Robotics: Mechanics & Control

Chapter 2: Motion Description

In this chapter we first briefly review the robot components, and then elaborate on spatial motion description. For this means, position and orientation representation by rotation matrix, screw axis, quaternions and Euler angles are introduces. General motion of a rigid body is then represented by Chasles's theorem, homogeneous transformation and screw axis representation.

Welcome

To Your Prospect Skills

On Robotics :

Mechanics and Control

01

About ARAS

ARAS Research group originated in 1997 and is proud of its 22+ years of brilliant background, and its contributions to the advancement of academic education and research in the field of Dynamical System Analysis and Control in the robotics application. ***ARAS*** are well represented by the industrial engineers, researchers, and scientific figures graduated from this group, and numerous industrial and R&D projects being conducted in this group. The main asset of our research group is its human resources devoted all their time and effort to the advancement of science and technology. One of our main objectives is to use these potentials to extend our educational and industrial collaborations at both national and international levels. In order to accomplish that, our mission is to enhance the breadth and enrich the quality of our education and research in a dynamic environment.

Contents

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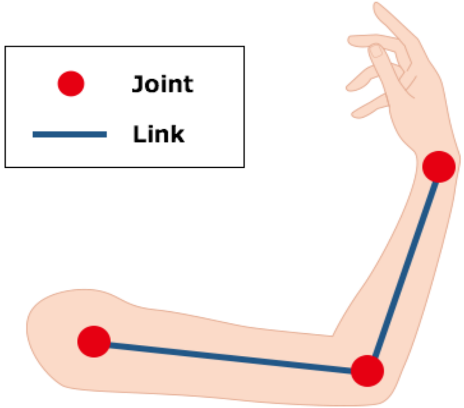
Motion Description

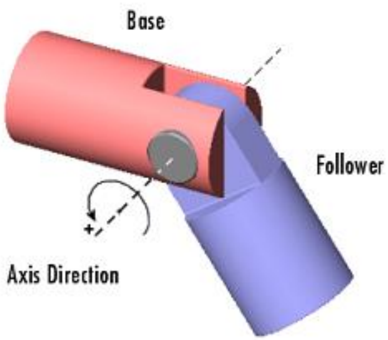
- Robot Components:
  - ✓ Links and Joints
    - Rigid Links
    - Flexible Links



Motion Description

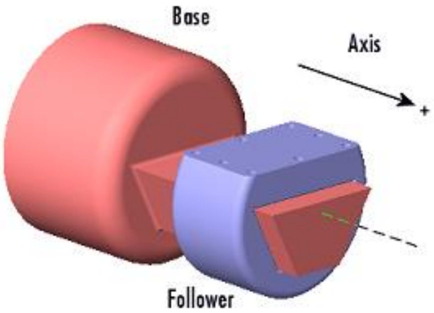
- Robot Components:
  - ✓ Primary Joints

Revolute (R)



1 Rotary DoF

Prismatic (P)

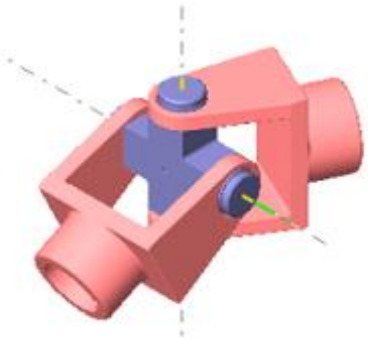


1 Translational DoF

Motion Description

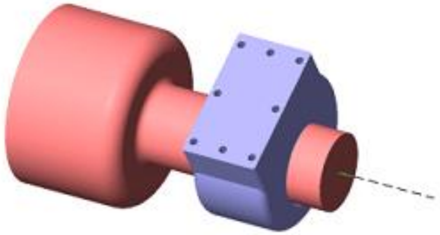
- Robot Components:
  - ✓ Compound Joints

Universal (U)



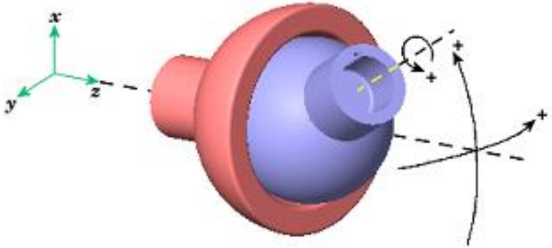
2R DoFs

Cylindrical (C)



PR: T1R DoF

Spherical (S)

