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FOR ENGINEERS AND SCIENTISTS

THIRD EDITION

Volume One

Part 1b

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To Susan Ohanian, writer, who gently tried to teach me some of her craft.—H.C.O.

To Frank D. Markert, a printer by trade; to Christiana Park, for her thirst for new knowledge; and to Erin, Ryan, Sean, and Gwen, for their wonder and clarity.—J.T.M.

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CHAPTER

7 Work and Energy

CONCEPTS IN CONTEXT

Concepts *in* Context

The high-speed and high-acceleration thrills of a roller coaster are made possible by the force of gravity. We will see that gravity does work on the roller-coaster car while it descends, increasing its kinetic energy.

To see how energy considerations provide powerful approaches for understanding and predicting motion, we will ask:

? What is the work done by gravity when the roller-coaster car descends along an incline? (Example 3, page 209)

? As a roller-coaster car travels up to a peak, over it, and then down again, does gravity do work? Does the normal force? (Checkup 7.1, question 1, page 210)

? For a complex, curving descent, how can the final speed be determined in a simple way? (Example 8, page 222; and Checkup 7.4, question 1, page 224)

Conservation laws play an important role in physics. Such laws assert that some quantity is conserved, which means that the quantity remains constant even when particles or bodies suffer drastic changes involving motions, collisions, and reactions. One familiar example of a conservation law is the conservation of mass. Expressed in its simplest form, this law asserts that the mass of a given particle remains constant, regardless of how the particle moves and interacts with other particles or other bodies. In the preceding two chapters we took this conservation law for granted, and we treated the particle mass appearing in Newton's Second Law ($m\mathbf{a} = \mathbf{F}$) as a constant, time-independent quantity. More generally, the sum of all the masses of the particles or bodies in a system remains constant, even when the bodies suffer transformations and reactions. In everyday life and in commercial and industrial operations, we always rely implicitly on the conservation of mass. For instance, in the chemical plants that reprocess the uranium fuel for nuclear reactors, the batches of uranium compounds are carefully weighed at several checkpoints during the reprocessing operation to ensure that none of the uranium is diverted for nefarious purposes. This procedure would make no sense if mass were not conserved, if the net mass of a batch could increase or decrease spontaneously.

This chapter and the next deal with the conservation of energy. This conservation law is one of the most fundamental laws of nature. Although we will derive this law from Newton's laws, it is actually much more general than Newton's laws, and it remains valid even when we step outside of the realm of Newtonian physics and enter the realm of relativistic physics or atomic physics, where Newton's laws fail. No violation of the law of conservation of energy has ever been discovered.

In mechanics, *we can use the conservation law for energy to deduce some features of the motion of a particle or of a system of particles* when it is undesirable or too difficult to calculate the full details of the motion from Newton's Second Law. This is especially helpful in those cases where the forces are not known exactly; we will see some examples of this kind in Chapter 11.

But before we can deal with energy and its conservation, we must introduce the concept of work. Energy and work are closely related. We will see that the work done by the net force on a body is equal to the change of the kinetic energy (the energy of motion) of the body.

7.1 WORK

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To introduce the definition of work done by a force, we begin with the simple case of motion along a straight line, with the force along the line of motion, and then we will generalize to the case of motion along some arbitrary curved path, with the force in some arbitrary direction at each point. Consider a particle moving along such a straight line, say, the x axis, and suppose that a constant force F_x, directed along the same straight line, acts on the particle. Then the **work done by the force** F_x *on the particle as it moves some given distance is defined as the product of the force and the displacement* Δx:

$$W = F_x \, \Delta x \tag{7.1}$$

work done by one constant force

This rigorous definition of work is consistent with our intuitive notion of what constitutes "work." For example, the particle might be a stalled automobile that you are pushing along a road (see Fig. 7.1). Then the work that you perform is proportional to the magnitude of the force you have to exert, and it is also proportional to the distance you move the automobile.

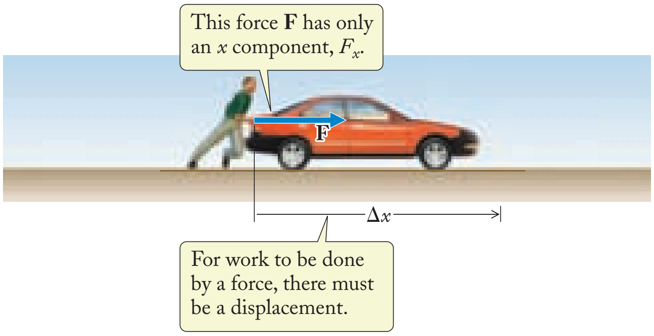


FIGURE 7.1 You do work while pushing an automobile along a road with a horizontal force **F**.

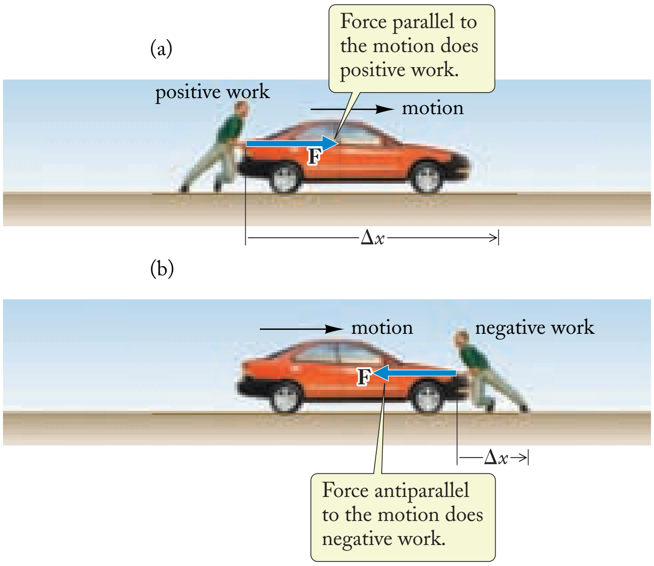


FIGURE 7.2 (a) The work you do on the automobile is positive if you push in the direction of motion. (b) The work you do on the automobile is negative if you push in the direction opposite to the motion.

Note that in Eq. (7.1), F_x is reckoned as positive if the force is in the positive x direction and negative if in the negative x direction. The subscript x on the force helps us to remember that F_x has a magnitude and a sign; in fact, F_x is the x component of the force, and this x component can be positive or negative. According to Eq. (7.1), *the work is positive if the force and the displacement are in the same direction* (both positive, or both negative), *and the work is negative if they are in opposite directions* (one positive, the other negative). When pushing the automobile, you do positive work on the automobile if you push in the direction of the motion, so your push tends to accelerate the automobile (Fig. 7.2a); but you do negative work on the automobile (it does work on you) if you push in the direction opposite to the motion, so your push tends to decelerate the automobile (Fig. 7.2b).

Equation (7.1) gives the work done by one of the forces acting on the particle. If several forces act, then Eq. (7.1) can be used to calculate the work done by each force. If we add the amounts of work done by all the forces acting on the particle, we obtain the net amount of work done by all these forces together. This net amount of work can be directly calculated from the net force:

$$W = F_{\text{net},x}\,\Delta x$$

In the SI system, *the unit of work is the* **joule** (J), which is the work done by a force of 1 N during a displacement of 1 m. Thus,

$$1\text{ joule} = 1\text{ J} = 1\text{ N}\cdot\text{m}$$

EXAMPLE 1 Suppose you push your stalled automobile along a straight road (see Fig. 7.1). If the force required to overcome friction and to keep the automobile moving at constant speed is 500 N, how much work must you do to push the automobile 30 m?

SOLUTION: With $F_x = 500$ N and $\Delta x = 30$ m, Eq. (7.1) gives

$$W = F_x\ \Delta x = 500\ \text{N} \times 30\ \text{m} = 15\,000\ \text{J} \tag{7.2}$$

EXAMPLE 2 A 1000-kg elevator cage descends 400 m within a skyscraper. (a) What is the work done by gravity on the elevator cage during this displacement? (b) Assuming that the elevator cage descends at constant velocity, what is the work done by the tension of the suspension cable?

SOLUTION: (a) With the x axis arranged vertically upward (see Fig. 7.3), the displacement is negative, $\Delta x = -400$ m; and the x component of the weight is also negative, $w_x = -mg = -1000\ \text{kg} \times 9.81\ \text{m/s}^2 = -9810$ N. Hence by the definition (7.1), the work done by the weight is

$$W = w_x\ \Delta x = (-9810\ \text{N}) \times (-400\ \text{m}) = 3.92 \times 10^6\ \text{J} \tag{7.3}$$

(b) For motion at constant velocity, the tension force must exactly balance the weight, so the net force $F_{\text{net},x}$ is zero. Therefore, the tension force of the cable has the same magnitude as the weight, but the opposite direction:

$$T_x = +mg = 9810\ \text{N}$$

The work done by this force is then

$$W = T_x\ \Delta x = 9810\ \text{N} \times (-400\ \text{m}) = -3.92 \times 10^6\ \text{J} \tag{7.4}$$

This work is negative because the tension force and the displacement are in opposite directions. Gravity does work on the elevator cage, and the elevator cage does work on the cable.

COMMENTS: (a) Note that the work done by gravity is completely independent of the details of the motion; the work depends on the total vertical displacement and on the weight, but not on the velocity or the acceleration of the motion. (b) Note that the work done by the tension is exactly the negative of the work done by gravity, and thus the net work done by both forces together is zero (we can also see this by examining the work done by the net force; since the net force $F_{\text{net},x} = w_x + T_x$ is zero, the net work $W = F_{\text{net},x}\ \Delta x$ is zero). However, the result (7.4) for the work done by the tension depends implicitly on the assumptions made about the motion. Only for unaccelerated motion does the tension force remain constant at 9810 N. For instance, if the elevator cage were allowed to fall freely with the acceleration of gravity, then the tension would be zero; the work done by the tension would then also be zero, whereas the work done by gravity would still be 3.92×10^6 J.

Although the rigorous definition of work given in Eq. (7.1) agrees to some extent with our intuitive notion of what constitutes "work," the rigorous definition clashes with our

JAMES PRESCOTT JOULE (1818–1889) *English physicist. He established experimentally that heat is a form of mechanical energy, and he made the first direct measurement of the mechanical equivalent of heat. By a series of meticulous mechanical, thermal, and electrical experiments, Joule provided empirical proof of the general law of conservation of energy.*

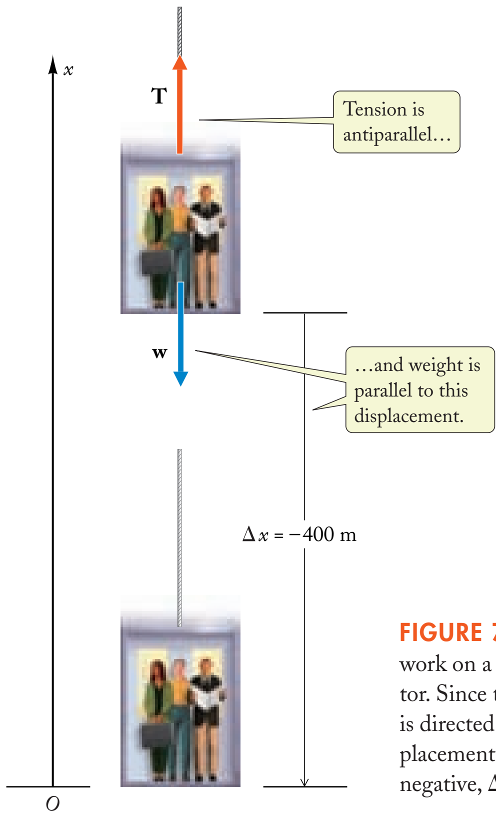


FIGURE 7.3 Gravity does work on a descending elevator. Since the positive x axis is directed upward, the displacement of the elevator is negative, $\Delta x = -400$ m.

