Rational Invariants of Algebraic Group Actions

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Partly based on joint works with Irina Kogan or George Labahn

Contents

1 Computing rational invariants

1.1 Action of an algebraic group

Rational action $\star$ of an affine algebraic group $\mathcal{G}$

$\mathcal{G} \subset \mathbb{K}^l$ an algebraic variety $\qquad G \subset \mathbb{K}[\lambda_1, \dots, \lambda_l]$ its ideal

Group structure:

$$m: \begin{array}{ccc} \mathcal{G} \times \mathcal{G} & \to & \mathcal{G} \\ (\lambda, \mu) & \mapsto & \lambda \cdot \mu \end{array} \qquad \text{and} \qquad i: \begin{array}{ccc} \mathcal{G} & \to & \mathcal{G} \\ \lambda & \mapsto & \lambda^{-1} \end{array}$$

Rational action of $\mathcal{G}$ on $\mathcal{Z} = \mathbb{K}^n$

$$\lambda \star z = \left(\frac{p_1(\lambda, z)}{q(\lambda, z)}, \dots, \frac{p_n(\lambda, z)}{q(\lambda, z)} \right)$$

$q, p_1, \dots, p_n \in \mathbb{K}[\lambda_1, \dots, \lambda_l, z_1, \dots, z_n]$

Orbit $\mathcal{O}_z$ of $z \in \mathcal{Z}$: the image of $\mathcal{G}$ under $\lambda \mapsto \lambda \star z$

Examples of Algebraic Groups

$\mathrm{SL}_n(\mathbb{K}) = \{A \in \mathbb{K}^{n \times n} \mid \det A = 1\}$

$\mathrm{O}_n(\mathbb{K}) = \left\{A \in \mathbb{K}^{n \times n} \mid A^{\,t} A = I_n\right\}$

$\mathrm{SO}_n(\mathbb{K}) = \mathrm{O}_n(\mathbb{K}) \cap \mathrm{SL}_n(\mathbb{K})$

$\mathrm{GL}_n(\mathbb{K}) = \{A \in \mathbb{K}^{n \times n} \mid \det A \neq 0\}$

naturally act on $\mathbb{K}^n$

Examples of rational actions of $\mathbf{SL}_2$

The action of $\mathrm{SL}_2(\mathbb{C})$ on *forms* $z_0x^2 + z_1xy + z_2y^2$ of degree 2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a^2 & ac & c^2 \\ 2ab & ad + bc & 2cd \\ b^2 & bd & d^2 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$$

Projective action of $\mathrm{SL}_2(\mathbb{R})$ on quadrapules of $\mathbb{R}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \frac{a\,z_0+b}{c\,z_0+d} \\ \frac{a\,z_1+b}{c\,z_1+d} \\ \frac{a\,z_2+b}{c\,z_2+d} \\ \frac{a\,z_3+b}{c\,z_3+d} \end{pmatrix}$$

Rational invariants

$\star: \ \mathcal{G} \times \mathcal{Z} \ \rightarrow \ \mathcal{Z}$ $\qquad$ $\mathcal{O}_z = \{\lambda \star z \mid \lambda \in \mathcal{G}\}$

Rational invariant: $f \in \mathbb{K}(z_1, \ldots, z_n)$ s.t. $f(\lambda \star z) = f(z),\ \forall \lambda \in \mathcal{G}$

Field of rational invariants: $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ $\qquad$ finitely generated

Thm: $\mathbb{K}(\mathcal{Z})^{\mathcal{G}} = \mathbb{K}(r_1, \ldots, r_k) \ \Leftrightarrow \ \{r_1, \ldots, r_k\}$ separating

[Rosenlicht 56]

Separation property: $\qquad$ For $z, z' \in \mathcal{Z} \setminus \mathcal{W}$

$$z' \in \mathcal{O}_z \quad \Leftrightarrow \quad g_1(z) = g_1(z'),\ \ldots,\ g_k(z) = g_k(z')$$

