Hindawi Publishing Corporation
Modelling and Simulation in Engineering
Volume 2016, Article ID 8464205, 7 pages
http://dx.doi.org/10.1155/2016/8464205

Research Article

On Torsion of Functionally Graded Elastic Beams

Marina Diaco

Department of Structures for Engineering and Architecture, University of Naples Federico II, via Claudio 21, 80125 Naples, Italy

Correspondence should be addressed to Marina Diaco; diaco@unina.it

Received 14 December 2015; Accepted 5 October 2016

Academic Editor: Theodoros C. Rousakis

The evaluation of tangential stress fields in linearly elastic orthotropic Saint-Venant beams under torsion is based on the solution of Neumann and Dirichlet boundary value problems for the cross-sectional warping and for Prandtl stress function, respectively. A skillful solution method has been recently proposed by Ecsedi for a class of inhomogeneous beams with shear moduli defined in terms of Prandtl stress function of corresponding homogeneous beams. An alternative reasoning is followed in the present paper for orthotropic functionally graded beams with shear moduli tensors defined in terms of the stress function and of the elasticity of reference inhomogeneous beams. An innovative result of invariance on twist centre is also contributed. Examples of functionally graded elliptic cross sections of orthotropic beams are developed, detecting thus new benchmarks for computational mechanics.

1. Introduction

Analyses of composite media are a well-investigated research field in structural mechanics. Theoretical noteworthy results also in the nonlinear range have recently contributed to several engineering applications, such as beam and plate theories [1–3], fracture mechanics [4–8], hyperelastic media [9–11], concrete systems [12–16], nonlocal models [17–19], homogenization [20–22], thermoelasticity [23–25], nanostructures [26–30], and limit analysis [31–33]. In the context of the classical theory of elasticity, an innovative methodology for the analysis of beams was proposed by Saint-Venant [34, 35], with the assumption that the normal interactions between longitudinal fibres vanish [36]. Basic results about this model are collected in classical treatments [37–44] with a coordinate approach. Coordinate-free investigations can be found in [45–48]. Nevertheless, analytical solutions of beams subjected to torsion can be obtained only for special cross-sectional geometries and shear moduli distributions. Exact solutions of functionally graded structures can be found in [49]. However, finite element strategies are often adopted in order to get effective numerical results when exact solutions are not available; see, for example, [50–53]. Alternatively, experimental methods are employed; see, for example, [54]. Recently Ecsedi showed that, for functionally graded cross sections under torsion, with shear modulus defined by a positive function of the Prandtl stress function of a corresponding homogeneous cross-section, the warping is invariant and the stress function is expressed in terms of the one associated with the reference homogeneous cross section [55, 56]. Ecsedi's treatment is based on an integral transformation proposed by Kirchhoff in nonlinear heat conduction [57]. An intrinsic reasoning is illustrated in the present paper, by performing a direct discussion of Neumann and Dirichlet boundary value problems for the cross-sectional warping and for the stress function of orthotropic composite beams under torsion. An invariance condition for the Cicala-Hodges centre is also assessed (see Section 3). Finally, new analytical solutions of functionally graded elliptic cross sections are constructed in Section 4. Basic results of Saint-Venant theory of linearly elastic orthotropic beams are collected in the next section.

2. Composite Saint-Venant Beams under Torsion

Let Ω be the simply or multiply connected cross section of an orthotropic and linearly elastic Saint-Venant composite beam under torsion. Position of a point in Ω, with respect to the centre $\mathbf{G}$ of the Young moduli $E : \Omega \mapsto \mathscr{R}$ of beam's longitudinal fibers, is denoted by $\mathbf{r}$. The tensor $\mathbf{R}$ is the rotation by $\pi/2$ counterclockwise in the cross-sectional plane

$\boldsymbol{\pi}_\Omega$. Hence $\mathbf{R}^T = \mathbf{R}^{-1} = -\mathbf{R}$ and $\mathbf{RR} = -\mathbf{I}$. Tangential stresses can be expressed in terms of the warping function [34] $\phi : \Omega \mapsto \mathscr{R}$ or of the stress function [58] $\Psi : \Omega \mapsto \mathscr{R}$ by the coordinate-free formulae [59]

$$\begin{aligned}\boldsymbol{\tau}(\alpha, \mathbf{r}) &= \boldsymbol{\Lambda}(\mathbf{r})\,\boldsymbol{\gamma}(\mathbf{r}) = \alpha\boldsymbol{\Lambda}(\mathbf{r})\left(\mathbf{Rr} + \nabla\phi(\mathbf{r})\right)\\ &= -\alpha\mathbf{R}\nabla\Psi(\mathbf{r}),\end{aligned} \tag{1}$$

where the scalar α is the twist, $\boldsymbol{\gamma} : \Omega \mapsto V$ is the elastic tangential strain, and $\boldsymbol{\Lambda} : \Omega \mapsto L(V; V)$ is the positive definite symmetric Lamé tensor field, V being the two-dimensional linear space of translations in $\boldsymbol{\pi}_\Omega$. The warping field is the solution of the following Neumann-like problem [60]:

$$\begin{aligned}\operatorname{div}\left(\boldsymbol{\Lambda}(\mathbf{r})\nabla\phi(\mathbf{r})\right) &= -\operatorname{div}\left(\boldsymbol{\Lambda}(\mathbf{r})\,\mathbf{Rr}\right), \quad \mathbf{r} \in \Omega,\\ \left(\boldsymbol{\Lambda}(\mathbf{r})\nabla\phi(\mathbf{r})\right)\cdot\mathbf{n}(\mathbf{r}) &= -\left(\boldsymbol{\Lambda}(\mathbf{r})\,\mathbf{Rr}\right)\cdot\mathbf{n}(\mathbf{r}), \quad \mathbf{r} \in \partial\Omega,\end{aligned} \tag{2}$$

where $\mathbf{n}$ is the unit outward normal to the domain Ω. Prandtl stress function is the solution of the Dirichlet problem