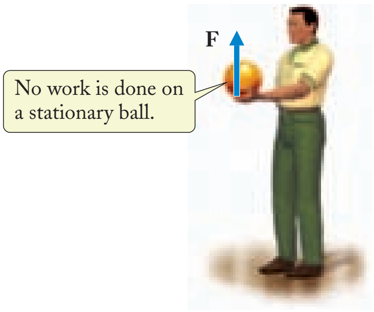


FIGURE 7.4 Man holding a ball. The displacement of the ball is zero; hence the work done on the ball is zero.

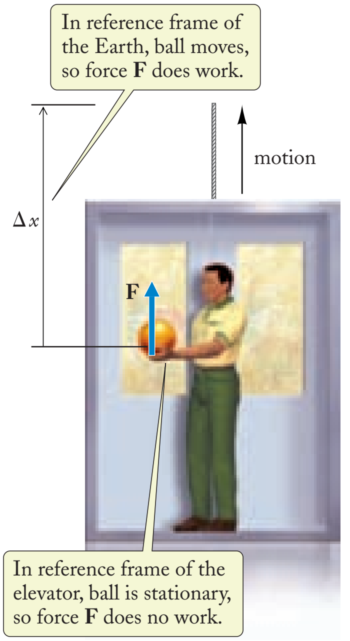


FIGURE 7.5 The man holding the ball rides in an elevator. The work done depends on the reference frame.

intuition in some instances. For example, consider a man holding a bowling ball in a fixed position in his outstretched hand (see Fig. 7.4). Our intuition suggests that the man does work—yet Eq. (7.1) indicates that no work is done on the ball, since the ball does not move and the displacement Δx is zero. The resolution of this conflict hinges on the observation that, although the man does no work *on the ball*, he does work *within his own muscles* and, consequently, grows tired of holding the ball. A contracted muscle is never in a state of complete rest; within it, atoms, cells, and muscle fibers engage in complicated chemical and mechanical processes that involve motion and work. This means that work is done, and wasted, internally within the muscle, while no work is done externally on the bone to which the muscle is attached or on the bowling ball supported by the bone.

Another conflict between our intuition and the rigorous definition of work arises when we consider a body in motion. Suppose that the man with the bowling ball in his hand rides in an elevator moving upward at constant velocity (Fig. 7.5). In this case, the displacement is not zero, and the force (push) exerted by the hand on the ball does work—the displacement and the force are in the same direction, and consequently the man continuously does positive work on the ball. Nevertheless, to the man the ball feels no different when riding in the elevator than when standing on the ground. This example illustrates that *the amount of work done on a body depends on the reference frame*. In the reference frame of the ground, the ball is moving upward and work is done on it; in the reference frame of the elevator, the ball is at rest, and no work is done on it. The lesson we learn from this is that before proceeding with a calculation of work, we must be careful to specify the reference frame.

If the motion of the particle and the force are not along the same line, then the simple definition of work given in Eq. (7.1) must be generalized. Consider a particle moving along some arbitrary curved path, and suppose that the force that acts on the particle is constant (we will consider forces that are not constant in the next section). The force can then be represented by a vector **F** (see Fig. 7.6a) that is constant in magnitude and direction. *The work done by this constant force during a (vector) displacement* **s** *is defined as*

$$W = Fs\cos\theta \tag{7.5}$$

where F is the magnitude of the force, s is the length of the displacement, and θ is the angle between the direction of the force and the direction of the displacement. Both F and s in Eq. (7.5) are positive; the correct sign for the work is provided by the factor $\cos\theta$. The work done by the force **F** is positive if the angle between the force and the displacement is less than 90°, and it is negative if this angle is more than 90°.

As shown in Fig. 7.6b, the expression (7.5) can be regarded as the magnitude of the displacement (s) multiplied by the component of the force along the direction of the displacement ($F\cos\theta$). If the force is parallel to the direction of the displacement ($\theta = 0$ and $\cos\theta = 1$), then the work is simply Fs; this coincides with the case of motion along a straight line [see Eq. (7.1)]. If the force is perpendicular to the direction of the displacement ($\theta = 90°$ and $\cos\theta = 0$), then the work vanishes. For instance, if a woman holding a bowling ball walks along a level road at constant speed, she does not do any work on the ball, since the force she exerts on the ball is perpendicular to the direction of motion (Fig. 7.7a). However, if the woman climbs up some stairs while holding the ball, then she does work on the ball, since now the force she exerts has a component along the direction of motion (Fig. 7.7b).

For two arbitrary vectors **A** and **B**, the product of their magnitudes and the cosine of the angle between them is called the **dot product** (or **scalar product**) of the vec-

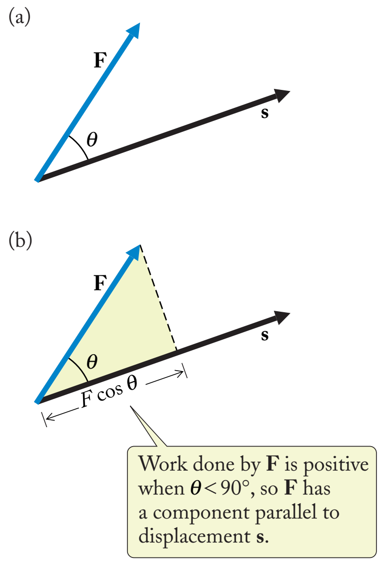


FIGURE 7.6 (a) A constant force **F** acts during a displacement **s**. The force makes an angle θ with the displacement. (b) The component of the force along the direction of the displacement is $F\cos\theta$.

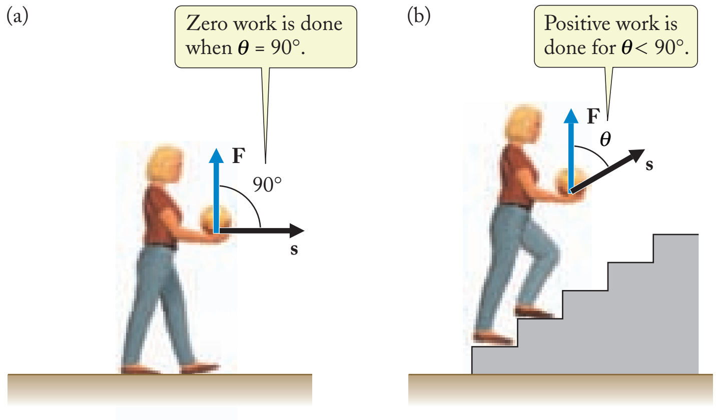


FIGURE 7.7 (a) The force exerted by the woman is perpendicular to the displacement. (b) The force exerted by the woman is now not perpendicular to the displacement.

tors (see Section 3.4). The standard notation for the dot product consists of the two vector symbols separated by a dot:

dot product (scalar product)

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta \tag{7.6}$$

Accordingly, the expression (7.5) for the work can be written as the dot product of the force vector **F** and displacement vector **s**,

$$W = \mathbf{F}\cdot\mathbf{s} \tag{7.7}$$

In Section 3.4, we found that the dot product is also equal to the sum of the products of the corresponding components of the two vectors, or

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z \tag{7.8}$$

If the components of **F** are F_x, F_y, and F_z and those of **s** are Δx, Δy, and Δz, then the second version of the dot product means that the work can be written

$$W = F_x\,\Delta x + F_y\,\Delta y + F_z\,\Delta z \tag{7.9}$$

Note that although this equation expresses the work as a sum of contributions from the x, y, and z components of the force and the displacement, the work does not have separate components. The three terms on the right are merely three terms in a sum. Work is a single-component, scalar quantity, not a vector quantity.

EXAMPLE 3 A roller-coaster car of mass m glides down to the bottom of a straight section of inclined track from a height h. (a) What is the work done by gravity on the car? (b) What is the work done by the normal force? Treat the motion as particle motion.

Concepts in Context

SOLUTION: (a) Figure 7.8a shows the inclined track. The roller-coaster car moves down the full length of this track. By inspection of the right triangle formed by the incline and the ground, we see that the displacement of the car has a magnitude

$$s = \frac{h}{\sin\phi} \tag{7.10}$$

[Here we use the label ϕ (Greek phi) for the angle of the incline to distinguish it from the angle θ appearing in Eq. (7.5).] Figure 7.8b shows a "free-body" diagram for the car; the forces acting on it are the normal force **N** and the weight **w**. The weight makes an angle $\theta = 90° - \phi$ with the displacement. According to Eq. (7.5), we then find that the work W done by the weight **w** is

$$W = ws\cos\theta = mg \times \frac{h}{\sin\phi} \times \cos(90° - \phi)$$

Since $\cos(90° - \phi) = \sin\phi$, the work is

$$W = mg \times \frac{h}{\sin\phi} \times \sin\phi = mgh \tag{7.11}$$

Alternatively, we can use components to calculate the work. For example, if we choose the x axis horizontal and the y axis vertical, the motion is two-dimensional, and we need to consider x and y components. The components of the weight are $w_x = 0$ and $w_y = -mg$. According to Eq. (7.9), the work done by the weight is then

$$W = w_x\,\Delta x + w_y\,\Delta y = 0 \times \Delta x + (-mg) \times \Delta y = 0 + (-mg) \times (-h) = mgh$$

Of course, this alternative calculation agrees with Eq. (7.11).

(b) The work done by the normal force is zero, since this force makes an angle of 90° with the displacement.

COMMENTS: (a) Note that the result (7.11) for the work done by the weight is independent of the angle of the incline—it depends only on the change of height, not on the angle or the length of the inclined plane. (b) Note that the result of zero work for the normal force is quite general. The normal force **N** acting on any body rolling or sliding on any kind of fixed surface never does work on the body, since this force is always perpendicular to the displacement.

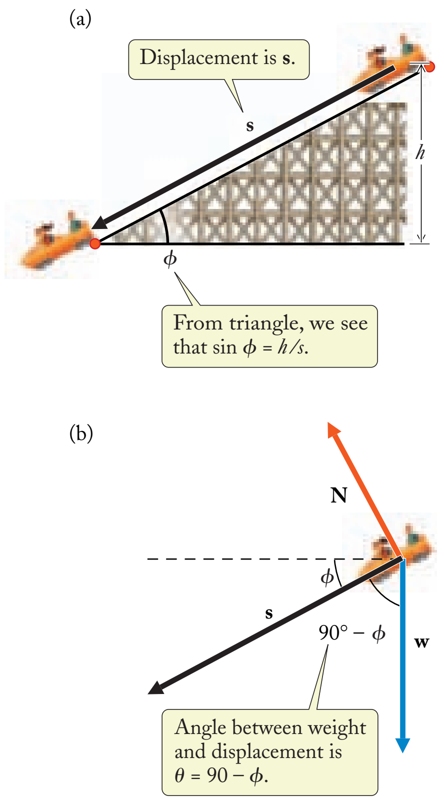


FIGURE 7.8 (a) A roller-coaster car undergoing a displacement along an inclined plane. (b) "Free-body" diagram showing the weight, the normal force, and the displacement of the car.

Checkup 7.1

Concepts in Context

QUESTION 1: Consider a frictionless roller-coaster car traveling up to, over, and down from a peak. The forces on the car are its weight and the normal force of the tracks. Does the normal force of the tracks perform work on the car? Does the weight?

