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Rational Invariants of Meta-abelian Groups of Linear Automorphisms*

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INTRODUCTION

Let k be an algebraically closed field of characteristic zero, G a finite group and V a finite-dimensional kG-module. Then G acts as a group of k-automorphisms on $k(V)$, the field of fractions of the symmetric k-algebra of V. Is the subfield $k(V)^G$ of $k(V)$ fixed by G rational (=purely transcendental) over k?

The case when G is abelian was completely settled by Fischer [3, 4] in 1916, and it is natural to consider next the "two-step abelian" (i.e., meta-abelian) case. We thus assume that G has a normal abelian subgroup N for which $H = G/N$ is abelian. For a cyclic H (of order n, say), Haeuslein [5] showed that $k(V)^G$ is indeed rational if n is prime and if the nth cyclotomic field has class number 1. This is the case precisely when n is a prime <23.

In Section 1 of this paper, we prove Haeuslein's result with the assumption that n is a prime relaxed. Thus $k(V)^G$ is rational if n is any of the 44 numbers listed in [8]. The difficulties arising from dropping the assumption that n is prime are discussed at the end of Section 1. For other values of n, the problem remains open. We note, however, the role played by $\dim_k(V)$ and we prove that $k(V)^G$ is rational whenever $\dim_k(V) < 23$, regardless of n.

The proof goes as follows. By [1, pp. 75–79], a meta-abelian group is an M-group. Actually, $k(V)$ has a base over k (i.e., a transcendence basis B for which $k(B) = k(V)$) on which G acts monomially and on which N acts diagonally (i.e., G acts on the subgroup of $k(V)^*$ generated by k^* and B, and $g(b)/b$ belongs to k^* for all b in B and all g in N). Using Fischer's method, one constructs a base of $L = k(V)^N$ on which H acts monomially. Thus our problem reduces to whether the abelian group H of monomial automorphisms has a rational fixed field. If the action of H could be linearized (i.e., if L has a base B over k for which H acts on the k-module generated by B), then Fischer's result would settle our problem. This can

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actually be done if H is cyclic (say $H = (h)$) of a prime order <23 [6]. If order(h) is not prime, the problem gets much harder, and it is in fact still unknown whether that can be done. Nor is it known whether all monomial automorphisms of order <23 have rational fixed fields. We were able, however, to linearize h using the fairly special form its characteristic polynomial turns to have (namely, $\prod(T^{s(i)} - 1)$), and imposing some restrictions on the sizes of the $s(i)$'s. These restrictions follow from either the hypothesis that order$(h) < 23$ or that $\dim_k(V) < 23$.

In Section 2, we drop the assumption that (the abelian) H is cyclic and we establish the rationality of $k(V)^G$ for $\dim_k(V) < 5$. In this case too, we do not know whether the action of H can be linearized.

The arithmetic version of this problem is often referred to as "Noether's Conjecture." It was first formulated by Noether in 1916 as a question of the rationality of $Q(V)^{C(r)}$ for the regular representation V of a cyclic group $C(r)$ of order r over the rational number field Q. The rationality was established by her [11] for $r = 3$, by Seidelmann [12] for $r = 4$, by Masuda [9] for $r < 8$ and later [10] for $r = 11$. In 1969, Swan [13] proved the surprising result that even in this simple case of a cyclic $C(r)$, $Q(V)^{C(r)}$ need not be rational, giving as an example the value $r = 47$. Further investigation of the problem was made by Endo and Miyata [2] and by Lenstra [7].

1. The Cyclic Case

Throughout this paper, k is an algebraically closed field of characteristic zero, G a finite group having a normal abelian subgroup N for which $H = G/N$ is abelian, V a finite-dimensional kG-module and $k(V)$ the field of fractions of the symmetric algebra of V over k. Let G act naturally as a group of k-automorphisms on $k(V)$. Our objective is to establish, under certain conditions, that $k(V)^G$ is rational (over k).

We adhere to the definitions made in the Introduction (noting especially the rather unstandard usage of the term "base"). If U is a group containing k^* as a subgroup such that U/k^* is $\mathbb{Z}$-free of finite rank and if f is an automorphism on U fixing k^*, then $\bar{U}$ denotes U/k^*, $\bar{f}$ the action on $\bar{U}$ induced by f, and $\chi(f, U)$ the characteristic polynomial of $\bar{f}$. The value of $\chi(f, U)$ at (an indeterminate) T is denoted by $\chi(f, U, T)$. The rth cyclotomic polynomial is denoted by ϕ_r.