$$\begin{aligned}\operatorname{div}\left(\mathbf{R}\boldsymbol{\Lambda}^{-1}\mathbf{R}\nabla\Psi\right)(\mathbf{r}) &= 2 \quad \text{on } \Omega,\\ \Psi(\mathbf{r}) &= 0 \quad \text{on } \partial\Omega_o,\\ \Psi(\mathbf{r}) &= c_i\\ &\quad \text{on } \partial\Omega_i,\ c_i \in \mathscr{R},\ i = 1, 2, \ldots, n,\end{aligned} \tag{3}$$

where Ω is a multiply connected cross-section, with $\partial\Omega_o$ exterior boundary and $\partial\Omega_i$ boundary of the ith hole, being $i = 1, \ldots, n$ and $n \geq 0$. The procedure for the evaluation of integration constants c_i is illustrated in [59]. Note that the warping function $\phi : \Omega \mapsto \mathscr{R}$ has been introduced above by assuming tacitly that the cross section undergoes a rotation about the pole $\mathbf{G}$. Denoting by $\phi_{\mathbf{C}} : \Omega \mapsto \mathscr{R}$ the warping function corresponding to a cross-sectional rotation with respect to a point $\mathbf{C} \in \boldsymbol{\pi}_\Omega$, we get the formula

$$\phi_{\mathbf{C}} = \phi(\mathbf{r}) + \left(\mathbf{Rr}_{\mathbf{C}}\right)\cdot\mathbf{r} - c, \tag{4}$$

with $\mathbf{r}_{\mathbf{C}}$ position vector of $\mathbf{C}$ and $c \in \mathscr{R}$. Tangential stress fields are independent of the rotation centre [40]. The twist centre $\mathbf{C}^{\text{tw}}$ and a particular value of the constant c were introduced in [61] by requiring that zeroth and first elastic moments of the scalar field $\phi_{\mathbf{C}} : \Omega \mapsto \mathscr{R}$ are zero

$$\begin{aligned}\int_\Omega E(\mathbf{r})\left(\phi(\mathbf{r}) - c\right)dA &= 0,\\ \int_\Omega E(\mathbf{r})\left(\phi(\mathbf{r}) + \left(\mathbf{Rr}_{\mathbf{C}}\right)\cdot\mathbf{r}\right)\mathbf{r}\,dA &= \mathbf{o}.\end{aligned} \tag{5}$$

The position of the twist centre is given by the formula

$$\mathbf{r}_{\mathbf{C}^{\text{tw}}} = \mathbf{R}\mathbf{J}_{\mathbf{G}}(E)^{-1}\int_\Omega E(\mathbf{r})\,\phi(\mathbf{r})\,\mathbf{r}\,dA, \tag{6}$$

with $\mathbf{J}_{\mathbf{G}}(E) := \int_\Omega E(\mathbf{r})\mathbf{r} \otimes \mathbf{r}\,dA$ bending stiffness and $\otimes$ tensor product. An equivalent definition of twist centre was proposed by Trefftz [62] in energetic terms. In [59] it was shown that the twist centre $\mathbf{C}^{\text{tw}}$ coincides with the shear centre $\mathbf{C}^{\text{sh}}_{\text{timo}}$ of Timoshenko beams [63], evaluated by the composite and orthotropic Saint-Venant beam theory. Hereafter, the point $\mathbf{C} := \mathbf{C}^{\text{tw}} \equiv \mathbf{C}^{\text{sh}}_{\text{timo}}$ will be named the Cicala-Hodges centre. The next section provides a family of composite beams, generated by a Lamé tensor field $\boldsymbol{\Lambda} : \Omega \mapsto L(V; V)$, for which the warping field and the Cicala-Hodges centre are invariant.

3. Invariances

Let us consider a sequence of tensor fields $\{\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \ldots, \boldsymbol{\Lambda}_n\}$ generated by a Lamé tensor field $\boldsymbol{\Lambda}_1 : \Omega \mapsto L(V; V)$ and by a sequence of positive scalar functions $\{h_1, h_2, \ldots, h_n\}$; that is, $h_i : \mathscr{X}_i \subseteq \mathscr{R} \mapsto]0, +\infty[$, according to the rule:

$$\boldsymbol{\Lambda}_n = \left(h_{n-1} \circ \Psi_{n-1}\right)\boldsymbol{\Lambda}_{n-1}, \quad n \geq 2, \tag{7}$$

with $\Psi_{n-1} : \Omega \mapsto \mathscr{R}$ Prandtl stress function associated with the torsion tangential stress field involving Lamé tensor field $\boldsymbol{\Lambda}_{n-1} : \Omega \mapsto L(V; V)$. The sequence of Lamé fields $\{\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \ldots, \boldsymbol{\Lambda}_n\}$ induces a sequence of Neumann-like PDE problems $\{\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_n\}$ for the warping field, defined by

$$\mathscr{P}_{n-1}) \begin{cases}\operatorname{div}\left(\boldsymbol{\Lambda}_{n-1}\boldsymbol{\gamma}_{n-1}\right)(\mathbf{r}) = 0, & \text{in } \Omega,\\ \left(\boldsymbol{\Lambda}_{n-1}(\mathbf{r})\,\boldsymbol{\gamma}_{n-1}(\mathbf{r})\right)\cdot\mathbf{n}(\mathbf{r}) = 0, & \text{on } \partial\Omega,\end{cases} \tag{8}$$

with $\boldsymbol{\gamma}_{n-1}(\mathbf{r}) = \boldsymbol{\Lambda}^{-1}_{n-1}(\mathbf{r})\boldsymbol{\tau}_{n-1}(\mathbf{r}) = \alpha(\mathbf{Rr} + \nabla\phi_{n-1}(\mathbf{r}))$. The following results hold true.