1.2 Generic orbit ideal

The ideal O of a generic orbit

$$\mathcal{O}_z = \{\lambda \star z \mid \lambda \in \mathcal{G}\}$$

$O = (G + (\,Z - \lambda \star z\,)) : q^{\infty} \cap \mathbb{K}(z)[Z]$ $\qquad$ $\mathcal{V}(()\, O_{|z}) = \bar{\mathcal{O}}_z$

$$\lambda \star z = \left(\frac{p_1(\lambda, z)}{q(\lambda, z)}, \ldots, \frac{p_n(\lambda, z)}{q(\lambda, z)}\right)$$

$$(Z - \lambda \star z) = (q(\lambda, z)\, Z_1 - p_1(\lambda, z), \ldots, q(\lambda, z)\, Z_n - p_n(\lambda, z)) \subset \mathbb{K}(z)[\lambda, Z]$$

$$\mathcal{O}_z = \mathcal{O}_{\lambda \star z}$$

Prop: $Q(z)$ a canonical representative of $O \ \Rightarrow \ Q(\lambda \star z) = Q(z)$

Generation & Rewriting

Q reduced Gröbner basis of O

$\{r_1, \ldots, r_k\}$ the coefficients of Q

Theorem:

$$\mathbb{K}(z)^{\mathcal{G}} = \mathbb{K}(r_1, \ldots, r_k)$$

Rewriting $\qquad \dfrac{p}{q} \in \mathbb{K}(z)^{\mathcal{G}}$

- $y_1, \ldots, y_k$ a new indeterminates
- $Q_y := Q(r_i \leftarrow y_i)$
- $p(Z) \longrightarrow^*_{Q_y} \sum_\alpha a_\alpha(y)\, Z^\alpha$
- $q(Z) \longrightarrow^*_{Q_y} \sum_\alpha b_\alpha(y)\, Z^\alpha$
- $\dfrac{p(z)}{q(z)} = \dfrac{a_\alpha(r)}{b_\alpha(r)}$

References

Rosenlicht 56

The coefficients of the Chow form of O are rational invariants and separate orbits

Popov & Vinberg 89

There exists a generating set Q of O the coefficients $\{r_1, \ldots, r_\kappa\}$ of which are in $\mathbb{K}(z)^{\mathcal{G}}$

$\{r_1, \ldots, r_\kappa\}$ separate orbits

Müller-Quade & Beth 99, Kemper 07: Case of linear group actions.

Proof: $(Q) = (Z - z) \cap \mathbb{K}(z)^{\mathcal{G}}[Z]$.

Derksen 99 : Polynomial invariants of reductive groups acting linearly

H. & Kogan 07, H. 16: this presentation and what follows.

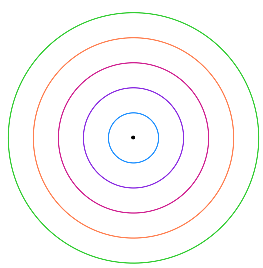
Linear actions in the plane

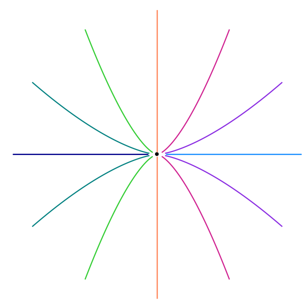
$$\lambda_1^2 + \lambda_2^2 = 1$$

$$(\lambda_1, \lambda_2) \mapsto \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$$

$$\mathbb{K}^*$$

$$\lambda \mapsto \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^3 \end{pmatrix}$$





$$Q = \{X^2 + Y^2 - (x^2 + y^2)\}$$

$$Q = \left\{Y^2 - \frac{y^2}{x^3} X^3\right\}$$

Example [Derksen 99]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \mapsto \begin{pmatrix} a & b & & & & & \\ c & d & & & & & \\ & & a & b & & & \\ & & c & d & & & \\ & & & & a^2 & ac & c^2 \\ & & & & 2ab & ad+bc & 2cd \\ & & & & b^2 & bd & d^2 \end{pmatrix}$$