Theorem 1 reduces our problem to one of the more classical type already encountered in the treatment of the Arithmetic Noether's Conjecture stated in the Introduction.

THEOREM 1. *Let $H = G/N$ be cyclic and let h be a generator of H. Then there exists a $\mathbb{Z}H$-module E containing k^* as an H-fixed submodule such that* (i) *$\bar{E}$ is $\mathbb{Z}$-free of a rank equal to* $\dim_k(V)$.

(ii) *$k(V)^N = k(E)$ (and hence $k(V)^G = k(E)^H$), where $k(E)$ is the canonically constructed field extension of k having as a base a set in E representing a $\mathbb{Z}$-basis of $\bar{E}$.*

(iii) *$\chi(h, E, T)$ is of the form $\prod_{i=1}^{r} (T^{s(i)} - 1)$.*

Proof. Let g be a pre-image of h in $G \to G/N = H$. Following Fischer's method of finding invariants of N, we form the N-eigen space decomposition $\oplus V_i$ of V having the minimal number of summands and we use the normality of N in G to prove that the action of each element of G on the set $\{V_i\}$ is a permutation. We then construct an N-eigen basis of V on which g acts as a permutation (up to multiplying by elements of k^*). This is done as follows. Let $(U_1, ..., U_t)$, where each U_i is some V_j, be a cycle in the decomposition into disjoint cycles of the permutational action of g on the set $\{V_i\}$. Since g^t acts on U_1, one can construct a g^t-eigen basis of U_1. Such a basis, combined with its images under powers of g, yields a basis of $\prod_{i=1}^{t} U_i$ on which the action of g is as desired. Doing the same on each cycle and combining the resulting bases, one gets the desired basis of V. Thus we have constructed a base B of $k(V)$ and a permutation p on the set B such that $g(b)/p(b)$ and $g'(b)/b$ belong to k^* for all b in B and all g' in N. Letting A be the subgroup of $k(V)^*$ generated by k^* and B, and letting E be the subgroup of A fixed by N, one easily sees (and it is the classical argument of Fischer) that $k(V)^N$ is rational and equals $k(E)$. Clearly, h acts on E making it a $\mathbb{Z}H$-module. Finally, the statement on $\chi(h, E)$ follows from observing the permutational action of $\bar{g}$ (and hence of $\bar{h}$) on A and noting that $\chi(g, A) = \chi(g, E)$, A/E being all torsion. ∎

THEOREM 2. *Let H be cyclic and let $n =$ order(H). If the class number of the nth cyclotomic field is 1, then $k(V)^G$ is rational over k.*

The idea of the proof is to subject the $\mathbb{Z}H$-module E obtained in Theorem 1 to a sequence of modifications within $k(E)^*$ that result in another $\mathbb{Z}H$-module F having all the properties of E and for which $\bar{F}$ is a permutation module. This will be accomplished after few preparatory lemmas have been proved.

Let E be as in Theorem 1, let $H = (h)$ and let $L = k(E)$. For a $\mathbb{Z}H$-submodule F of L^* containing k^*, let $\bar{F}$ denote F/k^* and let $\bar{h}$ (resp. $\bar{H}$) denote the action induced by h (resp. H) on $\bar{L}^*$. We refer to both $\mathbb{Z}H$- and $\mathbb{Z}\bar{H}$-modules simply as modules, the context making it clear which of the two rings is meant. A submodule F of L^* containing k^* and for which $\bar{F}$ is $\mathbb{Z}$-free of finite rank is called a monomial module. In all that follows, F and F_i's stand for such modules. An element u of L^* (resp. of L^*/k^*) is said to be annihilated by a polynomail P if $P(h)u$ belongs to k^* (resp. if $P(\bar{h}) = 1$). We say that $F_1 \sim F_2$ if $k(F_1) = k(F_2)$ and rank$(\bar{F}_1) =$ rank$(\bar{F}_2)$. One should note, however, that for each pair F_1 and F_2 of equivalent modules encountered below, $\chi(h, F_1) = \chi(h, F_2)$. The pushout of the diagram