QUESTION 2: While cutting a log with a saw, you push the saw forward, then pull backward, etc. Do you do positive or negative work on the saw while pushing it forward? While pulling it backward?

QUESTION 3: While walking her large dog on a leash, a woman holds the dog back to a steady pace. Does the dog's pull do positive or negative work on the woman? Does the woman's pull do positive or negative work on the dog?

QUESTION 4: You are trying to stop a moving cart by pushing against its front end. Do you do positive or negative work on the cart? What if you pull on the rear end?

QUESTION 5: You are whirling a stone tied to a string around a circle. Does the tension of the string do any work on the stone?

QUESTION 6: Figure 7.9 shows several equal-magnitude forces **F** and displacements **s**. For which of these is the work positive? Negative? Zero? For which of these is the work largest?

QUESTION 7: To calculate the work performed by a known constant force **F** acting on a particle, which two of the following do you need to know? (1) The mass of the particle; (2) the acceleration; (3) the speed; (4) the displacement; (5) the angle between the force and the displacement.

(A) 1 and 2 (B) 1 and 5 (C) 2 and 3
(D) 3 and 5 (E) 4 and 5

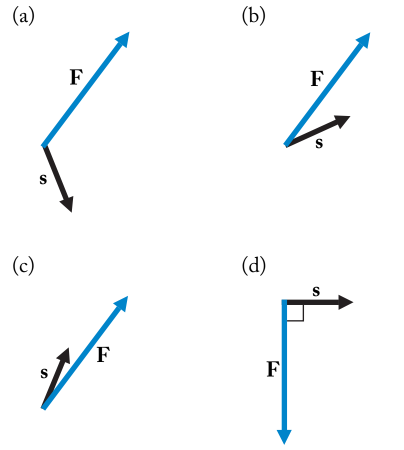


FIGURE 7.9 Several equal-magnitude forces and displacements.

7.2 WORK FOR A VARIABLE FORCE

Online Concept Tutorial

The definition of work in the preceding section assumed that the force was constant (in magnitude and in direction). But many forces are not constant, and we need to refine our definition of work so we can deal with such forces. For example, suppose that you push a stalled automobile along a straight road, and suppose that the force you exert is not constant—as you move along the road, you sometimes push harder and sometimes less hard. Figure 7.10 shows how the force might vary with position. (The reason why you sometimes push harder is irrelevant—maybe the automobile passes through a muddy portion of the road and requires more of a push, or maybe you get impatient and want to hurry the automobile along; all that is relevant for the calculation of the work is the value of the force at different positions, as shown in the plot.)

Such a variable force can be expressed as a function of position:

$$F_x = F_x(x)$$

(here the subscript indicates the x component of the force, and the x in parentheses indicates that this component is a function of x; that is, it varies with x, as shown in the diagram). To evaluate the work done by this variable force on the automobile, or on a particle, during a displacement from $x = a$ to $x = b$, we divide the total displacement into a large number of small intervals, each of length Δx (see Fig. 7.11). The beginnings and ends of these intervals are located at $x_0, x_1, x_2, \ldots, x_n$, where the first location x_0 coincides with a and the last location x_n coincides with b. Within each of the small intervals, the force can be regarded as approximately constant—within the interval x_{i-1} to x_i (where $i = 1$, or 2, or 3, . . . , or n), the force is approximately $F_x(x_i)$. This approximation is at its best if we select Δx to be very small. The work done by this force as the particle moves from x_{i-1} to x_i is then

$$W_i = F_x(x_i)\,\Delta x \tag{7.12}$$

and the total work done as the particle moves from a to b is simply the sum of all the small amounts of work associated with the small intervals:

$$W = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} F_x(x_i)\,\Delta x \tag{7.13}$$

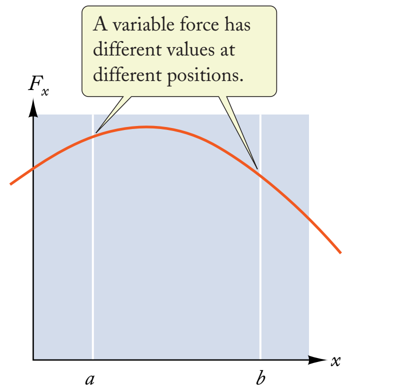


FIGURE 7.10 Plot of F_x vs. x for a force that varies with position.

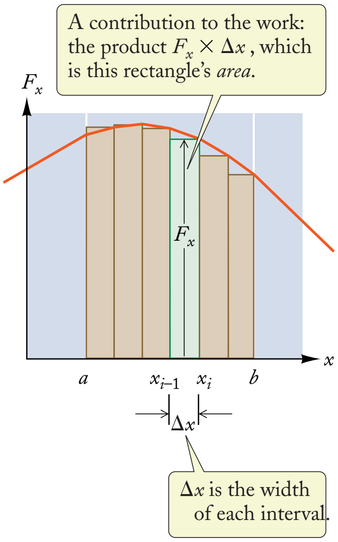


FIGURE 7.11 The curved plot of F_x vs. x has been approximated by a series of horizontal and vertical steps. This is a good approximation if Δx is very small.

Note that each of the terms $F_x(x_i)\,\Delta x$ in the sum is the area of a rectangle of height $F_x(x_i)$ and width Δx, highlighted in color in Fig. 7.11. Thus, Eq. (7.13) gives the sum of all the rectangular areas shown in Fig. 7.11.

Equation (7.13) is only an approximation for the work. In order to improve this approximation, we must use a smaller interval Δx. In the limiting case $\Delta x \to 0$ (and $n \to \infty$), the width of each rectangle approaches zero and the number of rectangles approaches infinity, so we obtain an exact expression for the work. Thus, the exact definition for the work done by a variable force is

$$W = \lim_{\Delta x \to 0} \sum_{i=1}^{n} F_x(x_i)\,\Delta x$$

This expression is called the **integral** of the function $F_x(x)$ between the limits a and b. The usual notation for this integral is

$$W = \int_a^b F_x(x)\,dx \qquad (7.14)$$

where the symbol $\int$ is called the integral sign and the function $F_x(x)$ is called the integrand. The quantity (7.14) is equal to *the area bounded by the curve representing* $F_x(x)$, *the x axis, and the vertical lines* $x = a$ *and* $x = b$ in Fig. 7.12. More generally, for a curve that has some portions above the x axis and some portions below, the quantity (7.14) is the net area bounded by the curve above and below the x axis, with areas above the x axis being reckoned as positive and areas below the x axis as negative.

We will also need to consider arbitrarily small contributions to the work. From Eq. (7.12), the infinitesimal work dW done by the force $F_x(x)$ when acting over an infinitesimal displacement dx is

$$dW = F_x(x)\,dx \qquad (7.15)$$

We will see later that the form (7.15) is useful for calculations of particular quantities, such as power or torque.

Finally, if the force is variable and the motion is in more than one dimension, the work can be obtained by generalizing Eq. (7.7):

work done by a variable force

$$W = \int \mathbf{F} \cdot d\mathbf{s} \qquad (7.16)$$

To evaluate Eq. (7.16), it is often easiest to express the integral as the sum of three integrals, similar to the form of Eq. (7.9). For now, we consider the use of Eq. (7.14) to determine the total work done by a variable force as it acts over some distance in one dimension.

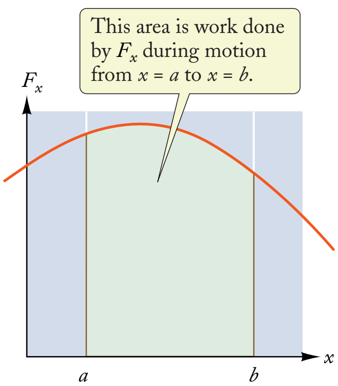


FIGURE 7.12 The integral $\int_a^b F_x(x)\,dx$ is the area (colored) under the curve representing $F_x(x)$ between $x = a$ and $x = b$.

EXAMPLE 4 A spring exerts a restoring force $F_x(x) = -kx$ on a particle attached to it (compare Section 6.2). What is the work done by the spring on the particle when it moves from $x = a$ to $x = b$?

SOLUTION: By Eq. (7.14), the work is the integral

$$W = \int_a^b F_x(x)\,dx = \int_a^b (-kx)\,dx$$

To evaluate this integral, we rely on a result from calculus (see the Math Help box on integrals) which states that the integral between a and b of the function x is the difference between the values of $\frac{1}{2}x^2$ at $x = b$ and $x = a$:

$$\int_a^b x\,dx = \tfrac{1}{2}x^2\Big|_a^b = \tfrac{1}{2}(b^2 - a^2)$$

where the vertical line | means that we evaluate the preceding function at the upper limit and then subtract its value at the lower limit. Since the constant $-k$ is just a multiplicative factor, we may pull it outside the integral and obtain for the work

$$W = \int_a^b (-kx)\,dx = -k\int_a^b x\,dx = -\tfrac{1}{2}k(b^2 - a^2) \qquad (7.17)$$

This result can also be obtained by calculating the area in a plot of force vs. position. Figure 7.13 shows the force $F(x) = -kx$ as a function of x. The area of the quadrilateral $aQPb$ that represents the work W is the difference between the areas of the two triangles OPb and OQa. The triangular area above the $F_x(x)$ curve between the origin and $x = b$ is $\frac{1}{2}$[base] × [height] = $\frac{1}{2}b \times kb = \frac{1}{2}kb^2$. Likewise, the triangular area between the origin and $x = a$ is $\frac{1}{2}ka^2$. The difference between these areas is $\frac{1}{2}k(b^2 - a^2)$. Taking into account that areas below the x axis must be reckoned as negative, we see that the work W is $W = -\frac{1}{2}k(b^2 - a^2)$, in agreement with Eq. (7.17).

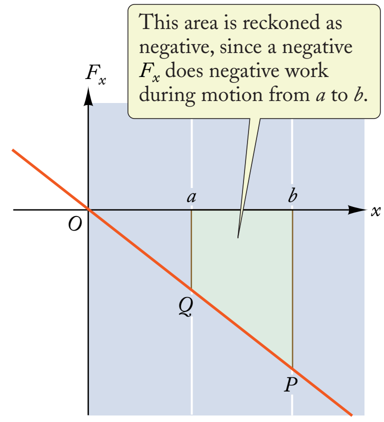


FIGURE 7.13 The plot of the force $F = -kx$ is a straight line. The work done by the force as the particle moves from a to b equals the (colored) quadrilateral area $aQPb$ under this plot.

MATH HELP INTEGRALS

The following are some theorems for integrals that we will frequently use.

The integral of a constant times a function is the constant times the integral of the function:

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

The integral of the sum of two functions is the sum of the integrals:

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

The integral of the function x^n (for $n \neq -1$) is

$$\int_a^b x^n\,dx = \frac{1}{n+1}x^{n+1}\Big|_a^b = \frac{1}{n+1}(b^{n+1} - a^{n+1})$$

In tables of integrals, this is usually written in the compact notation

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \qquad (\text{for } n \neq -1)$$

where it is understood that the right side is to be evaluated at the upper and at the lower limits of integration and then subtracted.

In a similar compact notation, here are a few more integrals of widely used functions (the quantity k is any constant):

$$\int \frac{1}{x}\,dx = \ln x$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx}$$

$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx)$$

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx)$$

Appendix 4 gives more information on integrals.

