Polya's Problem Solving Techniques

In 1945 George Polya published a book *How To Solve It,* which quickly became his most prized publication. It sold over one million copies and has been translated into 17 languages. In this book he identifies four basic principles of problem solving.

Polya's First Principle: *Understand the Problem*

This seems so obvious that it is often not even mentioned, yet students are often stymied in their efforts to solve problems simply because they don't understand it fully, or even in part. Polya taught teachers to ask students questions such as:

- *Do you understand all the words used in stating the problem?*
- *What are you asked to find or show?*
- *Can you restate the problem in your own words?*
- *Can you think of a picture or diagram that might help you understand the problem?*
- *Is there enough information to enable you to find a solution?*

Polya's Second Principle: *Devise a Plan*

Polya mentions that there are many reasonable ways to solve problems. The skill at choosing an appropriate strategy is best learned by solving many problems. You will find choosing a strategy increasingly easy. A partial list of strategies is included:

*Guess and check	*Look for a pattern
*Make an orderly list	*Draw a picture
*Eliminate the possibilities	*Solve a simpler problem
*Use symmetry	*Use a model
*Consider special cases	*Work backwards
*Use direct reasoning	*Use a formula
*Solve an equation	*Be ingenious

Polya's Third Principle: *Carry Out the Plan*

This step is usually easier than devising the plan. In general, all you need is care and patience, given that you have the necessary skills. Persist with the plan that you have chosen. If it continues not to work, discard it and choose another. Don't be misled, this is how mathematics is done, even by professionals.

Polya's Fourth Principle: *Look Back*

Polya mentions that much can be gained by taking the time to reflect and look back at what you have done, what worked, and what didn't. Doing this will enable you to predict what strategy to use to solve future problems.

(*How to Solve It* by George Polya, 2nd ed., Princeton University Press, 1957)

1. *Understand the Problem*

- **First.** You have to *understand* the problem.
- What is the unknown? What are the data? What is the condition?
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
- Draw a figure. Introduce suitable notation.
- Separate the various parts of the condition. Can you write them down?

2. Devising a Plan

- **Second.** Find the connection between the data and the unknown. You may be obligated to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.
- Have you seen it before? Or have you seen the same problem in a slightly different form?
- *Do you know a related problem?* Do you know a theorem that could be useful?
- *Look at the unknown!* Try to think of a familiar problem having the same or a similar unknown.
- *Here is a problem related to yours and solved before. Could you use it?* Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?
- Could you restate the problem? Could you restate id still differently? Go back to definitions.
- If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

3. Carrying Out The Plan

- **Third.** *Carry out* your plan.
- Carry out your plan of the solution, *check each step.* Can you see clearly that the step is correct? Can you prove that it is correct?

4. Looking Back

- **Fourth.** *Examine* the solution obtained.
- Can you *check the result?* Can you check the argument?
- Can you derive the solution differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

WARN!NG S!GNS!

Recognize three common instructional moves that are generally followed by taking over children's thinking.

By Victoria R. Jacobs, Heather A. Martin, Rebecca C. Ambrose, and Randolph A. Philipp

STUDIOARZ/THINKSTOCK

Have you ever finished working with a child and realized that *you* solved the problem and are uncertain what the child does or does not understand? Unfortunately, we have! When engaging in a problem-solving conversation with a child, our goal goes beyond helping the child reach a correct answer. We want to learn about the child's mathematical thinking, support that thinking, and extend it as far as possible. This exploration of children's thinking is central to our vision of both productive individual mathematical conversations and overall classroom mathematics instruction (Carpenter et al. 1999), but in practice, we find that simultaneously respecting children's mathematical thinking and accomplishing curricular goals is challenging.

In this article, we use the metaphor of traveling down a road that has as its destination children engaging in rich and meaningful problem solving like that depicted in the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010). This road requires opportunities for children to pursue their own ways of reasoning so that they can construct their own mathematical understandings rather than feeling as if they are mimicking their teachers' thinking. Knowing how to help children engage in these experiences is hard. For example, how can teachers effectively navigate situations in which a child has chosen a time-consuming strategy, seems puzzled, or is going down a path that appears unproductive?

Drawing from a large video study of 129 teachers ranging from prospective teachers to practicing teachers with thirty-three years of experience, we found that even those who are committed to pointing students to the rich, problem-solving road often struggle when trying to support and extend the thinking of individual children. After watching teachers and children engage in one-on-one conversations about 1798 problems, we identified three common teaching moves that generally preceded a teacher's taking over a child's thinking:

1. Interrupting the child's strategy
2. Manipulating the tools
3. Asking a series of closed questions

When teachers took over children's thinking with these moves, it had the effect of transporting children to the answer without engaging them in the reasoning about mathematical ideas that is a major goal of problem solving. We do not believe that any specific teaching move is always productive or always problematic, because, to be effective, a teaching move must be in response to a particular situation. However, because these three teaching moves were almost always followed by the taking over of a child's thinking, we came to view them as *warning signs*, analogous to signs a motorist might see when a potentially dangerous obstacle lies in the road ahead. By identifying these warning signs, we hope that teachers will learn to recognize them so that they can carefully examine these challenging situations before deciding how to proceed.


