrac{1}{(z^2-1)^2} = \dfrac{1}{(z-1)^2(z+1)^2}$ which has zeros of order 2. Here we prefer to use Cauchy's formula, if γ is a small disc enclosing only one of ± 1:

$$\operatorname{Res}(f;1) = \frac{1}{2\pi i}\oint_\gamma \frac{(z+1)^{-2}}{(z-1)^{1+1}}\mathrm{d}z = \frac{\mathrm{d}}{\mathrm{d}z}\left[(z+1)^{-2}\right]_{z=1} = \frac{-2}{2^3} = -\frac{1}{4}$$
$$\operatorname{Res}(f;-1) = \frac{1}{2\pi i}\oint_\gamma \frac{(z-1)^{-2}}{(z+1)^{1+1}}\mathrm{d}z = \frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^{-2}\right]_{z=-1} = \frac{-2}{(-2)^3} = \frac{1}{4}$$

c) Consider $\dfrac{1}{\sin z}$. Note that $\sin z = 0 \iff z = k\pi$ for $k \in \mathbb{Z}$. Hence, the ratio will have poles at $z = k\pi$. Taking a small disc enclosing at most one, we see

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n (z-k\pi)^{2n+1}}{(2n+1)!}, \quad |z-k\pi| \leq R < \rho$$

Or, near this singularity:

$$\sin z = (z-k\pi)\sum_{n=0}^{\infty} \frac{(-1)^n (z-k\pi)^{2n}}{(2n+1)!} =: (z-k\pi)g(z)$$

this series has the same radius of convergence and hence represents a holomorphic function there. Note also that $g(k\pi) \neq 0$ and so we may shrink the neighbourhood so that $g \neq 0$ in this small ball. It follows from that each zero of $\sin z$ is of order one and thus we may compute:

$$\operatorname{Res}(\sin z, k\pi) = \lim_{z\to k\pi} \frac{z-k\pi}{\sin z} = \frac{1}{\cos k\pi} = (-1)^k$$

We briefly digress to prove that this (simple) case of L'Hôpital's rule may be used.

Theorem 1. *Let f, g be holomorphic at z_0 with $\lim_{z\to z_0} f(z) = \lim_{z\to z_0} g(z) = 0$ and assume that $g'(z_0) \neq 0$. Then,*

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \tag{16}$$

Proof. Write for $z \neq z_0$ near z_0:

$$\frac{f(z)}{g(z)} = \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}}$$

and let $z \to z_0$.

∎

d) $f(z) = \cot z$. $\cot z$ shares poles with $\dfrac{1}{\sin z}$. Hence,

$$\operatorname{Res}(f; z = k\pi) = \lim_{z\to k\pi} (z-k\pi)\frac{\cos z}{\sin z} = \lim_{z\to k\pi} \cos z \cdot \lim_{z\to k\pi} \frac{(z-k\pi)}{\sin z} = 1$$

e) $f(z) := \dfrac{1}{\sin^2 z}$. We employ a similar strategy, since we may write

$$\sin z = (z - k\pi)\left(\underbrace{1 - \frac{(z-k\pi)^2}{3!} + \ldots + \frac{(-1)^n (z-k\pi)^{2n}}{(2n+1)!} + \ldots}_{g(z)}\right)$$

for $g(z)$ holomorphic and non vanishing in a neighbourhood of $k\pi$ with $g(k\pi) = 1$. Moreover, we have $g'(k\pi) = 0$. Then, $g(z)^{-2}$ satisfies the same properties and if γ is a small circle centred at $k\pi$ we have by Cauchy's integral formula

$$\operatorname{Res}(f, k\pi) = \frac{1}{2\pi i}\oint_\gamma \frac{g(z)^{-2}}{(z-k\pi)^2}\mathrm{d}z = \frac{\mathrm{d}}{\mathrm{d}z}\left[g(z)^{-2}\right]_{z=k\pi} = -2\frac{g'(k\pi)}{g(k\pi)^3} = 0$$

Problem 3.

