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Citation: Am. J. Phys. **81**, 33 (2013); doi: 10.1119/1.4755780
View online: http://dx.doi.org/10.1119/1.4755780
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v81/i1
Published by the American Association of Physics Teachers

Radiometry and the Friis transmission equation

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(Received 1 July 2011; accepted 13 September 2012)

To more effectively tailor courses involving antennas, wireless communications, optics, and applied electromagnetics to a mixed audience of engineering and physics students, the Friis transmission equation—which quantifies the power received in a free-space communication link—is developed from principles of optical radiometry and scalar diffraction. This approach places more emphasis on the physics and conceptual understanding of the Friis equation than is provided by the traditional derivation based on antenna impedance. Specifically, it shows that the wavelength-squared dependence can be attributed to diffraction at the antenna aperture and illustrates the important difference between the throughput (product of area and solid angle) of a single antenna or telescope and the throughput of a transmitter-receiver pair.
[http://dx.doi.org/10.1119/1.4755780]

I. INTRODUCTION

It is becoming increasingly common for physics and electrical engineering students to take courses together in optics, photonics, and topics in applied electromagnetics such as wireless communications, antennas, or remote sensing. For effective learning with students from different disciplines, it is helpful to identify examples where principles from one subject area can be applied to another. Exactly such an opportunity arises when students encounter principles of radiometry in optics courses or the free-space Friis transmission equation in antennas or wireless communications courses.

The Friis transmission equation relates the received power to the transmitted power, antenna-separation distance, and antenna gains in a free-space communication link.[1] However, students first encountering the usual textbook derivation, based on equivalent dipole antenna impedance, tend to gain little physical insight into the Friis equation. Even for students familiar with impedance concepts, the physical basis of the Friis equation becomes more apparent with a derivation based on radiometry[2–4] and scalar diffraction theory.[5–8] Radiometry is the field of science devoted to quantifying the amount of power carried by a beam of electromagnetic radiation using geometric optics, areas, and solid angles. Scalar diffraction theory allows us to use simple Fourier transform relationships to account for the beam-spreading effects induced by the finite size of antenna or optical apertures.[5–8] (Vector diffraction theory does the same thing more accurately by using Maxwell's equations without the simplifying scalar-field assumptions.[9])

Textbooks on antennas and communications[10–13] often write the Friis equation as

$$\frac{P_r}{P_t} = e_r e_t \left(\frac{\lambda}{4\pi R}\right)^2 D_r D_t = \left(\frac{\lambda}{4\pi R}\right)^2 G_{or} G_{ot}, \tag{1}$$

where P is power in Watts, e is the antenna efficiency, λ is the electromagnetic wavelength, R is the far-field line-of-sight distance between the antennas, D is the antenna directivity relative to isotropic, G is the antenna gain ($G = eD$), and the subscripts r and t denote the receiver and transmitter, respectively. Friis actually developed the transmission equation in terms of effective areas A_r and A_t, of receiving and transmitting antennas, respectively, resulting in

$$\frac{P_r}{P_t} = \frac{A_r A_t}{\lambda^2 R^2}. \tag{2}$$

Following his derivation of the transmission equation from geometry and antenna impedance,[1] Friis proposed that henceforth antennas should be described in terms of their effective area rather than antenna gain or radiation resistance, and that antenna radiation should be expressed as power flow per unit area rather than field strength in Volts per meter. In doing so, he effectively proposed moving from a traditional circuit viewpoint, which deals with current and voltage, to a radiometric one, which deals with power, power density, etc. However, in textbooks and classroom presentations, the Friis equation is normally derived or explained in terms of antenna impedance, leading to Eq. (1). Unfortunately, without a deeper understanding of how the antenna gain depends on wavelength, Eq. (1) can be mistakenly interpreted to mean that the received power increases with wavelength, whereas in fact the opposite occurs. Additional confusion often arises because the antenna impedance derivation is based on the effective area of an infinitesimal dipole, but applied broadly to antennas of any shape.[10,11]

Several authors have discussed alternate derivations of the Friis equation. Friis himself presented an argument that resembles a simpler form of the antenna derivation that is common in modern antenna textbooks, but he left the result in terms of antenna areas as seen in Eq. (2). Hogg[14] derived the Friis transmission equation using three different approaches—Fresnel zones, Gaussian beams, and modes—all of which lead to Eq. (2). Bush[15] outlined a derivation based on Fraunhofer diffraction integrals but did not combine diffraction with radiometry. Shortly thereafter, Heald[16] pointed out that the most significant benefit of Bush's paper was that it removed the oft-confusing antenna impedance from the derivation.