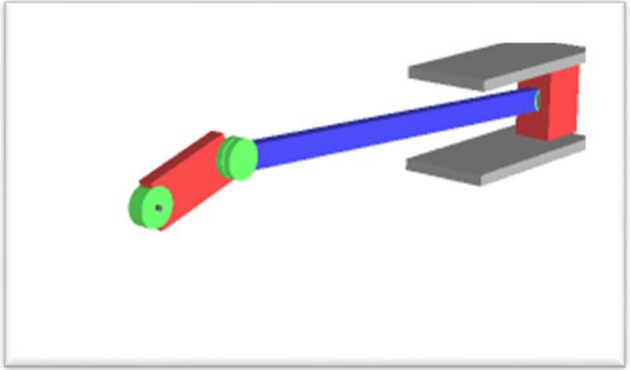


3R DoFs

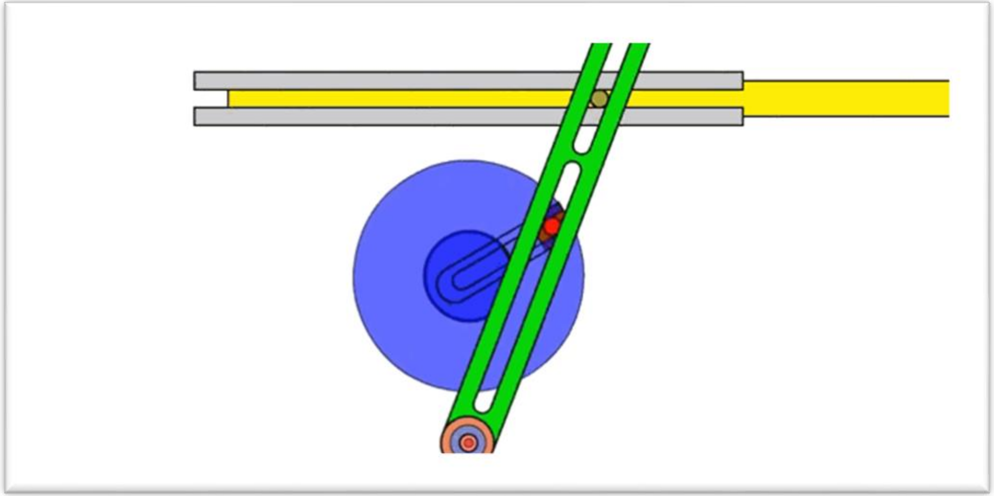
Motion Description

- Robot Kinematic Structure
  - ✓ Crank and Piston (Planar)
    - ▪ RRRP

RRPP

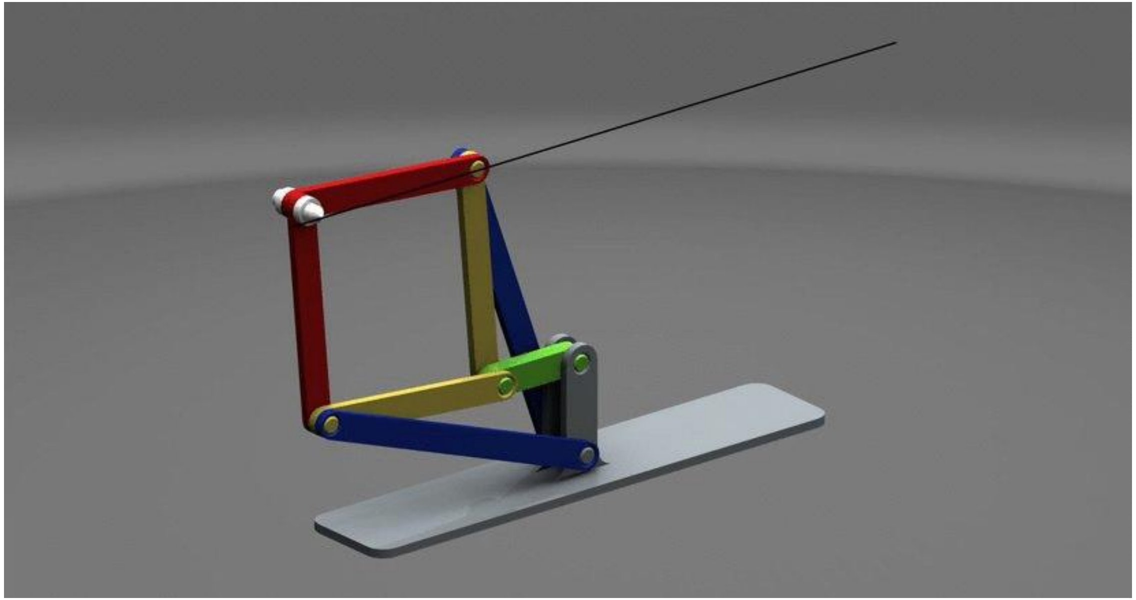


Single Kinematic Loop



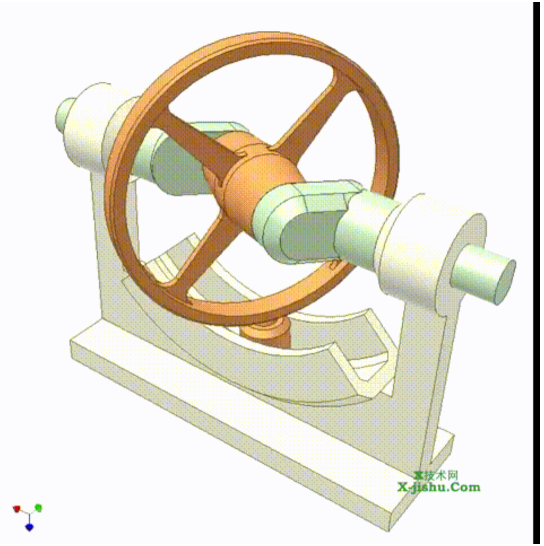
Motion Description

- Robot Kinematic Structure
  - ✓ Chain of Kinematic Loops (Planar)



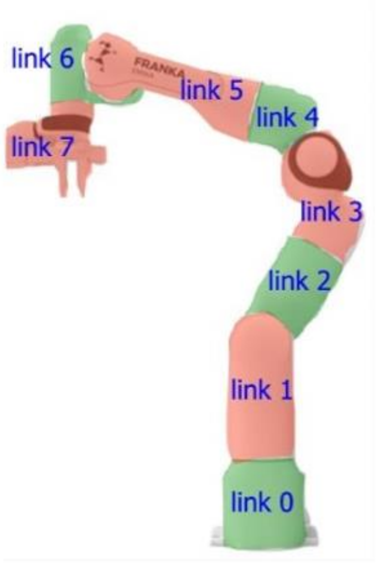
Motion Description

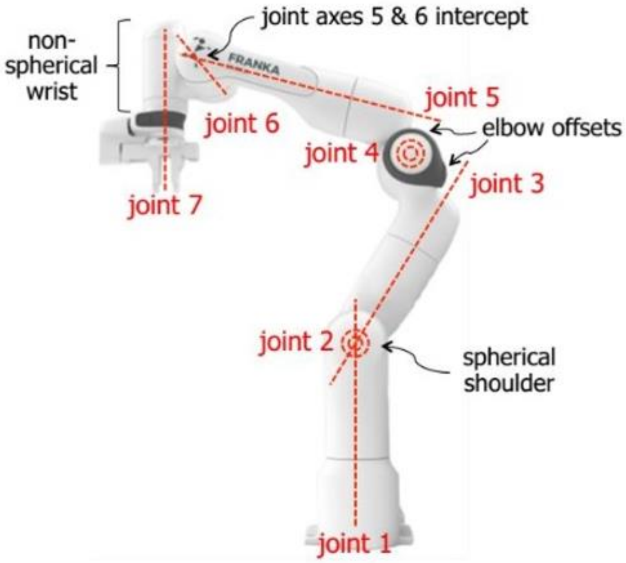
- Robot Kinematic Structure
  - ✓ Single Kinematic Chain (Spatial)



Motion Description

- Robot Kinematic Structure
  - ✓ Open Kinematic Chain (Spatial)

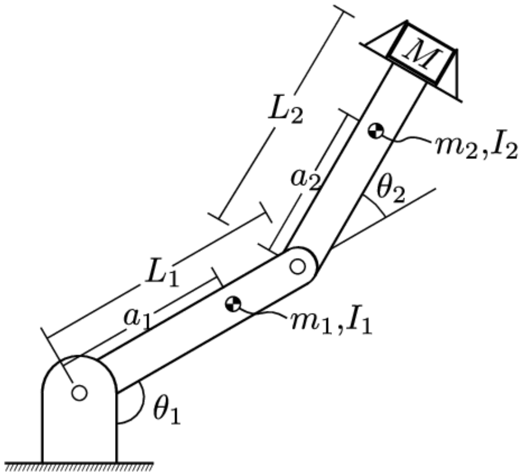




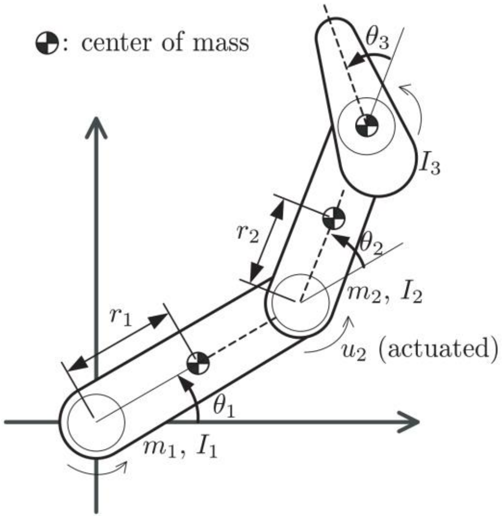
Motion Description

- Robot Kinematic Structure
  - ✓ Different Serial Robots (Planar)

RR (x_e, y_e)



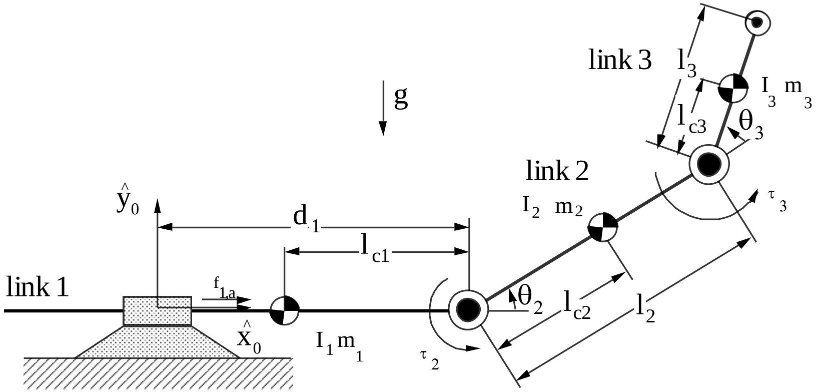
RRR (x_e, y_e, θ_e)



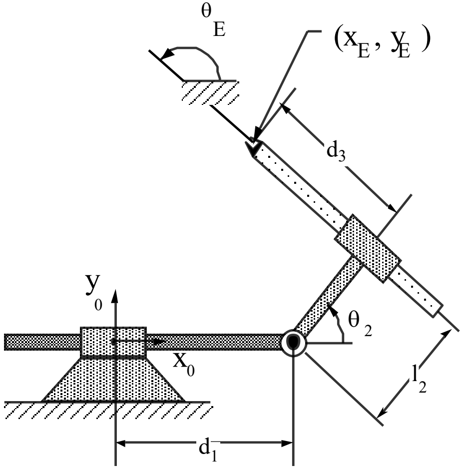
Motion Description

- Robot Kinematic Structure
  - ✓ Different Serial Robots (Planar)

PRR (x_e, y_e, θ_e)