Here we only evaluate two of the more difficult ones, namely *e*) and *h*). *f*) is the same as *e*) after an application of Jordan's lemma, which we have not covered and so I will omit it.

First consider $\displaystyle\int_{\mathbb{R}_+} \frac{\cos x}{x^2+a^2}dx = \frac{1}{2}\int_{\mathbb{R}} \frac{\cos x}{x^2+a^2}dx$ and moreover

$$\int_{\mathbb{R}} \frac{\cos x}{x^2+a^2}dx = \int_{\mathbb{R}} \frac{\Re(e^{ix})}{x^2+a^2}dx = \Re\left(\int_{\mathbb{R}} \frac{e^{ix}}{x^2+a^2}dx\right)$$

Consider the semi circle lying on the real axis with radius $R \gg 0$ centred at the origin with upper-arc lying in $\Im(z) >\geq 0$. For all $R \gg 0$, if we denote this semi by Γ and the upper arc by γ we have by the Calculus of Residues

$$\oint_\Gamma \frac{e^{iz}}{(z-ai)(z+ai)}\mathrm{d}z = 2\pi i\left(\frac{e^{i(ai)}}{2ai}\right) = \frac{\pi e^{-a}}{a}$$

$$\left|\int_\gamma \frac{e^{iz}}{(z-ia)(z+ai)}\mathrm{d}z\right| \leq \int_0^{2\pi} \frac{e^{iR\cos\theta - R\sin\theta}}{|R^2 e^{i\theta 2} + a^2|}Rd\theta \leq \int_0^{2\pi} \frac{\left|e^{-R\sin\theta}\right|}{R^2 - a^2}Rd\theta$$
$$\leq 2\pi\frac{R}{R^2-a^2} \xrightarrow{R\to\infty} 0$$

Thus, in the limit as $R \to \infty$ we recover

$$\frac{\pi e^{-a}}{a} = \oint_\gamma \frac{e^{iz}}{z^2+a^2}\mathrm{d}z = \int_{\mathbb{R}} \frac{e^{ix}}{x^2+a^2}\mathrm{d}x$$

Ergo, $\displaystyle\int_{\mathbb{R}_+} \frac{\cos x}{x^2+a^2}dx = \frac{1}{2}\Re\left(\frac{\pi e^{-a}}{a}\right) = \frac{\pi e^{-a}}{2a}$.

Finally, we examine $\displaystyle\int_0^\infty \frac{\log x}{1+x^2}\mathrm{d}x$. Consider the usual key hole with $R \gg 0$ and $\epsilon \gtrapprox 0$. Choosing a branch of the logarithm with $\arg z \in \left(-\dfrac{3\pi}{2}, -\dfrac{\pi}{2}\right]$ (why?) note that the integral over this curve Γ is simply

$$\oint_\Gamma \frac{\log z}{z^2+1}\mathrm{d}z = 2\pi i\sum_{z^\star}\operatorname{Res}(f; z^\star) = 2\pi i\left(\frac{\log i}{2i}\right) = \pi\log i$$

Note that if $e^z = i$ then $i = \cos\theta + i\sin\theta$ for $\theta = \frac{\pi}{2}$. Thus, in our branch of the logarithm we have $\log i = \dfrac{i\pi}{2}$. Thus,

$$\oint_\Gamma \frac{\log z}{z^2+1}\mathrm{d}z = i\frac{\pi^2}{2}$$

We now examine the behaviour along a semi-circle γ of radius R:

$$\begin{aligned}\left|\int_\gamma \frac{\log z}{z^2+1}\mathrm{d}z\right| \le \int_\gamma \frac{|\log z|}{|z^2|+1}\mathrm{d}z = \int_0^\pi \frac{|\log|R| + i\pi|}{R^2-1}R\mathrm{d}\theta \\ \le \int_0^\pi \frac{\log R + \pi}{R^2-1}R\mathrm{d}\theta\end{aligned}$$