Proposition 1. *Neumann-like PDE problems $\{\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_n\}$ provide, to within an additive constant, the same solution $\phi : \Omega \mapsto \mathscr{R}$.*

Proof. Let ϕ_{n-1} be the solution, to within a constant, of the problem $\mathscr{P}_{n-1}$. Since $\boldsymbol{\Lambda}_n = (h_{n-1} \circ \Psi_{n-1})\boldsymbol{\Lambda}_{n-1}$, problem $\mathscr{P}_n$ takes the form:

$$\mathscr{P}_{n}) \begin{cases}\operatorname{div}\left(\left(h_{n-1} \circ \Psi_{n-1}\right)\boldsymbol{\Lambda}_{n-1}\boldsymbol{\gamma}_{n}\right)(\mathbf{r}) = 0, & \text{in } \Omega,\\ \left(\boldsymbol{\Lambda}_{n-1}(\mathbf{r})\,\boldsymbol{\gamma}_{n}(\mathbf{r})\right)\cdot\mathbf{n}(\mathbf{r}) = 0, & \text{on } \partial\Omega.\end{cases} \tag{9}$$

Resorting to the formula

$$\begin{aligned}&\operatorname{div}\left(\left(h_{n-1} \circ \Psi_{n-1}\right)\boldsymbol{\Lambda}_{n-1}\boldsymbol{\gamma}_n\right)(\mathbf{r})\\ &\quad = \left(h_{n-1} \circ \Psi_{n-1}\right)(\mathbf{r})\operatorname{div}\left(\boldsymbol{\Lambda}_{n-1}\boldsymbol{\gamma}_n\right)(\mathbf{r})\\ &\qquad + dh\left(\Psi_{n-1}(\mathbf{r})\right)\nabla\Psi_{n-1}(\mathbf{r})\cdot\left(\boldsymbol{\Lambda}_{n-1}\boldsymbol{\gamma}_n(\mathbf{r})\right)\end{aligned} \tag{10}$$

and setting $\phi_n(\mathbf{r}) = \phi_{n-1}(\mathbf{r}) + c$ with $c \in \mathscr{R}$, we get $\boldsymbol{\gamma}_n = \boldsymbol{\gamma}_{n-1}$, so that the problem $\mathscr{P}_n$ may be rewritten as

$$\mathscr{P}_n) \begin{cases} (h_{n-1} \circ \Psi_{n-1})(\mathbf{r}) \operatorname{div}(\mathbf{\Lambda}_{n-1}\boldsymbol{\gamma}_{n-1})(\mathbf{r}) + dh(\Psi_{n-1}(\mathbf{r})) \nabla\Psi_{n-1}(\mathbf{r}) \cdot (\mathbf{\Lambda}_{n-1}\boldsymbol{\gamma}_{n-1}(\mathbf{r})) = 0, & \text{in } \Omega, \\ (\mathbf{\Lambda}_{n-1}(\mathbf{r})\boldsymbol{\gamma}_{n-1}(\mathbf{r})) \cdot \mathbf{n}(\mathbf{r}) = 0, & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Recalling the relation $\boldsymbol{\tau}_{n-1}(\alpha, \mathbf{r}) = \mathbf{\Lambda}_{n-1}(\mathbf{r})\boldsymbol{\gamma}_{n-1}(\mathbf{r}) = -\alpha\mathbf{R}\nabla\Psi_{n-1}(\mathbf{r})$, we infer that $\nabla\Psi_{n-1}(\mathbf{r}) \cdot (\mathbf{\Lambda}_{n-1}\boldsymbol{\gamma}_{n-1}(\mathbf{r})) = 0$ and the problem $\mathscr{P}_n$ collapses into the one $\mathscr{P}_{n-1}$. The result follows. □

Proposition 2. *The relationship between Prandtl stress functions corresponding to Lamé tensor fields $\mathbf{\Lambda}_{n-1}$ and $\mathbf{\Lambda}_n = (h_{n-1} \circ \Psi_{n-1})\mathbf{\Lambda}_{n-1}$ is expressed by the formula $\Psi_n = H_{n-1} \circ \Psi_{n-1}$, with H_{n-1} antiderivative of h_{n-1} such that Ψ_n is identically zero on the cross-sectional exterior boundary $\partial\Omega_o$.*

Proof. Resorting to Proposition 1 we get $\boldsymbol{\gamma}_{n-1} = \boldsymbol{\gamma}_n$. Then the equivalences hold

$$\begin{aligned} &\mathbf{\Lambda}_{n-1}^{-1}\boldsymbol{\tau}_{n-1} = \mathbf{\Lambda}_n^{-1}\boldsymbol{\tau}_n \Longleftrightarrow \\ &\mathbf{\Lambda}_{n-1}^{-1}(-\alpha\mathbf{R}\nabla\Psi_{n-1}) \\ &\quad = ((h_{n-1} \circ \Psi_{n-1})\mathbf{\Lambda}_{n-1})^{-1}(-\alpha\mathbf{R}\nabla\Psi_n), \end{aligned} \tag{12}$$

whence $(h_{n-1} \circ \Psi_{n-1})\nabla\Psi_{n-1} = \nabla(H_{n-1} \circ \Psi_{n-1}) = \nabla\Psi_n$ which gives $\Psi_n = H_{n-1} \circ \Psi_{n-1}$, with H_{n-1} antiderivative of h_{n-1} such that Ψ_n is identically zero on the cross-sectional exterior boundary $\partial\Omega_o$. □

Proposition 3. *Let $E : \Omega \mapsto]0, +\infty[$ be the Euler moduli scalar field of orthotropic and composite beams whose Lamé fields are described by the sequence $\{\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \ldots, \mathbf{\Lambda}_n\} = \{\mathbf{\Lambda}_1, (h_1 \circ \Psi_1)\mathbf{\Lambda}_1, \ldots, (h_{n-1} \circ \Psi_{n-1})\mathbf{\Lambda}_{n-1}\}$. For these beams, the location of the Cicala-Hodges centre is invariant.*

Proof. The result follows by the formula providing the twist centre position $\mathbf{r}_{\mathbf{C}^{\mathrm{tw}}} = \mathbf{R}\mathbf{J}_{\mathbf{G}}(E)^{-1} \int_\Omega E(\mathbf{r})\phi(\mathbf{r})\mathbf{r}\, dA$ and by Proposition 1. □