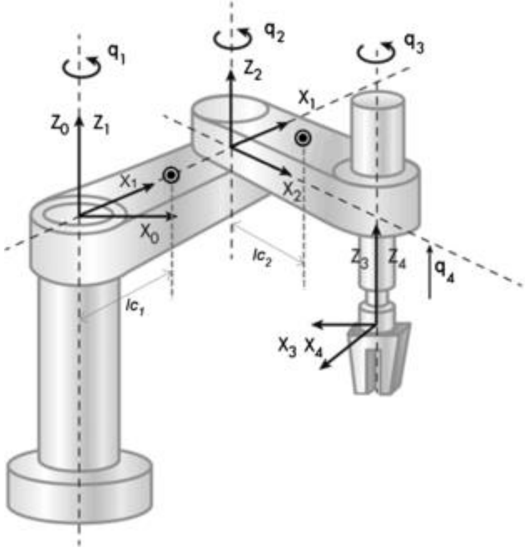
PRP (x_E, y_E, θ_E)



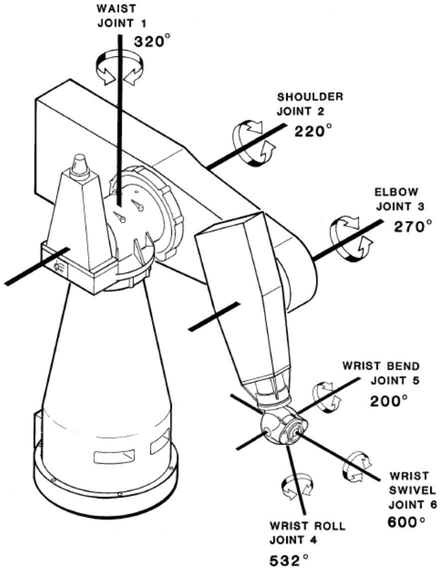
Motion Description

- Robot Kinematic Structure
  - ✓ Different Serial Robots (Spatial)

SCARA RRRP



PUMA 6R



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Motion Description

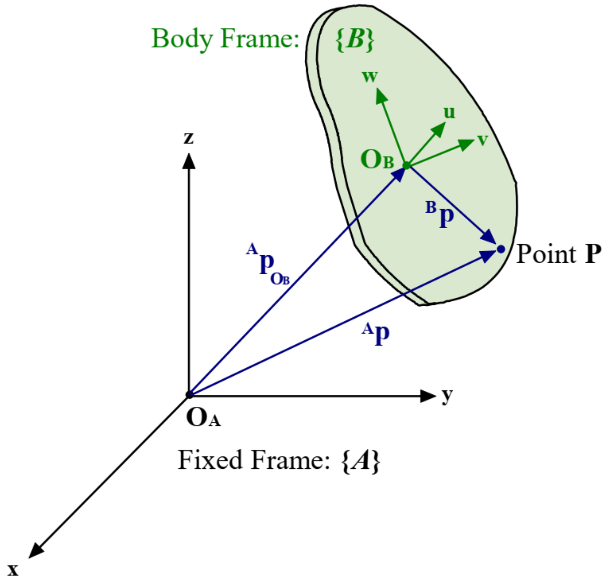
- Coordinate Systems
  - ✓ Cartesian coordinate
    - Reference Frame $\{A\}$

      With origin @ $\boldsymbol{O}_A$

      and $\{\hat{x}, \hat{y}, \hat{z}\}$ unit direction in space.
    - Moving Frame $\{B\}$

      With origin @ $\boldsymbol{O}_B$

      and $\{\hat{u}, \hat{v}, \hat{w}\}$ unit direction in space.
  - ✓ Cylindrical or Spherical

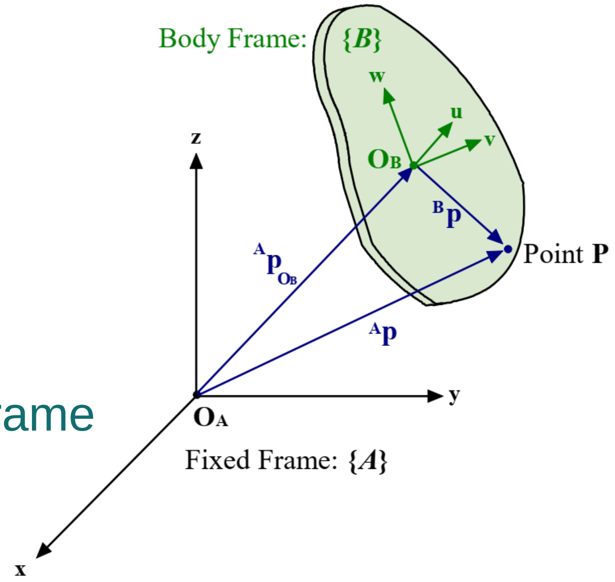


Motion Description

- Spatial Motion Description
  - ✓ Position of a point in rigid body
    - With reference to frame $\{A\}$

$$^{A}\boldsymbol{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

    - Superscript A denote the reference frame
    - Index $\{x, y, z\}$ denote the Cartesian component of vector $\boldsymbol{P}$.



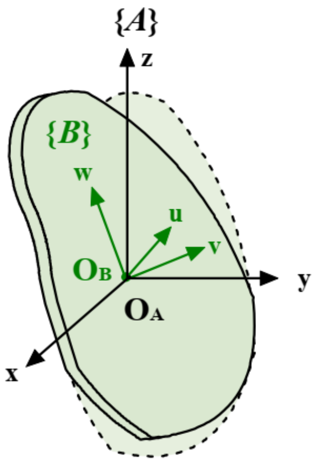
Motion Description

- Spatial Motion Description
  - ✓ Orientation of the whole rigid body
    - Consider pure rotation
    - Affix a moving frame $\{B\}$ to the rigid body
    - Represent the rotation of two frames
  - ✓ Rotation Matrix

$$^{A}\mathbf{R}_B = [{}^{A}\hat{x}_B \mid {}^{A}\hat{y}_B \mid {}^{A}\hat{z}_B] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

  - ✓ Or

$$^{A}\mathbf{R}_B = [{}^{A}\hat{u} \mid {}^{A}\hat{v} \mid {}^{A}\hat{w}] = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}.$$



Motion Description

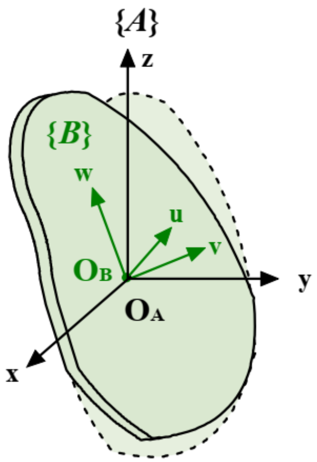
✓ Rotation Matrix

- Component wise:

$$^A\hat{x}_B = {}^A\hat{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k},$$
$$^A\hat{y}_B = {}^A\hat{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k},$$
$$^A\hat{z}_B = {}^A\hat{w} = w_x\hat{i} + w_y\hat{j} + w_z\hat{k},$$

- Use dot product $u_x = \hat{x}_B \cdot \hat{x}_A$ (in any frame)

$$^A\mathbf{R}_B = [{}^A\hat{x}_B \mid {}^A\hat{y}_B \mid {}^A\hat{z}_B] = \begin{bmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{bmatrix}.$$



- Direction Cosine representation
- The leading superscript is intentionally omitted, since the dot product is a scalar that can be evaluated in any frame $\{A\}$ or $\{B\}$.