The usual limit rules show that this tends to 0 as $R \to 0$ or $R \to \infty$. Hence we may shrink the keyhole into our usual contour preserving the value of the integral above and seeing

$$\begin{aligned}i\frac{\pi^2}{2} = \int_{-\infty}^{\infty} \frac{\log z}{z^2+1}\mathrm{d}z == \int_0^\infty \frac{\log x}{x^2+1}\mathrm{d}x \\ + \int_{-\infty}^0 \frac{\log z}{z^2+1}\mathrm{d}z\end{aligned}$$

On the otherhand,

$$\int_{-\infty}^0 \frac{\log z}{z^2+1}\mathrm{d}z = \int_{-\infty}^0 \frac{\log|z| + i\pi}{z^2+1}\mathrm{d}z = \int_0^\infty \frac{\log x}{x^2+1}\mathrm{d}x + i\pi\int_0^\infty \frac{1}{1+x^2}dx$$

Equating real parts we see

$$2\int_0^\infty \frac{\log x}{1+x^2}\mathrm{d}x = 0$$

Page 154

Problem 1.

Define $f(z) = z^7 - 2z^5 + 6z^3 - z + 1$. How many roots have their modulus < 1?

Define the auxiliary function $g(z) := 6z^3$, and note that on the circle $|z| = 1$ $|g(z)| \equiv 6$. On one hand,

$$|f(z) - g(z)| \le \left|z^7\right| + \left|2z^5\right| + |z| + 1 = 1 + 2 + 1 + 1 = 6 < 7$$

Hence, $|f| - |g| < |g|$ and thus $|f| < 2|g|$. Thus, f has as many roots inside $|z| < 1$ as $2g$ by *Rouche's* Theorem. Now, $g = 0$ only at $z = 0$, multiplicity 3. Thus, we see f has three such roots.

Problem 2.

How many roots does $z^4 - 6z + 3$ have their moduli between 1 and 2? We mimic the procedure above. Define $g(z) = 6x$ and write

$$|f - g| = \left|z^4 + 3\right| \le \left|z^4\right| + 3 = 4 < |g| = 6, \quad |z| = 1$$

Hence, we see that f has 1 root with $|z| < 1$. Now, on the boundary of the annulus $|z| = 2$ define $h(z) = z^4$, considering

$$|f - h| \leq |6z| + 3 = 12 + 3 < 2^4 = 16$$

Using now that z^4 has 4 roots we see by Rouche that f has 4 roots inside $|z| < 2$ and thus has 3 inside $1 \leq |z| < 2$. We claim it has no roots on $z = 1$. To see this, note that for such z

$$|f| \geq |6z| - \left|z^4\right| - |3| = 6 - 1 - 3 = 2$$

Page 133

Problem 1.

Set $f(z) = z^2 + z$ and hence $f'(z) = 2z + 1$ which vanishes at $z = -\dfrac{1}{2}$. Thus we cannot have a disc with radius larger than $1/2$. We now have to show that f is injective in $|z| < \frac{1}{2}$. Suppose that for z, w in this disc we have $w \neq z$ and $f(z) = f(w)$:

$$0 = (z^2 - w^2) + (z - w) = (z - w)(z + w + 1) = 0$$

Since by hypothesis we have $z \neq w$ it follows that $z + w + 1 = 0$ and thus

$$\begin{aligned} |z + w| = 1 &\implies |z| + |w| \geq 1 \\ &\iff |z| \geq 1 - |w| > \frac{1}{2} \end{aligned}$$

which is absurd.

Problem 2.

We find the largest neighbourhood of the origin in which $w := e^z$ is injective. Recall that e^z is *periodic* with period $2\pi i$, consequently it is natural to as whether e^z is injective in the disk $|z| < \pi$. Indeed, if we take z, w satisfying $\max(|z|, |w|) < \pi$ and assume

$$e^z = e^w \iff e^{z-w} = 1 \iff z - w = 2\pi i n, \quad n \in \mathbb{Z}$$

Hence, $|z - w| = 2\pi n$ with $n \in \mathbb{N}_0$. Hence, $|z| + |w| \geq 2\pi n$ implies