$G \;= (ad - bc - 1)$

$Z \; - \lambda \star z :$ $\;Z_1 - (az_1 + bz_2),\; Z_2 - (cz_1 + dz_2),\; Z_3 - (az_3 + bz_4),\; Z_4 - (cz_3 + dz_4)$

$$Z_5 - (a^2 z_5 + 2abz_6 + b^2 z_7), \quad Z_7 - (c^2 z_5 + 2cdz_6 + d^2 z_7)$$

$$Z_6 - (caz_5 + ad + bcz_6 + bdz_7),$$

$O \;= (G + (Z - \lambda \star z)) \cap \mathbb{K}(z)[Z]$

$$Z_6{}^2 - Z_7 Z_5 + g_1 g_3 - g_4{}^2, \quad Z_6 Z_4 + g_3 g_2 Z_2 - g_4 Z_4 - Z_3 Z_7,$$

$$Z_5 Z_4 - Z_3 Z_6 + g_3 g_2 Z_1 - g_4 Z_3, \quad Z_3 Z_2 - Z_1 Z_4 - g_2,$$

$$Z_2 Z_6 - Z_1 Z_7 + g_4 Z_2 - \frac{g_1}{g_2} Z_4, \quad Z_2 Z_5 + Z_1 g_4 - Z_6 Z_1 - \frac{g_1}{g_2} Z_3,$$

$$Z_2{}^2 + \frac{g_1}{g_3 g_2{}^2} Z_4{}^2 - \frac{1}{g_3} Z_7 - 2 \frac{g_4}{g_3 g_2} Z_4 Z_2,\; Z_1{}^2 - \frac{1}{g_3} Z_5 - 2 \frac{g_4}{g_3 g_2} Z_3 Z_1 + \frac{g_1}{g_3 g_2{}^2} Z_3{}^2$$

$$Z_2 Z_1 - \frac{g_4}{g_3} - \frac{Z_6}{g_3} + \frac{g_1}{g_3 g_2{}^2} Z_4 Z_3 - 2 \frac{g_4}{g_3 g_2} Z_4 Z_1,$$

1.3 Section to the orbits

Essential geometric ingredient: section of degree e

An irreducible variety $\mathcal{P}$ that intersects generic orbits in e points.

A generic affine space of dimension $n-d$ is a section

where $d =$ dimension of generic orbits

Essential algebraic ingredient: intersection ideal I

$$I \subset \mathbb{K}(z_1,\ldots,z_n)[Z_1,\ldots,Z_n] \qquad \dim_{\mathbb{K}} \mathbb{K}(z)[Z]/I = e$$

Under specialization $z_i \mapsto \bar{z}_i \in \mathbb{K}$

$$I_{\bar{z}} \subset \mathbb{K}[Z] \text{ is the ideal of } \mathcal{O}_{\bar{z}} \cap \mathcal{P}$$

for $\bar{z} \in \mathcal{Z} \setminus \mathcal{W}$

Prp: $$I_{\lambda\star\bar{z}} = I_{\bar{z}}$$

Intersection ideal as an elimination ideal

$$I = (P + (Z - \lambda \star z) + G) : q^{\infty} \cap \mathbb{K}(z)[Z]$$

P a prime ideal in $\mathbb{K}[Z]$ $\qquad \mathcal{P} = \mathcal{V}(P)$

$$\lambda \star z = \left(\frac{p_1(\lambda,z)}{q(\lambda,z)},\ldots,\frac{p_n(\lambda,z)}{q(\lambda,z)}\right)$$

$$(Z - \lambda \star z) = (q(\lambda,z)\,Z_1 - p_1(\lambda,z),\ldots,q(\lambda,z)\,Z_n - p_n(\lambda,z))$$

an ideal of $\mathbb{K}(z)[\lambda, Z]$

Invariants from the reduced Gröbner basis

$$I = (P + (Z - \lambda \star z) + G) : q^{\infty} \cap \mathbb{K}(z)[Z]$$

Q reduced Gröbner basis of I

$\{r_1, \ldots, r_k\}$ its coefficients

$I_{\lambda \star z} = I_z \;\Rightarrow\; r_i \subset \mathbb{K}(z)^{\mathcal{G}}$