4. Examples

Let us provide some analytical solutions of functionally graded orthotropic beams under torsion with elliptic cross sections. Inertia principal axes $\{x, y\}$ with origin in the centre $\mathbf{G}$ of the Euler moduli field $E : \Omega \mapsto \mathscr{R}$ will be adopted in the sequel. Position vector $\mathbf{r}$ and rotation $\mathbf{R}$ are written as

$$\begin{aligned} \mathbf{r} &= \begin{bmatrix} x \\ y \end{bmatrix}, \\ \mathbf{R} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \end{aligned} \tag{13}$$

whence $\mathbf{R}\mathbf{r} = [-y, x]^T$. Torsional warping of elliptic composite beams with Lamé tensor field,

$$\begin{aligned} \mathbf{\Lambda}_1(x, y) &= \begin{bmatrix} \mu_x & \mu_{xy} \\ \mu_{xy} & \mu_y \end{bmatrix} \\ &= \left(-k_1 k_2 \frac{a^2 y^2 + b^2 x^2 - a^2 b^2}{a^2 k_2 + b^2 k_1} + k_3\right) \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \end{aligned} \tag{14}$$

is provided by the formula [40] $\phi(x, y) = -((a^2 k_2 - b^2 k_1)/(a^2 k_2 + b^2 k_1))xy + c$, with $c \in \mathscr{R}$, $k_1, k_2, k_3 \in]0, +\infty[$ and a, b lengths of the ellipse semidiameters. Plots of the shear modulus μ_x and of the warping ϕ are provided in Figures 1 and 2.

Cartesian components of the tangential strain field $\boldsymbol{\gamma}$ and stress function Ψ_1 are given by the formulae

$$\begin{aligned} \boldsymbol{\gamma}(\alpha, x, y) &= \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \alpha(\mathbf{R}\mathbf{r} + \nabla\phi(\mathbf{r})) = \frac{2\alpha}{a^2 k_2 + b^2 k_1} \\ &\quad \cdot \begin{bmatrix} -a^2 k_2 y \\ b^2 k_1 x \end{bmatrix}, \\ \Psi_1(x, y) &= k_1 k_2 \\ &\quad \cdot \frac{a^2 y^2 + b^2 x^2 - a^2 b^2}{a^2 k_2 + b^2 k_1} \left(\frac{1}{2} k_1 k_2 \frac{a^2 y^2 + b^2 x^2 - a^2 b^2}{a^2 k_2 + b^2 k_1} \right. \\ &\quad \left. - k_3\right), \end{aligned} \tag{15}$$

as depicted in Figures 3 and 4. As shown in Section 3, Lamé tensor field $\mathbf{\Lambda}_1$ generates a sequence of composite beams under torsion for which warping function and Cicala-Hodges centre are invariant, and the relevant stress functions are given by Proposition 2. Analytical solutions of the following composite elliptic beams under torsion are discussed. The former is characterized by Lamé shear moduli described by the tensor field $\mathbf{\Lambda}_2 = (\Psi_1 + k)\mathbf{\Lambda}_1$, with $k \in]0, +\infty[$. The corresponding stress function is given by the formula $\Psi_2 = (1/2)\Psi_1^2 + k\Psi_1$, as assessed in Proposition 2. The latter is characterized by Lamé shear moduli described by the tensor field $\mathbf{\Lambda}_3 = (\exp \circ \Psi_2)\mathbf{\Lambda}_2$, where exp denotes the exponential function. The corresponding stress function is given by the formula $\Psi_3 = (\exp \circ \Psi_2) - 1$, as assessed in Proposition 2. Stress functions and tangential stresses are depicted in Figures 5, 6, 7, and 8.

It is worth noting that if the Euler moduli scalar field E is assumed to be the same in the examples discussed above, then

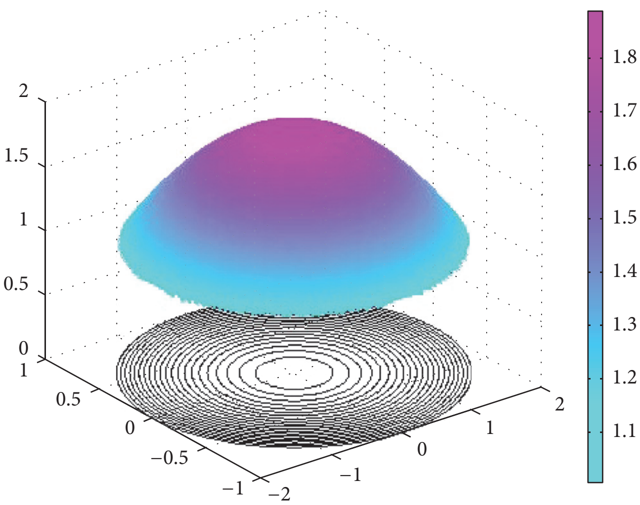


Figure 1: Shear modulus μ_x; $a = 2, b = 1$; $k_1 = k_3 = 1, k_2 = 2$. Color spectrum: μ_x.

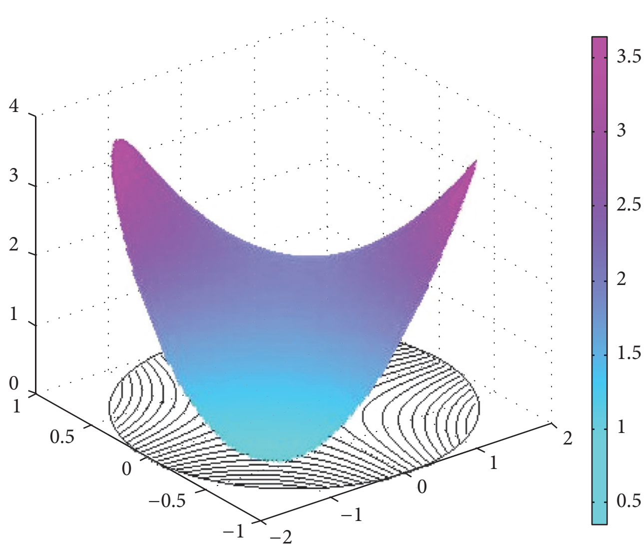


Figure 2: Cross-sectional torsional warping ϕ; $a = 2, b = 1$; $k_1 = 1$, $k_2 = c = 2$. Color spectrum: ϕ.