$$|z| \geq 2\pi n - |w| > 2\pi n - \pi \geq 2\pi - \pi = \pi$$

Contradicting our choice of z yielding then that $z = w$ proving injectivity. It now remains to show that this is the largest disk in which e^z is one to one. To see this, fix $\epsilon > 0$ and use the periodicity of e^z to pick elements in this disc with $z_2 - z_1 = 2\pi i$. We may do so as follows:

$$z_1 := -i\pi + \frac{\epsilon}{2}, \quad z_2 := i\pi + \frac{\epsilon}{2}$$

(try to visualize these points in $\mathbb{C}$). Then, $z_2 - z_1 = 2\pi i$ and hence $e^{z_2} = e^{z_1}$.

Problem 3.

Recall the half angle formulae

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{17}$$

Note that $\cos z - 1$ has a 0 of order $n = 2$ at $z = 0$. Thus,

$$\begin{aligned}\cos\theta - 1 = -(1-\cos\theta) = -\left(2\sin^2\left(\frac{\theta}{2}\right)\right) &= -2\sin^2\left(\frac{\theta}{2}\right)\\ &= \left(\sqrt{2}i\sin\left(\frac{\theta}{2}\right)\right)\end{aligned}$$

Problem 4.

This is a little more difficult, although **very instructive**. We will state it as a theorem, as it is of great importance.

Theorem 2. *Let $f(z)$ be holomorphic in a neighbourhood of the origin with $f'(0) \neq 0$. Then, there exists a (possibly smaller) neighbourhood or the origin and a function $g(z)$ holomorphic in this neighbourhood so that*

$$f(z) = f(0) + g(z)^n \tag{18}$$

in this neighbourhood.

Proof. We may expand f as a Taylor polynomial near the origin so that for all $|z|$ sufficiently small

$$f(z) - f(0) = a_1 z + a_2 z^2 + \ldots + a_k z^k + \ldots, \quad a_1 = f'(0) \neq 0 \tag{19}$$

$$f(z^n) - f(0) = a_1 z^n + a_2 z^{2n} + \ldots + a_k z^{nk} + \ldots \tag{20}$$

This last line is allowed in a smaller neighbourhood of the origin, for if $|z| < 1$ then $|z|^n < |z| < 1$ and hence the series is convergent. Now, writing

$$f(z) - f(0) = z^n\left(a_1 + a_2 z^n + \ldots\right) = z^n h(z)$$

where $h(z)$ is analytic near the origin and has $h(0) = 1$. Moreover, we may shrink the ball so that $h(z)$ is non vanishing in this ball. It hence has a logarithm in a closed ball contained in this neighbourhood (see the *logarithm* section) and consequently we may define an analytic **sheet** of the n^{th} root

$$g(z) = (h(z))^{\frac{1}{n}} = e^{\dfrac{\log h(z)}{n}} \tag{21}$$

This concludes the proof.

■

The Logarithm and a Useful Construction

We begin by considering the following problem, given a non-zero complex number z does there exist a unique complex number w such that $e^w = z$? Yes, however the answer is not unique. Moreover, in general we cannot simply choose a single determination. We may write for $\arg z \in [0, 2\pi)$

$$z = |z|\, e^{i\arg z} = |z|\left(\cos\arg z + i\sin\arg z\right)$$

Thus, if we write $w = x + iy$

$$e^w = e^{x+iy} e^x e^{iy} = |z|\, e^{i\arg z}$$

Equating now real and imaginary parts yields $e^w = |z|$ and $e^{iy} = e^{i \arg z}$. The former has a former *unique* solution $x = \ln |z|$ and the latter implies

$$y = \arg z$$

The trouble here is that the argument of z not **not** uniquely determined, indeed simply the complex number 1 may be expressed as

$$1 = 1 \cdot e^0 = 1 \cdot e^{2\pi i} = \ldots = 1 \cdot 1^{2\pi i n}, \quad n \in \mathbb{Z}$$