Thm [H. Kogan 07]

$$\mathbb{K}(z)^{\mathcal{G}} = \mathbb{K}(r_1, \ldots, r_k)$$

Pf: Rewriting $\quad \frac{p}{q} \in \mathbb{K}(z)^{\mathcal{G}}$

$y_1, \ldots, y_k$ a new indeterminates

$Q_y := Q(r_i \leftarrow y_i)$

$p(Z) \longrightarrow^*_{Q_y} \sum_\alpha a_\alpha(y)\, Z^\alpha$

$q(Z) \longrightarrow^*_{Q_y} \sum_\alpha b_\alpha(y)\, Z^\alpha$

$$\frac{p(z)}{q(z)} = \frac{a_\alpha(r)}{b_\alpha(r)}$$

Rational section = section of degree 1

$\mathcal{P}$ is of degree 1 $\quad\Leftrightarrow\quad I = (Z_1 - r_1(z), \ldots, Z_n - r_n(z))$

For $z \in \mathcal{Z} \setminus \mathcal{W}$, $\quad \mathcal{O}_z \cap \mathcal{P} = \{(r_1(z), \ldots, r_n(z))\}$

Then, for f invariant

$$f(z_1, \ldots, z_n) = f(r_1, \ldots, r_n)$$

.

A simple substitution!

A clear analogue of the local m. f. construction in [Fels Olver 99]

Example [Derksen 99]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \mapsto \begin{pmatrix} a & b & & & & & \\ c & d & & & & & \\ & & a & b & & & \\ & & c & d & & & \\ & & & & a^2 & ac & c^2 \\ & & & & 2ab & ad+bc & 2cd \\ & & & & b^2 & bd & d^2 \end{pmatrix}$$

$I \;= (P + (Z - \lambda \star z)) \cap \mathbb{K}(z)[Z]$

$$Q = \{Z_1 - 1, Z_2, Z_3, Z_4 - g_2, Z_5 - g_3, Z_6 - g_4, Z_7 - g_1\}$$

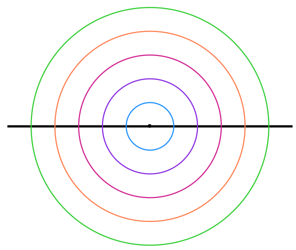
$$g_1 = z_7\, {z_1}^2 - 2\, z_2\, z_6\, z_1 + {z_2}^2 z_5, \quad g_2 = z_3\, z_2 - z_1\, z_4,$$
$$g_3 = \frac{{z_3}^2 z_7 - 2\, z_6\, z_4\, z_3 + z_5\, {z_4}^2}{(z_1\, z_4 - z_3\, z_2)^2},$$
$$g_4 = \frac{z_1\, z_6\, z_4 - z_1\, z_3\, z_7 + z_3\, z_2\, z_6 - z_2\, z_5\, z_4}{z_1\, z_4 - z_3\, z_2}$$

$f(z) \in \mathbb{K}(z)^{\mathcal{G}} \;\Rightarrow\; f(z_1, z_2, z_3, z_4, z_5, z_6, z_7) = f(1, 0, 0, g_2, g_3, g_4, g_1)$