The development presented here builds on this background to provide an even simpler path to the Friis equation based on optical radiometry and a simple result of scalar diffraction theory. Many undergraduate and graduate students have been exposed to enough basic optical principles that this development is intuitively appealing, particularly in a mixed class of physics and engineering students. This paper presents a combined discussion of the Friis transmission equation and optical radiometry, relying on the geometric optics propagation of radiance (power per unit area per unit solid angle) through free space. Beam spreading is introduced via scalar diffraction at the antenna aperture. This approach still relies on aperture areas, but in a much more direct manner and without any reference to antenna impedance. In this development, it becomes clear that the source of the wavelength-squared dependence is diffraction at the antenna aperture. This discussion also conveniently merges several topics from antenna theory and radiometry. Therefore, for students familiar with radiometry this approach builds on familiar principles to enhance understanding of the Friis equation, and for students not familiar with radiometry this approach teaches radiometry in the practical context of a free-space communication link.

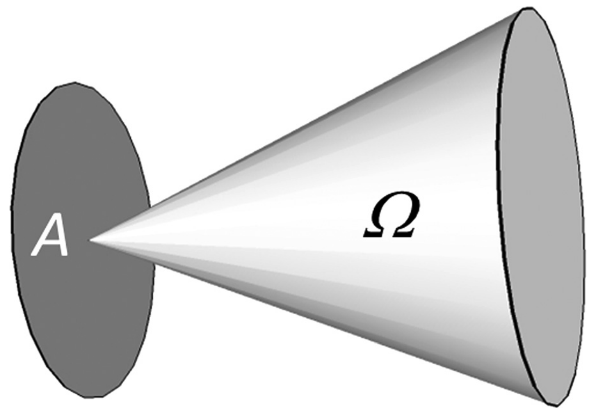

Fig. 1. Geometry for radiometric calculations, showing that a solid angle always opens away from the area (A) in which incident power is calculated.

II. BASIC CONCEPTS OF OPTICAL RADIOMETRY

The radiometric development of the Friis transmission equation relies on the concepts of radiance and throughput, both central to radiometry.[2,3] As indicated in Table I, radiance is the power incident on (emitted from) a surface per unit area, from (into) a given solid angle, with units of W/(m^2 sr). Suggestive of its physical meaning, radiance often is called brightness in the microwave radiometry and radio astronomy communities.[17,18] The concepts of radiance and related radiometric quantities can be introduced and used in courses ranging from antennas to photonics and optical design because they provide a consistent framework for calculating power transmitted from or received by an optical detector or antenna. While the five quantities in Table I are sufficient for power flow calculations, radiance is the most fundamental because it is invariant in a lossless system. Similarly, conservation of energy requires that the product of area and solid angle is invariant, a quantity referred to by optical scientists and engineers as *throughput* (sometimes called *étendue*). Antenna texts[10,11] and optical texts discussing coherent receivers[19] provide an added quantification of this concept by showing that the area-solid-angle product for a single-spatial-mode, diffraction-limited system is equal to the wavelength squared. This provides a theoretical limit to the field-of-view solid angle achievable with such systems for a given aperture size and wavelength.

Whether or not the classroom discussion extends to a full coverage of all the radiometric quantities in Table I, the most basic of radiometric principles can be taught and used to derive the Friis transmission equation,[1] the radar range equation,[20] and the lidar equation.[21] One of the most important of these radiometric principles is that the throughput is the product of an area and a solid angle that always opens away from the area, as shown in Fig. 1. Students often need to be reminded that the steradian is a dimensionless unit of solid angle and can be thought of in a manner similar to the more common planar angle. As shown in Fig. 2, a planar angle in radians is defined as the ratio of (circular) arc length to radius (s/r), whereas a solid angle in steradians is defined as the ratio of (spherical) surface area to the square of the radius (A_{sph}/r^2). In a more complicated situation, we can find a solid angle ω by integrating in spherical coordinates over the appropriate range of polar and azimuthal angles

$$\omega = \oiint \sin\theta \, d\theta \, d\phi. \tag{3}$$

Although many antenna and physics textbooks use a capital omega (Ω) for all solid angles, in radiometry a careful distinction is made between solid angle ω and projected solid angle Ω. For a projected solid angle, we replace the spherical surface area with a flat cross-sectional area, such as the projected aperture area of a lens, mirror, or antenna. The integral definition of the projected solid angle Ω is the same as Eq. (3) but includes a factor of $\cos(\beta)$ in the integrand, where β is the angle between the area's surface normal and the viewing direction. The small-angle approximation allows us to see that the projected solid angle subtended by an object can be estimated simply as the ratio of the object's projected cross-sectional area divided by the square of the distance R from the observer to the object

$$\Omega = \frac{A}{R^2}. \tag{4}$$

From dimensional analysis, the power collected by a receiver with area A_r from a transmitter subtending solid angle

Table I. Radiometric quantities and antenna equivalents.