Nonetheless, we may say several nice things about the complex logarithm. If f is holomorphic and non vanishing inside a domain $\Omega \subset \mathbb{C}$ and we have a continuous function w there such that

$$f(z) = e^{w(z)}$$

then w is also holomorphic with $w'(z) = \dfrac{f'(z)}{f(z)}$. Indeed, if we employ the holomorphic property of f and the exponential we may expand

$$f(z + \Delta z) - f(z) = e^{w(z+\Delta z)} - e^{w(z)}$$

so that

$$\begin{aligned} f(z + \Delta z) - f(z) &= e^{w(z)} \left(e^{w(z+\Delta z) - w(z)} - 1 \right) \\ &= f(z) \left(e^{\Delta w} - 1 \right), \quad \Delta w = w(z + \Delta z) - w(z) \end{aligned}$$

Dividing through by Δz

$$\begin{aligned} \frac{f(z + \Delta z) - f(z)}{\Delta z} &= \frac{f(z)}{\Delta z} \left(\Delta w + \frac{(\Delta w)^2}{2!} + \ldots \right) \\ &= f(z) \cdot \frac{w(z + \Delta z) - w(z)}{\Delta z} \left(1 + \frac{(\Delta w)}{2!} + \ldots \right) \end{aligned}$$

We see that by continuity of $w(z)$ the power series tends to 1 as $\Delta z \to 0$. Hence, letting $z \to z_0$ we see

$$w'(z) = \frac{f'(z)}{f(z)}$$

and hence that w is holomorphic. With this we can now construct a logarithm.

Theorem 3. *Let $\Gamma \subset \mathbb{C}$ be a simple closed curve and f be holomorphic on and inside Γ with $f \neq 0$ everywhere. There exists a holomorphic* single valued *function w so that*

$$f(z) = e^{w(z)} \quad \textit{inside and on } \Gamma \tag{22}$$

Proof. We proceed in 4 steps.

Step 1. Here we define the logarithm. Fix an interior point of the curve Γ and let $z_1, z_2 \in \Gamma$ be points such that the segments

$$\begin{aligned} \lambda &: [0, 1] \to \mathbb{C}, \quad z_1 \to z_0 \\ \lambda' &: [0, 1] \to \mathbb{C}, \quad z_2 \to z_0 \end{aligned}$$

lie in inside the curve Γ, except the points $z_1, z_2 \in \Gamma$. We will begin by showing that

$$f(z_1) \exp \int_\lambda \frac{f'(\xi)}{f(\xi)} d\xi = f(z_2) \exp \int_{\lambda'} \frac{f'(\xi)}{f(\xi)} d\xi \tag{23}$$

Along the arc $\overline{z_1 z_2} \subset \Gamma$ with positive orientation consider the function

$$\frac{1}{f(\zeta)} \exp \int_{z_1}^{\zeta} \frac{f'(\xi)}{f(\xi)} d\xi$$

where the integral is taken along this same curve. This is defined as f is holomorphic and non vanishing on Γ. Taking the derivative,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\zeta} \left(\frac{1}{f(\zeta)} \exp \int_{z_1}^{\zeta} \frac{f'(\xi)}{f(\xi)} d\xi \right) \\ &= -\frac{f'(\zeta)}{f(\zeta)^2} \exp \int_{z_1}^{\zeta} \frac{f'(\xi)}{f(\xi)} d\xi + \frac{1}{f(\zeta)} \cdot \exp \int_{z_1}^{\zeta} \frac{f'(\xi)}{f(\xi)} d\xi \cdot \frac{f'(\zeta)}{f(\zeta)} \equiv 0 \end{aligned}$$

for any ζ chosen along this sub-path of Γ. Thus, it is constant in ζ, it takes value $\dfrac{1}{f(z_1)}$ at $\zeta = z_1$, and must also assume this value at $\zeta = z_2$, that is:

$$\frac{1}{f(z_1)} = \frac{1}{f(z_2)} \exp \int_{z_1}^{z_2} \frac{f'(\xi)}{f(\xi)} d\xi$$

and consequently

$$\frac{f(z_2)}{f(z_1)} = \exp \int_{z_1}^{z_2} \frac{f'(\xi)}{f(\xi)} d\xi =: \exp \int_{\overline{z_1 z_2}} \frac{f'(\xi)}{f(\xi)} d\xi \tag{24}$$