Linear actions in the plane

$$\lambda_1^2+\lambda_2^2=1$$

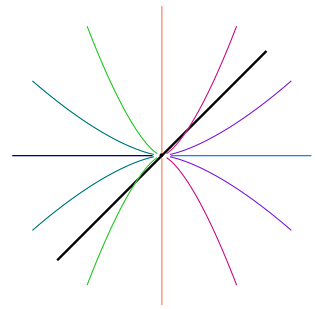
$$(\lambda_1,\lambda_2)\mapsto\begin{pmatrix}\lambda_1 & -\lambda_2\\ \lambda_2 & \lambda_1\end{pmatrix}$$



$$Q=\{Y,X^2-(x^2+y^2)\}$$

$$\mathbb{K}^*$$

$$\lambda\mapsto\begin{pmatrix}\lambda^2 & 0\\ 0 & \lambda^3\end{pmatrix}$$



$$Q=\left\{X-\frac{x^3}{y^2},Y-\frac{x^3}{y^2}\right\}$$

Rational sections of $\mathbf{SL}_2$

- The action of $\mathrm{SL}_2(\mathbb{C})$ on *forms* $z_0x^2+z_1xy+z_2y^2$ of degree 2

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\star\begin{pmatrix}z_0\\ z_1\\ z_2\end{pmatrix}=\begin{pmatrix}a^2 & ac & c^2\\ 2ab & ad+bc & 2cd\\ b^2 & bd & d^2\end{pmatrix}\begin{pmatrix}z_0\\ z_1\\ z_2\end{pmatrix}$$

$$I=\left(\underbrace{Z_0-1,\ Z_1}_{P},Z_2+\frac{1}{4}\left(z_1^2-4\,z_0z_2\right)\right)$$

- Projective action of $\mathrm{SL}_2(\mathbb{R})$ on quadrapules of $\mathbb{R}$:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\star\begin{pmatrix}z_0 & z_1 & z_2 & z_3\end{pmatrix}=\begin{pmatrix}\frac{a\,z_0+b}{c\,z_0+d} & \frac{a\,z_1+b}{c\,z_1+d} & \frac{a\,z_2+b}{c\,z_2+d} & \frac{a\,z_3+b}{c\,z_3+d}\end{pmatrix}$$

$$I=\left(\underbrace{Z_0,Z_1-1,Z_2^{-1}}_{P},Z_3-\frac{(z_3-z_0)(z_1-z_2)}{(z_2-z_3)(z_0-z_1)}\right)$$

2 Scalings = diagonal representations of tori

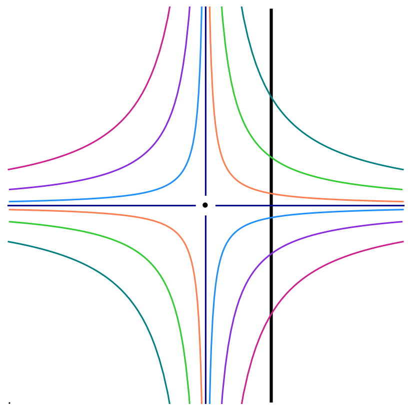
2.1 Scalings in the plane

Scalings in the plane

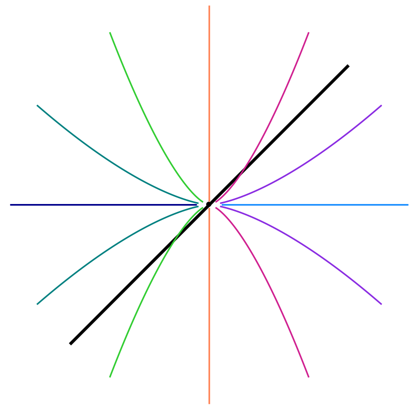
$$\begin{array}{rccc}\rho: & \mathbb{K}^* & \to & \mathrm{GL}(2)\\ & \lambda & \mapsto & \begin{pmatrix}\lambda^a & \cdot\\ \cdot & \lambda^b\end{pmatrix}\end{array}$$

$$\begin{array}{rccc}\star: & \mathbb{K}^*\times\mathbb{K}^2 & \to & \mathbb{K}^2\\ & (\lambda,(x,y)) & \mapsto & (\lambda^a x,\lambda^b y)\end{array}$$

$$A=\begin{bmatrix}a & b\end{bmatrix}$$



$A = \begin{bmatrix} 1 & -1 \end{bmatrix}$



$A = \begin{bmatrix} 2 & 3 \end{bmatrix}$

Bezout: $\alpha\, a - \beta\, b = 1$

$P = (X^{\alpha} - Y^{\beta}) : (XY)^{\infty}$

$$I = \left(X - \left(\frac{y^a}{x^b}\right)^{\beta},\ Y - \left(\frac{y^a}{x^b}\right)^{\alpha} \right)$$

