Probability

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Outline

Probability

2.1 Sample Space

2.2 Events

2.3 Counting Sample Points

2.4 Probability of an Event

2.5 Additive Rules

2.6 Conditional Probability, Independence, and the Product Rule

2.7 Bayes' Rule

SAMPLE SPACE

Definition 2.1:

The set of all possible outcomes of a statistical experiment is called the **sample space and is represented by the symbol *S*.**

Each outcome in a sample space is called an **element or a member of the** sample space, or simply a **sample point.**

SAMPLE SPACE

- Thus, the sample space S, *of possible outcomes when a coin is flipped, may be* written

$$S = \{H, T\}$$

where H *and* T *correspond to heads and tails, respectively.*

Example 2.1

- Consider the experiment of tossing a die. If we are interested in the number that shows on the top face, the sample space is

$$S1 = \{1, 2, 3, 4, 5, 6\}.$$

- If we are interested only in whether the number is even or odd, the sample space is simply

$$S2 = \{even, odd\}.$$

Example 2.2

- An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once.

- To list the elements of the sample space providing the most information, we construct the tree diagram of Figure 2.1.

Example 2.2

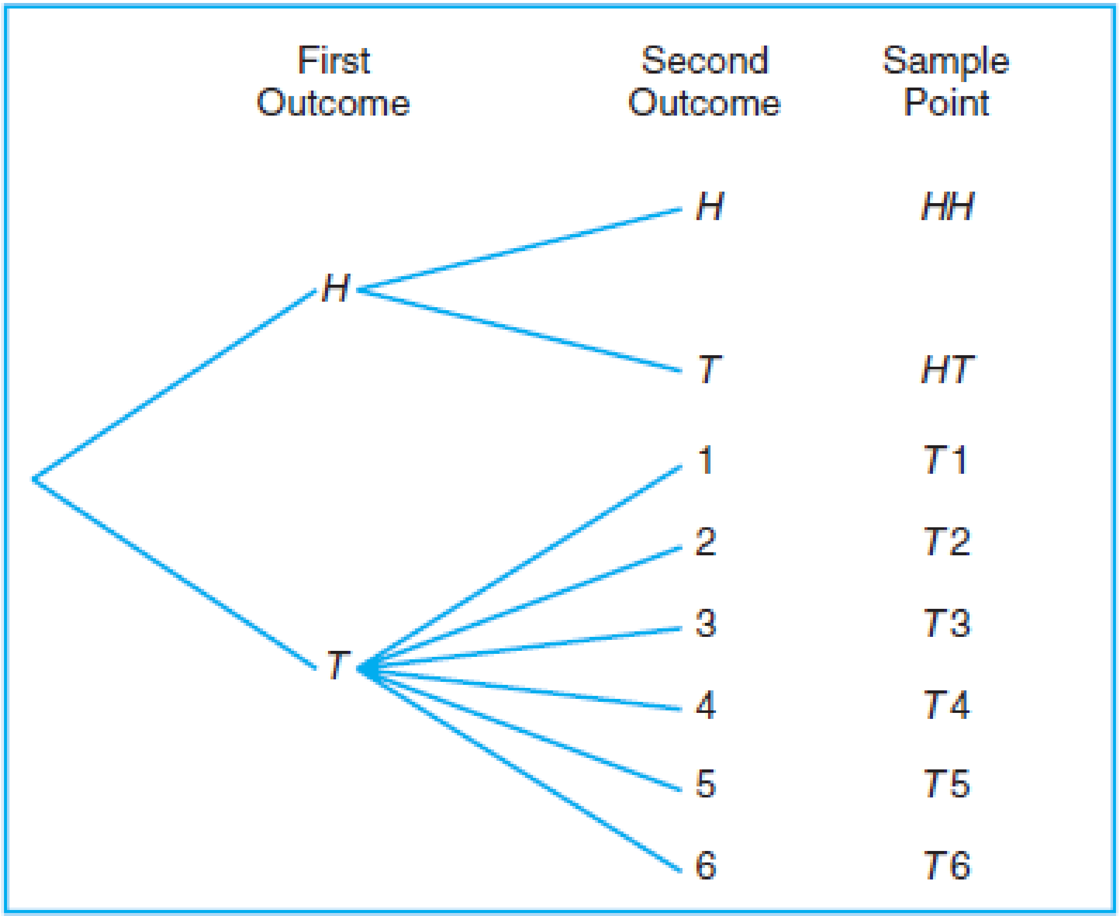


Figure 2.1: Tree diagram for Example 2.2.

Example 2.2

- By proceeding along all paths, we see that the sample space is

$$S = \{HH, HT, T1, T2, T3, T4, T5, T6\}.$$

Example 2.2

Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified defective, D, or non defective, N.

To list the elements of the sample space providing the most information, we construct the tree diagram of Figure 2.2.