Motion Description

✓ Rotation Matrix

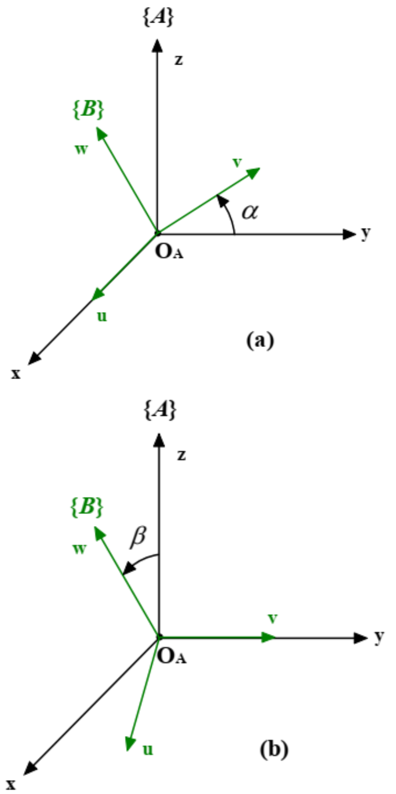
- Example 1:

a) Rotation about x axis with an angle of α

$$^{A}R_{B}=R_{x}(\alpha)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix},$$

b) Rotation about y axis with an angle of β

$$^{A}R_{B}=R_{y}(\beta)=\begin{bmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{bmatrix},$$



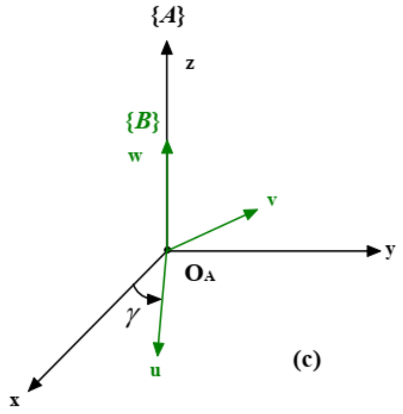
Motion Description

✓ Rotation Matrix

- Example 1: (Cont.)

c) Rotation about z axis with an angle of γ

$$^{A}R_{B} = R_{z}(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



✓ Rotation Matrix Properties

- Property 1: Orthonormal

Rotation matrix is an orthonormal matrix, i.e.

$$u \cdot u = v \cdot v = w \cdot w = 1,$$
$$u \cdot v = v \cdot w = w \cdot u = 0.$$

Rotation matrix: Nine Parameters with Six constraints.

Furthermore: $u \times v = w, \quad v \times w = u, \quad w \times u = v.$

✓ Rotation Matrix Properties

- ### Property 2: Transposition

By inspection: Rows of the rotation matrix ${}^{A}\mathbf{R}_{B}$ are the unit vectors of frame $\{A\}$ expressed in frame $\{B\}$:

$$ {}^{A}\mathbf{R}_{B} = \left[{}^{A}\hat{x}_{B} \mid {}^{A}\hat{y}_{B} \mid {}^{A}\hat{z}_{B} \right] = \begin{bmatrix} {}^{B}\hat{x}_{A}^{T} \\ {}^{B}\hat{y}_{A}^{T} \\ {}^{B}\hat{z}_{A}^{T} \end{bmatrix} \Rightarrow \quad {}^{B}\mathbf{R}_{A} = {}^{A}\mathbf{R}_{B}^{T}. $$

- ### Property 3: Inverse

The inverse of the rotation matrix is equal to its transpose

$$ {}^{B}\mathbf{R}_{A} = {}^{A}\mathbf{R}_{B}^{-1} = {}^{A}\mathbf{R}_{B}^{T}. $$

Note on the inverse of the rotation map.

Motion Description

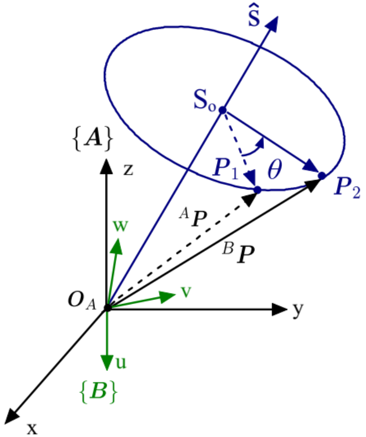
✓ Rotation Matrix Properties

- Property 4: Pure rotation map

For a pure rotation:

$$^A\boldsymbol{P} = {^A\boldsymbol{R}_B}\, {^B\boldsymbol{P}}.$$

See the proof in reference book [4].



- Example 2: Consider

$$^A\boldsymbol{R}_B = \begin{bmatrix} 0.933 & -0.067 & -0.354 \\ -0.067 & 0.933 & -0.354 \\ 0.354 & 0.354 & 0.866 \end{bmatrix} \quad \text{and} \quad {^B\boldsymbol{P}} = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}^T.$$

Then:

$$^A\boldsymbol{P} = {^A\boldsymbol{R}_B}\, {^B\boldsymbol{P}}$$

$$= \begin{bmatrix} 0.933 & -0.067 & -0.354 \\ -0.067 & 0.933 & -0.354 \\ 0.354 & 0.354 & 0.866 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.93 \\ -1.07 \\ 0.35 \end{bmatrix}.$$

Norm of the vector in $\{A\}$ and $\{B\}$ frames are both $\sqrt{5}$, they are just rotated.

Motion Description

✓ Rotation Matrix Properties

- Property 5: Determinant

$$\det({}^A\boldsymbol{R}_B) = 1. \tag{2.19}$$

Proof:

$$\begin{aligned}\det({}^A\boldsymbol{R}_B) &= w_x(u_y v_z - v_y u_z) - w_y(u_x v_z - v_x u_z) + w_x(u_x v_y - v_x u_y)\\ &= \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})\\ &= \boldsymbol{w} \cdot \boldsymbol{w}\\ &= 1.\end{aligned}$$

Norm of the vector under pure rotational map is preserved.

Motion Description

✓ Rotation Matrix Properties

- Property 6: Eigenvalues

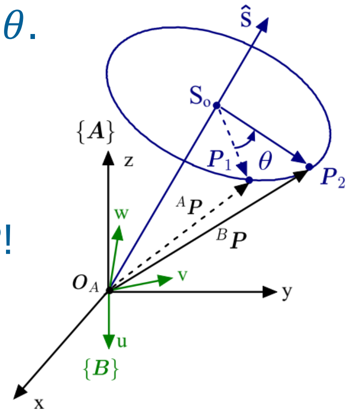
Eigenvalues of a rotation matrix are: $1, e^{i\theta}, e^{-i\theta}$, where,

$$\theta = \cos^{-1}\frac{\mathrm{tr}({}^{A}\boldsymbol{R}_{B})-1}{2}. \qquad (2.20)$$

The angle of rotation in a pure rotational map is denoted by θ.

What about the axis of Rotation?

It might be related to the real eigenvector of rotation matrix?!



Motion Description

- Orientation Representation

  ✓ Euler Angle-Axis of Rotation: Screw Axis

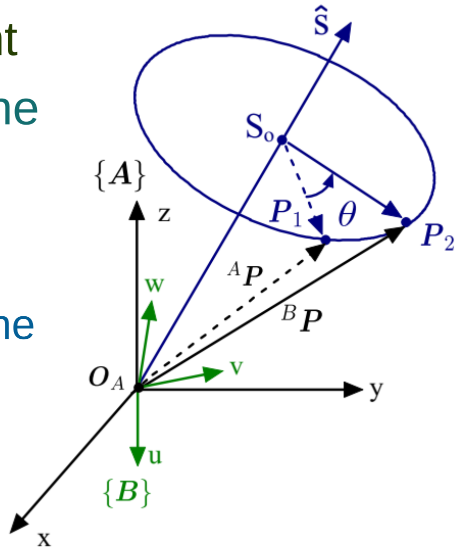
  - In rotation matrix representation some invariant parameters remain unchanged regardless of the frame of representation.

The pure rotation angle θ found by (2.21)

The Euler axis of rotation is the real eigenvalue of the rotation matrix, represented by $\hat{s}$.

Euler or Screw Axis Representation $\boldsymbol{s} = \theta\hat{s}$

Four parameters: $\{\hat{s}_x, \hat{s}_y, \hat{s}_z, \theta\}$, while $\|\hat{s}\| = 1$.