Hermite normal form

$$\underbrace{\begin{bmatrix} a & b \end{bmatrix}}_{\text{scaling}} \underbrace{\begin{bmatrix} \alpha & -b \\ -\beta & a \end{bmatrix}}_{\text{multiplier}} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\text{Hermite form}} .$$

2.2 Parameter reduction in models

Réduction par les symétries *de dilatation*

Modèle proie-prédateur

$$\left\{ \begin{array}{rcl} \dot{n} & = & \left(\left(1 - \frac{n}{K}\right) r - k \frac{p}{n+e} \right) n, \\ \dot{p} & = & s \left(1 - h \frac{p}{n}\right) p. \end{array} \right.$$

Paramètres : r, s, e, h, K, k

$$\left\{ \begin{array}{rcl} \dot{\mathfrak{n}} & = & \left(1 - \frac{\mathfrak{n}}{\mathfrak{k}} - \mathfrak{h} \frac{\mathfrak{p}}{\mathfrak{n}+1}\right) \mathfrak{n}, \\ \dot{\mathfrak{p}} & = & \mathfrak{s} \left(1 - \frac{\mathfrak{p}}{\mathfrak{n}}\right) \mathfrak{p}. \end{array} \right.$$

Paramètres : $\mathfrak{s}, \mathfrak{h}, \mathfrak{k}$

Symétrie: $\begin{array}{lll} t \to \lambda^{-1} t, & r \to \lambda r, & h \to \nu h, \\ n \to \mu n, & s \to \lambda s, & K \to \mu K, \\ p \to \mu\nu^{-1} p, & e \to \mu e, & k \to \lambda\nu k \end{array}$

Invariants : $\mathfrak{t} = r\, t,\ \mathfrak{n} = \frac{1}{e} n,\ \mathfrak{p} = \frac{h}{e} p,\ \mathfrak{s} = \frac{s}{r},\ \mathfrak{h} = \frac{k_2}{rh}, \mathfrak{k} = \frac{k_1}{e}.$

Réécriture: $t \mapsto \mathfrak{t},\ n \mapsto \mathfrak{n},\ p \mapsto \mathfrak{p},\ s \mapsto \mathfrak{s},\ k \mapsto \mathfrak{k},\ d \mapsto \mathfrak{d},\ r,\ h,\ K \mapsto 1.$

Réduction par les symétries *de dilatation*

- Determination des symétrie *de dilatation* Algèbre linéaire

- Calcul des invariants — Algèbre linéaire
- Réécriture en ces nouvelles variables — Algèbre linéaire
- Solutions à partir de celles du système réduit — Algèbre linéaire

sur les entiers.

[HL12] E. Hubert and G. Labahn. Rational invariants of scalings from Hermite normal forms. In ISSAC 2012.

[HL13] E. Hubert & G. Labahn. Foundations of Computational Mathematics, 2013

[HL16] E. Hubert and G. Labahn. Mathematics of Computations, 2016.

3 Symmetrization of polynomial systems for algebraic groups

3.1 Finite groups

Symmetrization

$f_1, \ldots, f_m \in \mathbb{K}[z_1, \ldots, z_n]$

$\mathcal{V}(f_1, \ldots, f_m)$ invariant :

$$f_1(z)=0,\ \ldots,\ f_m(z)=0 \ \Rightarrow\ f_1(\lambda\star z)=0,\ \ldots,\ f_m(\lambda\star z)=0,\ \forall\lambda\in\mathcal{G}$$

Or equivalently:

$$f_1(z)=0,\ \ldots,\ f_m(z)=0 \ \Rightarrow\ \lambda\star f_1(z)=0,\ \ldots,\ \lambda\star f_m(z)=0,\ \forall\lambda\in\mathcal{G}$$