$$\begin{array}{ccc} k^* & \longrightarrow & F_1 \\ \downarrow & & \\ F_2 & & \end{array}$$

which contains both F_1 and F_2 is denoted by $F_1 * F_2$. Thus $k(F_1 * F_2)$ is the composite field $k(F_1)\,k(F_2)$. Finally we define the subset $S(T)$ of $\mathbb{Z}[T] \times \mathbb{Z}[T]$ to be the set of all pairs $(P(T), Q(T))$ such that for some positive integers p and q, $P(T)$ divides $(T^p - 1)$, $Q(T)$ divides $(T^q - 1)$ and the $\gcd(T^p - 1, Q(T)) = 1$.

PROPOSITION 3. *Let $(P(T), Q(T))$ be in $S(T)$. If an element u of F is annihilated by Q, then there exists an f in $k(F)^*$ such that $(P(h)f)/u$ belongs to k^*.*

Proof (Due to the referee). Let p and q be as in the definition of $S(T)$, let $n = \text{order}(H)$ and let $Q_1(T) = (T^{pqn} - 1)/(T^p - 1)$. Let H_1 be the subgroup of H generated by $h_1 = h^p$ and let s be the sum of its elements. Then $Q_1(h)u = (s(u))^m$, where $m = nq/\text{order}(H_1)$. Since Q divides Q_1, then $(s(u))^m$ is in k^*. Since k is algebraically closed, then $s(u)$ is in k^*. Let a be the element of k^* for which $s(au) = 1$. Then by Hilbert's Theorem 90, there exists an f_1 in $k(F)^*$ such that $au = (h_1 - 1)f_1 = (h^p - 1)f_1$. It is now clear that $((h^p - 1)/P(h))f_1$ has the properties required of f. ∎

LEMMA 4. *Let $(P(T), Q(T))$ be in $S(T)$, and let F_2 be a submodule of F_1. If F_1/F_2 is the direct sum of cyclic modules annihilated by P, and if Q annihilates F_2, then $F_1 \sim F_2 * \bar{F}$, where $\bar{F}$ is isomorphic to F_1/F_2.*

Proof. Let $x_1,\dots, x_s$ be elements of F_1 representing generators of the cyclic summands of F_1/F_2 (one x_i for each summand), and set $u_i = P_i(h)x_i$ where P_i is the annihilator of x_i. Then $Q(h)u_i$ is in k^* and Proposition 3 guarantees the existence of an f_i in $k(F_2)^*$ such that $(P_i(h)f_i)/u_i$ is in k^*. It is now easy to see that the module generated by $\{x_i f_i : i = 1,\dots, s\}$ has the properties required of $\bar{F}$. ∎

COROLLARY 5. *Let E be as in Theorem* 1, *and let* $n = \text{order}(H)$. *If the* nth *cyclotomic field has class number* 1, *then $E \sim F_1 * F_2 * \cdots * F_s$, where each F_i is annihilated by a cyclotomic polynomial and where the sum of the $\bar{F}_i$'s is a direct sum.*

Proof. Let $(P_i)_{i=1}^m$ be the subsequence of $(\phi_i)_{i=1}^{\infty}$ consisting of the (cyclotomic) factors of $\chi(h, E)$, and set $Q_i = \prod_{j=i}^m P_j$. Let E_i be the submodule of E annihilated by Q_i. Clearly, E_i/E_{i+1} is annihilated by $Q_i/Q_{i+1} = P_i$. Since P_i is a factor of $T^n - 1$ and since the class number of the nth cyclotomic field (and hence that of the dth cyclotomic field for every factor d of n [8]) is assumed to be 1, then $\mathbb{Z}[T]/P_i(T)$ is a P.I.D. and hence E_i/E_{i+1} (being a $(\mathbb{Z}[T]/P_i(T))$-module) is the direct sum of cyclic modules. Thus, Lemma 4 applies to each pair (E_i, E_{i+1}). We apply Lemma 4 m times, letting the role of (F_1, F_2) in that lemma be played by (E_1, E_2), (E_2, E_3),..., (E_m, E_{m+1}) in this order and denoting by $F_1, F_2,..., F_m$ the modifications thus obtained. ∎