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Three warning signs

Consider the following interaction in which Penny, a third grader, is solving this problem:

> The teacher wants to pack 360 books in boxes. If 20 books can fit in each box, how many boxes does she need to pack all the books?

Penny pauses after initially hearing the problem, and the teacher supports her by discussing the problem situation, highlighting what she is trying to find:

***Teacher* [*T*]:** So, she has 360 books and 20 books in each box. So, we're trying to find how many boxes 360 books will fill.
***Penny* [*P*]:** Hmm …
T: So, you have 360 books, right? And what do you want to do with them?
P: Put them in each boxes of 20.
T: Boxes of 20; so you want to separate them into 20, right?
P: Mmm-hmm.
T: Into groups of 20. So, what are you trying to find?

Warning! Even with the best of intentions, some teacher efforts to move students' thinking forward can actually stifle it.

P: Trying to find how many go in each—well, you already finded out that, but you need to find how …
T: How many boxes, right?
P: Right.
T: So, you're trying to find out how many groups of 20 there are?
P: Mmm-hmm.
T: In 360?

After discussing the problem situation, Penny develops an approach, writes 360, and starts incrementing by twenties, writing 20 and 40. At this point, she whispers, "It's gonna take too long," but the teacher encourages Penny to continue by asking about her strategy. "Are you counting by twenties? Is that what you're doing there?"

Penny confirms and resumes her strategy, writing multiples of 20 through 140. Then, from the beginning of her list of numbers, she makes a mark under each one, apparently tallying the number of boxes she has made so far. At the end of her list, she resumes her strategy by writing the next number, 160, and making a mark under it (see **fig. 1**). When Penny pauses briefly before writing the next number, the teacher interrupts Penny's strategy to introduce her own by asking, "Do you know how many times two goes into thirty-six?"

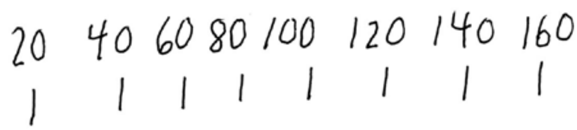
Here we see the first warning sign: *interrupting the child's strategy*. The teacher then picks up a pen and writes the problem 36 ÷ 2 as the standard division algorithm, and we see the second warning sign: *manipulating the tools*. Penny responds, "Twenty," and the teacher invites her to follow the steps to complete the algorithm (e.g., "How many times does two go into three?") but then changes the conversation slightly to consider the original numbers in the problem, writing the division problem 360 ÷ 20 as the standard division algorithm. The teacher completes the first part of the algorithm for this problem herself and then guides Penny through the rest of the steps by asking a series of closed questions, requiring only agreement ("Mmm-hmm") or short answers (e.g., "Eight")—illustrating the third warning sign: *asking a series of closed questions*.

T: Do you know how many times 20 goes into 160? [*Penny does not respond.*] Do you know how many times 2 goes into 16?
P: Two times sixteen? Times?

FIGURE 1

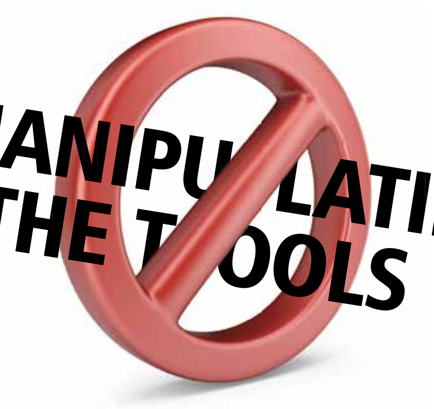
Penny's strategy was to count by twenties.

(a) She recorded each number, placed a mark under it, and then tallied the marks.



(b) When Penny paused, her teacher interrupted and introduced a different approach.

MANIPULATING THE TOOLS



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T: Well, if you go, how many 2s are in 16—so, 2, 4, 6, 8, 10, 12, 14, 16 [*writing the numbers while she counts by twos*]. How many is that? [*The teacher points along the list of numbers while she counts aloud.*] 1, 2, 3, 4, 5, 6, 7, 8, right?
P: Mmm-hmm.
T: So, 20 goes into 160, which is just [attaching] a zero. [*The teacher points at the appropriate spot on the paper for Penny to write.*]
P: [*writing*] Eight.
T: Mmm-hmm. Twenty times 8. Yes, 'cause 20 times 8 is 160, so this would be an 8, right?
P: Mmm-hmm.

With the answer of 18 now written, the teacher checks Penny's understanding of what they have just done with another series of closed questions.