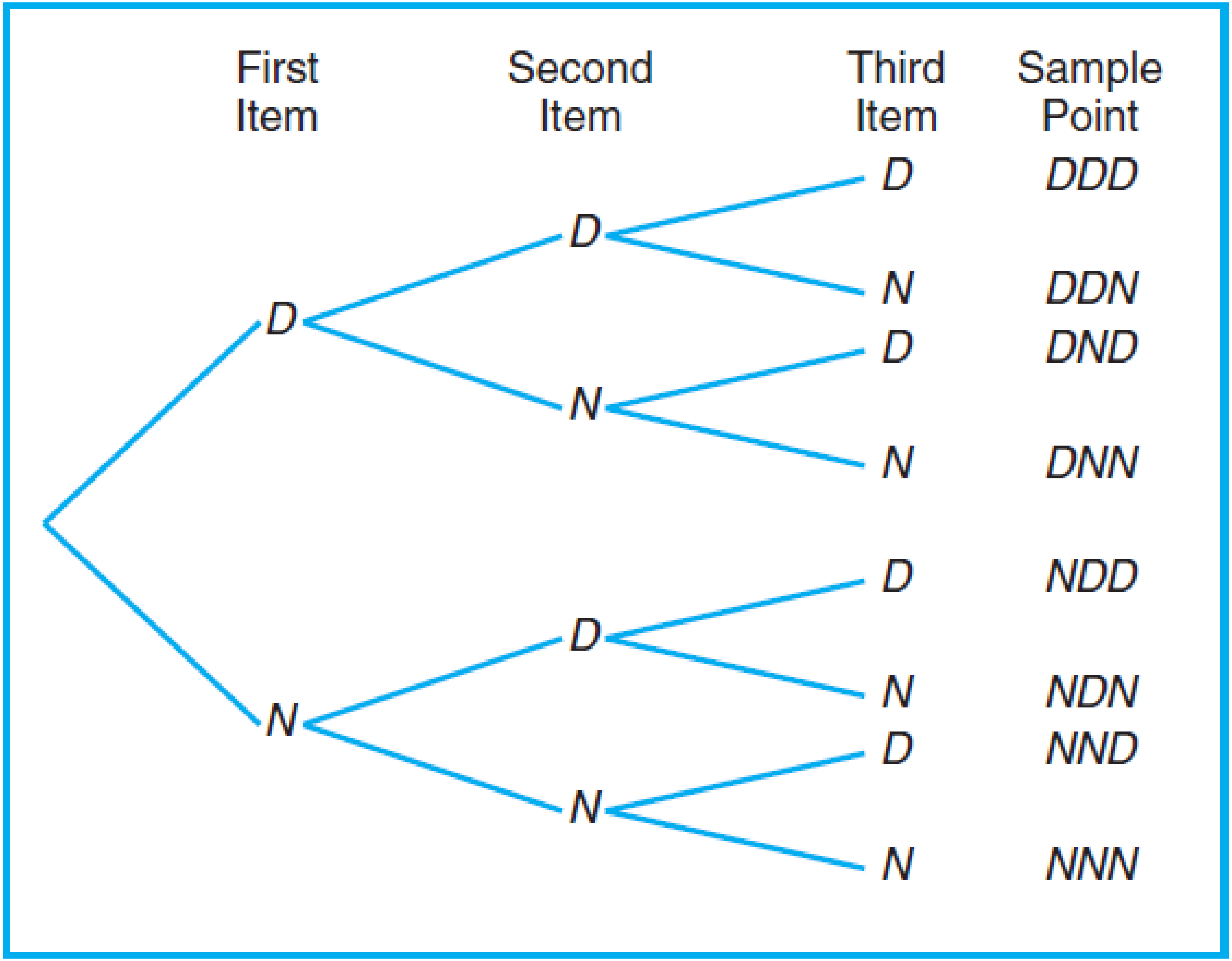


Figure 2.2: Tree diagram for Example 2.3.

Example 2.2

we see that the sample space is

$$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$$

Example 2.3

- Sample spaces with a large or infinite number of sample points are best described by a **statement or rule method. For example, if the possible outcomes** of an experiment are the set of cities in the world with a population over 1 million, our sample space is written

S = {x | x is a city with a population over 1 million},

which reads "S is the set of all x such that x is a city with a population over 1 million."

Definition 2.2

An **event** is a subset of a sample space.

Definition 2.2

Example 2.4: Given the sample space S = *{t | t ≥ 0}, where t is the life in years of a certain* electronic component,

then the event *A that the component fails before the end of* the fifth year is the subset $A = \{t \mid 0 \leq t < 5\}$.

Definition 2.2

It is conceivable that an event may be a subset that includes the entire sample space S *or a subset of S called the* ***null set and denoted by the symbol φ, which*** contains no elements at all. For instance, if we let *A be the event of detecting a* microscopic organism by the naked eye in a biological experiment, then $A = \varphi$.

Also, if $B = \{x \mid x \text{ is an even factor of } 7\}$, then *B must be the null set, since the only possible factors of 7 are the odd numbers* 1 and 7.

Definition 2.2

- Consider an experiment where the smoking habits of the employees of a manufacturing firm are recorded.

- A possible sample space might classify an individual as a nonsmoker, a light smoker, a moderate smoker, or a heavy smoker. Let the subset of smokers be some event. Then all the nonsmokers correspond to a different event, also a subset of *S, which is called the* ***complement of the set of smokers.***

Definition 2.3

- The **complement of an event *A* with respect to S is the subset of all elements** of S *that are not in A. We denote the complement of A by the symbol A′.*

Definition 2.3

- **Example 2.5: Let *R be the event that a red card is selected from an ordinary deck of 52 playing*** cards, and let *S be the entire deck.*

- *Then R′ is the event that the card selected* from the deck is not a red card but a black card.

Definition 2.3

- **Example 2.6: Consider the sample space**
- $S = \{book, cell\ phone, mp3, paper, stationery, laptop\}$.
- Let $A = \{book, stationery, laptop, paper\}$. *Then the complement of* A *is* $A' = \{cell\ phone, mp3\}$.

Definition 2.4

- **Definition 2.4: The intersection of two events *A and B, denoted by the symbol A* ∩ *B, is the*** event containing all elements that are common to *A and B.*

Definition 2.4

- **Example 2.7: Let *E be the event that a person selected at random in a classroom is majoring in*** engineering, and let *F be the event that the person is female.*
- *Then $E \cap F$ is the* event of all female engineering students in the classroom.

Definition 2.4

- **Example 2.8: Let *V = {a, e, i, o, u}* and *C = {l, r, s, t}*; *then it follows that V* ∩ *C = φ.***

- ***That is,*** *V and C have no elements in common and, therefore, cannot both simultaneously* occur.

Definition 2.5

- **Definition 2.5: Two events *A* *and B are mutually exclusive, or disjoint, if* $A \cap B = \varphi$*, that*** is, if *A and B have no elements in common.*

Definition 2.5

Example 2.9: A cable television company offers programs on eight different channels, three of which are affiliated with ABC, two with NBC, and one with CBS. The other two are an educational channel and the ESPN sports channel. Suppose that a person subscribing to this service turns on a television set without first selecting the channel.

Let:

- *A be the event that the program belongs to the NBC network and*
- *B the event that it belongs to the CBS network.*

Since a television program cannot belong to more than one network, the events *A and B have no programs in common.* Therefore, the intersection *$A \cap B$ contains no programs, and consequently the* events *A and B are mutually exclusive.* Often one is interested in the occurrence of at least one of two events associated with an experiment.

Definition 2.5

Thus, in the die-tossing experiment:

if $A = \{2, 4, 6\}$ *and* $B = \{4, 5, 6\}$, we might be interested in either *A or B occurring or both A and B occurring.*

- *Such* an event, called the **union of *A and B, will occur if the outcome is an element of*** the subset $\{2, 4, 5, 6\}$.

Definition 2.6

- **Definition 2.6: The union of the two events *A and B, denoted by the symbol A∪B, is the event*** containing all the elements that belong to *A or B or both.*

Definition 2.6

- **Example 2.10:**
- **Let *A* = *{a, b, c}* and *B* = *{b, c, d, e}*;**
- ***then A* $\cup$ *B* =**