Motion Description

- Orientation Representation

✓ Rotation Matrix from Screw Axis

- Given

$$^{A}\boldsymbol{R}_{B} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Euler or Screw Axis Representation $\boldsymbol{s} = \theta\hat{s}$, where

$$\theta = \cos^{-1}\frac{\mathrm{tr}(^{A}\boldsymbol{R}_{B}) - 1}{2}$$

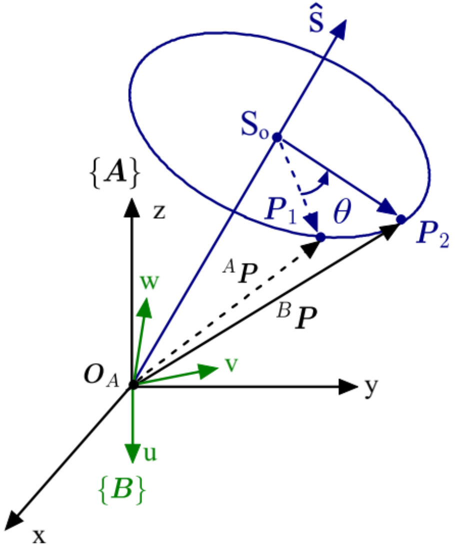
And

$$s_x = \frac{r_{32} - r_{23}}{2s\theta},$$

$$s_y = \frac{r_{13} - r_{31}}{2s\theta},$$

$$s_z = \frac{r_{21} - r_{12}}{2s\theta}.$$

where, $\sin\theta = s\theta$.



Motion Description

- Orientation Representation
  - ✓ Screw Axis from Rotation Matrix
    - Given the rotation axis and angle: $\boldsymbol{s} = \theta\hat{s}$

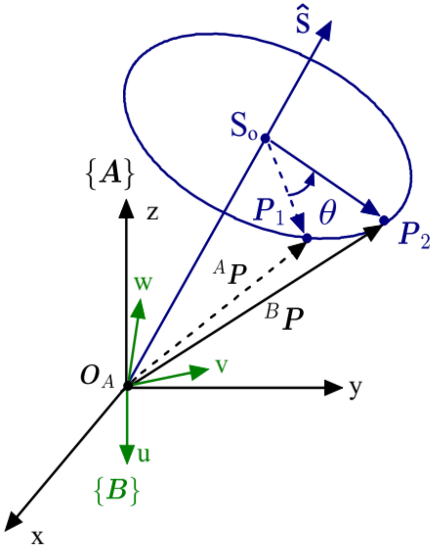
Using Rodrigues's rotation formula

$$\boldsymbol{P}_2 = \boldsymbol{P}_1 \cos\theta + (\hat{\mathbf{s}} \times \boldsymbol{P}_1)\sin\theta + (\boldsymbol{P}_1 \cdot \hat{\mathbf{s}})(1-\cos\theta)\hat{\mathbf{s}}$$

Then the rotation matrix is found by:

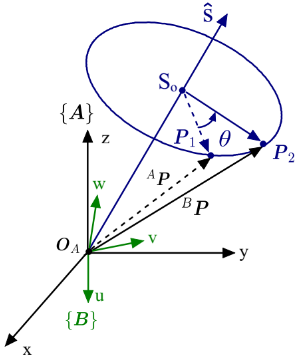
$${}^{A}\mathbf{R}_B = \begin{bmatrix} s_x^2 v\theta + c\theta & s_x s_y v\theta - s_z s\theta & s_x s_z v\theta + s_y s\theta \\ s_y s_x v\theta + s_z s\theta & s_y^2 v\theta + c\theta & s_y s_z v\theta - s_x s\theta \\ s_z s_x v\theta - s_y s\theta & s_z s_y v\theta + s_x s\theta & s_z^2 v\theta + c\theta \end{bmatrix}$$

where, $s\theta = \sin\theta$, $c\theta =$, $v\theta = 1 - \cos\theta$.



Motion Description

- Orientation Representation
  - ✓ Euler Parameters or Unit Quaternion
    - Given the rotation axis and angle: $\boldsymbol{s} = \theta \hat{s}$
    - Four Euler parameters are defined as:



$$\epsilon_1 = s_x \sin(\theta/2); \quad \epsilon_2 = s_y \sin(\theta/2); \quad \epsilon_3 = s_z \sin(\theta/2); \quad \epsilon_4 = \cos(\theta/2). \qquad (2.77)$$

- Unit Quaternion: 4×1 tuple

$$\epsilon = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \end{bmatrix}^T \text{ in which } \|\epsilon\|^2 = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$$

Very common in instrumentation systems.

Motion Description

- Orientation Representation

✓ Euler Parameters or Unit Quaternion

- Given the quaternions: $\epsilon = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \end{bmatrix}^T$
- The rotation matrix is found:

$$^{A}R_{B} = \begin{bmatrix} 1 - 2\epsilon_2^2 - 2\epsilon_3^2 & 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_3^2 & 2(\epsilon_2\epsilon_3 - \epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_2^2 \end{bmatrix}. \tag{2.78}$$

- Given The rotation matrix, the quaternions are found:

$$\epsilon_1 = \frac{(r_{32} - r_{23})}{4\epsilon_4}; \qquad \epsilon_3 = \frac{(r_{21} - r_{12})}{4\epsilon_4};$$

$$\epsilon_2 = \frac{(r_{13} - r_{31})}{4\epsilon_4}; \qquad \epsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}.$$

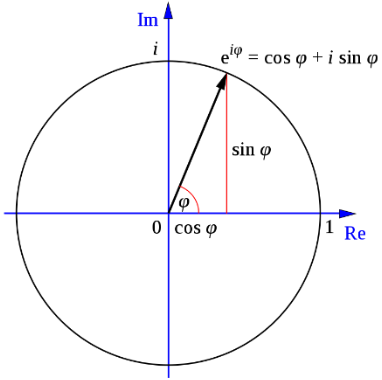
Scientist Bio

Leonhard Euler

(15 April 1707 – 18 September 1783)

Was a Swiss mathematician, physicist, astronomer, geographer, logician and engineer who made important and influential discoveries in many branches of mathematics, such as infinitesimal calculus and graph theory, while also making pioneering contributions to several branches such as topology and analytic number theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function. He is also known for his work in mechanics, fluid dynamics, optics, astronomy and music theory rigid bodies.

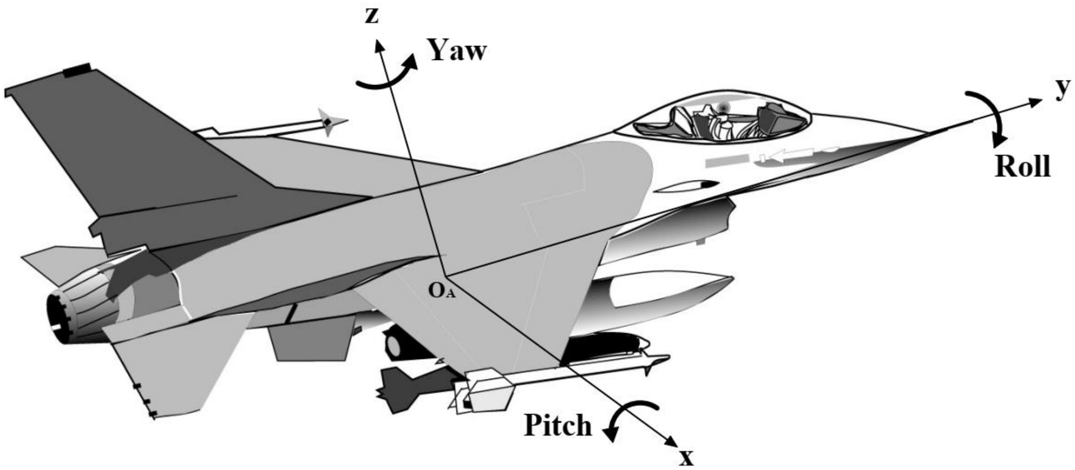
Euler was one of the most eminent mathematicians of the 18th century and is held to be one of the greatest in history. He is also widely considered to be the most prolific, as his collected works fill 92 volumes, more than anyone else in the field.