T: So, how many boxes do we need? [*When Penny does not respond, the teacher points to the answer of 18.*] What does this represent? Do you know?
P: Eighteen.
T: Mmm-hmm, but do you know like in this problem how we would …
P: Eighty-one? I mean …
T: Do you know what this [18] represents? Like this 20 represents the 20 books that can fit in each box.
P: Mmm-hmm.
T: And 360 represents the total number of books. So, 18 represents …
P: The boxes.
T: How many boxes?
P: Eighteen.
T: There you go. Does that make sense?
P: Mmm-hmm.
T: 'Cause you just have to divide them into the different boxes.

In this example, the teacher began the interaction with moves that supported Penny's thinking (e.g., probing her initial strategy and understanding of the problem) and then helped her reach a correct answer. However, we share this illustration because it also highlights the three moves that should serve as warning signs because they often, and in this case did, lead to taking over the child's thinking: interrupting the child's strategy, manipulating the tools, and asking a series of closed questions.

1. Interrupting the child's strategy

When a teacher interrupts a child's strategy to suggest a different direction, the teacher's thinking becomes privileged because the child's thinking—which was "in process"—is halted. This interruption may involve talking over a child who is already speaking, or jumping in when a child is working silently. In both cases, this warning sign generally accompanies the hazard of breaking the child's train of thought—the child may struggle to regain momentum in solving the problem or may lose the thread of his or her idea altogether. Additionally, the teacher may introduce a strategy that does not make sense to the child. In the example above, Penny had a viable strategy and was in the process of executing it when her strategy was interrupted with a different approach proposed by her teacher. Perhaps the teacher thought that Penny's strategy of counting up by twenties would take too long or that she would struggle too much to find each multiple. Or perhaps the teacher had expected (or hoped) that Penny would use the standard division algorithm. In any case, Penny had no opportunity to return to her original strategy and complete it. Furthermore, Penny was making sense of the problem situation with her original strategy, but this sense making disappeared when the teacher introduced the algorithmic strategy.

In our larger study, we observed that some children, like Penny, had viable strategies for solving their problems, whereas other children's strategies and intent were unclear. However, in all cases, their thinking was "in process" in that they were writing, counting aloud, moving fingers while working silently, and so on. The teachers' interruptions sometimes introduced completely new strategies (as in Penny's case) and other times pushed children to engage with their partial strategies in specific ways that changed children's problem-solving approaches and were inconsistent with their reasoning. In each case, teachers risked impeding or aborting children's thinking by inserting and privileging their own ideas while halting the children's in-process thinking.

2. Manipulating the tools

Another warning sign teachers should notice is when they visibly take control of the interaction by manipulating the pen, cubes, or other tools. In the example above, Penny had a written recording of her strategy in progress at the top of the page when the teacher's writing of the standard division algorithm shifted Penny's focus to the teacher's strategy. The teacher then retained control of the pen for much of the interaction while she wrote and talked her way through this algorithm. In doing so, she changed the representation of the problem from Penny's written recording of the multiples of twenty and the accompanying tallying of boxes to an approach that was abstract for Penny and not a good match for her thinking—as evidenced in Penny's struggles to make sense of both the calculation and the result.

In our larger study, we observed teachers writing things or moving manipulatives, although sometimes they did so without changing the course of conversations so completely. However, taking over tools was inherently risky because doing so sent children a message about who owned the thinking. Teachers also risked altering problem representations to representations unclear to children—teachers and children may be thinking differently, even when looking at the same manipulatives or written representations (Ball 1992).

3. Asking a series of closed questions

This third warning sign highlights a situation that may begin nonhazardously—when the teacher asks a question with a simple and often obvious answer. The danger arises when this question is followed by another and another and another such question. The net effect of a series of closed questions is that the problem gets broken down for the child into tiny steps that require minimal effort and little understanding of the problem situation. Such was the case for Penny after the standard division algorithm was introduced because the teacher asked questions that required little more than Penny's agreement ("Mmm-hmm"). Penny did not have to think about the underlying ideas of division, and the problem-solving endeavor was instead reduced to following directions.

In our larger study, we observed teachers giving directions that were sometimes phrased as questions and other times as steps to follow. In either case, when the answer was finally reached, the children had often forgotten the

TABLE 1

Become aware of teaching moves and of potentially taking over students' thinking.

Warning signs for taking over children's thinking

Warning signs	Questions to consider before proceeding	Potential alternative moves
1. Interrupting the child's strategy	Do I understand how the child is thinking and will my ideas interfere with that thinking? Will the child be able to make sense of my ideas?	• Slow down: Allow the child to finish before intervening. • Encourage the child to talk about his or her strategy so far. • Ask questions to ensure that the child understands the problem situation and how the strategy relates to that situation. • Ask whether trying another tool or strategy would help.
2. Manipulating the tools	Will the child still be in control of the problem solving? Will my problem representation make sense to the child?	
3. Asking a series of closed questions	Will my questions be about the child's thinking or my thinking? Will the child still have an opportunity to engage with substantive mathematics, or will my questions prevent him or her from doing so?	