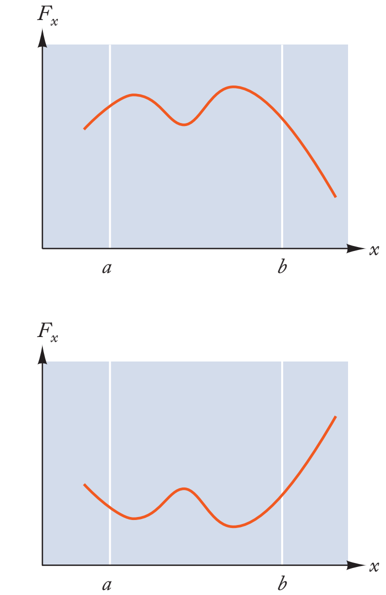


FIGURE 7.14 Two examples of plots of variable forces.

Checkup 7.2

QUESTION 1: Figure 7.14 shows two plots of variable forces acting on two particles. Which of these forces will perform more work during a displacement from a to b?

QUESTION 2: Suppose that a spring exerts a force $F_x(x) = -kx$ on a particle. What is the work done by the spring as the particle moves from $x = -b$ to $x = +b$?

QUESTION 3: What is the work that *you* must do to pull the end of the spring described in Example 4 from $x = a$ to $x = b$?

QUESTION 4: An amount of work W is performed to stretch a spring by a distance d from equilibrium. How much work is performed to further stretch the spring from d to $2d$?

(A) $\frac{1}{2}W$ (B) W (C) $2W$ (D) $3W$ (E) $4W$

7.3 KINETIC ENERGY

In everyday language, energy means a capacity for vigorous activities and hard work. Likewise, in the language of physics, energy is a capacity for performing work. Energy is "stored" work, or latent work, which can be converted into actual work under suitable conditions. *A body in motion has energy of motion, or kinetic energy*. For instance, a speeding arrow has kinetic energy that will be converted into work when the arrow strikes a target, such as a the trunk of a tree. The tip of the arrow then performs work on the wood, prying apart and cutting the wood fibers. The arrow continues to perform work and to penetrate the wood for a few centimeters, until all of its kinetic energy has been exhausted. A high-speed arrow has a deeper penetration and delivers a larger amount of work to the target than a low-speed arrow. Thus, we see that the kinetic energy of the arrow, or the kinetic energy of any kind of particle, must be larger if the speed is larger.

We now examine how work performed by or on a particle is related to changes of the speed of the particle. For clarity, we consider the work done *on* a particle by the net external force F_{net} acting on it (rather than the work done *by* the particle). When the force F_{net} acts on the particle, it accelerates the particle; if the acceleration has a component along the direction of motion of the particle, it will result in a change of the speed of the particle. The force does work on the particle and "stores" this work in the particle; or, if this force decelerates the particle, it does negative work on the particle and removes "stored" work.

We can establish an important identity between the work done by the net force and the change of speed it produces. Let us do this for the simple case of a particle moving along a straight line (see Fig. 7.15). If this straight line coincides with the x axis, then the work done by the net force $F_{net,x}$ during a displacement from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F_{net,x}\, dx \tag{7.18}$$

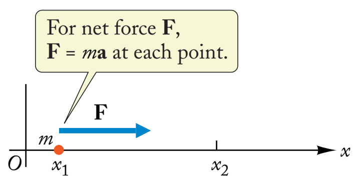


FIGURE 7.15 A particle moves on a straight line from x_1 to x_2 while a force **F** acts on it.

By Newton's Second Law, the net force equals the mass m times the acceleration $a = dv/dt$, and therefore the integral equals

$$\int_{x_1}^{x_2} F_{net,x}\, dx = \int_{x_1}^{x_2} ma\, dx = m \int_{x_1}^{x_2} \frac{dv}{dt}\, dx \tag{7.19}$$

The velocity v is a function of time; but in the integral (7.19) it is better to regard the velocity as a function of x, and to rewrite the integrand as follows:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = v\frac{dv}{dx} \tag{7.20}$$

Consequently, the work becomes

$$m\int_{x_1}^{x_2}\frac{dv}{dt}\,dx = m\int_{x_1}^{x_2} v\frac{dv}{dx}\,dx = m\int_{v_1}^{v_2} v\,dv = m\,\tfrac{1}{2}\,v^2\Big|_{v_1}^{v_2}$$
$$= \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \tag{7.21}$$

or

$$W = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \tag{7.22}$$

This shows that the change in the square of the speed is proportional to the work done by the force.

Although we have here obtained the result (7.22) for the simple case of motion along a straight line, it can be shown that the same result is valid for motion along a curve, in three dimensions.

According to Eq. (7.22), whenever we perform positive work on the particle, we increase the "amount of $\frac{1}{2}mv^2$" in the particle; and whenever we perform negative work on the particle (that is, when we let the particle perform work on us), we decrease the "amount of $\frac{1}{2}mv^2$" in the particle. Thus, *the quantity* $\frac{1}{2}mv^2$ *is the amount of work stored in the particle, or the* **kinetic energy** *of the particle*. We represent the kinetic energy by the symbol K:

$$K = \tfrac{1}{2}mv^2 \tag{7.23}$$

kinetic energy

With this notation, Eq. (7.22) states that *the change of kinetic energy equals the net work done on the particle;* that is,

$$K_2 - K_1 = W \tag{7.24}$$

or

$$\Delta K = W \tag{7.25}$$

work–energy theorem

This result is called the **work–energy theorem**. Keep in mind that the work in Eqs. (7.22), (7.24), and (7.25) must be evaluated with the *net* force; that is, all the forces that do work on the particle must be included in the calculation.

When a force does positive work on a particle initially at rest, the kinetic energy of the particle increases. The particle then has a capacity to do work: if the moving particle subsequently is allowed to push against some obstacle, then this obstacle does negative work on the particle and simultaneously the particle does positive work on the obstacle. When the particle does work, its kinetic energy decreases. The total amount of work the particle can deliver to the obstacle is equal to its kinetic energy. Thus, *the kinetic energy represents the capacity of a particle to do work by virtue of its speed.*

The acquisition of kinetic energy through work and the subsequent production of work by this kinetic energy are neatly illustrated in the operation of a waterwheel driven by falling water. In a flour mill of an old Spanish Colonial design, the water runs down from a reservoir in a steep, open channel (see Fig. 7.16). The motion of the water particles is essentially that of particles sliding down an inclined plane. If we

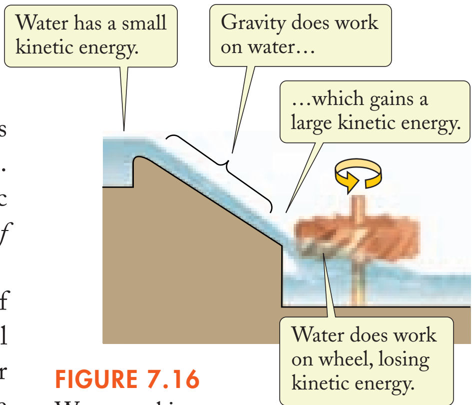


FIGURE 7.16
Water pushing on a horizontal waterwheel.

ignore friction, then the only force that does work on the water particles is gravity. This work is positive, so the kinetic energy of the water increases and it attains a maximum value at the lower end of the channel (where its speed is greatest). The stream of water emerges from this channel with high kinetic energy and hits the blades of the waterwheel. The water pushes on the wheel, turns it, and gives up its kinetic energy while doing work—and the wheel runs the millstones and does useful work on them. Thus, the work that gravity does on the descending water is ultimately converted into useful work, with the kinetic energy playing an intermediate role in this process.

The unit of kinetic energy is the joule, the same as the unit of work. Table 7.1 lists some typical kinetic energies.

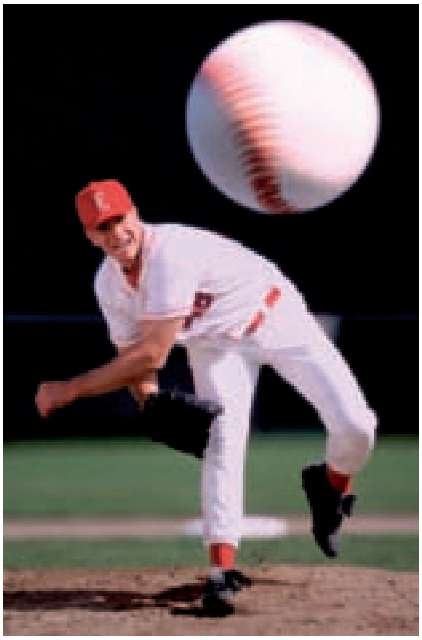


FIGURE 7.17 Pitcher throwing a ball. The ball leaves his hand with a speed of 30 m/s.

EXAMPLE 5 During a baseball game, the pitcher throws the ball with a speed of 30 m/s (Fig. 7.17). The mass of the ball is 0.15 kg. What is the kinetic energy of the ball when it leaves his hand? How much work did his hand do on the ball during the throw?

SOLUTION: The final speed of the ball, when it leave the hand at the end of the throwing motion, is $v_2 = 30$ m/s. The final kinetic energy of the ball is

$$K_2 = \tfrac{1}{2}mv_2^2 = \tfrac{1}{2} \times 0.15 \text{ kg} \times (30 \text{ m/s})^2 = 68 \text{ J} \tag{7.26}$$

According to the work–energy theorem [Eq. (7.25)], the work done by the hand on the ball equals the change of kinetic energy. Since the initial kinetic energy at the beginning of the throwing motion is zero ($v_1 = 0$), the change of kinetic energy equals the final kinetic energy, and the work is

$$W = K_2 - K_1 = 68 \text{ J} - 0 = 68 \text{ J}$$

Note that for this calculation of the work we did not need to know the (complicated) details of how the force varies during the throwing motion. The work–energy theorem gives us the answer directly.

TABLE 7.1 SOME KINETIC ENERGIES

Orbital motion of Earth	2.6×10^{33} J
Ship *Queen Elizabeth* (at cruising speed)	9×10^{9} J
Jet airliner (Boeing 747 at maximum speed)	7×10^{9} J
Automobile (at 90 km/h)	5×10^{5} J
Rifle bullet	4×10^{3} J
Person walking	60 J
Falling raindrop	4×10^{-5} J
Proton from large accelerator (Fermilab)	1.6×10^{-7} J
Electron in atom (hydrogen)	2.2×10^{-18} J
Air molecule (at room temperature)	6.2×10^{-21} J

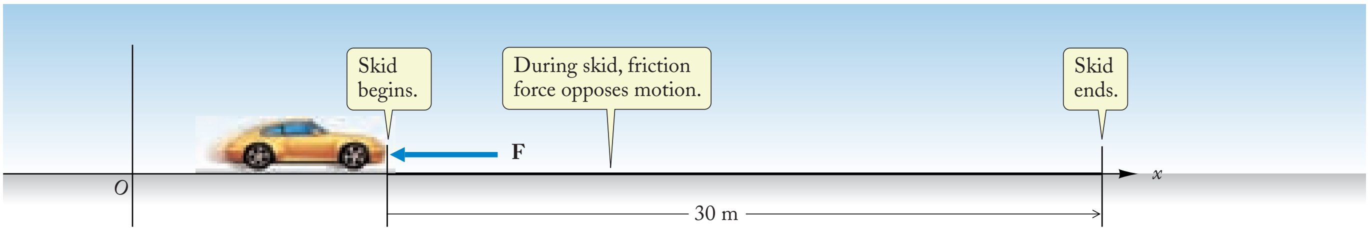


FIGURE 7.18 Automobile skidding on a street.

EXAMPLE 6 While trying to stop his automobile on a flat street, a drunk driver steps too hard on the brake pedal and begins to skid. He skids for 30 m with all wheels locked, leaving skid marks on the pavement, before he releases the brake pedal and permits the wheels to resume rolling (see Fig. 7.18). How much kinetic energy does the automobile lose to friction during this skid? If you find skid marks of 30 m on the pavement, what can you conclude about the initial speed of the automobile? The mass of the automobile is 1100 kg, and the coefficient of sliding friction between the wheels and the street is $\mu_k = 0.90$.