Quantity	Symbol	Units	Antenna equivalent
Power	P	W	Power, P
Irradiance	E	W/m^2 incident	Power density, W
Exitance	M	W/m^2 leaving surface	Power density, W
Intensity	I	W/sr	Radiation intensity, U
Radiance	L	W/(m^2 sr)	Brightness, B (in radio astronomy)

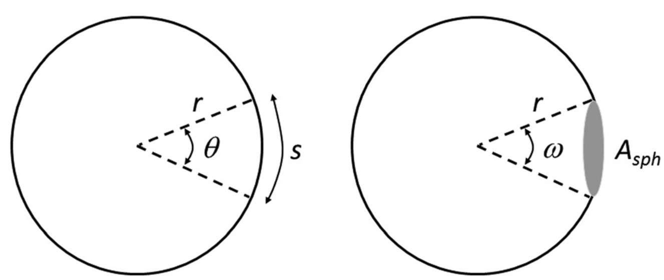

Fig. 2. In plane geometry an angle θ is defined as the ratio of circular arc length s to radius r (left); in spherical geometry a solid angle ω is the ratio of spherical surface area A to the square of the radius r^2. The projected solid angle Ω results from considering the projected area instead of the actual spherical area.

$\Omega_{t,r}$ as seen from the receiver, is given by the product of the source radiance L (W m^{-2} sr^{-1}) and the geometric throughput $A_r\Omega_{t,r}$ (m^2 sr)

$$P = LA_r\Omega_{t,r} \cong L\frac{A_r A_t}{R^2}. \tag{5}$$

For example, in a Friis transmission scenario the area A_r represents the effective area of a receiving antenna and $\Omega_{t,r}$ is the projected solid angle subtended by the transmitting antenna with effective area A_t located a distance R from the receiver ($\Omega_{t,r} = A_t/R^2$ is a common approximation where the flat cross-sectional area A_t is used instead of integrating over the projected spherical area). Because the resulting throughput is equal to the product of the two areas divided by the square of the distance between them, it is an invariant of the system regardless of which area is the receiver and which is the transmitter. This invariance means that as the beam is compressed into a smaller area, its solid angle expands proportionally. An important consequence of this is that light focused onto a tiny single-mode fiber, for example, often has a solid angle that exceeds the maximum angle that can propagate in the fiber, resulting in a very low light-coupling efficiency. A similar concern exists when one attempts to focus any electromagnetic energy into a small area.

When discussing antennas it is important to emphasize that the receiver solid angle used in the Friis transmission equation is not the "beam solid angle" of antenna theory, except in the unlikely case of the transmit antenna beam exactly filling the receiver field of view at distance R; rather, we simply use the solid angle subtended by one antenna as seen from the other, a solid angle that usually is smaller than the antenna beam solid angle. One final note is that all solid angles discussed in the remainder of this paper are projected solid angles (the difference between solid angles and projected solid angles is tiny for small angles).

III. FRIIS EQUATION DEVELOPED FROM RADIOMETRY AND SCALAR DIFFRACTION

To develop the Friis transmission equation from optical radiometry and scalar diffraction theory, we begin with a common antenna calculation to incorporate the concept of directivity into radiometry. The directivity D is an important measure of how an antenna radiates preferentially in a given direction relative to an isotropic antenna that radiates uniformly in all directions (see Fig. 3). Directivity can be defined as the ratio of an antenna's radiation intensity U (W sr^{-1}) in a given direction, usually the direction of maximum radiation, to the isotropic radiation intensity U_0 (W sr^{-1}). For total radiated power P, the radiation intensity can be written as $U = P/\Omega$, while the isotropic radiation intensity is $U_0 = P/(4\pi)$. Therefore, the directivity can be written as either a ratio of antenna radiation intensity U to isotropic radiation intensity U_0 or a ratio of isotropic solid angle 4π to the antenna's beam solid angle Ω

$$D = \frac{U}{U_0} = \left(\frac{P}{\Omega}\right)\left(\frac{4\pi}{P}\right) = \frac{4\pi}{\Omega}. \tag{6}$$