A geometric interpretation of Euler's Formula

Motion Description

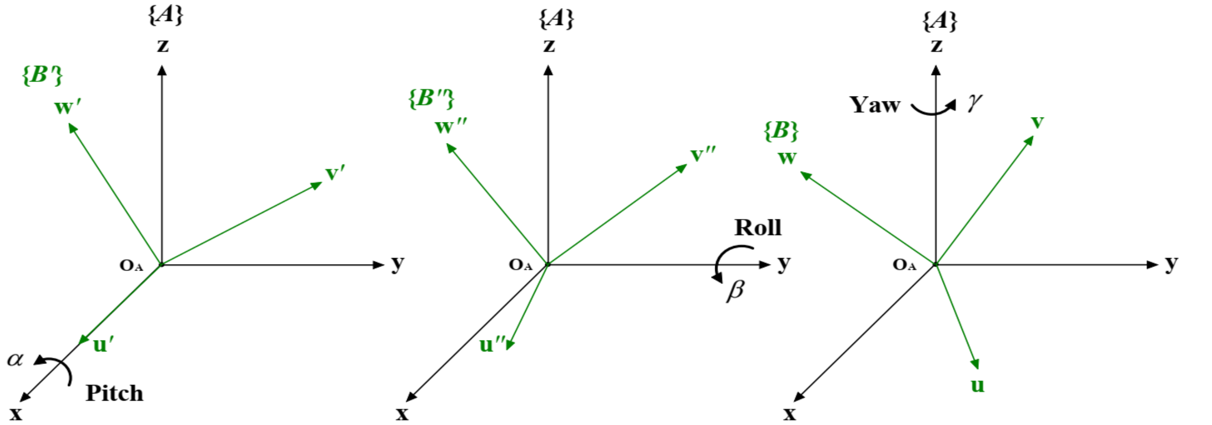
- Orientation Representation
  - ✓ Euler Angles: Three parameter representation
    - Pitch-Roll-Yaw



Motion Description

✓ Euler Angles:

- Pitch-Roll-Yaw: rotation about fixed frame axes



Rotate moving frame about x with angle α (pitch angle), then rotate about y with angle β (roll angle), and then rotate about z with angle γ (yaw angle)

Motion Description

✓ Fixed Axes Euler Angles:

- Pitch-Roll-Yaw: Pre-Multiplication

$$R_{PRY}(\alpha,\beta,\gamma) = R_z(\gamma)R_y(\beta)R_x(\alpha)$$

$$= \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix}, \qquad (2.37)$$

$$= \begin{bmatrix} c\beta c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ c\beta s\gamma & s\alpha s\beta s\gamma + c\alpha c\gamma & c\alpha s\beta s\gamma - s\alpha c\gamma \\ -s\beta & s\alpha c\beta & c\alpha c\beta \end{bmatrix}. \qquad (2.38)$$

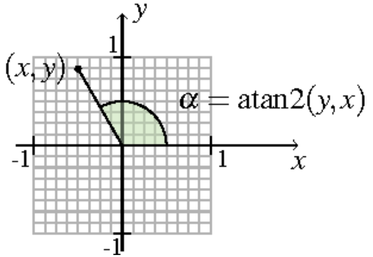
- Inverse Map:

(if $c\beta \neq 0$)

$$\beta = \text{Atan2}\left(-r_{31}, \pm\sqrt{r_{11}^2 + r_{21}^2}\right),$$

$$\gamma = \text{Atan2}\left(r_{21}/c\beta, r_{11}/c\beta\right),$$

$$\alpha = \text{Atan2}\left(r_{32}/c\beta, r_{33}/c\beta\right).$$



where $\text{atan}(1.0/-1.0) = -\pi/4$ while $\text{atan2}(1.0, -1.0) = 3\pi/4$.

Motion Description

✓ Fixed Axes Euler Angles:

- Pitch-Roll-Yaw:
- Inverse Map: (if $c\beta = 0$ or $\beta = \pm\,\pi/2$)

The rotation matrix is reduced to:

$$R_{PRY} = \begin{bmatrix} 0 & \pm s\alpha c\gamma - c\alpha s\gamma & \pm s\alpha c\gamma + c\alpha s\gamma \\ 0 & \pm s\alpha c\gamma + c\alpha s\gamma & \pm s\alpha c\gamma - c\alpha s\gamma \\ \mp 1 & 0 & 0 \end{bmatrix}$$

The inverse solution degenerates

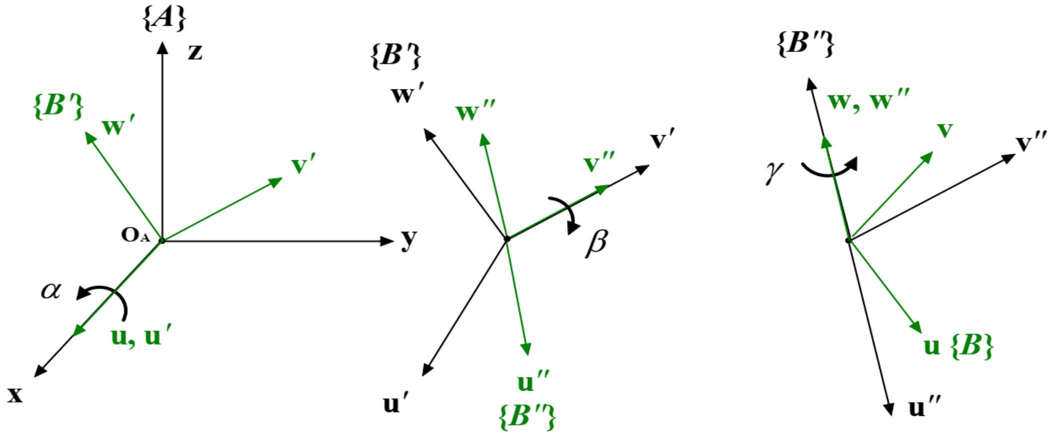
Only the sum or difference of α and γ may be computed:

$$\begin{cases} \beta = \pi/2 \\ \alpha - \gamma = \text{Atan2}\,(r_{12}, r_{22}) \end{cases} \quad \text{or} \quad \begin{cases} \beta = -\pi/2 \\ \alpha + \gamma = \text{Atan2}\,(-r_{12}, r_{22}) \end{cases}$$

Motion Description

✓ Euler Angles:

- $u - v - w$: Rotation about moving frame axes



Rotate moving frame about u with angle α, then rotate about v with angle β, and then rotate about w with angle γ.

Motion Description

✓ Moving Axes Euler Angles:

- $u - v - w$: Post-Multiplication

$$^{A}R_{B}(\alpha,\beta,\gamma) = R_{u}(\alpha)R_{v}(\beta)R_{w}(\gamma). \quad (2.42)$$

$$R_{uvw}(\alpha,\beta,\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix}. \quad (2.43)$$

- Inverse Map:

(if $c\beta \neq 0$)

$$\beta = \text{Atan2}\,(r_{13}, \pm\sqrt{r_{11}^2 + r_{12}^2}),$$

$$\alpha = \text{Atan2}\,(-r_{23}/c\beta, r_{33}/c\beta),$$

$$\gamma = \text{Atan2}\,(-r_{12}/c\beta, r_{11}/c\beta).$$

Motion Description

✓ Moving Axes Euler Angles:

- $u-v-w$: Post-Multiplication
- Inverse Map: (if $c\beta = 0$)

  The inverse solution degenerates

  Only the sum or difference of α and γ may be computed:

$$\begin{cases} \beta = \pi/2 \\ \alpha + \gamma = \text{Atan2}\,(r_{32}, r_{22}) \end{cases} \quad \text{or} \quad \begin{cases} \beta = -\pi/2 \\ \alpha - \gamma = \text{Atan2}\,(r_{32}, r_{22}) \end{cases}$$

- Example:

For $R = \begin{bmatrix} 0 & 0 & 1.0000 \\ 0.9658 & -0.2588 & 0 \\ 0.2588 & 0.9658 & 0 \end{bmatrix} \rightarrow c\beta = 0$, and $\begin{cases} \beta = \pi/2 \\ \alpha + \gamma = 5\pi/12 \end{cases}$ or $\begin{cases} \beta = -\pi/2 \\ \alpha - \gamma = 5\pi/12. \end{cases}$

Motion Description

✓ Other Euler Angles: (24 Angle Set)

- Moving frame: $w - v - w$

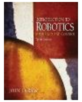
$$R_{wvw}(\alpha,\beta,\gamma) = R_w(\alpha)R_v(\beta)R_w(\gamma)$$

$$= \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}.$$

- Moving frame: $w - u - w$

$$R_{wuw}(\alpha,\beta,\gamma) = R_w(\alpha)R_u(\beta)R_w(\gamma)$$

$$= \begin{bmatrix} c\alpha c\gamma - s\alpha c\beta s\gamma & -c\alpha s\gamma - s\alpha c\beta c\gamma & s\alpha s\beta \\ s\alpha c\gamma + c\alpha c\beta s\gamma & -s\alpha s\gamma + c\alpha c\beta c\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix}.$$



- All other set components in Appendix B.