SOLUTION: The magnitude of the sliding friction force is $f_k = \mu_k N = \mu_k mg$. With the x axis along the direction of motion, the x component of this friction force is negative:

$$F_x = -\mu_k mg$$

Since the force is constant, the work done by this force is

$$W = F_x \,\Delta x = -\mu_k mg \times \Delta x$$

$$= -0.90 \times 1100 \text{ kg} \times 9.81 \text{ m/s}^2 \times 30 \text{ m} = -2.9 \times 10^5 \text{ J}$$

According to the work–energy theorem, this work equals the change of kinetic energy:

$$\Delta K = W = -2.9 \times 10^5 \text{ J}$$

Since the kinetic energy of the automobile decreases by 2.9×10^5 J, its initial kinetic energy must have been at least 2.9×10^5 J. Hence its initial speed must have been at least large enough to provide this kinetic energy; that is,

$$\tfrac{1}{2}mv_1^2 \geq 2.9 \times 10^5 \text{ J}$$

and so

$$v_1 \geq \sqrt{\frac{2 \times 2.9 \times 10^5 \text{ J}}{m}} = \sqrt{\frac{2 \times 2.9 \times 10^5 \text{ J}}{1100 \text{ kg}}} = 23 \text{ m/s} = 83 \text{ km/h}$$

Checkup 7.3

QUESTION 1: Two automobiles of equal masses travel in opposite directions. Can they have equal kinetic energies?

QUESTION 2: A car is traveling at 80 km/h on a highway, and a truck is traveling at 40 km/h. Can these vehicles have the same kinetic energy? If so, what must be the ratio of their masses?

QUESTION 3: Consider a golf ball launched into the air. The ball rises from the ground to a highest point, and then falls back to the ground. At what point is the kinetic energy largest? Smallest? Is the kinetic energy ever zero?

QUESTION 4: A horse is dragging a sled at steady speed along a rough surface, with friction. The horse does work on the sled, but the kinetic energy of the sled does not increase. Does this contradict the work–energy theorem?

QUESTION 5: If you increase the speed of your car by a factor of 3, from 20 km/h to 60 km/h, by what factor do you change the kinetic energy?

(A) $\frac{1}{9}$ (B) $\frac{1}{3}$ (C) 1 (D) 3 (E) 9

PROBLEM-SOLVING TECHNIQUES — CALCULATION OF WORK

In calculations of the work done by a force acting on a body, keep in mind that

- A force that has a component in the direction of the displacement does positive work; a force that has a component in the direction opposite to the displacement does negative work.
- A force perpendicular to the displacement does no work [examples: the normal force acting on a body sliding on a surface, the centripetal force acting on a body in circular motion (uniform or not)].
- For a constant force, the work can be calculated either from Eq. (7.5) or from Eq. (7.9); use the former if you know the magnitude of the force and the angle, and use the latter if you know the components.
- For a variable force, the calculation of the work involves integration along the path [Eq. (7.14)]; also, Eq. (7.15) can be used for the work during an infinitesimal displacement.
- The work–energy theorem is valid only if the work is calculated with the *net* force. When two of the three quantities (work done, initial kinetic energy, and final kinetic energy) are known, the theorem can be applied to determine the third: $W = K_2 - K_1$.

Online Concept 9 Tutorial

7.4 GRAVITATIONAL POTENTIAL ENERGY

As we saw in the preceding section, the kinetic energy represents the capacity of a particle to do work by virtue of its speed. We will now become acquainted with another form of energy that represents the capacity of the particle to do work by virtue of its position in space. This is the **potential energy**. In this section, we will examine the special case of gravitational potential energy for a particle moving under the influence of the constant gravitational force near the surface of the Earth, and we will formulate a law of conservation of energy for such a particle. In the next chapter we will examine other cases of potential energy and formulate the General Law of Conservation of Energy.

The gravitational potential energy represents the capacity of the particle to do work by virtue of its height above the surface of the Earth. When we lift a particle to some height above the surface, we have to do work against gravity, and we thereby store work in the particle. Thus, a particle high above the surface is endowed with a large amount of latent work, which can be exploited and converted into actual work by allowing the particle to push against some obstacle as it descends. A good example of such an exploitation of gravitational potential energy is found in a grandfather clock, where a weight hanging on a cord drives the wheel of the clock (Fig. 7.19). The weight does work on the wheel, and gradually converts all of its gravitational potential energy into work as it descends (in a typical grandfather clock, the weight takes about a week to sink down from the top to the bottom, and you must then rewind the clock, by lifting the weight).

FIGURE 7.19 The descending weights of the grandfather clock pull on the cords and do work on the wheel of the clock.

To obtain a general expression for the gravitational potential energy of a particle moving on a straight or a curving path, we first consider a particle moving on an inclined plane. According to Eq. (7.11), when a particle of mass m descends a distance h along an inclined plane, the work done by gravity is

$$W = mgh \tag{7.27}$$

As already remarked on in Example 3, this result is independent of the angle of inclination of the plane—it depends only on the change of height. More generally, for a curved path, the result is independent of the shape of the path that the particle follows from its starting point to its endpoint. For instance, the curved path and the straight sloping path in Fig. 7.20a lead to exactly the same result (7.27) for the work done by gravity. To recognize this, we simply approximate the curved path by small straight segments (see Fig. 7.20b). Each such small segment can be regarded as a small inclined plane, and therefore the work is mg times the small change of height. The net amount of work for all the small segments taken together is then mg times the net change of height, in agreement with Eq. (7.27).

If the vertical coordinate of the starting point is y_1 and the vertical coordinate of the endpoint is y_2 (see Fig. 7.20), then $h = y_1 - y_2$ and Eq. (7.27) becomes

$$W = mg(y_1 - y_2) \qquad \text{or} \qquad W = -(mgy_2 - mgy_1) \tag{7.28}$$

According to Eq. (7.28), whenever gravity performs positive work on the particle ($y_1 > y_2$, a descending particle), the "amount of mgy" of the particle decreases; and

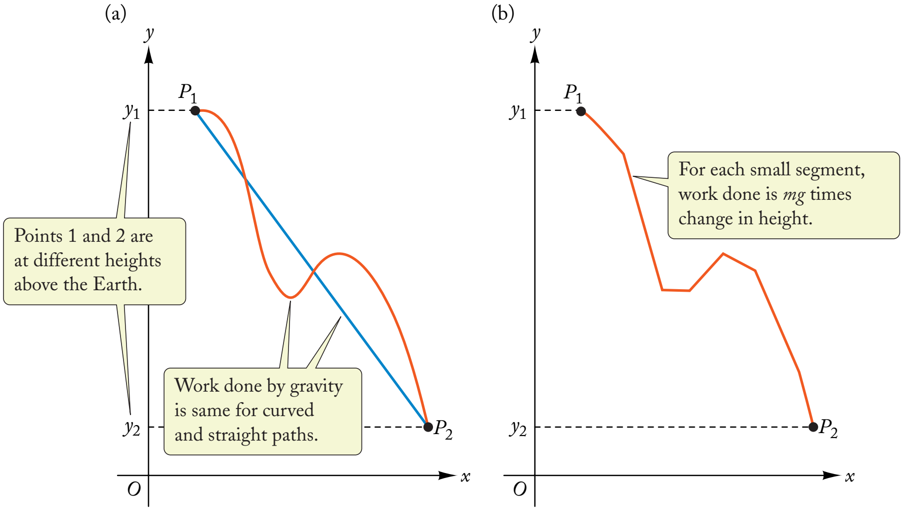


FIGURE 7.20 (a) A curved path (red) and a straight path (blue) from point P_1 to point P_2. (b) The curved path can be approximated by short straight segments.

whenever gravity performs negative work on the particle ($y_1 < y_2$, an ascending particle), the "amount of mgy" increases. Thus, *the quantity mgy represents the amount of stored, or latent, gravitational work; that is, it represents the gravitational potential energy*. We will adopt the notation U for the **gravitational potential energy**:

gravitational potential energy

$$U = mgy \tag{7.29}$$

This potential energy is directly proportional to the height y, and it has been chosen to be zero at $y = 0$ (see Fig. 7.21).

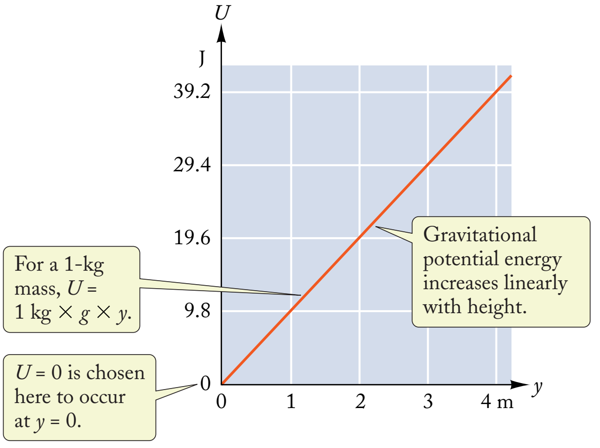


FIGURE 7.21 Plot of the gravitational potential energy of a mass of 1 kg as a function of height y.

In terms of the gravitational potential energy, Eq. (7.28) for the work done by gravity becomes

$$W = -U_2 + U_1 \tag{7.30}$$

Since $\Delta U = U_2 - U_1$ is the change in potential energy, Eq. (7.30) says that the work equals the negative of the change in potential energy,

$$W = -\Delta U \tag{7.31}$$

EXAMPLE 7 What is the kinetic energy and what is the gravitational potential energy (relative to the ground) of a jet airliner of mass 73 000 kg cruising at 240 m/s at an altitude of 9000 m?

SOLUTION: The kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2} \times 7.3 \times 10^4 \text{ kg} \times (240 \text{ m/s})^2 = 2.1 \times 10^9 \text{ J}$$

The gravitational potential energy is $U = mgy$. If we measure the y coordinate from the ground level, then $y = 9000$ m for our airliner, and

$$U = mgy = 7.3 \times 10^4 \text{ kg} \times 9.81 \text{ m/s}^2 \times 9.0 \times 10^3 \text{ m} = 6.4 \times 10^9 \text{ J}$$

We see that the airliner has about three times more potential energy than kinetic energy.

If we let the particle push or pull on some obstacle (such as the wheel of the grandfather clock) during its descent from y_1 to y_2, then the total amount of work that we can extract during this descent is equal to the work done by gravity; that is, it is equal to $-U_2 + U_1 = -(U_2 - U_1) = -\Delta U$, or the negative of the change of potential energy. Of course, the work extracted in this way really arises from the Earth's gravity—the particle can do work on the obstacle because gravity is doing work on the particle. Hence *the gravitational potential energy is really a joint property of the particle and the Earth; it is a property of the configuration of the particle–Earth system.*

If the only force acting on the particle is gravity, then by combining Eqs. (7.24) and (7.30) we can obtain a relation between potential energy and kinetic energy. According to Eq. (7.24), the change in kinetic energy equals the work, or $K_2 - K_1 = W$; and according to Eq. (7.30), the negative of the change in potential energy also equals the work: $W = -U_2 + U_1$. Hence the change in kinetic energy must equal the negative of the change in potential energy:

$$K_2 - K_1 = -U_2 + U_1$$

We can rewrite this as follows:

$$K_2 + U_2 = K_1 + U_1 \tag{7.32}$$

This equality indicates that the quantity $K + U$ is a constant of the motion; that is, it has the same value at the endpoint as it had at the starting point. We can express this as

$$K + U = [\text{constant}] \tag{7.33}$$

The sum of the kinetic and potential energies is called the **mechanical energy** *of the particle*. It is usually designated by the symbol E:

$$E = K + U \tag{7.34}$$

mechanical energy

This energy represents the total capacity of the particle to do work by virtue of both its speed and its position.