Definition 2.6

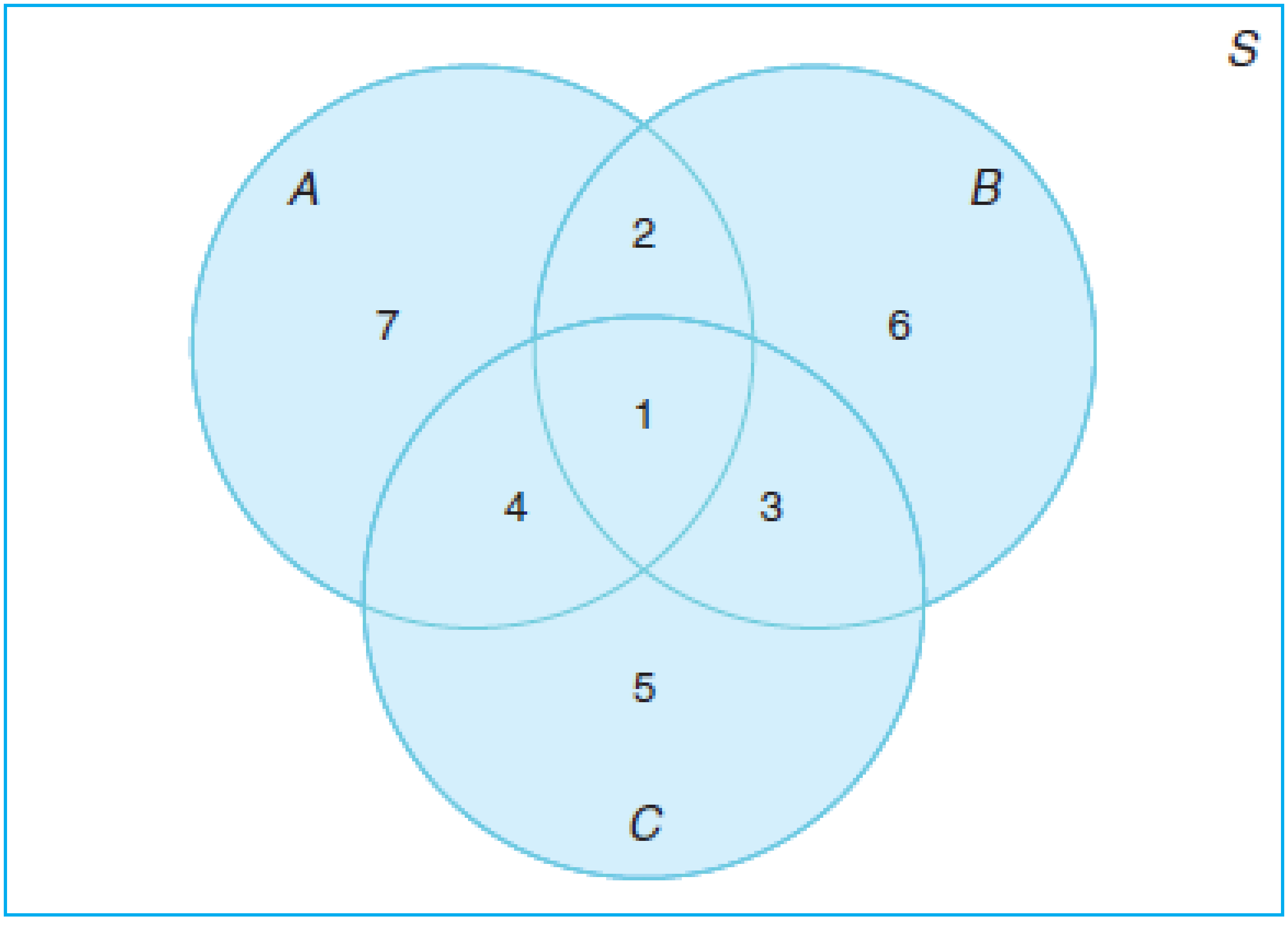
- **Example 2.12:**
- **If** $\boldsymbol{M = \{x \mid 3 < x < 9\}}$ ***and*** $\boldsymbol{N = \{y \mid 5 < y < 12\}}$**,**
- ***then*** $M \cup N =$

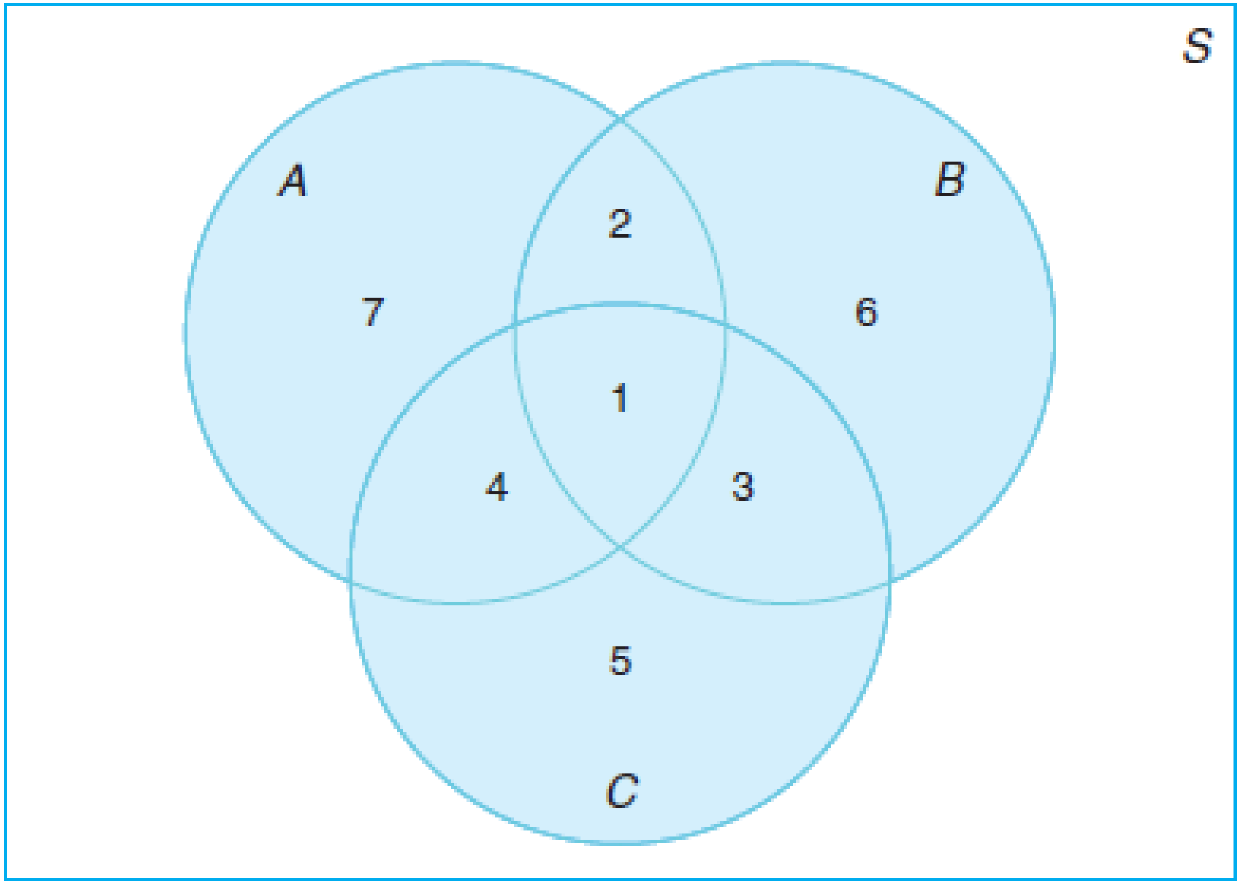
Definition 2.6

- The relationship between events and the corresponding sample space can be illustrated graphically by means of **Venn diagrams.**

- **In a Venn diagram we let** the sample space be a rectangle and represent events by circles drawn inside the rectangle.