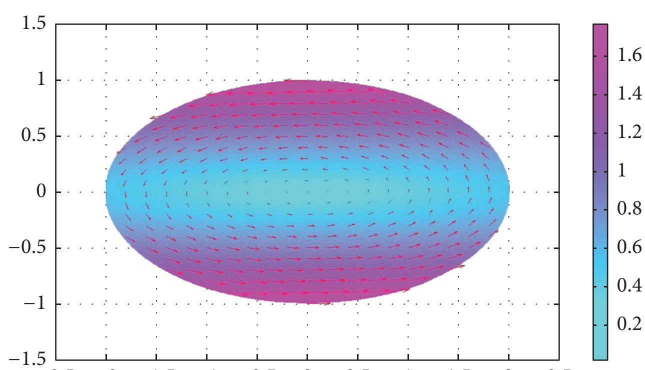


Figure 3: Tangential strain field per unit twist $\boldsymbol{\gamma}/\alpha$; $a = 2, b = 1$; $k_1 = k_3 = 1, k_2 = 2$. Color spectrum: $\|\boldsymbol{\gamma}(x, y)\| := [\gamma_{xz}^2(x, y) + \gamma_{yz}^2(x, y)]^{1/2}$ for $\alpha = 1$.

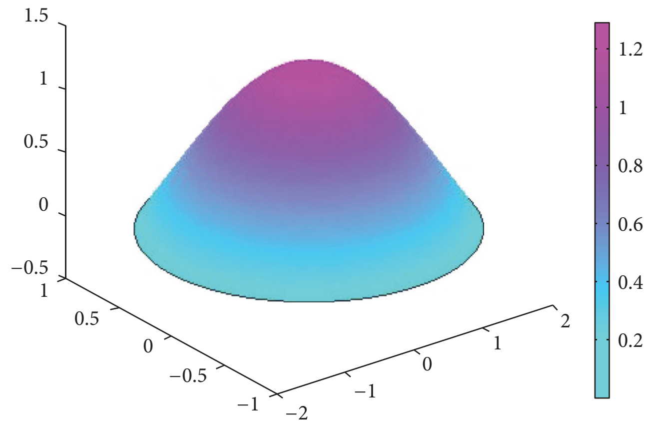


Figure 4: Stress function Ψ_1; $a = 2, b = 1$; $k_1 = k_3 = 1, k_2 = c = 2$. Color spectrum: Ψ_1.

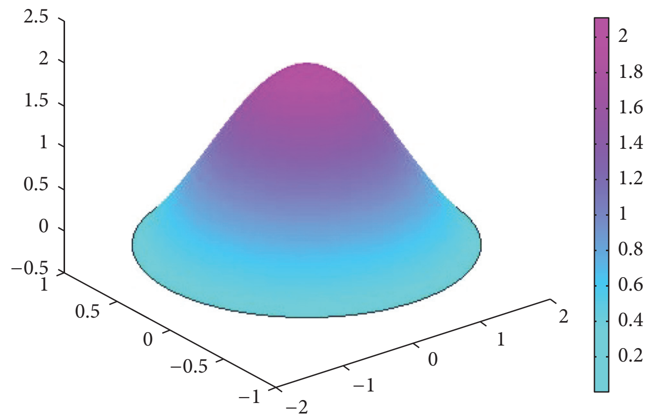


Figure 5: Stress function Ψ_2; $a = 2, b = 1$; $k_1 = k_3 = 1, k_2 = c = 2$. Color spectrum: Ψ_2.

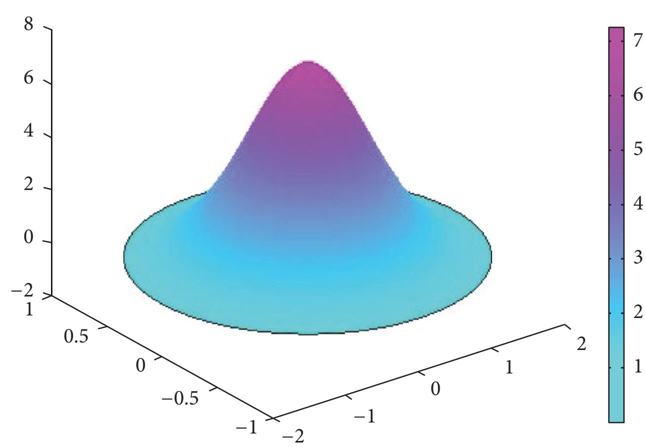


Figure 6: Stress function Ψ_3; $a = 2, b = 1$; $k_1 = k_3 = 1, k_2 = c = 2$. Color spectrum: Ψ_3.

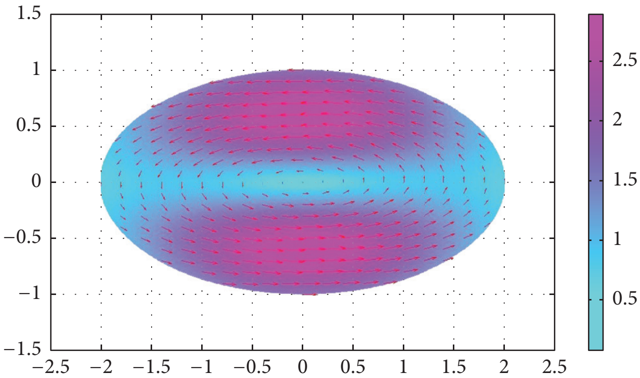


FIGURE 7: Tangential stress field per unit twist $\boldsymbol{\tau}_2/\alpha$; $a = 2$, $b = 1$; $k_1 = k_3 = 1$, $k_2 = 2$. Color spectrum: $\|\boldsymbol{\tau}_2(x, y)\|$ for $\alpha = 1$.

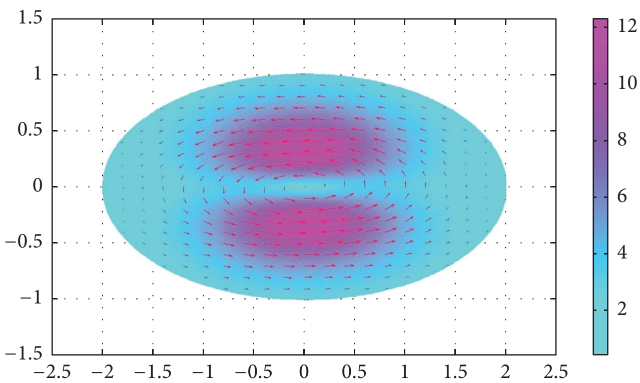


FIGURE 8: Tangential stress field per unit twist $\boldsymbol{\tau}_3/\alpha$; $a = 2$, $b = 1$; $k_1 = k_3 = 1$, $k_2 = 2$. Color spectrum: $\|\boldsymbol{\tau}_3(x, y)\|$ for $\alpha = 1$.

warping functions and Cicala-Hodges centres are invariant, as prescribed by Propositions 1 and 3.