Equation (7.33) shows that if the only force acting on the particle is gravity, then the mechanical energy remains constant:

$$E = K + U = [\text{constant}] \tag{7.35}$$

Law of Conservation of Mechanical Energy

This is the **Law of Conservation of Mechanical Energy**.

Since the sum of the potential and kinetic energies must remain constant during the motion, an increase in one must be compensated by a decrease in the other; this means that *during the motion, kinetic energy is converted into potential energy and vice versa*. For instance, if we throw a baseball straight upward from ground level ($y = 0$), the initial kinetic energy is large and the initial potential energy is zero. As the baseball rises, its potential energy increases and, correspondingly, its kinetic energy decreases, so as to keep the sum of the kinetic and potential energies constant. When the baseball reaches its maximum height, its potential energy has the largest value, and the kinetic energy is (instantaneously) zero. As the baseball falls, its potential energy decreases, and its kinetic energy increases (see Fig. 7.22).

Apart from its practical significance in terms of work, the mechanical energy is very helpful in the study of the motion of a particle. If we make use of the formulas for K and U, Eq. (7.35) becomes

$$E = \tfrac{1}{2}mv^2 + mgy = [\text{constant}] \tag{7.36}$$

CHRISTIAAN HUYGENS (1629–1695) *Dutch mathematician and physicist. He invented the pendulum clock, made improvements in the manufacture of telescope lenses, and discovered the rings of Saturn. Huygens investigated the theory of collisions of elastic bodies and the theory of oscillations of the pendulum, and he stated the Law of Conservation of Mechanical Energy for motion under the influence of gravity.*

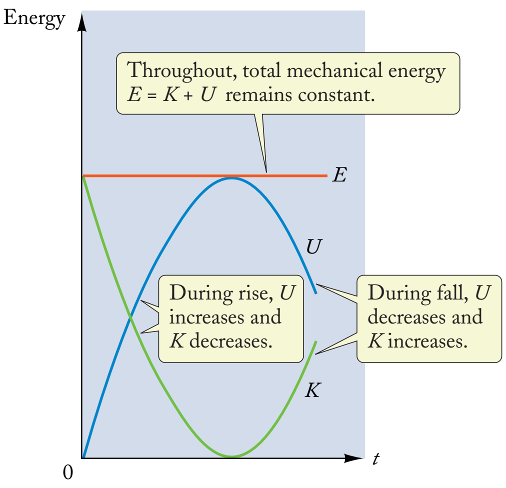


FIGURE 7.22 Kinetic energy K, potential energy U, and mechanical energy $E = K + U$ as functions of time during the upward and downward motions of a baseball.

This shows explicitly how the baseball, or any other particle moving under the influence of gravity, trades speed for height during the motion: whenever y increases, v must decrease (and conversely) so as to keep the sum of the two terms on the left side of Eq. (7.36) constant.

If we consider the vertical positions (y_1 and y_2) and speeds (v_1 and v_2) at two different times, we can equate the total mechanical energy at those two times:

$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2$$

Rearranging, we immediately obtain

$$-g\,\Delta y = \frac{1}{2}(v_2^2 - v_1^2) \qquad (7.37)$$

where $\Delta y = y_2 - y_1$. We recognize Eq. (7.37) as the same form that we obtained when studying the equations of motion [see Eq. (2.29)]. Here, however, the result follows directly from conservation of mechanical energy; we did not need to determine the detailed time dependence of the motion.

An important aspect of Eq. (7.36) is that it is valid not only for a particle in free fall (a projectile), but also for a particle sliding on a surface or a track of arbitrary shape, provided that there is no friction. Of course, under these conditions, besides the gravitational force there also acts the normal force; but this force does no work, and hence does not affect Eq. (7.28), or any of the equations following after it. The next example illustrates how these results can be applied to simplify the study of fairly complicated motions, which would be extremely difficult to investigate by direct calculation with Newton's Second Law. This example gives us a glimpse of the elegance and power of the Law of Conservation of Mechanical Energy.

Concepts in Context

EXAMPLE 8 A roller-coaster car descends 38 m from its highest point to its lowest. Suppose that the car, initially at rest at the highest point, rolls down this track without friction. What speed will the car attain at the lowest point? Treat the motion as particle motion.

SOLUTION: The coordinates of the highest and the lowest points are $y_1 = 38$ m and $y_2 = 0$, respectively (see Fig. 7.23). According to Eq. (7.36), the energy at the start of the motion for a car initially at rest is

$$E = \frac{1}{2}mv_1^2 + mgy_1 = 0 + mgy_1 \qquad (7.38)$$

and the energy at the end of the motion is

$$E = \frac{1}{2}mv_2^2 + mgy_2 = \frac{1}{2}mv_2^2 + 0 \qquad (7.39)$$

The conservation of energy implies that the right sides of Eqs. (7.38) and (7.39) are equal:

$$\frac{1}{2}mv_2^2 = mgy_1 \qquad (7.40)$$

Solving this for v_2, we find

$$v_2 = \sqrt{2gy_1} \qquad (7.41)$$

(a)

(b)

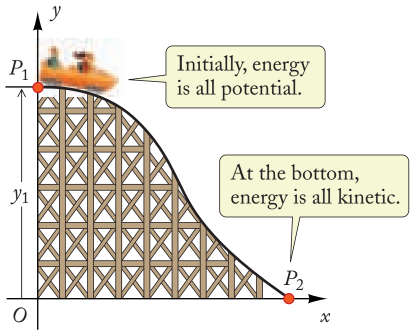


FIGURE 7.23 (a) A roller coaster. (b) Profile of a roller coaster. The roller-coaster car descends from P_1 to P_2.

which gives

$$v_2 = \sqrt{2 \times 9.81\ \text{m/s}^2 \times 38\ \text{m}} = 27\ \text{m/s}$$

Note that according to Eq. (7.41) the final velocity is independent of the mass of the car; since both the kinetic energy and the gravitational potential energy are proportional to mass, the mass cancels in this calculation.

COMMENT: This example illustrates how energy conservation can be exploited to answer a question about motion. To obtain the final speed by direct computation of forces and accelerations would have been extremely difficult—it would have required detailed knowledge of the shape of the path down the hill. With the Law of Conservation of Energy we can bypass these complications.

PROBLEM-SOLVING TECHNIQUES — ENERGY CONSERVATION IN ANALYSIS OF MOTION

As illustrated by the preceding example, the use of energy conservation in a problem of motion typically involves three steps:

1. First write an expression for the energy at one point of the motion [Eq. (7.38)].
2. Then write an expression for the energy at another point [Eq. (7.39)].
3. And then rely on energy conservation to equate the two expressions [Eq. (7.40)]. This yields one equation, which can be solved for the unknown final speed or the unknown final position (if the final speed is known).

Note that the value of the gravitational potential energy $U = mgy$ depends on the level from which you measure the y coordinate. However, the *change* in the potential energy does not depend on the choice of this level, and therefore any choice will lead to the same result for the change of kinetic energy. Thus, you can make any choice of zero level, but you must continue to use this choice throughout the entire calculation. You will usually find it convenient to place the zero level for the y coordinate either at the final position of the particle (as in the preceding example), or at the initial position, or at some other distinctive height, such as the bottom of a hill or the ground floor of a building. And always remember that the formula $U = mgy$ for the gravitational potential energy assumes that the y axis is directed vertically upward.

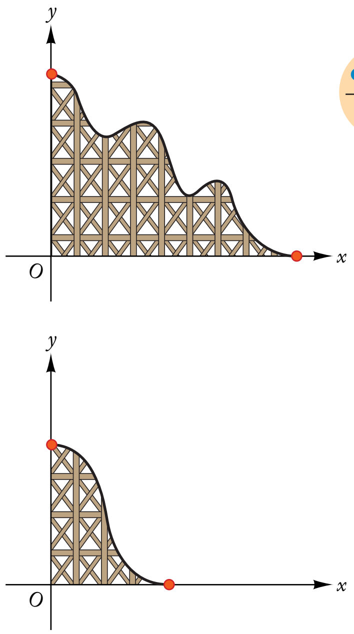


FIGURE 7.24 Two roller-coaster profiles. The two plots have the same vertical scale.

Concepts in Context

Checkup 7.4

QUESTION 1: Figure 7.24 shows two roller coasters in profile. Cars are released at the top of each, from rest. Which, if either, of these roller coasters gives the car a larger speed at the bottom? Neglect friction.

QUESTION 2: A piano is being moved from the second floor of one house to the second floor of another, nearby house. Describe the changes in the gravitational potential energy of the piano during this move.

QUESTION 3: A skidding truck slides down a mountain road, at constant speed. Is the mechanical energy $E = K + U$ conserved?

QUESTION 4: At an amusement park, a girl jumps off a high tower and lands in a pool. Meanwhile, a boy slides down a (frictionless) water slide that also takes him from the tower into the pool. Who reaches the pool with the higher speed? Who reaches the pool first?

QUESTION 5: A bicyclist rolls down a hill without braking, starting at the top, from rest. A second bicyclist rolls down the same hill, starting at one-half the height, from rest. By what factor will the speed of the first bicyclist be larger than that of the second, at the bottom? Ignore friction.

(A) $\sqrt{2}$ (B) 2 (C) $2\sqrt{2}$ (D) 4

SUMMARY

MATH HELP Integrals **(page 213)**

PROBLEM-SOLVING TECHNIQUES Calculation of Work **(page 218)**

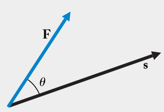
PROBLEM-SOLVING TECHNIQUES Energy Conservation in Analysis of Motion **(page 223)**

SI UNIT OF WORK (Unit of energy) joule = J = N·m

WORK DONE BY A CONSTANT FORCE

Parallel to a displacement Δx $W = F_x \Delta x$ **(7.1)**

Not parallel to a displacement **s** $W = Fs\cos\theta = \mathbf{F}\cdot\mathbf{s}$ **(7.6; 7.5)**