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original goal and were rarely able to relate the solution to the problem situation. We saw this confusion with Penny when she guessed, "Eighty-one?" in response to a question about how many boxes were needed. This apparent stab in the dark was a signal that the teacher's sequence of closed questions did not help Penny make sense of the teacher's algorithmic strategy or relate it to the original problem.

Heeding the warning signs

The warning signs exemplified in Penny's interaction arose often in our study, sometimes in isolation and sometimes as a set. So, what can teachers do? When possible, we encourage teachers to heed the warning signs by choosing alternative moves that are more likely to preserve children's thinking. The questions in **table 1** are designed to help teachers consider alternative moves. We do not suggest that these alternative moves are foolproof—unfortunately, no moves are. Engaging with children's thinking is a constant negotiation, fraught with trial and error, as teachers work to find ways to elicit and respect children's thinking while nudging that thinking toward reasoning that is more sophisticated. However, in analyzing our data, we were struck with how often the three warning signs were unproductive in achieving this goal, thus prompting us to consider alternative moves.

For example, how might the interaction have been different if Penny had not been interrupted and had been able to complete her initial strategy? The teacher could have probed Penny's completed strategy, validating and eliciting her ways of thinking about the problem. If the teacher still wondered about efficiency, she might have asked if Penny could think of another way of solving the problem, perhaps in a way that was more efficient. This approach would have built on Penny's ways of thinking about the problem while still preserving the goal of efficiency. Alternatively, if the teacher did choose to suggest the division algorithm, she could have left Penny in control of the pen and posed some open-ended questions to explore Penny's understanding of the algorithm and its connection to the problem situation. Another option would have been to ask Penny to consider efficiency while she was still solving the problem with her original strategy. After Penny had completed 160 books (8 boxes) by counting by 20s, the teacher could have asked her to reflect on what she had done so far and if that work could help her proceed more quickly. (This question might prompt Penny to recognize that doubling 160 books [and 8 boxes] would be close to the needed 360 books, but she would also have the option of continuing with her original strategy.) Although there is no perfect move in any situation, these types of alternative moves might have increased the likelihood that the teacher would have supported and extended Penny's thinking without taking over that thinking. (See Jacobs and Ambrose [2008–2009] for more on alternative moves.)

Are these moves ever productive?

Our data convinced us that the warning signs were generally unproductive moves, but we wondered if these same moves could ever be productive. After all, teaching moves need to be considered in context because the same move can be productive in one situation but unproductive in another. We found that the three warning signs were occasionally used productively but, to us, they almost seemed like different moves because, although they looked similar on the surface, they were coupled with the preservation of children's thinking.

For example, teachers sometimes productively interrupted a child going far off track or engaging in an extremely inefficient strategy by discussing with the child how he or she was thinking. This move was not, as we saw with Penny, used to immediately suggest a different direction but instead deepened the child's (and teacher's) understanding of how the child was thinking about the problem. Similarly, teachers sometimes productively manipulated the tools to help organize the workspace by removing "extra" cubes after ensuring that they were considered "extra" by the child (versus, for example, removing cubes to ensure that the correct quantities were represented). This move provided some organizational scaffolding while preserving the child's way of thinking about the problem. Finally, teachers sometimes productively asked a series of closed questions to check on their understanding of a child's strategy. This move kept the focus on the student's thinking by putting the child in position to confirm or deny what he or she had already done, said, or thought. Thus, we are not suggesting that the three warning signs can *never* be used productively. However, our data overwhelmingly showed that these moves typically led to taking over children's thinking and thus should be used with caution.

Good intentions

All four authors have had the experience of solving a problem for a child without gaining any idea what the child does or does not understand. We always begin these interactions with good intentions, but other pressures (e.g., shortness of time) or goals (e.g., desire to see the child use a more sophisticated strategy) often derail our efforts. Our data also showed that taking over a child's thinking was not linked to any particular tone or interaction style. In other words, in any given situation, any of us can be tempted to take over a child's thinking.

In summary, avoiding the impulse to take over a child's thinking in one-on-one conversations (either inside or outside the classroom) is challenging. We also recognize that the task becomes even more challenging in social situations like small-group work or whole-class discussions. Nonetheless, in all these instructional situations, the same goals exist: eliciting, supporting, and extending children's thinking. Further, the moves identified as warning signs are likely to thwart efforts to achieve these goals because children get transported to the answer without actually engaging in problem solving. In identifying the warning signs, our hope is that teachers will be more likely to pause and consider alternative moves to avoid the dangers of taking over children's thinking. As a first step, we invite readers to go online (see the **More4U** box to the right) to practice recognizing these warning signs in an interaction with a first grader.

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This research was supported in part by a grant from the National Science Foundation (ESI0455785). The opinions expressed in this article do not necessarily reflect the position, policy, or endorsement of the supporting agency.