5. Conclusions

The outcomes of the present paper may be summarized as follows.

(i) Neumann and Dirichlet boundary value problems for the cross-sectional warping and for Prandtl stress function of linearly elastic, orthotropic composite beams under torsion have been examined.

(ii) Invariance conditions for the warping and for the Cicala-Hodges shear centre of simply and multiply connected cross sections have been established.

(iii) The relationship between Prandtl stress functions of orthotropic composite beams with invariant warping has been assessed.

(iv) Examples have been developed for orthotropic composite beams with elliptic cross section, providing thus also new benchmarks for numerical analyses.

Competing Interests

The author declares no competing interests.

References

[1] D. H. Hodges, *Nonlinear Composite Beam Theory*, AIAA, Reston, Va, USA, 2006.

[2] F. Marmo and L. Rosati, "Analytical integration of elasto-plastic uniaxial constitutive laws over arbitrary sections," *International Journal for Numerical Methods in Engineering*, vol. 91, no. 9, pp. 990–1022, 2012.

[3] F. Marmo and L. Rosati, "The fiber-free approach in the evaluation of the tangent stiffness matrix for elastoplastic uniaxial constitutive laws," *International Journal for Numerical Methods in Engineering*, vol. 94, no. 9, pp. 868–894, 2013.

[4] A. M. Tarantino, "Nonlinear fracture mechanics for an elastic Bell material," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 50, no. 3, pp. 435–456, 1997.

[5] A. M. Tarantino, "The singular equilibrium field at the notch-tip of a compressible material in finite elastostatics," *Journal of Applied Mathematics and Physics*, vol. 48, no. 3, pp. 370–388, 1997.

[6] A. M. Tarantino, "On extreme thinning at the notch tip of a neo-Hookean sheet," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 51, no. 2, pp. 179–190, 1998.

[7] A. M. Tarantino, "On the finite motions generated by a mode I propagating crack," *Journal of Elasticity*, vol. 57, no. 2, pp. 85–103, 1999.

[8] A. M. Tarantino, "Crack propagation in finite elastodynamics," *Mathematics and Mechanics of Solids*, vol. 10, no. 6, pp. 577–601, 2005.

[9] A. M. Tarantino, "Asymmetric equilibrium configurations of symmetrically loaded isotropic square membranes," *Journal of Elasticity*, vol. 69, no. 1–3, pp. 73–97, 2002.

[10] A. M. Tarantino, "Homogeneous equilibrium configurations of a hyperelastic compressible cube under equitriaxial dead-load tractions," *Journal of Elasticity*, vol. 92, no. 3, pp. 227–254, 2008.

[11] F. Marmo and L. Rosati, "A general approach to the solution of Boussinesq's problem for polynomial pressures acting over polygonal domains," *Journal of Elasticity*, vol. 122, no. 1, pp. 75–112, 2016.

[12] G. Dinelli, G. Belz, C. E. Majorana, and B. A. Schrefler, "Experimental investigation on the use of fly ash for lightweight precast structural elements," *Materials and Structures*, vol. 29, no. 194, pp. 632–638, 1996.

[13] V. A. Salomoni, C. E. Majorana, G. M. Giannuzzi, and A. Miliozzi, "Thermal-fluid flow within innovative heat storage concrete systems for solar power plants," *International Journal of Numerical Methods for Heat and Fluid Flow*, vol. 18, no. 7-8, pp. 969–999, 2008.

[14] V. A. Salomoni, G. Mazzucco, C. Pellegrino, and C. E. Majorana, "Three–dimensional modelling of bond behaviour between concrete and FRP reinforcement," *Engineering Computations*, vol. 28, no. 1, pp. 5–29, 2011.

[15] V. A. Salomoni, C. E. Majorana, B. Pomaro, G. Xotta, and F. Gramegna, "Macroscale and mesoscale analysis of concrete as a multiphase material for biological shields against nuclear radiation," *International Journal for Numerical and Analytical Methods in Geomechanics*, vol. 38, no. 5, pp. 518–535, 2014.

[16] G. Xotta, G. Mazzucco, V. A. Salomoni, C. E. Majorana, and K. J. Willam, "Composite behavior of concrete materials under high temperatures," *International Journal of Solids and Structures*, vol. 64, pp. 86–99, 2015.

[17] F. Marotti de Sciarra, "A general theory for nonlocal softening plasticity of integral-type," *International Journal of Plasticity*, vol. 24, no. 8, pp. 1411–1439, 2008.

[18] F. Marotti de Sciarra, "Variational formulations and a consistent finite-element procedure for a class of nonlocal elastic continua," *International Journal of Solids and Structures*, vol. 45, no. 14-15, pp. 4184–4202, 2008.

[19] F. Marotti de Sciarra, "Variational formulations, convergence and stability properties in nonlocal elastoplasticity," *International Journal of Solids and Structures*, vol. 45, no. 7-8, pp. 2322–2354, 2008.

[20] F. Greco and R. Luciano, "A theoretical and numerical stability analysis for composite micro-structures by using homogenization theory," *Composites Part B: Engineering*, vol. 42, no. 3, pp. 382–401, 2011.

[21] A. Caporale, L. Feo, R. Luciano, and R. Penna, "Numerical collapse load of multi-span masonry arch structures with FRP reinforcement," *Composites Part B: Engineering*, vol. 54, no. 1, pp. 71–84, 2013.

[22] A. Caporale, L. Feo, and R. Luciano, "Damage mechanics of cement concrete modeled as a four-phase composite," *Composites Part B: Engineering*, vol. 65, pp. 124–130, 2014.

[23] F. Marotti de Sciarra and M. Salerno, "On thermodynamic functions in thermoelasticity without energy dissipation," *European Journal of Mechanics—A/Solids*, vol. 46, pp. 84–95, 2014.

