Finance 30210
Solutions to Problem Set #8: Introduction to Game Theory

1) Consider the following version of the prisoners dilemma game (Player one's payoffs are in bold):

		Player Two	
		Cooperate	Cheat
Player One	Cooperate	**$10** $10	**$0** $12
	Cheat	**$12** $0	**$5** $5

a) What is each player's dominant strategy? Explain the Nash equilibrium of the game.

Start with player one:

- *If player two chooses cooperate, player one should choose cheat ($12 versus $10)*
- *If player two chooses cheat, player one should also cheat ($0 versus $5).*

Therefore, the optimal strategy is to always cheat (for both players) this means that (cheat, cheat) is the only Nash equilibrium.

b) Suppose that this game were played three times in a row. Is it possible for the cooperative equilibrium to occur? Explain.

If this game is played multiple times, then we start at the end (the third playing of the game). At the last stage, this is like a one shot game (there is no future). Therefore, on the last day, the dominant strategy is for both players to cheat. However, if both parties know that the other will cheat on day three, then there is no reason to cooperate on day 2. However, if both cheat on day two, then there is no incentive to cooperate on day one.

2) Consider the familiar "Rock, Paper, Scissors" game. Two players indicate either "Rock", "Paper", or "Scissors" simultaneously. The winner is determined by

- Rock crushes scissors
- Paper covers rock
- Scissors cut paper

Indicate a -1 if you lose and +1 if you win. Write down the strategic (matrix) form of the game. What is the Nash equilibrium of the game?

Here's the strategic form of the game (a description of the payouts from each combination of moves) – Player One's payouts are in bold.

		Player Two		
		Rock	***Paper***	***Scissors***
Player One	***Rock***	***0*** *0*	***-1*** *1*	***1*** *-1*
	Paper	***1*** *-1*	***0*** *0*	***-1*** *1*
	Scissors	***-1*** *1*	***1*** *-1*	***0*** *0*

Note that neither player has a dominant strategy.

- *If Player one chooses rock, Player two should play paper*
- *If Player one chooses paper, Player two responds with scissors*
- *If Player one chooses scissors, Player two chooses rock*

Further, this game is symmetric, so Player two's optimal responses are the same. Both players randomly select rock, paper, or scissors

In an episode of Seinfeld, Kramer played a version of this game with his friend Mickey except that the rules were a little different:

- Rock crushes scissors
- Rock Flies Right through paper
- Scissors cut paper

How does this modification alter the Nash equilibrium of the game?
Here's the strategic form of the game (a description of the payouts from each combination of moves) – Player One's payouts are in bold.

		Player Two		
		Rock	**Paper**	**Scissors**
Player One	**Rock**	**0** 0	**1** -1	**1** -1
	Paper	**-1** 1	**0** 0	**-1** 1
	Scissors	**-1** 1	**1** -1	**0** 0

Note that both players have a dominant strategy.

- *If Player one chooses rock, Player two should choose rock*
- *If Player one chooses paper, Player two responds with rock (or paper)*

- *If Player one chooses scissors, Player two responds with rock*

Notice that playing rock is a dominant strategy for both players (i.e. its best to choose rock, regardless of what your opponent is playing!

Therefore, the equilibrium for this game is unique:

Both players always select rock.

This was confirmed in Seinfeld.

3) Consider the following game (Player One's Payouts in bolds):

Player 1		Player 2		
		Left	Middle	Right
	Up	**1**, 2	**3**, 5	**2**, 1
	Middle	**0**, 4	**2**, 1	**3**, 0
	Down	**-1**, 1	**4**, 3	**0**, 2

a) Does either player have a dominant strategy? Explain.

Player 1*:*

- *If Player 2 plays Left, Play up*
- *If Player 2 plays Middle, Play down*
- *If Player 2 plays Right, play middle*

Player 2*:*

- *If Player 2 plays Up, Play middle*
- *If Player 2 plays Middle, Play left*
- *If Player 2 plays Down, play middle*

Neither player has a dominant strategy

b) Does either player have a dominated strategy? Explain.

Yes, player 2's dominated strategy is playing right (he will never play right)

c) Solve the equilibrium for this game.

Once we eliminate right as a strategy for player 2,

Player 1	Player 2		
		Left	Middle
	Up	**1**, 2	**3**, 5
	Middle	**0**, 4	**2**, 1
	Down	**-1**, 1	**4**, 3

Now, player 1 has a dominated strategy. Player one will never play middle. So, let's delete that

Player 1	Player 2		
		Left	Middle
	Up	**1**, 2	**3**, 5
	Down	**-1**, 1	**4**, 3

Now, player 2 has a dominated strategy…left. Let's eliminate that.

Player 1	Player 2		
			Middle
	Up		**3**, 5
	Down		**4**, 3

So, player 2 chooses middle and player one chooses down.

4) Consider the game of chicken. Two players drive their cars down the center of the road directly at each other. Each player chooses SWERVE or STAY. Staying wins you the admiration of your peers (a big payoff) only if the other player swerves. Swerving loses face if the other player stays. However, clearly, the worst output is for both players to stay! Specifically, consider the following payouts.

(Player one's payoffs are in bold):

		Player Two	
		Stay	Swerve
Player One	Stay	**-6** -6	**2** -2
	Swerve	**-2** 2	**1** 1

a) Does either player have a dominant strategy? Explain.

In this case, neither player has a dominant strategy. Suppose player two chooses to stay. Then player one's best response is to swerve (-6 vs. -2). However, if player two swerves, then player one should stay (2 vs. 1).

b) Suppose that Player B has adopted the strategy of Staying 1/5 of the time and swerving 4/5 of the time. Show that Player A is indifferent between swerving and staying.