DOT PRODUCT (OR SCALAR PRODUCT) OF TWO VECTORS

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z \qquad (7.6; 7.8)$$

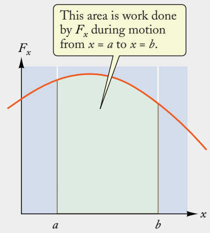
WORK DONE BY CONSTANT GRAVITATIONAL FORCE (Descending from a height h)

$$W = mgh \qquad (7.11)$$

WORK DONE BY A VARIABLE FORCE

In one dimension

$$W = \int_a^b F_x(x)\,dx \qquad (7.15)$$



In two or three dimensions

$$W = \int \mathbf{F}\cdot d\mathbf{s} \qquad (7.16)$$

WORK DONE BY A SPRING (Moving from $x = a$ to $x = b$)

$$W = -\tfrac{1}{2}k(b^2 - a^2) \qquad (7.17)$$

KINETIC ENERGY

$$K = \tfrac{1}{2}mv^2 \qquad (7.23)$$

WORK–ENERGY THEOREM

$$\Delta K = W \qquad (7.25)$$

GRAVITATIONAL POTENTIAL ENERGY

$$U = mgy \qquad (7.29)$$

RELATION BETWEEN WORK AND CHANGE IN POTENTIAL ENERGY

$$W = -\Delta U \qquad (7.31)$$

MECHANICAL ENERGY

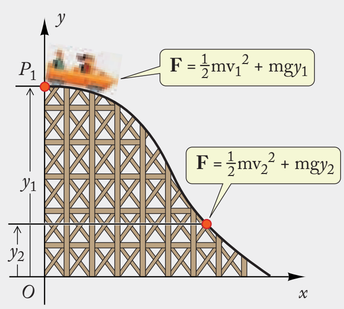
$$E = K + U \qquad (7.34)$$

CONSERVATION OF MECHANICAL ENERGY

$$E = K + U = [\text{constant}] \qquad (7.35)$$

CONSERVATION OF MECHANICAL ENERGY AT TWO POINTS

$$\tfrac{1}{2}mv_1^2 + mgy_1 = \tfrac{1}{2}mv_2^2 + mgy_2 \qquad (7.37)$$



QUESTIONS FOR DISCUSSION

1. Does the work of a force on a body depend on the frame of reference in which it is calculated? Give some examples.
2. Does your body do work (external or internal) when standing at rest? When walking steadily along a level road?
3. Consider a pendulum swinging back and forth. During what part of the motion does the weight do positive work? Negative work?
4. Since $v^2 = v_x^2 + v_y^2 + v_z^2$, Eq.(7.23) implies $K = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2$. Does this mean that the kinetic energy has x, y, and z components?
5. Consider a woman steadily climbing a flight of stairs. The external forces on the woman are her weight and the normal force of the stairs against her feet. During the climb, the weight does negative work, while the normal force does no work. Under these conditions how can the kinetic energy of the woman remain constant? (Hint: The entire woman cannot be regarded as a particle, since her legs are not rigid; but the upper part of her body can be regarded as a particle, since it is rigid. What is the force of her legs against the upper part of her body? Does this force do work?)
6. An automobile increases its speed from 80 to 88 km/h. What is the percentage of increase in kinetic energy? What is the percentage of reduction of travel time for a given distance?
7. Two blocks in contact slide past one another and exert friction forces on one another. Can the friction force *increase* the kinetic energy of one block? Of both? Does there exist a reference frame in which the friction force decreases the kinetic energy of both blocks?
8. When an automobile with rear-wheel drive is accelerating on, say, a level road, the horizontal force of the road on the rear wheels does not give the automobile any energy because the point of application of this force (point of contact of wheel on ground) is instantaneously at rest if the wheel is not slipping. What force gives the body of the automobile energy? Where does this energy come from? (Hint: Consider the force that the rear axle exerts against its bearings.)
9. Why do elevators have counterweights? (See Fig. 5.40.)
10. A parachutist jumps out of an airplane, opens a parachute, and lands safely on the ground. Is the mechanical energy for this motion conserved?
11. If you release a tennis ball at some height above a hard floor, it will bounce up and down several times, with a gradually decreasing amplitude. Where does the ball suffer a loss of mechanical energy?
12. Two ramps, one steeper than the other, lead from the floor to a loading platform (Fig. 7.25). It takes more force to push a (frictionless) box up the steeper ramp. Does this mean it takes more work to raise the box from the floor to the platform?

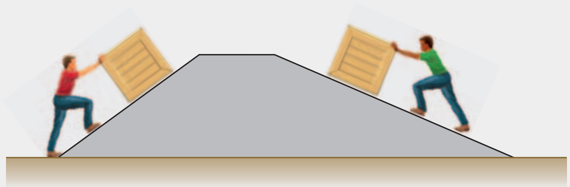


FIGURE 7.25 Two ramps of different steepness.

13. Consider the two ramps described in the preceding question. Taking friction into account, which ramp requires less work for raising a box from the floor to the platform?
14. A stone is tied to a string. Can you whirl this stone in a vertical circle with constant speed? Can you whirl this stone with constant energy? For each of these two cases, describe how you must move your hand.

PROBLEMS

7.1 Work†

1. If it takes a horizontal force of 300 N to push a stalled automobile along a level road at constant speed, how much work must you do to push this automobile a distance of 5.0 m?
2. In an overhead lift, a champion weight lifter raises 254 kg from the floor to a height of 1.98 m. How much work does he do?
3. Suppose that the force required to push a saw back and forth through a piece of wood is 35 N. If you push this saw back and forth 30 times, moving it forward 12 cm and back 12 cm each time, how much work do you do?
4. It requires 2200 J of work to lift a 15-kg bucket of water from the bottom of a well to the top. How deep is the well?
5. A child drags a 20-kg box across a lawn for 10 m and along a sidewalk for 30 m; the coefficient of friction is 0.25 for the first part of the trip and 0.55 for the second. If the child always pulls horizontally, how much work does the child do on the box?
6. A man moves a vacuum cleaner 1.0 m forward and 1.0 m back 300 times while cleaning a floor, applying a force of 40 N during each motion. The pushes and pulls make an angle of 60° with the horizontal. How much work does the man do on the vacuum cleaner?

† For help, see Online Concept Tutorial 9 at www.wwnorton.com/physics

7. A record for stair climbing was achieved by a man who raced up the 1600 steps of the Empire State Building to a height of 320 m in 10 min 59 s. If his average mass was 75 kg, how much work did he do against gravity? At what average rate (in J/s) did he do this work?

8. Suppose you push on a block sliding on a table. Your push has a magnitude of 50 N and makes a downward angle of 60° with the direction of motion. What is the work you do on the block while the block moves a distance of 1.6 m?

9. Consider the barge being pulled by two tugboats, as described in Example 4 of Chapter 5. The pull of the first tugboat is 2.5×10^5 N at 30° to the left, and the pull of the second tugboat is 1.0×10^5 N at 15° to the right (see Fig. 7.26). What is the work done by each tugboat on the barge while the barge moves 100 m forward (in the direction of the x axis in Fig. 7.26)? What is the total work done by both tugboats on the barge?

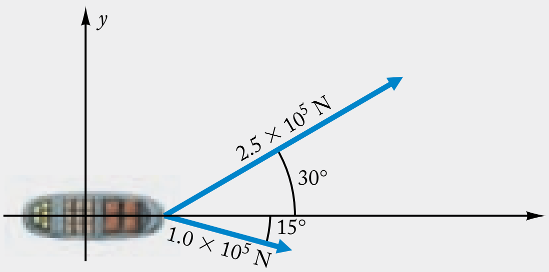


FIGURE 7.26 A barge pulled by two tugboats.

10. A 2.0-kg stone thrown upward reaches a height of 4.0 m at a horizontal distance of 6.0 m from the point of launch. What is the work done by gravity during this displacement?

*11. A man pushes a heavy box up an inclined ramp making an angle of 30° with the horizontal. The mass of the box is 60 kg, and the coefficient of kinetic friction between the box and the ramp is 0.45. How much work must the man do to push the box to a height of 2.5 m at constant speed? Assume that the man pushes on the box in a direction parallel to the surface of the ramp.

12. The driver of a 1200-kg automobile notices that, with its gears in neutral, it will roll downhill at a constant speed of 110 km/h on a road of slope 1:20. Draw a "free-body" diagram for the automobile, showing the force of gravity, the normal force (exerted by the road), and the friction force (exerted by the road and by air resistance). What is the magnitude of the friction force on the automobile under these conditions? What is the work done by the friction force while the automobile travels 1.0 km down the road?

13. Driving an automobile down a slippery, steep hill, a driver brakes and skids at constant speed for 10 m. If the automobile mass is 1700 kg and the angle of slope of the hill is 25°, how much work does gravity do on the car during the skid? How much work does friction do on the car?

14. The automobile in Example 6 of Chapter 6 is traveling on a flat road. For a trip of length 250 km, what is the total work done against air friction when traveling at 20 m/s? At 30 m/s?

15. A constant force of 25 N is applied to a body while it moves along a straight path for 12 m. The force does 175 J of work on the body. What is the angle between the force and the path of the body?

*16. A strong, steady wind provides a force of 150 N in a direction 30° east of north on a pedestrian. If the pedestrian walks first 100 m north and then 200 m east, what is the total work done by the wind?

*17. A man pulls a cart along a level road by means of a short rope stretched over his shoulder and attached to the front end of the cart. The friction force that opposes the motion of the cart is 250 N.

(a) If the rope is attached to the cart at shoulder height, how much work must the man do to pull the cart 50 m at constant speed?

(b) If the rope is attached to the cart below shoulder height so it makes an angle of 30° with the horizontal, what is the tension in the rope? How much work must the man now do to pull the cart 50 m? Assume that enough mass was added so the friction force is unchanged.

*18. A particle moves in the $x-y$ plane from the origin $x = 0$, $y = 0$ to the point $x = 2$, $y = -1$ while under the influence of a force $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j}$. How much work does this force do on the particle during this motion? The distances are measured in meters and the force in newtons.

*19. An elevator consists of an elevator cage and a counterweight attached to the ends of a cable that runs over a pulley (Fig. 7.27). The mass of the cage (with its load) is 1200 kg, and the mass of the counterweight is 1000 kg. The elevator is driven by an electric motor attached to the pulley. Suppose that the elevator is initially at rest on the first floor of the building and the motor makes the elevator accelerate upward at the rate of 1.5 m/s^2.

(a) What is the tension in the part of the cable attached to the elevator cage? What is the tension in the part of the cable attached to the counterweight?

(b) The acceleration lasts exactly 1.0 s. How much work has the electric motor done in this interval? Ignore friction forces and ignore the mass of the pulley.

(c) After the acceleration interval of 1.0 s, the motor pulls the elevator upward at constant speed until it reaches the third floor, exactly 10.0 m above the first floor. What is the total amount of work that the motor has done up to this point?

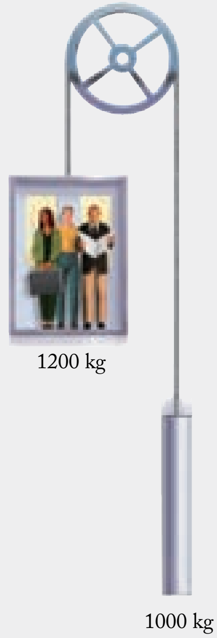


FIGURE 7.27 Elevator cage and counterweight.

*20. By means of a towrope, a girl pulls a sled loaded with firewood along a level, icy road. The coefficient of friction between the sled and the road is $\mu_k = 0.10$, and the mass of the sled plus its load is 150 kg. The towrope is attached to the front end of the sled and makes an angle of 30° with the horizontal. How much work must the girl do on the sled to pull it 1.0 km at constant speed?

*21. During a storm, a sailboat is anchored in a 10-m-deep harbor. The wind pushes against the boat with a steady horizontal force of 7000 N.

(a) The anchor rope that holds the boat in place is 50 m long and is stretched straight between the anchor and the boat (Fig. 7.28a). What is the tension in the rope?

(b) How much work must the crew of the sailboat do to pull in 30 m of the anchor rope, bringing the boat nearer to the anchor (Fig. 7.28b)? What is the tension in the rope when the boat is in this new position?

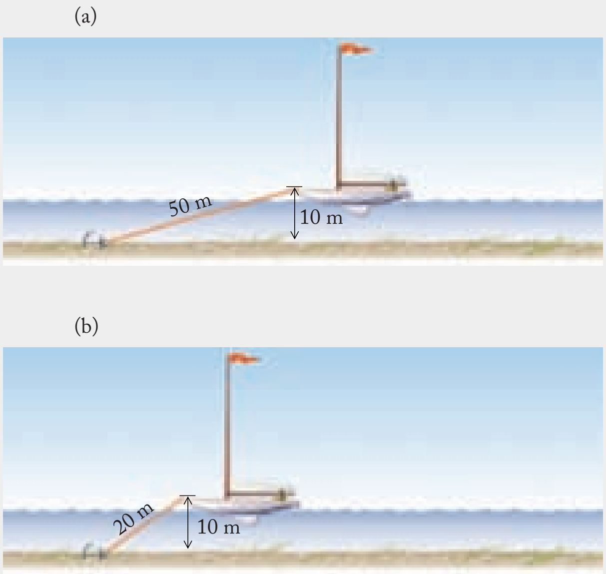


FIGURE 7.28 A sailboat at anchor.