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MAKING THE MOST OF

STORY PROBLEMS

Honoring students' solution approaches helps teachers capitalize on the power of story problems. No more elusive train scenarios!

By Victoria R. Jacobs and Rebecca C. Ambrose

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Story problems are an important component of the mathematics curriculum, yet many adults shudder to remember their own experiences with them, often recalling the elusive train problems from high school algebra. In contrast, research shows that story problems can be powerful tools for engaging young children in mathematics, and many students enjoy making sense of these situations (NCTM 2000; NRC 2001). Honoring children's story problem approaches is of critical importance so that they construct strategies that make sense to them rather than parrot strategies they do not understand.

To explore how teachers can capitalize on the power of story problems, we chose to study teacher-student conversations in problem-solving interviews in which a K–3 teacher worked one-on-one with a child. The skills needed for productive interviewing are the same as those needed in the classroom: Teachers must observe, listen, question, design follow-up tasks, and so on. We focused our investigation on interviews because interviews isolate these important teacher-student conversations from other aspects of classroom life.

Supporting and Extending Mathematical Thinking

After analyzing videotaped problem-solving interviews conducted by 65 teachers interviewing 231 children solving 1,018 story problems, we identified eight categories of teacher moves (i.e., intentional actions) that, when timed properly, were productive in advancing mathematical conversations. We separately considered (a) the supporting moves that a teacher used before a student arrived at a correct answer and (b) the extending moves that a teacher used after the child gave a correct answer. We want to be clear that the eight categories of teacher moves we present are not intended to be a checklist that a teacher executes on every problem. Instead, we consider these moves to be a toolbox from which a teacher can draw, after considering the specific situation and instructional goals. In the midst of instruction, the most effective moves arise in response to what a child says or does and, therefore, cannot be preplanned. Because strategically responding to children's mathematical thinking is challenging, we identified our eight categories of teacher moves in an effort to assist teachers in this decision making.

Before a correct answer is given

When a child struggles or has the wrong answer, a teacher must determine how and when to intervene in order to facilitate moving the child forward without taking over the child's thinking. Supporting a student's mathematical thinking requires the teacher to "enter the child's mind" (Ginsburg 1997) as much as possible to determine what the source of difficulty might be. Then the teacher's hypotheses about a child's thinking should drive the choices made. Because "entering the child's mind" can be quite difficult, a teacher needs to be flexible and prepared to explore various supportive approaches. In our analysis, we identified four categories of moves that teachers regularly used to support a child's thinking before the student arrived at a correct answer (see **table 1**).

Table 1

Teacher Moves to Support a Child's Thinking before a Correct Answer Is Given

Category	Sample Teacher Moves
Ensure that the child understands the problem.	Ask him to explain what he knows about the problem. Rephrase or elaborate the problem. Use a more familiar or personalized context in the problem.
Change the mathematics in the problem to match the child's level of understanding.	Change the problem to use easier numbers. Change the problem to use an easier mathematical structure.
Explore what the child has already done.	Ask him to explain a partial or incorrect strategy. Ask specific questions to explore how what he has already done relates to the quantities and relationships in the problem.
Remind the child to use other strategies.	Ask him to consider using a different tool. Ask him to consider using a different strategy. Remind him of relevant strategies he has used before.

Ensure that the child understands the problem. A teacher can provide support by helping a child develop an understanding of the problem to be solved. Typical teacher moves include rereading a problem multiple times and asking a child about specific quantities in a problem (e.g., "How many puppies are in the park?"). A twist on this repetition is to ask children to explain problems in their own words. In listening to them describe a story problem in its entirety, a teacher can pinpoint what children do and do not understand.

Rephrasing or elaborating on a story can also help to engage a child. Often, this elaboration involves using a more familiar context or personalization so that the child and her friends are characters in the story. For example, a kindergartner was asked to solve the following problem:

> The teacher has twelve pencils and three baskets. If she wants to put the same number of pencils in each basket, how many pencils should she put in each basket?

The child made a pile of fifteen cubes and kept rearranging them. In response, the teacher, Mr. Reynolds, decided to elaborate and personalize the problem by involving their classroom and making himself the teacher in the story:

A teacher's most effective teaching arises in response to what a child says or does.

> Let me change it a little bit. Let's try this. Mr. Reynolds has three baskets. I have three baskets, and I have twelve pencils in my hand, and I say, "I've got to do something with these pencils. I can't walk around with them all day! What am I going to do with these pencils? Oh, here's what I'll do. I'll put some in each basket so the kids can come get them." But then I think, "I'd better put the same number in each basket. Because if I put, like, two in one basket and ten in one basket, that's not fair. So I have to put the same number of pencils in each basket." How many pencils would I put in each one of those baskets so that all the baskets would have the same number of pencils inside?