Definition 2.6

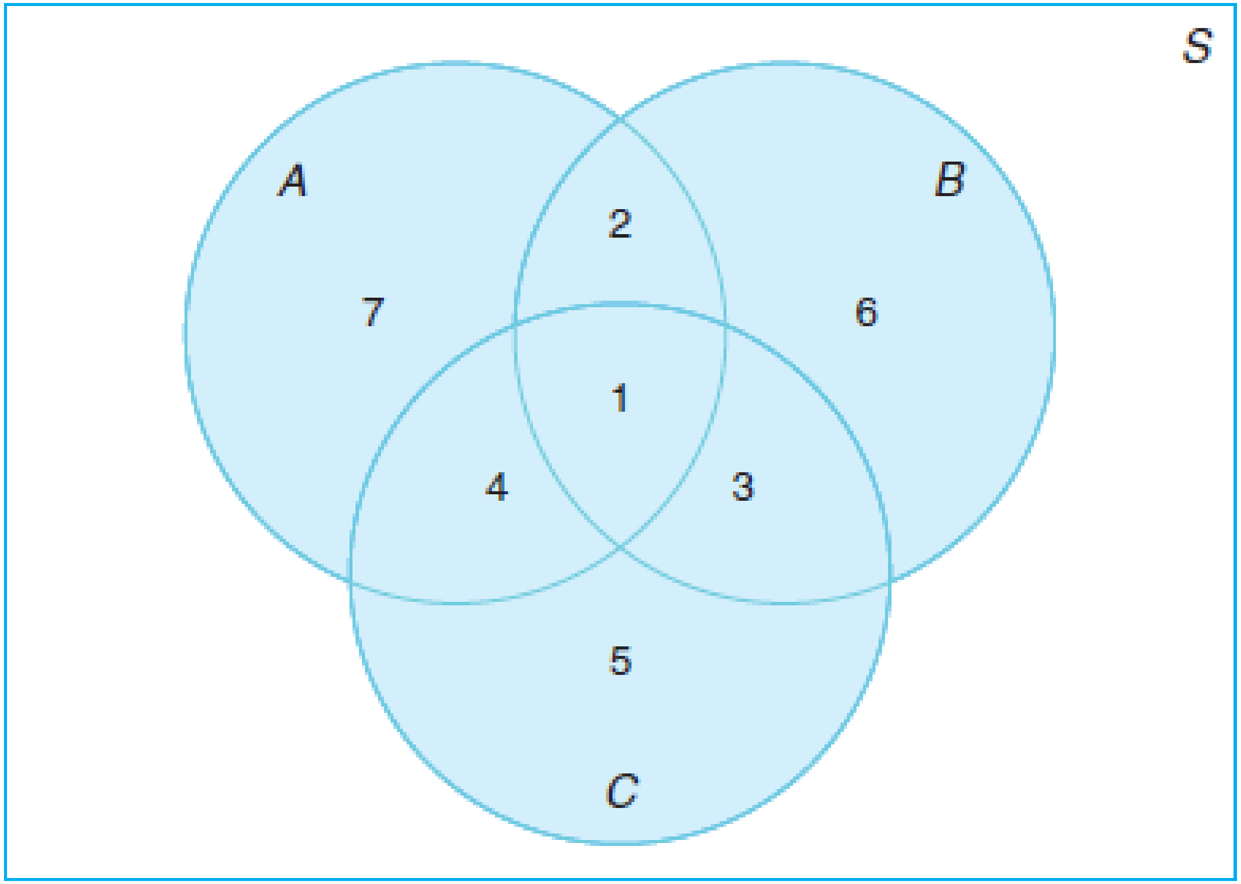




Thus, in Figure 2.3, we see that:

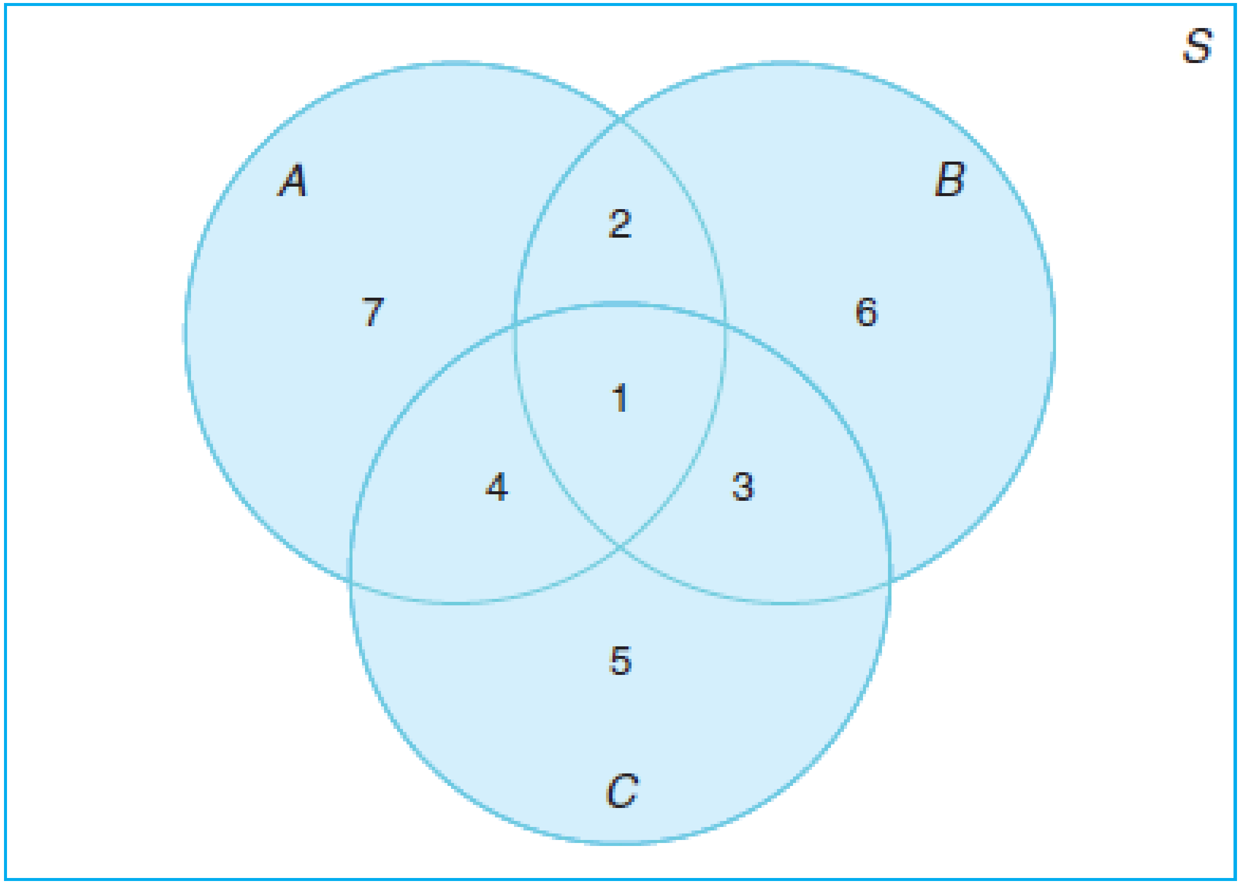
$A \cap B$ = *regions 1 and 2,*

$B \cap C$ = *regions 1 and 3,*



Thus, in Figure 2.3, we see that:

- $A \cup C$ = *regions 1, 2, 3, 4, 5, and 7,*
- $B' \cap A$ = *regions 4 and 7,*



Thus, in Figure 2.3, we see that:

- $A \cap B \cap C$ = *region 1,*
- $(A \cup B) \cap C'$ = regions 2, 6, and 7,

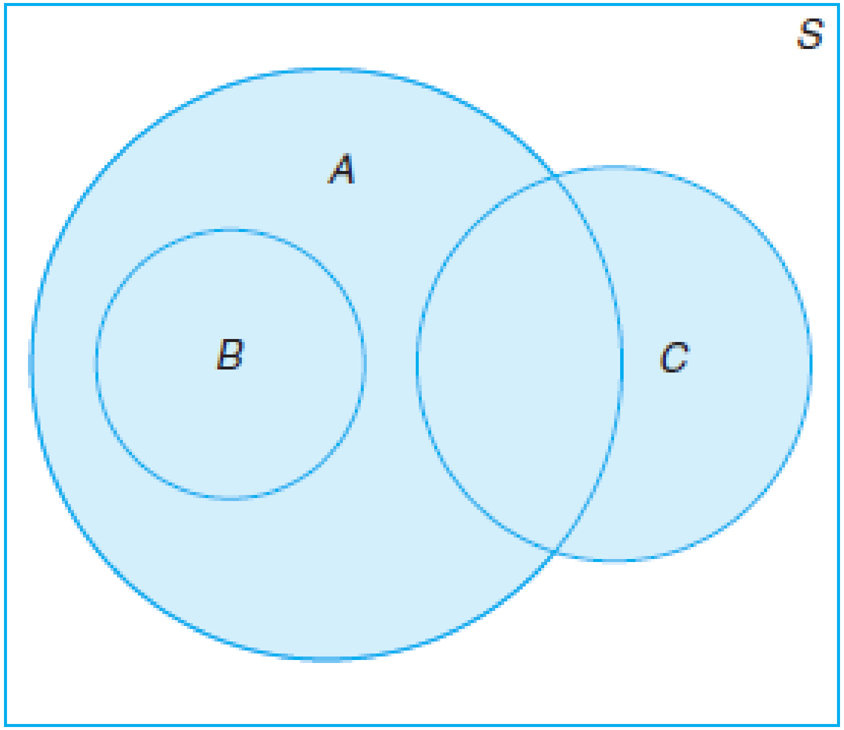


Figure 2.4: Events of the sample space S.

- In Figure 2.4, we see that events *A, B, and C are all subsets of the sample* space *S.*
- *It is also clear that event B is a subset of event A; event* $B \cap C$ *has no* elements and hence *B and C are mutually exclusive;*
- *Event* $A \cap C$ *has at least one* element; and
- Event $A \cup B = A$.

Definition 2.6

- Figure 2.4 might, therefore, depict a situation where we select a card at random from an ordinary deck of 52 playing cards and observe whether the following events occur:

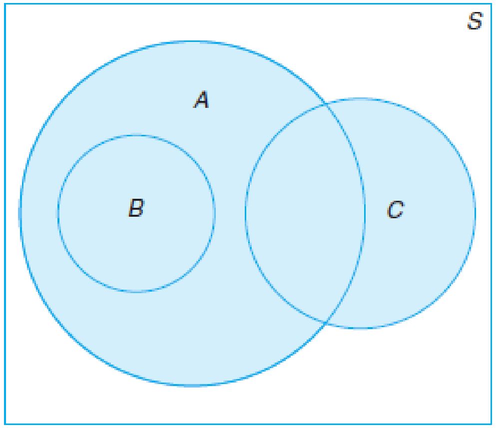


Figure 2.4: Events of the sample space S.

A: the card is red,
B: the card is the jack, queen, or king of diamonds,
C: the card is an ace.

Clearly, the event $A \cap C$ *consists of only the two red aces.*

ASSIGNMENT

2.1 List the elements of each of the following sample spaces:

(a) the set of integers between 1 and 50 divisible by 8;

(b) the set $S = \{x \mid x2 + 4x - 5 = 0\}$;

(c) the set of outcomes when a coin is tossed until a tail or three heads appear;

(d) the set $S = \{x \mid x \text{ is a continent}\}$;

(e) the set $S = \{x \mid 2x - 4 \geq 0 \text{ and } x < 1\}$.

ASSIGNMENT

2.3 Which of the following events are equal?

(a) $A = \{1, 3\}$;

(b) $B = \{x \mid x \text{ is a number on a die}\}$;

(c) $C = \{x \mid x2 - 4x+3 = 0\}$;

(d) $D = \{x \mid x$ *is the number of heads when six coins* are tossed$\}$.

2.3 Counting Sample Points RULE 2.1

- **Rule 2.1: If an operation can be performed in n_1 *ways, and if for each of these ways a second*** operation can be performed in n_2 *ways, then the two operations can be performed* together in $n_1 n_2$ *ways.*

2.3 Counting Sample Points

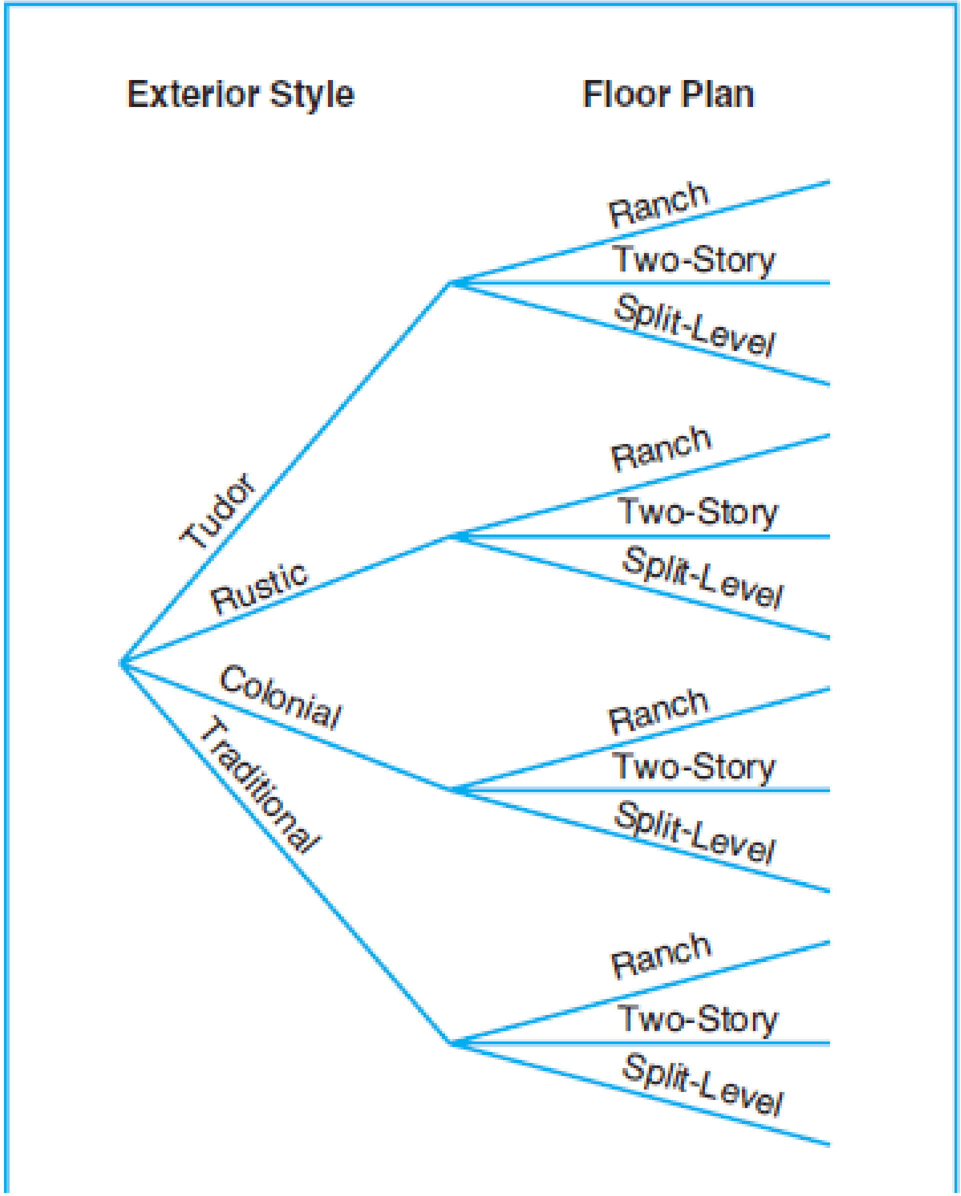
- **Example 2.13: How many sample points are there in the sample space when a pair of dice is** thrown once?