LEMMA 6. *Let $(P(T), Q(T))$ be in $S(T)$. If $\bar{F} = \bar{F}_1 \oplus \bar{F}_2$ and if $\bar{F}_1$ and $\bar{F}_2$ are cyclic modules annihilated by P and Q (resp.), then $F \sim F'$ for some cyclic $\bar{F}'$.*

Proof. Let x and u be elements of F representing generators of $\bar{F}_1$ and $\bar{F}_2$ (resp.), and let P_1 be the annihilator of F_1. Then by Proposition 3 there exists an f in $k(F_2)^*$ such that $(P_1(h)f)/u$ is in k^*. Now take F' to be the cyclic module generated by xf. ∎

COROLLARY 7. *If $\bar{F}$ is the direct sum of cyclic modules annihilated by distinct cyclotomic polynomials, then $F \sim F'$ for some cyclic $\bar{F}'$.*

Proof. Ler $\bar{F}_i$ $(i = 1,..., r)$ be the cyclic summands of $\bar{F}$, and let P_i be the cyclotomic polynomial annihilating F_i. Set $Q_i = \prod_{j=i+1}^{r} P_j$. Let $W_r = F_r$ and define W_{r-i} $(i = 1, 2,..., r-1)$ to be the module equivalent to $W_{r-i+1} * F_{r-i}$ obtained by applying Lemma 6 to $\bar{W}_{r-i+1} \oplus \bar{F}_{r-i}$. Then W_1 has the properties required of F'. ∎

Proof of Theorem 2. In virtue of Corollary 5, one can assume that the module $\bar{E}$ obtained in Theorem 1 is the direct sum of modules annihilated by cyclotomic factors of $\chi(h, E)$. Each of these summands is in turn the direct sum of cyclic modules. (This is because the nth cyclotomic field has class number 1, and therefore $\mathbb{Z}[T]/f(T)$ is a P.I.D. for every cyclotomic factor f of $T^n - 1$.) We index these cyclic summands of $\bar{E}$ by the set $D = \{(i, j)\colon i$ divides $s(j)$; $j = 1,..., r\}$ and we set $D(j) = \{(a, b) \in D\colon b = j\}$. Then $\bar{E} = \oplus_{t \in D} \bar{E}_t = \oplus_{j=1}^{r} \oplus_{t \in D(j)} \bar{E}_t$, where for $t = (a, b)$, $\bar{E}_t$ is cyclic annihilated by ϕ_a. We now apply Corollary 7 to $\oplus_{t \in D(j)} \bar{E}_t$ (for each j) to obtain F_j with $\bar{F}_j$ cyclic annihilated by $T^{s(j)} - 1$. Thus $E \sim F_1 * F_2 * \cdots * F_r$ and it is obvious that $F_1 * F_2 * \cdots * F_r$ is a permutation module. Therefore, $k(E)^H$, $=k(V)^G$, is rational. ∎

THEOREM 8. *Let H be cyclic. If* $\dim_k(V) < 23$, *then $k(V)^G$ is rational over k.*

Proof. Follows from the fact that $\dim_k(V)$ is the sum of the $s(i)$'s (appearing in Theorem 1) and therefore the class number of the $s(i)$th cyclotomic field is 1 for all i. ∎

Note. With the added hypothesis that n is prime, Theorem 2 was proved by Haeuslein [5]. The main ingredients in that proof are two statements (A) and (B) that are known to be true only if n is prime:

(A) $\mathbb{Z}[T]/(T^n - 1)$ is a semi-P.I.R. (For definition and reference, see [6, Theorem 0.4].)

(B) If $f(T)$ is a prime factor of $(T^n - 1)$ and if E is a cyclic module over $\mathbb{Z}[t] = \mathbb{Z}[T]/f(T)$, then $k(E) = k(F)$ for some permutation (actually trivial) t-module $\bar{F}$ [6, Theorem 1.1(iii)].

When n is not prime, (A) is false and (B) is still an open statement. This twofold difficulty is removed by Corollaries 5 and 7 above.

We finally remark that knowledge of both order(H) and $\dim_k(V)$ may yield the rationality of $k(V)^G$ when Theorems 2 and 8 fail to. As an example, $k(V)^G$ is rational when order$(H) = 39 > \dim_k(V)$.

2. The Klein Case

In this section, we drop the assumption that (the abelian) H is cyclic and we prove the rationality of $k(V)^G$ for $\dim_k V < 5$.