Notice that a narrower beam solid angle Ω results in a larger directivity D. This concept of solid angle ratio is used

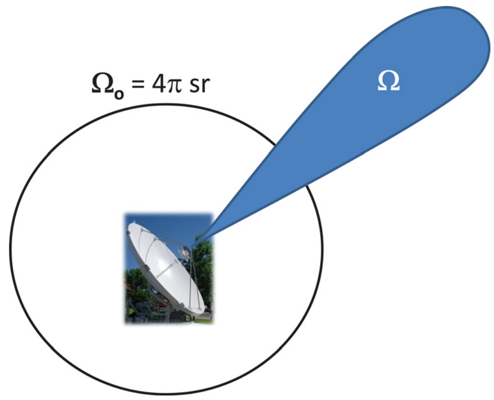

Fig. 3. Directivity is a concept that describes how much of an antenna's radiated power is transmitted into a given direction (per unit solid angle Ω) relative to the amount radiated by an isotropic antenna radiating uniformly into solid angle $\Omega_o = 4\pi$ sr.

often in optical radiometry but is not usually referred to as directivity, a useful term that probably should be adopted by the optics community. We can incorporate directivity into the present discussion by writing the solid angle of the transmitted beam as the cross-sectional beam area A_b divided by the distance squared

$$\Omega_t = \frac{A_b}{R^2}. \tag{7}$$

Using this result in Eq. (6) gives a transmitter directivity of

$$D_t = \frac{4\pi R^2}{A_b}. \tag{8}$$

It is here that diffraction must be invoked to bring the wavelength into the equation. Even for students who have not studied scalar diffraction theory,[5–8] an explanation can be offered that diffraction causes a beam of electromagnetic energy to spread into an angle θ_d that is proportional to the ratio of wavelength λ over the size d of the aperture or obstruction causing the diffraction

$$\theta_d \cong \frac{\lambda}{d}. \tag{9}$$

This result is exactly true for a square aperture, but if d is the radius of a circular aperture, Eq. (9) must be multiplied by 1.22 to obtain the Airy disk radius.[5–8] This is a detail that can be discussed or not, depending on student background, available time, and instructor interest. At distance R sufficiently large that the beam dimension is much greater than the original aperture area, Eq. (9) and a small-angle approximation results in an estimate for the transverse dimension of the beam

$$y \cong R\theta_d \cong \frac{R\lambda}{d}. \tag{10}$$

Now the beam area can be approximated as

$$A_b \cong y^2 = \frac{R^2\lambda^2}{d^2} = \frac{R^2\lambda^2}{A_t}, \tag{11}$$

where A_t is the effective area of the transmit antenna. Substituting this beam area into Eq. (8), we obtain a wavelength-dependent expression for the transmit antenna directivity

$$D_t = \frac{4\pi R^2 A_t}{\lambda^2 R^2} = \frac{4\pi A_t}{\lambda^2}, \tag{12}$$

which can be solved to obtain the standard relationship expressing the antenna effective area in terms of transmitter directivity

$$A_t = \left(\frac{\lambda^2}{4\pi}\right) D_t. \tag{13}$$

In this development students see directly that the wavelength-squared term expresses the diffraction-induced beam spreading of an electromagnetic wave launched from a finite aperture. In fact, Eqs. (6) and (13) together reproduce the traditional antenna result

$$A\Omega = \lambda^2, \tag{14}$$

which says that the beam throughput is equal to the wavelength squared.[10,11,19] In both the antenna and radiometric discussions of the Friis equation, this expression plays a central role because it demonstrates that as the wavelength increases, either the antenna area must increase even faster (as the square of the wavelength) or the beam will spread into a proportionally larger solid angle. We arrive at this result using only simple geometric radiometry and a basic result of scalar diffraction theory, without employing radiation resistances, equivalent circuits, power-transfer relations, or anything unique to an infinitesimal dipole. The derivation using those quantities can be useful, but the radiometric method is an alternative that enhances understanding of the physics behind the equation and makes the material more accessible to a wider range of students.

The Friis equation derivation continues with the recognition that, in radiometric terminology, the transmit antenna emits a radiance given by the ratio of transmitted power to the transmitter throughput

$$L_t = \frac{P_t}{A_t \Omega_b} \cong \frac{P_t R^2}{A_t A_b}. \tag{15}$$

Using Eq. (11) for the beam area A_b, we find that

$$L_t = \frac{P_t R^2}{A_t} \frac{A_t}{R^2 \lambda^2} = \frac{P_t}{\lambda^2}. \tag{16}$$

Here we see that the transmitted radiance can be written simply as the ratio of transmitted power to the wavelength squared, again a result of the throughput equality expressed in Eq. (14). Equation (16) is another relatively common antenna concept that would be useful in discussions of diffraction-limited optical systems.