Solution : The first die can land face-up in any one of n1 = 6 ways. For each of these 6 ways, the second die can also land face-up in *n2 = 6 ways. Therefore, the pair of dice* can land in *n1n2 = (6)(6) = 36 possible ways.*

2.3 Counting Sample Points

- **Example 2.14: A developer of a new subdivision offers prospective home buyers** a choice of Tudor, rustic, colonial, and traditional exterior styling in *ranch, two-story, and split-level floor plans*.

- In how many different ways can a buyer order one of these homes?



Solution : ***Since n1 = 4 and n2 = 3, a buyer must choose from*** *n1n2 = (4)(3) = 12 possible homes.*

2.3 Counting Sample Points RULE 2.2

Rule 2.2: If an operation can be performed in *n1 ways, and if for each of these a second* operation can be performed in *n2 ways, and for each of the first two a third* operation can be performed in *n3 ways, and so forth, then the sequence of k* operations can be performed in *n1n2 · · · nk ways.*

2.3 Counting Sample Points RULE 2.2

Example 2.16: Sam is going to assemble a computer by himself. He has the choice of:

• chips from two (2) brands,
• a hard drive from four (4),
• memory from three (3), and
• an accessory bundle from five (5) local stores.

How many different ways can Sam order the parts?

Solution : Since n1 = 2, n2 = 4, n3 = 3, and n4 = 5, there are *nl × n2 × n3 × n4 = 2× 4 × 3 × 5 = 120* different ways to order the parts.

2.3 Counting Sample Points

- **Definition 2.7: A permutation is an arrangement of all or part of a set of objects.**

2.3 Counting Sample Points PERMUTATION

- Consider the three letters *a, b, and c.*
- *The possible permutations are abc, acb, bac, bca, cab, and cba.*
- Thus, we see that there are 6 distinct arrangements.
- No matter which two letters are chosen for the first two positions, there is only $n_3 = 1$ *choice for the last position, giving a total of* $n_1 n_2 n_3 = (3)(2)(1) = 6$ *permutations*

2.3 Counting Sample Points PERMUTATION

- **Definition 2.8: For any non-negative integer *n, n!, called "n factorial," is defined as:***

$$n! = n(n-1)\cdots(2)(1),$$

with special case $0! = 1$.

Theorem 2.1: The number of permutations of *n objects is n!.*

2.3 Counting Sample Points PERMUTATION

- The number of permutations of the four letters *a, b, c, and d will be 4! = 24.*
- Now consider the number of permutations that are possible by taking two letters at a time from four. These would be *ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, and dc.*
- *Using Rule 2.1 again, we have two positions to fill, with $n1 = 4$ choices for the* first and then *$n2 = 3$ choices for the second, for a total of:*

$$n1n2 = (4)(3) = 12 \text{ permutations}$$

2.3 Counting Sample Points PERMUTATION

- In general, *n distinct objects taken r at a time can be arranged in:*
  $n(n-1)(n-2)\cdots(n-r+1)$ ways.

- We represent this product by the symbol:

$$ {}_nP_r = \frac{n!}{(n-r)!}. $$

2.3 Counting Sample Points PERMUTATION

Example 2.18: In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

Solution : Since the awards are distinguishable, it is a permutation problem.
The total number of sample points is:

$$_{25}P_3 = \frac{25!}{(25-3)!}$$

PROBABILITY

This is an alternate lecture…

1. Introduction

- Probability theory is devoted to the study of uncertainty and variability
- Tasks:
  - 1. Basic of probability: rules, terminology, the basic calculus of probability
  - 2. Random variables, expectations, variances
  - 3. Simulation

Main Concepts

1) Probability in this course represents the relative frequency of outcomes after a great many (infinity many) repetitions.

2) We study the probability because it is a tool that let us make an inference from a sample to a population

3) Probability is used to understand what patterns in nature are “real” and which are due to chance

4) Independent is the fundamental concept of probability & statistics

5) Conditional probability is also fundamental importance in part because it help us understand independence

Sample Space

- Probability: quantify the variability in the outcome of any experiment whose exact outcome cannot be predicted with certainty.
- The Space of outcome!!
- Sample space: a set of all possible outcomes of an experiment. Usually denoted by S.

Example

- Throw a coin
  - S = {H, T}
- Throw a coin twice
  - S={HH, HT, TH, TT}
- 7 race horses {1, 2, 3, 4, 5, 6,7}
  - S={all 7! Permutation of {1,...,7}

Sample Space

- Finite sample space: finite number of elements in the space S.
- Countable infinite sample space: ex. natural numbers.
- Discrete sample space: if it has finite many or countable infinity of elements.
- Continuous sample space: If the elements constitute a continuum. Ex. All the points in a line.

Event

- Event: subset of a sample space. In words, an **event** A is a set (or group) of possible outcomes of an uncertain process e.g. {Heads in a single coin toss}, {rain}.

- Example: A government agency must decide where to locate two new computer research facilities (Vancouver , Toronto).

- C={(1,0), (0,1)} is the event that between them, Vancouver and Toronto will get one.