Find $p_1, \ldots, p_\ell$ in $\mathbb{K}[z]^{\mathcal{G}}$ s.t. $\mathcal{V}(f_1, \ldots, f_m) = \mathcal{V}(p_1, \ldots, p_\ell)$

Case of a finite group $\mathcal{G}$

$f \in \mathbb{K}[z]$ $\qquad g = |\mathcal{G}|$

Reynolds operator: $\frac{1}{g}\sum_{\lambda\in\mathcal{G}} \lambda\star f \ \in \mathbb{K}[z]^{\mathcal{G}}$

For $j = 1, \ldots, g$ consider the element of $\mathbb{K}[z]^{\mathcal{G}}$

$f^{(j)} =$ the j^{th} symmetric function in $\{\lambda\star f \mid \lambda\in\mathcal{G}\}$.

$$\prod_{\lambda\in\mathcal{G}}(\zeta - \lambda\star f) = \zeta^g - f^{(1)}(z)\,\zeta^{g-1} + \ldots + (-1)^g f^{(g)}$$

Prop: $\left(\forall\lambda\in\mathcal{G},\ \lambda\star f\,(z) = 0\right) \ \Leftrightarrow\ \left(f^{(j)}(z) = 0,\ \forall 1\leq j\leq g\right)$

Def: $f^{(1)}, \ldots, f^{(g)}$ are the *symmetrizations* of f

Symmetrization for a finite group $\mathcal{G}$

Thm: If $\mathcal{V}(f_1, \ldots, f_m)$ is invariant then

$$\mathcal{V}\left(f_i^{(j)}\,|\,1 \leq i \leq m,\, 1 \leq j \leq g\right) = \mathcal{V}(f_1, \ldots, f_m)\,.$$

[Sturmfels 93]

For semi-algebraic set [Cimpric, Kulmann, Scheiderer 09]

What to expect for an algebraic group $\mathcal{G}$

For $f \in \mathbb{K}[z]$ and $1 \leq j \leq e$

$f^{(j)}(z_1, \ldots, z_n) = p_j\left(r_1(z),\, \ldots, r_k(z)\right)$

$\mathbb{K}(z)^{\mathcal{G}} = \mathbb{K}\left(r_1, \ldots, r_k\right)$

3.2 Algebraic groups

Symmetrization

$$f_1, \ldots f_m \in \mathbb{K}[z_1, \ldots, z_n] \qquad \begin{array}{ccc} \mathcal{G} \times \mathcal{Z} & \rightarrow & \mathcal{Z} \\ (\lambda, z) & \mapsto & \lambda \star z \end{array}$$

$\mathcal{F} = \mathcal{V}(f_1, \ldots, f_m)$ invariant : $\quad z \in \mathcal{F} \;\; \Rightarrow \;\; \mathcal{O}_z \in \mathcal{F}$

Pb1: Find $F \subset \mathbb{K}[x]^{\mathcal{G}}$ s.t. $\mathcal{V}(F) = \mathcal{V}(f_1, \ldots, f_m)$

Pb2: Find $F \subset \mathbb{K}[r_1, \ldots, r_k]$ s.t. $\mathcal{V}(F) \setminus \mathcal{V}(h) = \mathcal{V}(f_1, \ldots, f_m) \setminus \mathcal{V}(h)$

where $\mathbb{K}(r_1, \ldots, r_k) = \mathbb{K}(z)^{\mathcal{G}}$

Case of a section of degree 1

$\mathcal{P}$ a section of degree 1

$$I = (Z_1 - r_1(z), \ldots, Z_n - r_n(z))$$

$$F = \{\, f_1\left(r_1, \ldots, r_n\right), \ldots, f_m\left(r_1, \ldots, r_n\right)\}$$

And the syszgies on the r_i are given by P

Case of a section of degree e

$I \subset \mathbb{K}(z)[Z]$ $\qquad\qquad$ $\dim_{\mathbb{K}(z)} \mathbb{K}(z)[Z] \,/\, I = e$