[24] L. Rosati and F. Marmo, "Closed-form expressions of the thermo-mechanical fields induced by a uniform heat source acting over an isotropic half-space," *International Journal of Heat and Mass Transfer*, vol. 75, pp. 272–283, 2014.

[25] M. Čanađija, R. Barretta, and F. Marotti de Sciarra, "A gradient elasticity model of Bernoulli-Euler nanobeams in non-isothermal environments," *European Journal of Mechanics—A/Solids*, vol. 55, pp. 243–255, 2016.

[26] F. Marotti de Sciarra and R. Barretta, "A new nonlocal bending model for Euler-Bernoulli nanobeams," *Mechanics Research Communications*, vol. 62, no. 1, pp. 25–30, 2014.

[27] F. Marotti de Sciarra, M. Canadija, and R. Barretta, "A gradient model for torsion of nanobeams," *Comptes Rendus—Mecanique*, vol. 343, no. 4, pp. 289–300, 2015.

[28] R. Barretta, R. Luciano, and F. Marotti De Sciarra, "A fully gradient model for euler-bernoulli nanobeams," *Mathematical Problems in Engineering*, vol. 2015, Article ID 495095, 8 pages, 2015.

[29] R. Barretta, L. Feo, R. Luciano, F. Marotti de Sciarra, and R. Penna, "Functionally graded Timoshenko nanobeams: a novel nonlocal gradient formulation," *Composites B*, vol. 100, pp. 208–219, 2016.

[30] R. Barretta, L. Feo, R. Luciano, and F. Marotti de Sciarra, "Application of an enhanced version of the Eringen differential model to nanotechnology," *Composites B*, vol. 96, pp. 274–280, 2016.

[31] A. Caporale, R. Luciano, and L. Rosati, "Limit analysis of masonry arches with externally bonded FRP reinforcements," *Computer Methods in Applied Mechanics and Engineering*, vol. 196, no. 1–3, pp. 247–260, 2006.

[32] A. Caporale and R. Luciano, "Limit analysis of masonry arches with finite compressive strength and externally bonded reinforcement," *Composites Part B: Engineering*, vol. 43, no. 8, pp. 3131–3145, 2012.

[33] A. Caporale, L. Feo, D. Hui, and R. Luciano, "Debonding of FRP in multi-span masonry arch structures via limit analysis," *Composite Structures*, vol. 108, no. 1, pp. 856–865, 2014.

[34] A. J. C. B. de Saint-Venant, "Mémoire sur la torsion des prismes," *Mémoires de l'Académie Royale des Sciences de Paris, Savants Étrangers*, vol. 14, pp. 233–560, 1855.

[35] A. J. C. B. de Saint-Venant, "Mémoire sur la exion des prismes," *Journal de Mathématiques Pures et Appliquées*, vol. 1, no. 2, pp. 89–189, 1856.

[36] A. Clebsh, *Theorie der Elasticität Fester Körper*, B.G. Teubner, Leipzig, Germany, 1862.

[37] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Dover, New York, NY, USA, 1944.

[38] N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Leningrad, Moscow, Russia, 3rd edition, 1953, Translated from the russian by J. R. M. Radok, P. Noordhoff, Holland, Groningen, Netherlands.

[39] S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York, NY, USA, 1951.

[40] I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, McGraw-Hill, New York, NY, USA, 1956.

[41] V. V. Novozhilov, *Theory of Elasticity*, Pergamon, London, UK, 1961.

[42] S. G. Lekhnitskii, *Theory of Elasticity of an Anisotropic Elastic Body*, Holden-Day, San Francisco, Calif, USA, 1963.

[43] L. Solomon, *Élasticité Lineaire*, Masson, Paris, France, 1968.

[44] A. I. Lurje, *Theory of Elasticity*, Izd. Nauka, Moscow, Russia, 1970 (Russian).

[45] R. Barretta and A. Barretta, "Shear stresses in elastic beams: an intrinsic approach," *European Journal of Mechanics, A/Solids*, vol. 29, no. 3, pp. 400–409, 2010.

[46] G. Romano, A. Barretta, and R. Barretta, "On torsion and shear of Saint-Venant beams," *European Journal of Mechanics—A/Solids*, vol. 35, pp. 47–60, 2012.

[47] R. Barretta, "On stress function in Saint-Venant beams," *Meccanica*, vol. 48, no. 7, pp. 1811–1816, 2013.

[48] R. Barretta, "Analogies between Kirchhoff plates and Saint-Venant beams under torsion," *Acta Mechanica*, vol. 224, no. 12, pp. 2955–2964, 2013.

[49] R. Barretta, "Analogies between Kirchhoff plates and Saint-Venant beams under flexure," *Acta Mechanica*, vol. 225, no. 7, pp. 2075–2083, 2014.

[50] R. Luciano and J. R. Willis, "Bounds on non-local effective relations for random composites loaded by configuration-dependent body force," *Journal of the Mechanics and Physics of Solids*, vol. 48, no. 9, pp. 1827–1849, 2000.

[51] R. Luciano and J. R. Willis, "Boundary-layer corrections for stress and strain fields in randomly heterogeneous materials," *Journal of the Mechanics and Physics of Solids*, vol. 51, no. 6, pp. 1075–1088, 2003.

[52] R. Luciano and J. R. Willis, "Hashin-Shtrikman based FE analysis of the elastic behaviour of finite random composite bodies," *International Journal of Fracture*, vol. 137, no. 1–4, pp. 261–273, 2006.

[53] F. Marotti de Sciarra, "Finite element modelling of nonlocal beams," *Physica E: Low-Dimensional Systems and Nanostructures*, vol. 59, pp. 144–149, 2014.

[54] A. Patti, R. Barretta, F. Marotti de Sciarra, G. Mensitieri, C. Menna, and P. Russo, "Flexural properties of multi-wall carbon nanotube/polypropylene composites: experimental investigation and nonlocal modeling," *Composite Structures*, vol. 131, pp. 282–289, 2015.

[55] I. Ecsedi, "Some analytical solutions for Saint-Venant torsion of non-homogeneous cylindrical bars," *European Journal of Mechanics—A/Solids*, vol. 28, no. 5, pp. 985–990, 2009.