Now consider the positive closed curve with positive orientation from by travelling along $\overline{z_1 z_2}$, along λ' and λ^-, which is λ with reverse orientation. Denoting this closed curve by γ we see by Cauchy's Theorem that

$$\begin{aligned} 0 = \oint_{\gamma} \frac{f'(\xi)}{f(\xi)} d\xi &= \int_{\overline{z_1 z_2}} \frac{f'(\xi)}{f(\xi)} d\xi + \int_{\lambda'} \frac{f'(\xi)}{f(\xi)} d\xi + \int_{\lambda^-} \frac{f'(\xi)}{f(\xi)} d\xi \\ &= \int_{\overline{z_1 z_2}} \frac{f'(\xi)}{f(\xi)} d\xi + \int_{\lambda'} \frac{f'(\xi)}{f(\xi)} d\xi - \int_{\lambda} \frac{f'(\xi)}{f(\xi)} d\xi \end{aligned}$$

Whence, by the above

$$\frac{f(z_2)}{f(z_1)} = \exp \int_{\lambda} \frac{f'(\xi)}{f(\xi)} d\xi - \int_{\lambda'} \frac{f'(\xi)}{f(\xi)} d\xi$$

Implying

$$f(z_2) \exp \int_{\lambda'} \frac{f'(\xi)}{f(\xi)} d\xi = f(z_1) \exp \int_{\lambda} \frac{f'(\xi)}{f(\xi)} d\xi \tag{25}$$

Step 2. Here we show uniqueness of representation, so to speak. Namely, that the above is equal to $f(z_0)$ and hence the function is independent of chosen path. This is a reprise of the argument above, where we may again show that

$$\frac{\mathrm{d}}{\mathrm{d}\zeta} \left[\frac{1}{f(\zeta)} \exp \left(-\int_{z_1}^{\zeta} \frac{f'(\xi)}{f(\xi)} d\xi \right) \right] = 0$$

as ζ moves from z_1 to z_0 on the curve λ. Then, taking $\zeta = z_0$ the result follows; i.e

$$f(z_0) = f(z_1) \exp \int_{\lambda} \frac{f'(\xi)}{f(\xi)} d\xi \tag{26}$$

the second equality follows from step 1.

We may now define for an interior point z_0 and such a boundary point z_1:

$$w(z_0) := w(z_1) + \int_\lambda \frac{f'(\mathfrak{z})}{f(\mathfrak{z})} \mathrm{d}\mathfrak{z} = w(z_2) + \int_{\lambda'} \frac{f'(\mathfrak{z})}{f(\mathfrak{z})} \mathrm{d}\mathfrak{z} \tag{27}$$

Ultimately, it is this *uniqueness* of representation that will yield continuity of this function, which is defined on and inside Γ.

Step 3. The function $w(z)$ is continuous where it is defined. Fix a point z_0 inside Γ and consider now a small disc centred about it contained inside the interior of Γ except for one point where it intersects a point z_1 on the boundary of Γ. Consider again an arc λ from z_1 to z_0 with $(z_1, z_0]$ contained in the interior of the curve. If we take z_0' near z_0 then it lies inside this same disc, which is convex. Consequently for all such z_0' we may choose a path λ' from $(z_1, z_0'])$ contained inside the interior of the disc and consequently inside the region enclosed by Γ.