- S={(0,0), (0,1), (0,2),(1,0),(1,1),(2,0)}

Mutually exclusive events

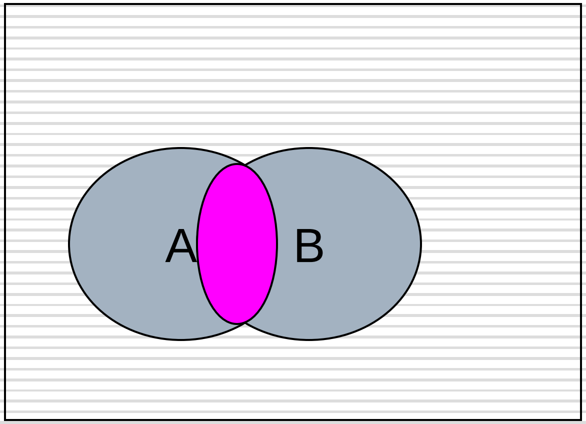
- Mutually exclusive: Two events have no elements in common.
- Ex. C={(1,0), (0,1)}, D={(0,0),(0,1), (0,2)}, E={(0,0), (1,1)}
- Then C and E are, while D and E are not.

Events

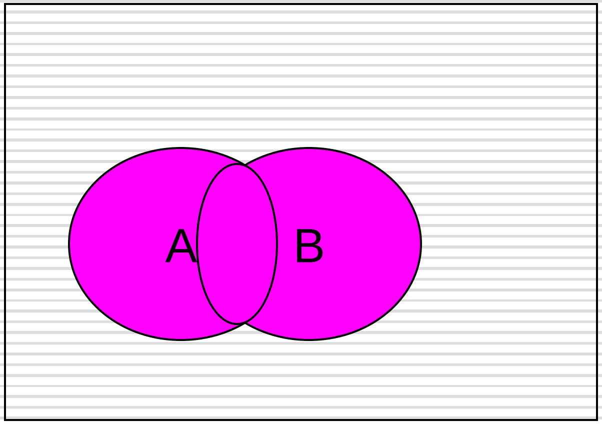
- *Union*: $A \cup B$ subset of S that contains all elements that are either in A *or* B, or both.
- *Intersection*: $A \cap B$ subset of S that contains all elements that are in both A *and* B
- *Complement*: $\bar{A}$ subset of S that contains elements that are *not in* A

Venn Diagrams

- Set A and B

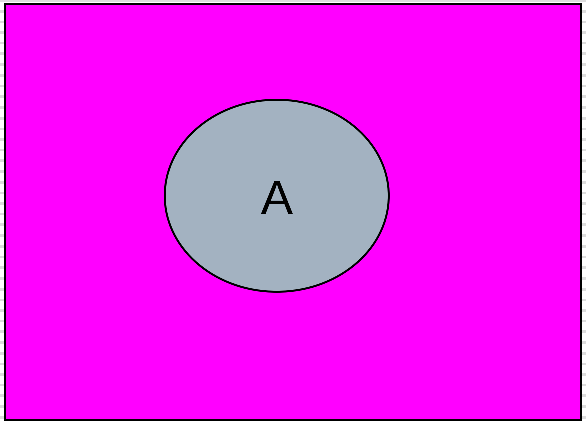


- Set A or B



Venn Diagrams

- Not A



DeMorgan's Law

$$\overline{\left(\bigcup_{i=1}^{n} E_i\right)} = \bigcap_{i=1}^{n} \overline{E}_i$$

$$\overline{\left(\bigcap_{i=1}^{n} E_i\right)} = \bigcup_{i=1}^{n} \overline{E}_i$$

What is Probability defined?

- Classical probability concept
- Frequency interpretation
- Subjective probability
- Axiom of probability

Probability

- **Classical probability concept**: If there are n equally likely possibilities, of which one must occur, and x are regarded as favorable, or as a "success", the probability of a "success" is given by x/n.
- Ex. The probability of drawing an ace from a well shuffled deck of 52 playing cards. 4/52.

Limitation

- Limited of classical probability: many situations in which the various possibility cannot be regarded as equally likely.
- Ex. Election.

Frequency interpretation

- The probability of an event (or outcome) is the proportion of times the event occur in a long run of repeated experiment.

Subjective probability

- Probabilities: personal or subjective evaluations.
- Express the strength of one’s belief with regard to the uncertainties that are involved.

Axiom of Probability

- Axiom 1. $0 \le P(A) \le 1$ for each event A in S.
- Axiom 2. P(S) = 1
- Axiom 3. If A and B are mutually exclusive events in S, then

$$P(A \cup B) = P(A) + P(B)$$

Checking probabilities

- Example: P69,
- An experiment has the three possible and mutually exclusive outcomes A, B, C. Check the assignment of probabilities is permissible:

$$P(A)=\frac{1}{3}, P(B)=\frac{1}{2}, P(C)=\frac{1}{3}$$

$$P(A)=0.57, P(B)=0.24, P(C)=0.19$$

Counting
-- Combinatorial analysis

- Goal: Determine the number of elements in a finite sample space (or in a event).

- Example: P58. A consumer testing service rates lawn mowers:

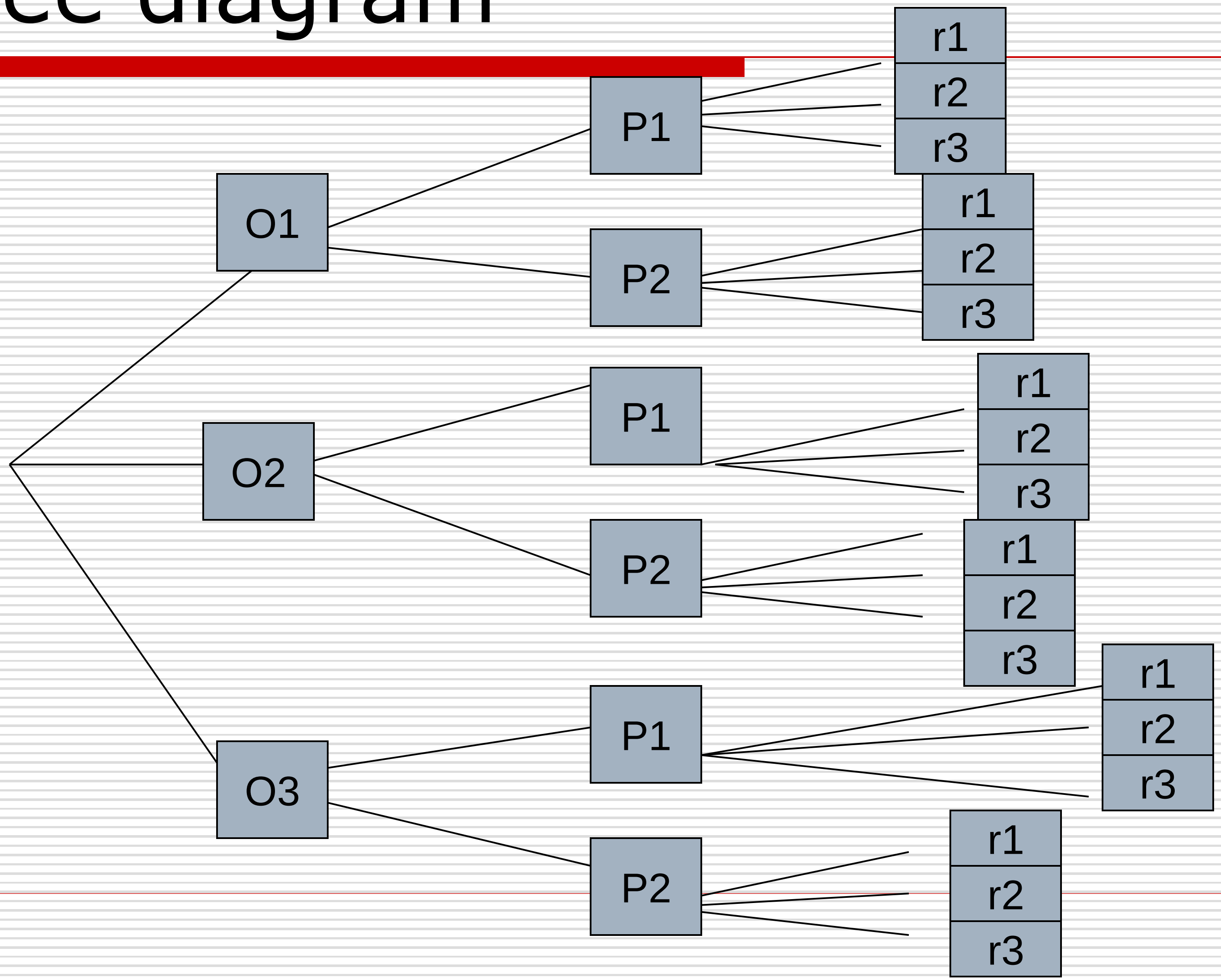
  1) operate: easy, average, difficult

  2) price: expensive, inexpensive

  3) repair: costly, average, cheap

Q: How many different ways can a law mower be rated by this testing service?

Tree diagram



Tree diagram

- A given path from left to right along the branches of the trees, we obtain an element of the sample space

- Total number of path in a tree is equal to the total number of elements in the sample space.

Multiplication of choice

- Theorem 3.1.

  If sets $A_1, A_2, \ldots, A_k$ contain, respectively, $n_1, n_2, \ldots, n_k$ elements, there are $n_1 n_2 \ldots n_k$ ways of choosing first an element of A_1, then an element of A_2, ..., and finally an element of A_k.

Permutation

- Permutation: r objects are chosen from a set of n distinct objects, any particular arrangement, or order of these objects.

- Total number of permutation r from n objects.

$$nP_r = n(n-1)(n-2)\ldots(n-r+1)$$

Factorial notation

- 1 ! = 1, 2! = 2*1 =2, 3!=3*2*1=6.

$$n! = n(n-1)\ldots 1$$

- Let 0!=1.

$$nP_r = \frac{n!}{(n-r)!}$$

Combinations

- Combinations of n objects taken r at a time.

$$\binom{n}{r} = \frac{nP_r}{r!} = \frac{n!}{(n-r)!r!}$$

- r objects from n, but don't care about the order of these r objects.

EX. contrast

$$12P_2 = 12*11 = 132, C_2^{12} = 12*11/2 = 66$$

Please calculate:

$$12P_3, 12P_4, 12P_5$$

How fast factorial grow and the impact that considering order has.

Examples

- **Urn Problem** : Suppose we have an urn with 30 blue balls and 50 red balls in it and that these balls are identical except for color. Suppose further the balls are well mixed and that we draw 3 balls, without replacement.
- Determine the probability that the balls are all of the same color.

Element Theorem

- Theorem 3.4.

If $A_1, A_2, \ldots, A_n$ are mutually exclusive events in a sample space S, then

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + \ldots + P(A_n)$$

Proposition 1.

$$P(A) = 1 - P(\bar{A})$$

Propositions

- Proposition 2.

  If $A \subset B$, then $P(A) \leq P(B)$

- Proposition 3. If A and B are any events in S, then

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof Sketch:
1. Apply the formula of exercise 3.13 (c) and (d)
2. Apply theorem 3.4

Example:

- Suppose that we toss two coins and suppose that each of the four points in the sample space S={(H,H), (H, T), (T, H), (T, T)} is equally likely and hence has probability ¼. Let

  E = {(H, H), (H, T)} and F ={(H, H), (T, H)}.

What is the probability of P(E or F)?

Extension

Discussion:

$$P(A \cup B \cup C) = ?$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

$$P(A \cup B \cup C \cup D \ldots) = ???$$

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \cdots + (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r})$$
$$+ \cdots + (-1)^{n+1} P(E_1 E_2 \cdots E_n)$$

Counting -- continuous

- Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Multinomial coefficient

  A set of n distinct item is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$

How many different divisions are possible?

Multinomial coefficients

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

$$(x_1 + x_2 + \ldots + x_r)^n = \sum_{n_1 + n_2 + \ldots + n_r = n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

Examples

- In the game of bridge the entire deck of 52 cards is dealt out to 4 players. What is the probability that
- (a) one of the player receives all 13 spades;
- (b) each player receives 1 ace?

Solution

- (a)

$$\frac{4\binom{39}{13,13,13}}{\binom{52}{13,13,13,13}} = 6.3*10^{-12}$$

(b)

$$\frac{4!\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}} = 0.105$$

EX.

- In the game of bridge the entire deck of 52 cards is dealt out to 4 players. What is the probability that
- the diamonds ◊ in 4 players are 6 4 2 1?

Examples

- A poker hand consists of 5 cards. What is the probability that one is dealt a straight?

$$\frac{10(4^5 - 4)}{\binom{52}{5}} = 0.0039$$

Examples

- What is the probability that one is dealt a full house?

$$\frac{13 \cdot 12 \cdot \binom{4}{2} \cdot \binom{4}{3}}{\binom{52}{5}} = 0.0014$$

Ex.

- If **n** people are presented in a room, what is probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than ½?

$$\frac{365 \cdot 364 \cdot 363 \cdots (365 - n + 1)}{365^n}$$

$$n = 23$$

Probability and a paradox

- Suppose we posses an infinite large urn and an infinite collection of balls labeled ball number 1, number 2, and so on.
- Experiment: At 1 minute to 12P.M., ball numbered 1 through 10 are placed in the urn, and ball number 10 is withdrawn. At ½ minute to 12 P.M., ball numbered 11 through 20 are placed in the urn, and ball number 20 is withdraw. At ¼ minute to 12 P.M., and so on.
- Question: how many balls are in the urn at 12 P.M.?

Paradox

Empty
Any number n is withdraw in $(1/2)^{(n-1)}$.

The balls are withdraw begins

Infinite of course !

What is case that

Define E_n to be the event that ball number 1 is still in the urn after the first n withdrawals have been made.

$$P(E_n) = \frac{9 \cdot 18 \cdot 27 \cdots (9n)}{10 \cdot 19 \cdot 28 \cdots (9n+1)}$$

Case study

- Randomized quick sort algorithm
- Expected number of comparisons

Summary

- Sample space specifies all possible outcomes.
- Always assign probabilities to events that satisfy the axioms of probability.

Homework

- Problems in Textbook (3.7,3.16,3.31,3.34,3.37)