[56] I. Ecsedi, "Some analytical solutions for Saint-Venant torsion of non-homogeneous anisotropic cylindrical bars," *Mechanics Research Communications*, vol. 52, pp. 95–100, 2013.

[57] H. S. Carslaw and J. C. Jager, *Conduction of Heat in Solids*, Clarendon Press, Oxford, UK, 1959.

[58] L. Prandtl, "Zur torsion von prismatischen staben," *Physikalische Zeitscrift*, vol. 4, pp. 758–770, 1903.

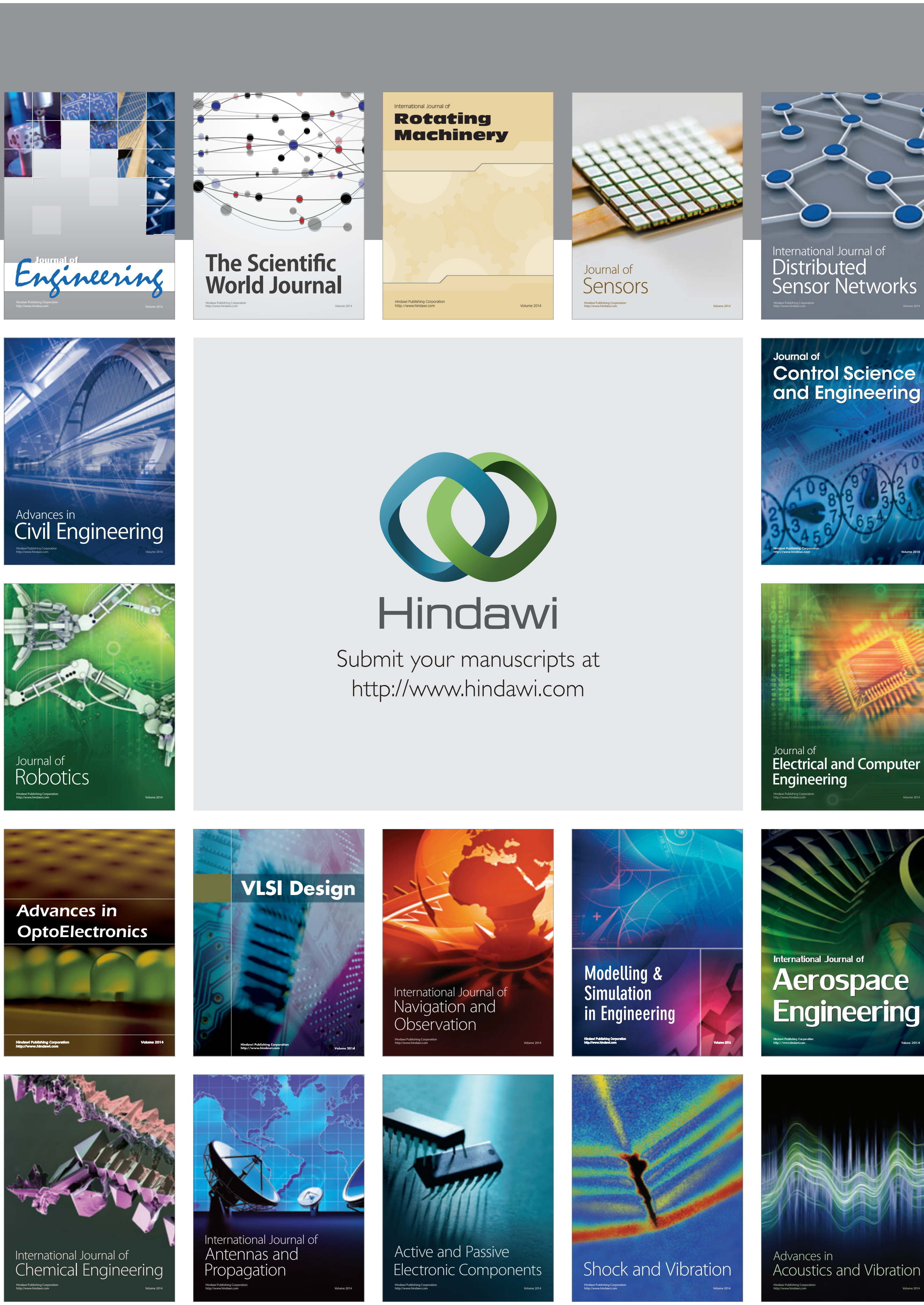
[59] R. Barretta, "On the relative position of twist and shear centres in the orthotropic and fiberwise homogeneous Saint-Venant beam theory," *International Journal of Solids and Structures*, vol. 49, no. 21, pp. 3038–3046, 2012.

[60] R. Barretta, "Cesàro-Volterra method in orthotropic Saint-Venant beam," *Journal of Elasticity*, vol. 112, no. 2, pp. 233–253, 2013.

[61] P. Cicala, "Il centro di taglio nei solidi cilindrici," *Atti Accademia delle Scienze di Torino (serie Fisica)*, vol. 70, pp. 356–371, 1935.

[62] E. Trefftz, "Über den Schubmittelpunkt in einem durch eine Einzellast gebogenen Balken," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 15, no. 4, pp. 220–225, 1935.

[63] W. Yu, D. H. Hodges, V. Volovoi, and C. E. S. Cesnik, "On Timoshenko-like modeling of initially curved and twisted composite beams," *International Journal of Solids and Structures*, vol. 39, no. 19, pp. 5101–5121, 2002.


Journal of Engineering
The Scientific World Journal
International Journal of Rotating Machinery
Journal of Sensors
International Journal of Distributed Sensor Networks
Advances in Civil Engineering
Hindawi
Submit your manuscripts at
http://www.hindawi.com
Journal of Control Science and Engineering
Journal of Robotics
Journal of Electrical and Computer Engineering
Advances in OptoElectronics
VLSI Design
International Journal of Navigation and Observation
Modelling & Simulation in Engineering
International Journal of Aerospace Engineering
International Journal of Chemical Engineering
International Journal of Antennas and Propagation
Active and Passive Electronic Components
Shock and Vibration
Advances in Acoustics and Vibration