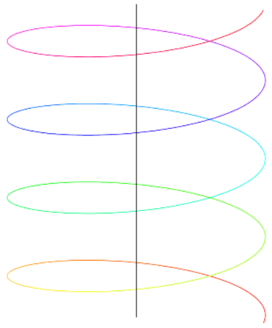
Scientist Bio

Michel Floréal Chasles

(15 November 1793 – 18 December 1880)

Was a French mathematician. In 1837 he published the book *Aperçu historique sur l'origine et le développement des méthodes en géométrie* ("Historical view of the origin and development of methods in geometry"), a study of the method of reciprocal polars in projective geometry. The work gained him considerable fame and respect and he was appointed Professor at the École Polytechnique in 1841, then he was awarded a chair at the Sorbonne in 1846. In 1841, Charles published an English version as *Two Geometrical Memoirs on the General Properties of Cones of the Second Degree and on the Spherical Conics*, adding a significant amount of original material.

In kinematics, Chasles's description of a Euclidean motion in space as screw displacement was seminal to the development of the theories of dynamics of rigid bodies.



Chasles' theorem

Contents

In this chapter we first briefly review the robot components, and then elaborate on spatial motion description. For this means, position and orientation representation by rotation matrix, screw axis, quaternions and Euler angles are introduces. General motion of a rigid body is then represented by Chasles's theorem, homogeneous transformation and screw axis representation.

Motion Description

- ## General Motion of a Rigid Body

Chasles' Theorem: The most general rigid body displacement can be produced by a translation along a line (called its screw axis) followed by a rotation about that same line.

✓ Simple Version:

- General Motion = Rotation + Translation

$$ {}^{A}P = {}^{A}R_{B}\,{}^{B}P + {}^{A}P_{O_B}. \tag{2.52}$$

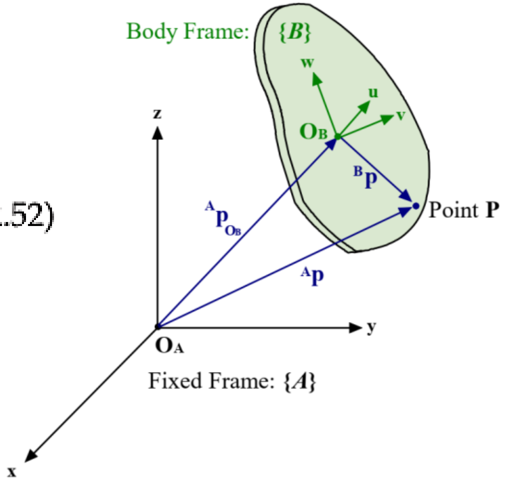
- Full matrix representation

Homogeneous Transformation Matrix (4 × 4):

$$ {}^{A}T_{B} = \left[\begin{array}{ccc|c} & {}^{A}R_{B} & & {}^{A}P_{O_B} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. $$

Then:

$$ {}^{A}P = {}^{A}T_{B}\,{}^{B}P, \tag{2.53}$$



Motion Description

- General Motion of a Rigid Body
  - ✓ Screw Displacement
    - General Motion =

      Rotation about $\hat{s}$ + Translation along $\hat{s}$

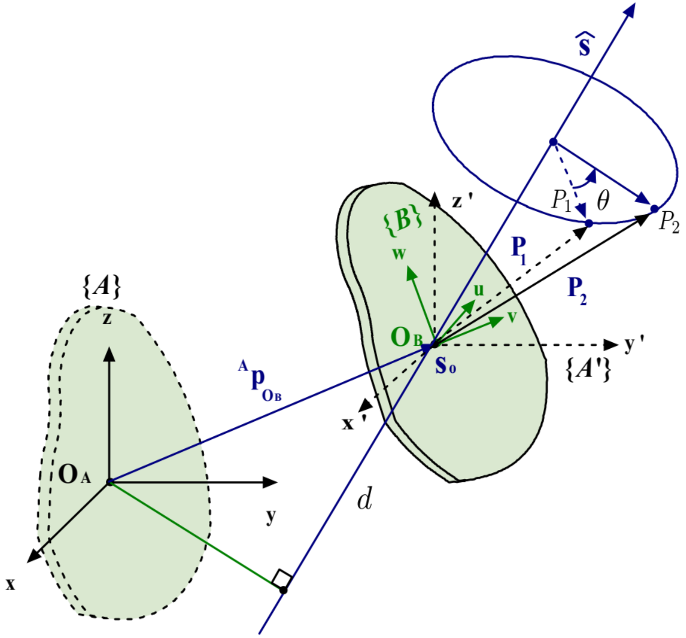
      $\{\hat{s}, \theta\} \quad + \quad \{s_0, d\}$
    - The frame displacement ${}^{A}\boldsymbol{P}_{O_B}$

      is represented by s_0, and

      The screw lead is denoted by d
    - Eight parameter representation

      With two constraints:

$$\hat{\mathbf{s}}^T \hat{\mathbf{s}} = 1; \quad \mathbf{s}_o^T \hat{\mathbf{s}} = d.$$



Motion Description

- General Motion of a Rigid Body
  - ✓ Given ${}^{A}\boldsymbol{T}_B$ Find Screw Parameters by:
    - Find $\{\hat{s}, \theta\}$ from Slide 28 and find $\{s_0, d\}$ by $s_o = {}^{A}\mathbf{P}_{O_B}; \quad d = \mathbf{s}_o^T \hat{\mathbf{s}}.$
  - ✓ Given Screw Parameters Find ${}^{A}\boldsymbol{T}_B$ by:

$$
{}^{A}\boldsymbol{T}_B = \left[\begin{array}{ccc|c}
s_x^2 v\theta + c\theta & s_x s_y v\theta - s_z s\theta & s_x s_z v\theta + s_y s\theta & p_x \\
s_y s_x v\theta + s_z s\theta & s_y^2 v\theta + c\theta & s_y s_z v\theta - s_x s\theta & p_y \\
s_z s_x v\theta - s_y s\theta & s_z s_y v\theta + s_x s\theta & s_z^2 v\theta + c\theta & p_z \\
\hline
0 & 0 & 0 & 1
\end{array}\right] \tag{2.64}
$$

while,

$$
\begin{aligned}
p_x &= d\, s_x - s_{o_x}(s_x^2 - 1)v\theta - s_{o_y}(s_x s_y v\theta - s_z s\theta) - s_{o_z}(s_x s_z v\theta + s_y s\theta) \\
p_y &= d\, s_y - s_{o_x}(s_y s_x v\theta + s_z s\theta) - s_{o_y}(s_y^2 - 1)v\theta - s_{o_z}(s_y s_z v\theta - s_x s\theta) \\
p_z &= d\, s_z - s_{o_x}(s_z s_x v\theta - s_y s\theta) - s_{o_y}(s_z s_y v\theta + s_x s\theta) - s_{o_z}(s_z^2 - 1)v\theta
\end{aligned} \tag{2.65}
$$

Motion Description

- General Motion of a Rigid Body
  - ✓ Example: Given

$$
{}^{A}\mathbf{T}_{B} = \left[\begin{array}{ccc|c} +0.7500 & 0.6124 & -0.2500 & +1.4142 \\ -0.6124 & 0.5000 & -0.6124 & -1.5764 \\ -0.2500 & 0.6124 & +0.7500 & -1.4142 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].
$$

  - The screw displacement is found as:

$$
\left\{\theta = \frac{\pi}{3}; \quad \hat{s} = \frac{1}{\sqrt{2}}[1, 0, -1]^T\right\}
$$

and

$$
\left\{d = 2; \quad s_o = [1.4142, -1.5764, -1.4142]^T\right\}.
$$

Contents

In this chapter we first briefly review the robot components, and then elaborate on spatial motion description. For this means, position and orientation representation by rotation matrix, screw axis, quaternions and Euler angles are introduces. General motion of a rigid body is then represented by Chasles's theorem, homogeneous transformation and screw axis representation.