We first prove the following simple lemma. The facts that $\mathbb{Z}[T]/(T^2 - 1)$ is a semi-P.I.R. and that a monomial automorphism of order 2 has a rational fixed field [6] are freely used.

Lemma 9. *Let E_1, E_2 be the endomorphisms on $\mathbb{Z}^4$ defined by*

$$E_1((a_1, a_2, a_3, a_4)) = (a_2, a_1, a_3, a_4),$$
$$E_2((a_1, a_2, a_3, a_4)) = (a_1, a_2, a_4, a_3).$$

Let U be a rank 4 subgroup of $\mathbb{Z}^4$ invariant under both E_1 and E_2. Then U has a system $\{u_1, u_2, u_3, u_4\}$ of generators such that both u_1 and u_2 are fixed by E_1 and by E_2 and such that either

(1) $E_1: \quad u_3 \to u_4 \to u_3,$

$$E_2: \begin{cases} u_3 \to -u_4 + \alpha u_1 + \beta u_2 \\ u_4 \to -u_3 + \alpha u_1 + \beta u_2, \end{cases}$$

or

(2) $E_1: \quad u_3 \to u_3;\ u_4 \to -u_4 + \alpha u_1 + \beta u_2,$

$\ E_2: \quad u_4 \to u_4;\ u_3 \to -u_3 + \mu u_1 + \nu u_2,$

where α, β, μ, ν are integers which are significant only up to their values mod 2.

Proof. Let u_1, u_2 be generators of the subgroup W of U consisting of the elements fixed by E_1 and E_2, and let e_1 be the endomorphism induced on U/W by E_1. It is easy to see that the minimal polynomial of e_1 is T^2-1; and thus the group-ring $R = \mathbb{Z}[e_1]$, being $\cong \mathbb{Z}[T]/(T^2-1)$, is a semi-P.I.R. Hence, the torsion-free R-module U/W decomposes into the direct sum of cyclic R-modules. Since $\text{rank}(U/W) = 2$, and since T^2-1 is the smallest polynomial that annihilates e_1, it follows immediately that there are only the following two possibilities:

(i) $U/W = R\bar{v}$, where $v \in U$, $\bar{v} = v + W$ and $\text{Ann}_R(\bar{v}) = e_1^2 - 1$.

(ii) $U/W = R\bar{v}_1 \oplus R\bar{v}_2$, where $v_j \in U$, $\bar{v}_j = v_j + W$, and $\text{Ann}_R(\bar{v}_j) = e_1 - (-1)^j$.

To obtain (1) from (i), set $u_3 = v$ and observe that

$$(E_2E_1 + id)v = (E_1 + E_2)v \in W.$$

To obtain (2) from (ii), set $u_3 = v_1$ and $u_4 = v_2$.

The last statement follows from observing the effect the change of the basis $\{u_1, u_2, u_3, u_4\}$ into $\{u_1, u_2, u_3', u_4'\}$ has on the equations in (1) and (2), where $u_3' - u_3$ and $u_4' - u_4$ are in W. ∎

We now return to our problem. We form the irredundant decomposition $\oplus_{i=1}^m V_i$ of V into N-eigen spaces and we use the normality of N in G to prove that each g in G acts as a permutation on the set $\{V_1, ..., V_m\}$. Assuming that N is its own centralizer, one sees that the elements of N are the only elements of G that act as the identity permutation. Thus $H = G/N$ is isomorphic to a subgroup of the symmetric group S_m. The only abelian non-cyclic subgroups of S_m ($m \leqslant \dim_k(V) < 5$) are the Klein subgroups of S_4. Thus we assume that $\dim_k(V) = m = 4$, that $\dim_k(V_i) = 1$ and that H is a Klein group. Let g_1 and g_2 be elements of G representing generators of H, and let p_1 and p_2 be the elements of S_4 corresponding to g_1 and g_2. Let y_i be a basis of (the one-dimensional) V_i, let P be the subgroup of $k(V)^*$ generated by k^* and $\{y_1, ..., y_4\}$, and let A be the subgroup of P fixed by N. By Fischer's argument, A is a rank 4 subgroup of P with $k(A) = k(V)^N$. If p_1 and p_2 are the transposition (1 2) and (3 4), then by Lemma 9 one constructs a base $\{x, y, z, w\}$ of $k(V)^N$ (over k) on which the action of g_1 and g_2 is either of the actions described in (V) and (VI) of Table I. If p_1 and p_2 are (1 2)(3 4) and (1 3)(2 4), then by a lemma parallel to Lemma 9, one constructs a base $\{x, y, z, w\}$ on which the action of g_1 and g_2 is one of the actions described in (I), (II), (III) and (IV) of Table I. We let $K = k(x, y, z, w) = k(V)^N$, $K_1 = K^{g_1}$ and $K_{12} = K_1^{g_2}$. We now establish the rationality of