This elaborated story did not change the mathematical structure of the problem but did make the problem more real for the child, and in this case, she solved the problem correctly by using trial and error to create three piles of four cubes each. Elaborating a story may seem counterintuitive because it goes against the traditional approach of helping children identify keywords or irrelevant information in story problems. However, when elaboration is designed to make a problem more meaningful, children are more likely to avoid mechanical problem-solving approaches and instead work to make sense of the problem situation.

Change the mathematics to match the child's level of understanding. When children do not understand a problem, even after attempts to rephrase or elaborate it, changing the problem itself can be productive. One type of change is to use easier numbers. Specifically, using smaller or friendlier numbers (e.g., decade numbers) can help them gain access to the mathematics underlying a problem. After making sense of an easier problem, students generally gain confidence and, in many cases, can then make sense of the original problem.

Similarly, because research shows that children have more difficulty with some problem structures than others, another type of change is to use an easier mathematical structure (Carpenter et al. 1999). For example, a first grader was asked to solve this problem:

> Twelve mice live in a house. Nine live upstairs. How many live downstairs?

Because part-whole problems such as this do not have an explicit joining or separating action, children often do not know how the quantities relate. This student made a set of nine cubes and a set of twelve cubes and joined them to get twenty-one. After several unsuccessful attempts to help the child understand the problem, the teacher chose to change the problem to include an explicit separating action. Specifically, the teacher explained, "Nine of those mice are going to go upstairs and watch TV." In response, the girl separated nine mice from her set of twelve, leaving a group of three. This change in mathematical structure did more than allow the student to solve a problem correctly. By providing her access to an easier but related problem, the teacher created opportunities for discussing the quantities and relationships in both problems. Thus, with further skilled questioning, the teacher could use the child's understanding of the second problem to help her understand the original problem and, more generally, problems with a part-whole structure.

Explore what the child has already done. When struggling with a problem, children can sometimes determine what went wrong if they are encouraged to articulate partial or incorrect strategies. General questions, such as "Can you tell me how you solved it" or "What did you do first?" can be helpful for starting conversations, but follow-up questions require a teacher to ask about the details of a child's strategy and thus cannot be preplanned. For example, a first grader was asked to solve this problem:

> One cat has four legs. How many legs do seven cats have?

The child (C) put out seven teddy-bear counters. He saw teddy-bear counters as having two legs and two arms and, therefore, counted only two legs on each teddy bear, answering ”Fourteen.” The teacher (T) recognized that his confusion was linked to the counters he had chosen, and she posed questions to clarify how his work related to the problem context:

T: How many legs on a bear?
C: Two.
T: How many legs on a cat?
C: Four.
T: How many did you count? How many legs *each* did you count?
C: Two.
T: Is that how many legs cats have?
C: No, cats have four, and bears have two.
T: OK, could you do that again for me?
C: First I get one cat [*puts out one teddy-bear counter*], and then I get a bear [*puts out another teddy-bear counter*], and this cat has four legs, and the bear has two legs.
T: Are there bears in the story?
C: No, there's cats.

This dialogue continued for some time before the child solved the problem correctly by counting four legs on each bear and then again by using a different tool. The support the teacher provided began with what the child had already done, and through specific questioning, she helped him make sense of how his initial strategy was related (and not related) to the problem. Note that she could not have preplanned this conversation, because it grew out of her careful observation of his way of using the teddy-bear counters.

Remind the child to use other strategies. Sometimes students get lost in a particular strategy, and instead of abandoning that strategy for a more effective one, they persist in using it in unproductive ways. A teacher can help by nudging them to think more flexibly and to try alternative approaches. A simple suggestion to try a different tool or a different strategy can sometimes give a child permission to move on and self-redirect. At times, a teacher may also find that suggesting a particular tool or reminding a child of strategies used in the past is beneficial. For example, a first-grade student was asked to solve the following problem:

> Let's pretend we're out at the snack tables, and four seagulls come to the snack tables. And then seven more seagulls come to the snack tables. How many seagulls are at the snack tables?

The child first counted to four, raising one finger with each count. She then put those four fingers down. Next, she counted to seven, raising one finger with each count. At this point, the child was baffled, staring at her fingers. The teacher suggested, “Want to try it with cubes?” The child immediately made a stack of four Unifix cubes and a stack of seven Unifix cubes and then counted them altogether to get an answer of eleven. She was confident and efficient once she started using the Unifix cubes. The teacher did not tell the child how to solve the problem but did encourage her to consider using a tool that was more conducive to representing both sets; the child did not have enough fingers to show seven and four at the same time. This support reflected the teacher's understanding of children's

Table 2

Teacher Moves to Extend a Child's Thinking after a Correct Answer Is Given

Category	Sample teacher moves
Promote reflection on the strategy the child just completed.	Ask her to explain her strategy. Ask specific questions to clarify how the details of her strategy are connected to the quantities and mathematical relationships in the problem.
Encourage the child to explore multiple strategies and their connections.	Ask her to try any second strategy. Ask her to try a second strategy connected to her initial strategy in deliberate ways (e.g., more efficient counting or abstraction of work with manipulatives). Ask her to compare and contrast strategies.
Connect the child's thinking to symbolic notation.	Ask her to write a number sentence that “goes with” the problem. Ask her to record her strategy.
Generate follow-up problems linked to the problem the child just completed.	Ask her to solve the same or a similar problem with numbers that are more challenging. Ask her to solve the same or a similar problem with numbers that are strategically selected to promote more sophisticated strategies.

direct-modeling strategies in which they represent both sets before combining them.