We need to show that if player B follows the strategy (stay = ¼, swerve = 5/4) then player A is indifferent between swerving and staying. If we calculate the expected reward to player A from staying/swerving, we get

E(stay) = (1/5)(-6) + (4/5)(2) = 2/5

E(swerve) = (1/5)(-2) + (4/5)(1) = 2/5

They are in fact equal.

c) If both player A and Player B use this probability mix, what is the chance that they crash?

Both players are staying 1/5 of the time. Therefore, the probability that the crash (stay, stay) is (1/5)(1/5) = 1/25 = 4%.

5) Consider the following game. Two criminals are thinking about pulling off a bank robbery. The take from the bank would be $20,000 each , but the job requires two people (one to rob the bank and one to drive the getaway car. Each criminal could instead rob a liquor store. The take from robing a liquor store is only $1000 but can be done with one person acting alone.

a) Write down the payoff matrix for this game.

		Player Two	
		Bank Job	Liquor Store
Player One	Bank Job	**20,000** 20,000	**0** 1,000
	Liquor Store	**1,000** 0	1,000 1,000

b) What are the strategies for this game?

Player 1: If player two does the bank job, do the bank job
Player 2: if player one does the liquor store, do the liquor store

c) What are the equilibria for this game?

There are two pure strategy equilibria here (bank job, bank job) and (liquor store, liquor store). Once in these equilibria, neither side has an incentive to change.

There is also a mixed strategy equilibria.

Let P_B, P_L *be the probabilities that player B chooses the bank job or liquor store. For player one, the expected return from the bank job and liquor store are as follows;*

$$EV_B = P_B\left(20,000\right) + P_L\left(0\right)$$
$$EV_L = 1,000$$

For Player A to be indifferent between the two, the expected values must be equal.

$$P_B\left(20,000\right) + P_L\left(0\right) = 1,000$$
$$P_B = \left(\frac{1,000}{20,000}\right) = .05$$
$$P_L = .95$$

The game is symmetric for player B, so both choose to rob the bank 5% of the time and rob the liquor store 95% of the time (assuming that they are risk neutral).

6) Consider the following bargaining problem: $20 dollars needs to be split between Jack and Jill. Jill gets to make an initial offer. Jack then gets to respond by either accepting Jill's initial offer or offering a counter offer. Finally, Jill can respond by either accepting Jakes offer or making a final offer. If Jake does not accept Jill's final offer both Jack and Jill get nothing. Jack discounts the future at 10% (i.e. future earnings are with 10% less than current earnings while Jill discounts the future at 20%. Calculate the Nash equilibrium of this bargaining problem.

The key to each of these games is as follows: At any stage, the offer made needs to be acceptable to both parties. We need to work backwards:

Stage 3*: Note that if Jack rejects Jill's offer at this stage, the money disappears. Therefore, Jack will accept anything positive.*

Jill offers: $20 to herself, $0 to Jack

Stage 2*: Now, Jack must make an offer that Jill will accept (if the game gets to stage three, Jack gets nothing). Jill is indifferent between $20 in one year and $16 today (she discounts the future at 20%).*

Jack offers: $16 to Jill, $4 to Himself

Stage 1*: Now, Jill must make an offer that Jack will accept (and is preferable to her – if this is not possible, then she will make an offer jack rejects and the game goes to stage 2). Jack is indifferent between $4 in one year and $3.60 today (he discounts the future at 10%). Note that $16.40 is preferred by Jill to $16 in one year.*

Jill offers: $16.40 to herself, $3.60 to Jack

7) Consider a variation on the previous problem:

You and your sister have just inherited $3M that needs to be split between the two of you.

The rules are the same as above (offer, counteroffer, and final offer) except that each period, $1M is removed from the total (each round of negotiation costs $1M in lawyers fees). Further, assume that both you and your sister value future payments just as much as current payments (i.e. no discount factor). Calculate the Nash equilibrium for this game.

Stage 3*: Note that if your sister rejects your offer at this stage, the money disappears. Therefore, your sister will accept anything positive.*

You offer: $1M to you, $0 to your sister

Stage 2*: Now, your sister must make an offer that you will accept (if the game gets to stage three, she gets nothing). If it gets to stage three, you get $1M.*

Your sister offers: $1M to you, $1M to her

Stage 1*: Now, you must make an offer that your sister will accept (and is preferable to you – if this is not possible, then you will make an offer she rejects and the game goes to stage 2). Your sister gets $1M if the game reaches stage two.*

You offer: $2M to you, $1M to your sister

8) Consider yet, another variation of the previous problem: Same rules as in (4), However, this time, you learn something about your sister: You discover that your sister has always hated you. All she cares about with regards to splitting the $3M is that she gets more than you do (i.e. an allocation of $500,000 for you and $1M for her is preferred by her to an allocation of $1.5M apiece!). Calculate the new Nash equilibrium of the game.

Stage 3*: Note that now, your sister's happiness is based on relative earnings (earnings relative to you). You must come up with an offer she will accept or you both get nothing.*

You offer: $500K to you, $500K to your sister

Stage 2*: Now, your sister must make an offer that you will accept (if the game gets to stage three, she gets $500K). If it gets to stage three, you get $500K.*

Your sister offers: $500K to you, $1.5M to her (three to one ratio)

Stage 1*: Now, you must make an offer that your sister will accept (and is preferable to you – if this is not possible, then you will make an offer she rejects and the game goes to stage 2). Your sister gets $1.5M if the game reaches stage two.*

You offer: $750,000 to you, $2.25M to your sister (three to one ratio)