We now employ a usual strategy. Consider the closed curve γ formed by traveling along $[z_0, z_1] \to [z_1, z_0'] \to [z_0', z_0]$ where this last path is taken along the radius of the disc. This is a closed curve (a triangle) on and inside which f is holomorphic and thus

$$0 = \oint_\gamma \frac{f'(\xi)}{f(\xi)} d\xi = \int_{\lambda-} \frac{f'(\xi)}{f(\xi)} d\xi + \int_{\lambda'} \frac{f'(\xi)}{f(\xi)} d\xi + \int_{[z_0', z_0]} \frac{f'(\xi)}{f(\xi)} d\xi \tag{28}$$

Therefore,

$$w(z_0) - w(z_0') = \int_\lambda \frac{f'(\xi)}{f(\xi)} d\xi - \int_{\lambda'} \frac{f'(\xi)}{f(\xi)} d\xi = \int_{[z_0', z_0]} \frac{f'(\xi)}{f(\xi)} d\xi$$

Now observe that since $f'(\zeta)$ and $f(\zeta)$ are jointly holomorphic with $f(\zeta) \neq 0$, their ratio is continuous (even holomorphic) inside and on Γ and must achieve a maximum, say, $M \geq 0$. In any case it is bounded and we may make the following estimate for all z_0' near z_0:

$$|w(z_0) - w(z_0')| \leq M \int_{z_0'}^{z_0} d\tau = M\,|z_0 - z_0'| \xrightarrow{z_0' \to z_0} 0$$

Hence, w is continuous.

Step 4. We now show the function $w(z)$ is holomorphic. This is really a consequence of the introductory result, as we have already shown continuity. This concludes the proof.

■

Principle of Argument We write

$$\begin{aligned} \frac{1}{2\pi i} \oint_\gamma \frac{f'(\mathfrak{z})}{f(\mathfrak{z})} d\mathfrak{z} &= \frac{1}{2\pi i} \oint_\gamma \frac{d}{d\mathfrak{z}} \log f(z) d\mathfrak{z} \\ &= \frac{1}{2\pi i} (\log|z| + i\theta_2 - \log|z| - i\theta_1) = \frac{\Delta\theta}{2} \end{aligned}$$

Question. How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane? Hint: Sketch the image of the imaginary axis and apply the argument principle to a large half disk.

We cal solve this equation with the interpretation of the argument principle above. Let for $R \gg 0$ $\Gamma(R)$ denote the half circle lying on the real axis pointing into the portion of $\mathbb{C}$ with $\Re(z) \geq 0$. Now, on the line $\Im(z)$ we see for y with $-R \leq y \leq R$ that:

$$\begin{aligned} (iy)^4 + 8(iy)^3 + 3(iy)^2 + 8(iy) + 3 &= y^4 - 8iy^3 - 3y^2 + 8iy + 3 \\ \Re(f(iy)) = y^4 - 3y^2 + 3, &\quad \Im(f(iy)) = -8y^3 + 8y \end{aligned}$$

At both $iy = \pm\infty$ it follows from

$$\frac{\Re(f(z))}{\Im(f(z))} = \mathcal{O}\left(\frac{1}{z}\right)$$

that as $R = \pm\infty$ $f(iy)$ goes the argument of $f(z)$ goes from 0 to 0 so to speak. We know that it doesn't go to a non-zero multiple of π as it cannot wind around the origin, for the real part $\Re(f(z)) > 0$ for all such y on this axis. We now study how the function behaves on the arc. Parametrize it by $z = Re^{i\theta}$, then

$$f(\varphi(\theta)) = R^4 e^{4i\theta} + 8R^3 e^{3i\theta} + 3R^2 e^{2i\theta} + 8Re^{i\theta} + 3 = 0$$

or,

$$R^4 e^{4i\theta} \left(1 + \mathcal{O}(1/R)\right)$$

consequently, we have as $R \to \infty$ that the argument of f along this arg, as $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ ranges by

$$i2\pi - (-i2\pi) = 4\pi i$$

hence, for R large, we have the total number of zeroes is $\frac{1}{2\pi i} \cdot 4\pi i = 2$.