After a correct answer is given

Solving a story problem correctly using a valid strategy is an important mathematical endeavor. However, we view problem solving as a context for having mathematical conversations, and this conversation need not end when the correct answer is reached. Instead, a teacher can pose additional questions to help students deepen their understanding of completed work and connect it to other mathematical ideas. We have identified four categories of moves that teachers regularly used to extend children's thinking after arrival at a correct answer (see **table 2**).

Promote reflection on the strategy just completed. Once a student has correctly solved a problem, a teacher can ask for a strategy explanation or for clarification about how the use of a particular strategy makes sense with the quantities and mathematical relationships expressed in the problem. Articulating these ideas can reinforce a child's understanding and give a teacher a window into that understanding. Again, attention to detail matters. Similar to the supporting questions intended to explore children's partial or incorrect strategies, teachers' extending questions were most productive when they were specific and in response to the details of what a student had already said or done. For example, a second grader was asked to solve this problem:

> This morning I had some candy. Then I gave you five pieces of candy. Now I have six pieces of candy left. How many pieces of candy did I have this morning before I gave some to you?

The student quickly solved this problem mentally and explained, "Five plus five, if you took one away, is ten and then one more is eleven, so you had eleven." Children often provide correct answers to problems with this structure, in which the initial quantity is unknown, without really understanding what they are finding. In this case, the teacher probed the child's thinking in relation to this issue:

T: So how did you know to add them together?
C: I don't know. I just added them, I guess.
T: Well, think about it. Why does that make sense for the problem?

The child thought about this question for some time and eventually used Unifix cubes to act out the story and convince himself (and the teacher) that eleven was the correct answer *and* made sense with the story. By asking him to reflect further on his strategy, the teacher ensured that he was making sense of the mathematics.

Encourage the child to explore multiple strategies and their connections. Children need opportunities to not only solve problems but also explore the mathematical connections among multiple strategies for the same problem. One approach is to ask them to generate a second strategy—any strategy—to a problem they have already solved. Another approach is to ask for a second strategy that is connected to their initial strategy in deliberate ways. For instance, a third grader using base-ten blocks to represent 12 pages of 10 spelling words per page put out 12 tens rods but counted all 120 blocks by ones! The teacher built on this initial strategy by asking her to count the blocks another way. The child responded by counting by tens and even shared that this second strategy was easier.

Another way a teacher can deliberately build on an initial strategy is to ask for a mental strategy that is an abstraction of work with manipulatives. For example, a third grader was asked to solve the following problem:

Although a teacher may initially support a student's step-by-step recording of a strategy, he should diligently support the *child's* thinking.

There are 247 girls on the playground and there are 138 boys on the playground. How many children are on the playground?

The student initially represented both quantities with base-ten flats, rods, and single cubes. Next he combined the hundred flats (3), combined the ten rods (7), combined some of the single cubes to make 10, traded the 10 single cubes for 1 tens rod, making a total of 8 tens rods, and finally counted the remaining single cubes (5) to answer 385. The teacher then asked, "Doing just what you did with the materials, could you solve that problem in your head?" The child looked at the numbers and abstracted what he had just done with the cubes. Specifically, he explained that he could add 100 to 200 to get 300 and then add 30 to 40 to get 70. Next he put 2 from the 7 with the 8 to get another 10, which made 80, and had 5 ones left, so the answer was 385. When executing this mental strategy, the child articulated the underlying mathematical idea of both strategies: combine like units and, when necessary, regroup (i.e., decompose the 7 into 5 and 2 so that the 2 can be combined with the 8 to make a new 10).

Through experiences with multiple strategies, children can gain the ability and flexibility to change strategies when one is unsuccessful. A teacher can also use multiple strategies to highlight underlying mathematical ideas by asking students to explicitly compare and contrast strategies. At times, a teacher may even ask a child to compare a successful strategy to a previously unsuccessful attempt, because, in many cases, the child will discover the reason the strategy failed.

Connect the child's thinking to symbolic notation. When solving a story problem by drawing, using manipulatives, or computing mentally, students may not use any symbolic notation. A teacher can encourage students to connect their work with mathematical symbols by asking them to either generate a number sentence that "goes with" the problem or record the strategy used to solve the problem. Although requesting a number sentence that "goes with" the problem is perhaps the more typical request, asking for a strategy representation can be powerful. Young children often begin recording their strategies in unconventional ways that include a mix of symbols and drawings. They might draw pictures of manipulatives they used and then add number labels to parts of those pictures. Over time, children's recordings become progressively more abstract until they are completely symbolic.