Motion Description

- General Motion of a Rigid Body
  - ✓ Transformation Arithmetic
    - Consecutive Transformation

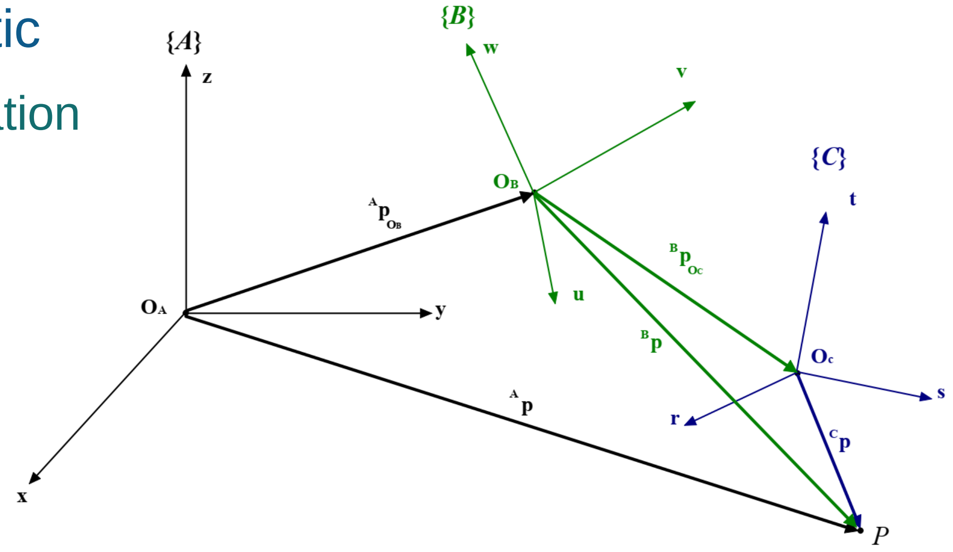
$$^{A}P = {}^{A}T_{B}\, {}^{B}T_{C}\, {}^{C}P.$$

Hence,

$$^{A}T_{C} = {}^{A}T_{B}\, {}^{B}T_{C}.$$

Rotation matrix:

$$^{A}T_{C} = \left[\begin{array}{ccc|c} & {}^{A}R_{B}\, {}^{B}R_{C} & & {}^{A}R_{B}\, {}^{B}P_{O_C} + {}^{A}P_{O_B} \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$



Motion Description

✓ Transformation Arithmetic

- Inverse Transformation

Direct inverse calculation of ${}^{A}\boldsymbol{T}_{B}$ might be computationally intensive.

$$ {}^{B}\boldsymbol{T}_{A} = {}^{A}\boldsymbol{T}_{B}^{-1} = \left[\begin{array}{ccc|c} & {}^{A}\boldsymbol{R}_{B}^{T} & & -{}^{A}\boldsymbol{R}_{B}^{T}\,{}^{A}\boldsymbol{P}_{O_B} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. $$

- Proof:

Recall ${}^{B}\boldsymbol{R}_{A} = {}^{A}\boldsymbol{R}_{B}^{T}$.

Furthermore,

$$ {}^{A}\boldsymbol{P}_{O_A} = -{}^{A}\boldsymbol{P}_{O_B} $$

$$ {}^{B}\boldsymbol{P}_{O_A} = {}^{B}\boldsymbol{R}_{A}\,{}^{A}\boldsymbol{P}_{O_A} = -{}^{B}\boldsymbol{R}_{A}\,{}^{A}\boldsymbol{P}_{O_B} $$

$$ = -{}^{A}\boldsymbol{R}_{B}^{T}\,{}^{A}\boldsymbol{P}_{O_B} $$

Motion Description

✓ Transformation Arithmetic

- Finite Angle Rotations

For finite angle of rotation the order of rotations are important.

- Example:

$$R_x(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{bmatrix},$$

$$R_z(\theta_2) = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Consider, first rotation about $\hat{x}$ with θ_1.

Then, rotate about $\hat{z}$ with θ_2.

The total rotation matrix is R_1:

$$R_1 = R_x(\theta_1)\, R_z(\theta_2) = \begin{bmatrix} c_2 & -s_2 & 0 \\ c_1 s_2 & c_1 c_2 & -s_1 \\ s_1 s_2 & s_1 c_2 & c_1 \end{bmatrix},$$

The order cannot be exchanged:

$$R_2 = R_z(\theta_2)\, R_x(\theta_1) = \begin{bmatrix} c_2 & -c_1 s_2 & s_1 s_2 \\ s_2 & c_1 c_2 & -s_1 c_2 \\ 0 & s_1 & c_1 \end{bmatrix}.$$

$$R_1 \neq R_2$$

Finite rotations obey matrix manipulation rules, and are not vector, and do not commute.

Motion Description

✓ Transformation Arithmetic

- Infinitesimal Angle Rotations

Infinitesimal angle of rotation act like vectors.

- Example:

Consider, first rotation about $\hat{x}$ with $\delta\theta_1$, then, rotate about $\hat{z}$ with $\delta\theta_2$.

The total rotation matrix is R_1:

$$R_1 = R_x(\theta_1)\, R_z(\theta_2) = \begin{bmatrix} 1 & -\delta\theta_2 & 0 \\ \delta\theta_2 & 1 & -\delta\theta_1 \\ \delta\theta_1\delta\theta_2 & \delta\theta_1 & 1 \end{bmatrix},$$

Exchange the order of rotation R_2:

$$R_2 = R_z(\theta_2)\, R_x(\theta_1) = \begin{bmatrix} 1 & -\delta\theta_2 & \delta\theta_1\delta\theta_2 \\ \delta\theta_2 & 1 & -\delta\theta_1 \\ 0 & \delta\theta_1 & 1 \end{bmatrix},$$

For infinitesimal $\delta\theta_i$'s, the higher orders are close to zero, and the order can be exchanged: $R_1 = R_2$

✓ Example:

- Consider the following rotation matrix is given

$$ {}^0\boldsymbol{R}_1 = \begin{bmatrix} 0.933 & 0.167 & 0.354 \\ 0.067 & 0.933 & -0.354 \\ -0.354 & 0.354 & 0.866 \end{bmatrix}. $$

While, ${}^0\boldsymbol{P}_{O_1} = [2 \quad -1 \quad 0]^T$; Find ${}^0\boldsymbol{T}_1$.

Solution: The given rotation matrix is not orthonormal, by inspection correct the mistyping error by: $r_{12} = 0.067$. Then

$$ {}^1\boldsymbol{T}_0 = {}^0\boldsymbol{T}_1^{-1} = \left[\begin{array}{ccc|c} & {}^0\boldsymbol{R}_1^T & & -{}^0\boldsymbol{R}_1^T\,{}^0\boldsymbol{P}_{O_1} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ = \left[\begin{array}{ccc|c} 0.933 & 0.067 & -0.354 & -1.80 \\ 0.067 & 0.933 & 0.354 & 0.80 \\ 0.354 & -0.354 & 0.866 & -1.06 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. $$

Hamid D. Taghirad
Professor

About Hamid D. Taghirad

Hamid D. Taghirad has received his B.Sc. degree in mechanical engineering from Sharif University of Technology, Tehran, Iran, in 1989, his M.Sc. in mechanical engineering in 1993, and his Ph.D. in electrical engineering in 1997, both from McGill University, Montreal, Canada. He is currently the University Vice-Chancellor for Global strategies and International Affairs, Professor and the Director of the Advanced Robotics and Automated System (ARAS), Department of Systems and Control, Faculty of Electrical Engineering, K. N. Toosi University of Technology, Tehran, Iran. He is a senior member of IEEE, and Editorial board of International Journal of Robotics: Theory and Application, and International Journal of Advanced Robotic Systems. His research interest is *robust* and *nonlinear control* applied to *robotic systems*. His publications include five books, and more than 250 papers in international Journals and conference proceedings.

Robotics: Mechanics & Control

Chapter 2: Motion Description

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