Some Additional Problems

1. A classic problem in Fourier analysis is the computation of $\displaystyle\int_0^\infty \frac{\log x}{x^2 + a^2} \mathrm{d}x$ for $a \in \mathbb{R}$. We will use an indented semi circle as a contour of integration. For $\epsilon > 0$ and $R > 0$ remove the small key-hole from the semi-circle. Now, chose a branch of the logarithm with $\arg z \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$, it is wise to chose this branch because it will include the contour and the boundaries. Note that the function

$$f(z) = \frac{\log z}{z^2 + a^2}$$

has a pole of order 1 inside the contour at the point ai. Thence, by the residue theorem, where $\Gamma(R, \epsilon)$ is the chosen contour

$$\oint_{\Gamma(R,\epsilon)} f(z)\mathrm{d}z = 2\pi i \left(\frac{\log ai}{2ai}\right) = \frac{\pi}{a} \log ai = \frac{\pi}{a} \left(\log |a| + i\pi\right)$$

Now, on any semi-circle (with radius, say ρ):

$$\left|\int_{C_\rho} \frac{\log z}{z^2 + a^2} \mathrm{d}z\right| \leq \int_{C_\rho} \frac{|\log z|}{|z^2| - a^2} \mathrm{d}z \leq \int_0^{2\pi} \frac{\log |z| + \pi}{|z^2| - a^2} \rho \mathrm{d}\theta$$
$$= 2\pi \frac{\log |\rho| + \pi}{\rho^2 - a^2} \rho \mathrm{d}\theta \xrightarrow{\rho \to 0, \quad \rho \to \infty} 0$$

Now, sending $\epsilon \to 0$ and $R \to \infty$ we recover

$$\int_{-\infty}^\infty \frac{\log z}{z^2 + a^2} \mathrm{d}z = \frac{\pi}{a} \left(\log |a| + i\pi\right)$$

On the other hand,

$$\int_{-\infty}^\infty \frac{\log z}{z^2 + a^2} \mathrm{d}z = \int_{-\infty}^0 \frac{\log z}{z^2 + a^2} \mathrm{d}z + \int_0^\infty \frac{\log x}{x^2 + a^2} \mathrm{d}x$$

And,

$$\int_{-\infty}^{0} \frac{\log z}{z^2+a^2}\mathrm{d}z = \int_{-\infty}^{0} \frac{\log|z|+i\pi}{z^2+a^2}\mathrm{d}z = \int_{0}^{\infty} \frac{\log x}{x^2+a^2}\mathrm{d}x + i\pi\int_{-\infty}^{0} \frac{1}{z^2+a^2}\mathrm{d}z$$

Thus,

$$\frac{\pi}{a}\left(\log|a|+i\pi\right) = 2\int_{0}^{\infty} \frac{\log x}{x^2+a^2}\mathrm{d}x + i\pi\int_{-\infty}^{0} \frac{1}{z^2+a^2}\mathrm{d}z$$

equating real and imaginary parts:

$$\int_{0}^{\infty} \frac{\log x}{x^2+a^2}\mathrm{d}x = \frac{\pi}{2a}\log|a| \tag{29}$$

We now calculate $\oint_{|z|=p} \frac{|dz|}{|z-a|^2}$. We begin with the observation

$$\begin{aligned}\oint_{|z|=p} \frac{|dz|}{|z-a|^2} &= \int_0^{2\pi} \frac{\left|pie^{i\theta}d\theta\right|}{(e^{i\theta}-a)\overline{(e^{i\theta}-a)}} = \int_0^{2\pi} \frac{\left|pie^{i\theta}d\theta\right|}{(e^{i\theta}-a)(e^{i-\theta}-\overline{a})}\\ &= \int_0^{2\pi} \frac{pd\theta}{(e^{i\theta}-a)(e^{i-\theta}-\overline{a})} = \int_0^{2\pi} \frac{pd\theta}{(e^{i\theta}-a)(e^{i-\theta}-\overline{a})}\cdot\frac{ie^{i\theta}}{ie^{i\theta}}\\ &= -i\int_0^{2\pi} \frac{pie^{i\theta}d\theta}{(e^{i\theta}-a)(1-e^{i\theta}\overline{a})} = -i\oint_{|z|=p} \frac{dz}{(z-a)(1-z\overline{a})}\\ &= i\oint_{|z|=p} \frac{dz}{(z-a)(z\overline{a}-1)}\end{aligned}$$