Generating a symbolic representation of a strategy can help children develop meaning for, and facility with, mathematical symbols because the representation is linked with their interpretation of the problem. For example, a second grader solved the problem about the number of legs on seven cats by first putting out seven tiles (cats). Next he moved two tiles to the side and said, "Four plus four equals eight." He then moved another tile to the side and said, "Eight plus four equals twelve." He continued moving one tile at a time until he had used them all, each time adding four more to his running total. When asked to write a number sentence to show what he had done, he wrote the following:

$4 + 4 \rightarrow 8 + 4 \rightarrow 12 + 4 \rightarrow 16 + 4 \rightarrow 20 + 4 \rightarrow 24 + 4 \rightarrow 28$

Unlike the number sentence that '"goes with" the problem ($7 \times 4 = 28$), his symbolic representation reflects how this student thought about and solved the problem. Note that his use of arrows instead of equal signs avoids the incorrect use of the equal sign between expressions.

Requesting links between strategies and symbolic notation is important so that children see the mathematics done on paper as connected to solving story problems. Moreover, once children become facile with symbolic notation, the notation itself can become a tool for problem solving and reflection. We offer a final note of caution: A teacher may initially need to support a student in recording each step of a strategy so that parts are not omitted. However, a teacher needs to be vigilant in providing support to record the child's—not the teacher's—ways of thinking about a problem.

Generate follow-up problems. By carefully sequencing problems, a teacher can create unique opportunities for mathematical discussions. Although we recognize the importance of practice, we are suggesting something beyond simply assigning additional problems to solve. We advocate that, in the midst of instruction, a teacher can consider a child's existing understanding and then modify the initial problem or create a new problem to add challenge or to encourage use of more sophisticated strategies. A first grader was asked to solve this problem:

The Kumyeey woman was collecting acorns. She had nine baskets, and she put ten acorns in each basket. So how many acorns did she have altogether?

The child quickly responded, "Ninety," explaining that he had counted, "Ten, one," putting up ten fingers and then one finger. He continued, "Twenty, two," again putting up ten fingers and this time, two fingers. He continued with this pattern of counting and finger use: "Thirty, three; forty, four; fifty, five; sixty, six; seventy, seven; eighty, eight; ninety, nine." The teacher then decided to extend the child's use of ten by posing a related problem and asking him to consider the connections:

T: So that's how you got ninety. What if she had nine baskets, but she put eleven in each basket instead of ten? [*Child thinks for a while*.] Could you use some of the work that you've already done—that we did during the afternoon—or would you have to start all over again? She still has nine baskets, and there are still ten acorns in each basket, and then she puts in one more so that each basket has eleven.
C: Ohhhhh! I get it. Well, there's already ten in each basket, so that's ninety. So I count up nine, one more nine. I mean nine ones. I'm going to add nine ones. So there's already ninety, so ninety-one, ninety-two, ninety-three, ninety-four, ninety-five, ninety-six, ninety-seven, ninety-eight, ninety-nine.

By strategically selecting numbers and by drawing attention to the link between problems, the teacher was able to further this child's base-ten understanding by helping him recognize and use the ten in the number eleven.

Summary

Our project builds on previous work on teacher questioning (see, for example, Mewborn and Huberty 1999; Stenmark 1991), which provide lists of potential questions. These lists can be important starting points for eliciting a student's thinking, but we hope that our eight categories of teacher moves (four designed to support children's thinking and four to extend it) can help teachers further customize questioning to make the most of story problems. These moves do not always lead to correct answers, and we reiterate that not all eight are intended to be used in every situation. However, together they form a toolbox from which teachers can select means to help students solve problems and explore connections among mathematical ideas. Engaging in mathematical discussions about story problems is challenging; we offer three final guidelines:

- **Elicit and respond to a child's ideas.** The most effective teacher moves cannot be preplanned. Instead, they must occur in response to a student's specific actions or ideas. Thus, expertise is tied less to planning before a student arrives and more to seeding conversations, finding the mathematics in children's comments and actions, and making in-the-moment decisions about how to support and extend children's thinking.
- **Attend to details in a child's strategy and talk.** Research on children's developmental trajectories shows that subtle differences in children's strategies and talk can reflect important distinctions in their mathematical understandings (NRC 2001). A teacher can customize instruction on the basis of these distinctions, and by attending to details of a child's explanations and comments, a teacher also communicates respect for a child's ideas.
- **Do not always end a conversation after a correct answer is given.** Important learning can occur after students give a correct answer if the teacher asks them to articulate, reflect on, and build on their initial strategies.

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This research was supported in part by National Science Foundation grant #ESI0455785. The views expressed are those of the authors and do not necessarily reflect the views of the National Science Foundation. ▲