TABLE I

	$g_1(x)$	$g_1(y)$	$g_1(z)$	$g_1(w)$	$g_2(x)$	$g_2(y)$	$g_2(z)$	$g_2(w)$	Where $r_i^2 = s_i^2 = 1$, where α, β, μ, ν are integers significant only up to their values mod 2 and where
(I)	x^{-1}	y^{-1}	$s_3 z x^\alpha y^\beta$	$s_4 w x^\mu y^\nu$	y	x	w	z	$\mu = \beta$ $\nu = \alpha$ $s_3 = s_4$
(II)	x^{-1}	y^{-1}	$s_3 z x^\alpha y^\beta$	$s_4 w x^\mu y^\nu$	y	x	$r_3 z$	w^{-1}	$\alpha = \beta$ $\nu = -\mu$
(III)	x^{-1}	y^{-1}	$s_3 z x^\alpha y^\beta$	$s_4 w x^\mu y^\nu$	$r_1 x$	$r_2 y^{-1}$	w	z	$s_3 r_1^\alpha r_2^\beta = s_4$ $\mu = \alpha$ $\nu = -\beta$
(IV)	x^{-1}	y^{-1}	$s_3 z x^\alpha y^\beta$	$s_4 w x^\mu y^\nu$	$r_1 x$	$r_2 y^{-1}$	$r z y^\beta$	$w^{-1} x^{-\mu}$	$r_1^\mu = r_2^\nu = 1$ $r_2^\beta r^2 = r_2^\beta r_1^\alpha = 1$
(V)	$s_1 x$	$s_2 y$	w	z	$r_1 x$	$r_2 y$	$w^{-1} x^\alpha y^\beta$	$dz^{-1} x^\alpha y^\beta$	$d = r_1^\alpha r_2^\beta = s_1^\alpha s_2^\beta$
(VI)	$s_1 x$	$s_2 y$	$s_3 z$	$w^{-1} x^\alpha y^\beta$	$r_1 x$	$r_2 y$	$z^{-1} x^\mu y^\nu$	$r_4 w$	$r_1^\alpha r_2^\beta = r_1^\mu r_2^\nu = 1$ $s_1^\alpha s_2^\beta = s_1^\mu s_2^\nu = 1$

TABLE II

(C1)	(I), (III), (II, $\mu = 0$)
(C2)	(II, $\mu = 1$)
(C3)	(IV, $\mu = 0$)
(C4)	(IV, $\mu = 1, \nu = 0$)
(C5)	(IV, $\mu = 1, \nu = 1, s_4 = 1$)
(C6)	(IV, $\mu = 1, \nu = 1, s_4 = -1$)
(C7)	(V)
(C8)	(VI, $\|\alpha\nu - \beta\mu\| = 1$)
(C9)	(VI, $\alpha = \beta = 0$)
(C10)	(VI, $\alpha = \mu = 1, \beta = \nu = 0$)

$k(V)^G$ by establishing the rationality of K_{12} for each of the actions of g_1 and g_2 of Table I.

THEOREM 10. *If G is meta-abelian and if* $\dim_k(V) < 5$, *then* $k(V)^G$ *is rational over k.*

Proof. We rearrange the six cases (I)–(VI) of Table I to form the ten cases (C1)–(C10) of Table II. Note that to obtain (VI) from (C8)–(C10), one might need to interchange the roles of g_1 and g_2, x and y, or x and xy.