Multiplication by $f \in \mathbb{K}[Z]$

$$\begin{array}{rccc} m_f: & \mathbb{K}(z)\mathbb{K}(z)^{\mathcal{G}}[Z]/I & \to & \mathbb{K}(z)\mathbb{K}(z)^{\mathcal{G}}[Z]/I \\ & \bar{g} & \mapsto & \overline{f\,g} \end{array}$$

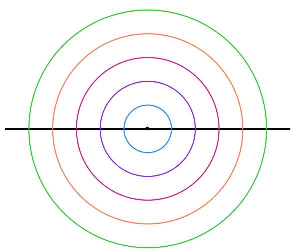
is a $\mathbb{K}(z)$-linear map.

The Gröbner basis of I provides a basis for $\mathbb{K}(z)^{\mathcal{G}}[Z]/I$ and the matrix for m_f. Its entries belong to $\mathbb{K}[r_1, \ldots, r_k]$.

Characteristic polynomial: $\qquad\qquad$ $f^{(j)} \in \mathbb{K}(z)^{\mathcal{G}}\mathbb{K}[r]$

$$\zeta^e - f^{(1)}(zr)\,\zeta^{e-1} + \ldots + (-1)^j\; f^{(j)}(zr)\,\zeta^{e-j} + \ldots + (-1)^e\, f^{(e)}(zr)$$

Example $\mathcal{G} = \mathbf{SO}_2$



$$\lambda \star z = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ with } c^2 + s^2 = 1$$

$[\,1,\, Y\,]$ basis for $\mathbb{K}(z)[Z]/I$ where $I = \left(X,\; Y^2 - (\underbrace{x^2 + y^2}_{r}) \right)$

$$f = \alpha\,1 + \beta\,X + \gamma\,Y$$

$1\,f \;\equiv\; \alpha 1 \,+\, \gamma\,Y \;\bmod I \;\; Y\,f \;\equiv\; \alpha\,Y \,+\, \gamma\,r$ mod I $\qquad\qquad$ $M_f = \begin{pmatrix} \alpha & \gamma\,r \\ \gamma & \alpha \end{pmatrix}$

$$f^{(1)} = 2\alpha, \qquad f^{(2)} = \alpha^2 - \gamma^2\,r$$

Symmetrization

Thm: There is an invariant hypersurface $\mathcal{W} = \mathcal{V}(h)$ s.t. if $\mathcal{F} = \mathcal{V}(f_1, \ldots f_m)$ is $\mathcal{G}$-invariant then

$$\mathcal{F} \setminus \mathcal{W} = \mathcal{V}\Big(\, f_i^{(j)} \,|\, 1 \le j \le e,\; 1 \le i \le m \Big) \setminus \mathcal{W}$$

Pf: Eigenvalues of m_f = evaluations of f at the roots of I.

If $z \in \mathcal{Z} \setminus \mathcal{W}$ then $\mathcal{V}(I_z) = \left\{z^{(1)}, \ldots, z^{(e)}\right\} = \mathcal{O}_z \cap \mathcal{P}$

$$\prod_{j=1}^{e}\left(\zeta - f\left(z^{(j)}\right)\right) = \zeta^e - f^{(1)}(z)\zeta^{e-1} + \ldots + \ldots + (-1)^e f^{(e)}(z)$$

$$\left(f(z^{(1)}) = 0, \ldots, f(z^{(e)}) = 0\right) \;\Leftrightarrow\; \left(f^{(j)}(z) = 0, \forall\, 1 \leq j \leq e\right)$$

$\mathcal{F}$ invariant implies :

$$z \in \mathcal{F} \setminus \mathcal{W} \;\Leftrightarrow\; \mathcal{O}_z \cap \mathcal{P} = \left\{z^{(1)}, \ldots, z^{(e)}\right\} \subset \mathcal{F} \setminus \mathcal{W}$$

Thanks Merci

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