Radiance propagates unchanged in a lossless medium, so all that is required to find the received power is to multiply the radiance by the appropriate receiver throughput. This is where students sometimes make the mistake of multiplying by the effective receiver area (correct) and the receive-antenna beam solid angle (incorrect). This mistake is equivalent to multiplying Eq. (16) by λ^2, which would result in 100% of the transmitted power being collected by the receiver (although this is true in the special case where the receiver beam exactly fills the area of the transmitter). As indicated in Fig. 1, the correct solid angle to use in the received-power calculation is the solid angle subtended by the transmit aperture at the receiver A_t/R^2. Multiplying this solid angle by the effective area of the receiver aperture gives the correct throughput, which is not equal to λ^2. Equation (16) is then transformed into a variation of Eq. (2)

$$P_r = L_t A_r \Omega_{tr} = \frac{P_t}{\lambda^2} \frac{A_r A_t}{R^2}. \tag{17}$$

The final step to obtain the Friis equation in the form of Eq. (1) is to use Eq. (13) to convert the effective receiver and transmitter areas into quantities involving directivities

$$\frac{P_r}{P_t} = \frac{1}{\lambda^2} \frac{\lambda^2 D_t}{4\pi} \frac{\lambda^2 D_r}{4\pi} \frac{1}{R^2} = \left(\frac{\lambda}{4\pi R}\right)^2 D_r D_t. \tag{18}$$

For non-ideal antennas, we can include efficiency factors and rewrite the resulting Friis transmission equation in terms of both directivity and gain

$$\frac{P_r}{P_t} = e_r e_t \left(\frac{\lambda}{4\pi R}\right)^2 D_r D_t = \left(\frac{\lambda}{4\pi R}\right)^2 G_r G_t. \tag{19}$$

It is useful to note that whenever irradiance is encountered in a class that also discusses electric fields, it is important to use notation that avoids confusion between the two. Otherwise there is minimal difficulty getting students to apply antenna and optical radiometry concepts together.

IV. DISCUSSION AND CONCLUSION

Students are likely to be confused with the two forms of the Friis equation given by Eqs. (1) and (2) because they appear to depend on wavelength in entirely different ways. In Eq. (2), the received-to-transmitted power ratio is inversely proportional to the wavelength squared, whereas in Eq. (1), the ratio appears to be directly proportional to the wavelength squared. The reason for this difference is that the directivities (or gains) in Eq. (1) depend inversely on the wavelength-squared. Thus, Eq. (1) has a factor of λ^4 effectively hidden from view and naively gives the wrong wavelength dependence. Conversely, Eq. (2) correctly predicts the appropriate diffraction behavior of lower received power at longer wavelengths when the antenna areas are unchanged.

In two offerings of a first-year graduate course on antennas and a graduate course on remote sensing systems, with both electrical engineering and physics students enrolled, the Friis transmission equation was presented with both the dipole effective area and radiometric derivations. Before the radiometric derivation was presented, many of the students could recognize and some could use the equation, but all were entirely unable to describe the physical meaning of the wavelength dependence of the Friis equation. In questioning immediately following the presentation of the dipole derivation, none of the students accurately identified the wavelength dependence as arising from diffraction. However, immediately following the radiometric discussion described here, 100% of the students identified the source of the wavelength-squared dependence as diffraction and could explain, in basic terms at least, how the wavelength entered in and flowed through the derivation. Additionally, all of the students selected the radiometric presentation as the one they felt was most physically intuitive.

The Friis transmission equation is a sufficiently simple and practical tool, and students generally enjoy using it as the basis of discussion of radiometric concepts. Students who have taken both antenna and optics classes have a particularly high level of appreciation for seeing the two fields brought together into a common discussion. There are several antenna concepts (e.g., directivity, throughput $= \lambda^2$, etc.) that enhance optical radiometry, just as the radiometric approach allows an alternate development of the wavelength-squared dependence that helps students better understand its diffraction roots.

ACKNOWLEDGMENT

The author gratefully acknowledges discussions with Dr. Richard Wolff and students at Montana State University. This material is based in part on research sponsored by the Air Force Research Laboratory, under Agreement No. FA9550-10-1-0115. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Research Laboratory or the U.S. Government.

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