Set: $\xi = (1 - x)/(1 + x)$, $\eta = (1 - y)/(1 + y)$, $\delta = (x - y)/(x + y)$,

$$\begin{aligned} \zeta &= z + g_1(z) && \text{if} \quad z + g_1(z) \neq 0 \\ &= z && \text{otherwise,} \\ \omega &= w + g_1(w) && \text{if} \quad w + g_1(w) \neq 0 \\ &= w && \text{otherwise} \end{aligned}$$

(C1) Here, $K = k(\xi, \eta, \zeta, \omega)$, and g_1 is homothetic, g_2 monomial.

(C2) Here, $K_1 = k(A, B, C, D)$, where $A = 1 - \delta^2$, $B = (1 + \delta\xi)/(1 - \delta^2)$, $C = \zeta\delta^t$, $t = 0$ or 1, and $D = \omega$. If $s_4 = 1$, then the action of g_2 on A, B, C, D is monomial. If $s_4 = -1$, then by examining the action of g_2 on A, B, C, D, one easily sees that K_{12} is generated by the five elements $D + g_2(D)$, $B + g_2(B)$, $(D - g_2(D))/(B - g_2(B))$, $(B - g_2(B))^2$, $C(B - g_2(B))^e$, where $e = 0$ or 1, and that in the algebraic dependence among the first four, the fourth is linear.

(C3) If $\mu = \nu = \alpha = 0$, then the proof of (C1) goes through. Otherwise, interchanging the roles of g_1 and g_2 reduces this case to a previous one.

(C4) Set

$$\begin{aligned} \bar{\zeta} &= \zeta + g_2(\zeta) && \text{if} \quad \zeta + g_2(\zeta) \neq 0 \\ &= \zeta && \text{otherwise.} \end{aligned}$$

Then $K_1 = k(s_4 + (1+\xi^2)/(1-\xi^2), \eta/\xi, \bar{\zeta}\xi^t, \omega)$, $t = 0$ or 1. If $r_2 = 1$, then g_2 acts monomially. Otherwise, using $g_1 g_2$ for g_2 and interchanging x and y reduces this case to (C3).

(C5) Define $\bar{\zeta}$ as in (C4). Then K_1 is generated by $(1-\xi^2)(1-\eta^2)$, η/ξ, $\omega(1-\xi^2)/(1-\xi\eta)$ and $\bar{\zeta}\xi^t$, $t = 0$ or 1; and g_2 is monomial.

(C6) Define $\bar{\zeta}$ as in (C4). K_1 is generated by $A = \xi^2$, $B = \eta/\xi$, $C = \bar{\zeta}\xi^t$ (where $t = 0, 1$) and $D = \omega\xi(1-\eta^2)/(\xi-\eta)$. K_{12} is then generated by the five elements: A, B^2, $D + g_2(D)$, $(D - g_2(D))/B$, CB^t; and the algebraic dependence among the first four is linear in the second.

(C7) If $d = 1$, take α and β so that $\alpha\beta \leqslant 0$, and choose t_1 and t_2 in $\{0, 1\}$ so that $\alpha t_1 + \beta t_2 = 0$. Then K_1 is generated by $z + w$, zw, $x(z-w)^{t_1}$, $y(z-w)^{t_2}$; and g_2 acts monomially.

If $d = -1$, interchange x and y, or replace x by x/y if necessary, so that $s_1 = 1$. Since $d = s_1^\alpha s_2^\beta$, then $s_2 = -1$ and $\beta = -1$. Then K_1 is generated by x, $(z-w)^2$, $(z+w)$ and $y(z-w)zwx^{-\alpha}$; and g_2 acts monomially.

(C8) Here, both g_1 and g_2 are homothetic relative to the base: $z + g_2(z)$, $z - g_2(z)$, $w + g_1(w)$, $w - g_1(w)$.

(C9) Relative to the base $\{x, y, z, (1-w)/(1+w)\}$, g_1 is homothetic, and g_2 is monomial.

(C10) Let $t_i = 1$ (resp. 0) if $s_i = -1$ (resp. 1) and let $w_1 = w + g_1(w)$, $w_2 = w - g_1(w)$, $A = x/w_1$, $B = y(w_2)^{t_2}$ and $C = z(w_2)^{t_3}/w_1$. Since $(A^{-1}Cg_2(C) + 4t_3A)(w_1)^{s_3}$ is in k^*, then $\{A, B, C + g_2(C), C - g_2(C)\}$ is a base of K_1 on which g_2 acts monomially. ▮

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