7.2 Work for a Variable Force†

22. The spring used in the front suspension of a Triumph sports car has a spring constant $k = 3.5 \times 10^4$ N/m. How much work must you do to compress this spring by 0.10 m from its relaxed condition? How much more work must you do to compress the spring a further 0.10 m?

23. A particle moving along the x axis is subjected to a force F_x that depends on position as shown in the plot in Fig. 7.29. From this plot, find the work done by the force as the particle moves from $x = 0$ to $x = 8.0$ m.

24. A 250-g object is hung from a vertical spring, stretching it 18 cm below its original equilibrium position. How much work was done by gravity on the object? By the spring?

† For help, see Online Concept Tutorial 9 at www.wwnorton.com/physics

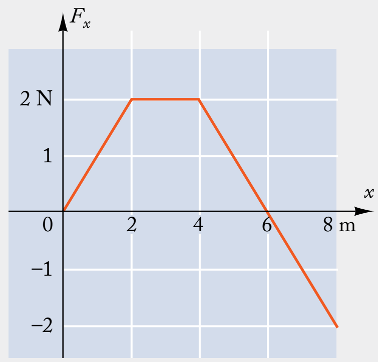


FIGURE 7.29 Position-dependent force.

25. When an ideal, horizontal spring is at equilibrium, a mass attached to its end is at $x = 0$. If the spring constant is 440 N/m, how much work does the spring do on the mass if the mass moves from $x = -0.20$ m to $x = +0.40$ m?

26. The spring on one kind of mousetrap has a spring constant of 4500 N/m. How much work is done to set the trap, by stretching the spring 2.7 cm from equilibrium?

*27. To stretch a spring a distance d from equilibrium takes an amount W_0 of work. How much work does it take to stretch the spring from d to $2d$ from equilibrium? From Nd to $(N + 1)d$ from equilibrium?

*28. A particular spring is not ideal; for a distance x from equilibrium, the spring exerts a force $F_x = -6x - 2x^3$, where x is in meters and F_x is in newtons. Compared with an ideal spring with a spring constant $k = 6.0$ N/m, by what factor does the work done by the nonideal spring exceed that done by the ideal spring when moving from $x = 0$ to $x = 0.50$ m? From $x = 1.0$ m to $x = 1.5$ m? From $x = 2.0$ m to $x = 2.5$ m?

*29. The ends of a relaxed spring of length l and force constant k are attached to two points on two walls separated by a distance l.

(a) How much work must you do to push the midpoint of the spring up or down a distance y (see Fig. 7.30)?

(b) How much force must you exert to hold the spring in this configuration?

*30. A particle moves along the x axis from $x = 0$ to $x = 2.0$ m. A force $F_x(x) = 2x^2 + 8x$ acts on the particle (the distance x is measured in meters, and the force in newtons). Calculate the work done by the force $F_x(x)$ during this motion.

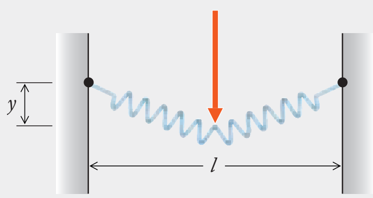


FIGURE 7.30 The midpoint of the spring has been pushed down a distance y. When the spring is relaxed, its length matches the distance l between the walls.

*31. Suppose that the force acting on a particle is a function of position; the force has components $F_x = 4x^2 + 1$, $F_y = 2x$, $F_z = 0$, where the force is measured in newtons and distance in meters. What is the work done by the force if the particle moves on a straight line from $x = 0$, $y = 0$, $z = 0$ to $x = 2.0$ m, $y = 2.0$ m, $z = 0$?

*32. A horse pulls a sled along a snow-covered curved ramp. Seen from the side, the surface of the ramp follows an arc of a circle of radius R (Fig. 7.31). The pull of the horse is always parallel to this surface. The mass of the sled is m, and the coefficient of sliding friction between the sled and the surface is μ_k. How much work must the horse do on the sled to pull it to a height $(1 - \sqrt{2}/2)R$, corresponding to an angle of 45° along the circle (Fig. 7.31)? How does this compare with the amount of work required to pull the sled from the same starting point to the same height along a straight ramp inclined at 22.5°?

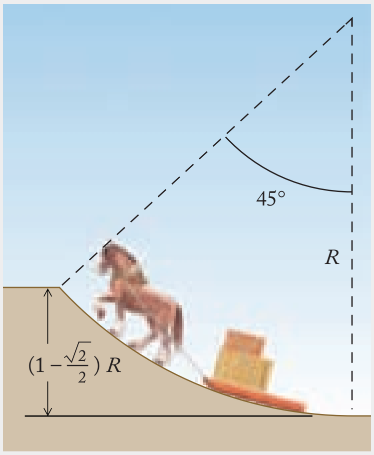


FIGURE 7.31 A horse pulling a sled along a curved ramp.

**33. The force between two inert gas atoms is often described by a function of the form

$$F_x = Ax^{-13} - Bx^{-7}$$

where A and B are positive constants and x is the distance between the atoms. Answer in terms of A and B.

(a) What is the equilibrium separation?

(b) What is the work done if the atoms are moved from their equilibrium separation to a very large distance apart?

7.3 Kinetic Energy

34. In a serve, a champion tennis player sends the ball flying at 160 km/h. The mass of the ball is 60 g. What is the kinetic energy of the ball?

35. Calculate the kinetic energy that the Earth has owing to its motion around the Sun.

36. The electron in a hydrogen atom has a speed of 2.2×10^6 m/s. What is the kinetic energy of this electron?

37. The fastest skier is Graham Wilkie, who attained 212.52 km/h on a steep slope at Les Arcs, France. The fastest runner is Robert Hayes, who briefly attained 44.88 km/h on a level track. Assume that the skier and the runner each have a mass of 75 kg. What is the kinetic energy of each? By what factor is the kinetic energy of the skier larger than that of the runner?

38. The Skylab satellite disintegrated when it reentered the atmosphere. Among the pieces that crashed down on the surface of the Earth, one of the heaviest was a lead-lined film vault of 1770 kg that had an estimated impact speed of 120 m/s on the surface. What was its kinetic energy? How many kilograms of TNT would we have to explode to release the same amount of energy? (One kilogram of TNT releases 4.6×10^6 J.)

39. An automobile of mass 1600 kg is traveling along a straight road at 80 km/h.

(a) What is the kinetic energy of this automobile in the reference frame of the ground?

(b) What is the kinetic energy in the reference frame of a motorcycle traveling in the same direction at 60 km/h?

(c) What is the kinetic energy in the reference frame of a truck traveling in the opposite direction at 60 km/h?

40. According to statistical data, the probability that an occupant of an automobile suffers lethal injury when involved in a crash is proportional to the square of the speed of the automobile. At a speed of 80 km/h, the probability is approximately 3%. What are the probabilities at 95 km/h, 110 km/h, and 125 km/h?

41. For the projectile described in Problem 47 of Chapter 2, calculate the initial kinetic energy ($t = 0$) and calculate the final kinetic energy ($t = 3.0$ s). How much energy does the projectile lose to friction in 3.0 s?

42. Compare the kinetic energy of a 15-g bullet fired at 630 m/s with that of a 15-kg bowling ball released at 6.3 m/s.

43. Compare the kinetic energy of a golf ball ($m = 45$ g) falling at a terminal velocity of 45 m/s with that of a person (75 kg) walking at 1.0 m/s.

44. A child's toy horizontally launches a 20-g ball using a spring that was originally compressed 8.0 cm. The spring constant is 30 N/m. What is the work done by the spring moving the ball from its compressed point to its relaxed position, where the ball is released? What is the kinetic energy of the ball at launch? What is the speed of the ball?

45. A mass of 150 g is held by a horizontal spring of spring constant 20 N/m. It is displaced from its equilibrium position and released from rest. As it passes through equilibrium, its speed is 5.0 m/s. For the motion from the release position to the equilibrium position, what is the work done by the spring? What was the initial displacement?

46. A 60-kg hockey player gets moving by pushing on the rink wall with a force of 500 N. The force is in effect while the skater extends his arms 0.50 m. What is the player's kinetic energy after the push? The player's speed?

47. A 1300-kg communication satellite has a speed of 3.1 km/s. What is its kinetic energy?

48. Suppose you throw a stone straight up so it reaches a maximum height h. At what height does the stone have one-half its initial kinetic energy?

49. The velocity of small bullets can be roughly measured with ballistic putty. When the bullet strikes a slab of putty, it penetrates a distance that is roughly proportional to the kinetic energy. Suppose that a bullet of velocity 160 m/s penetrates 0.80 cm into the putty and a second, identical bullet fired from a more powerful gun penetrates 1.2 cm. What is the velocity of the second bullet?

50. A particle moving along the x axis is subject to a force

$$F_x = -ax + bx^3$$

where a and b are constants.

(a) How much work does this force do as the particle moves from x_1 to x_2?

(b) If this is the only force acting on the particle, what is the change of kinetic energy during this motion?

*51. In the "tapping mode" used in atomic-force microscopes, a tip on a cantilever taps against the atoms of a surface to be studied. The cantilever acts as a spring of spring constant 2.5×10^{-2} N/m. The tip is initially displaced away from equilibrium by 3.0×10^{-8} m; it accelerates toward the surface, passes through the relaxed spring position, begins to slow down, and strikes the surface as the displacement approaches 2.5×10^{-8} m. What kinetic energy does the tip have just before striking the surface?

*52. With the brakes fully applied, a 1500-kg automobile decelerates at the rate of 8.0 m/s^2.

(a) What is the braking force acting on the automobile?

(b) If the initial speed is 90 km/h, what is the stopping distance?

(c) What is the work done by the braking force in bringing the automobile to a stop from 90 km/h?

(d) What is the change in the kinetic energy of the automobile?

*53. A box of mass 40 kg is initially at rest on a flat floor. The coefficient of kinetic friction between the box and the floor is $\mu_k = 0.60$. A woman pushes horizontally against the box with a force of 250 N until the box attains a speed of 2.0 m/s.

(a) What is the change of kinetic energy of the box?

(b) What is the work done by the friction force on the box?

(c) What is the work done by the woman on the box?

7.4 Gravitational Potential Energy†

54. It has been reported that at Cherbourg, France, waves smashing on the coast lifted a boulder of 3200 kg over a 6.0-m wall. What minimum energy must the waves have given to the boulder?

55. A 75-kg man walks up the stairs from the first to the third floor of a building, a height of 10 m. How much work does he do against gravity? Compare your answer with the food energy he acquires by eating an apple (see Table 8.1).

† For help, see Online Concept Tutorial 9 at www.wwnorton.com/physics

56. What is the kinetic energy and what is the gravitational potential energy (relative to the ground) of a goose of mass 6.0 kg soaring at 30 km/h at a height of 90 m?

57. Surplus energy from an electric power plant can be temporarily stored as gravitational energy by using this surplus energy to pump water from a river into a reservoir at some altitude above the level of the river. If the reservoir is 250 m above the level of the river, how much water (in cubic meters) must we pump in order to store 2.0×10^{13} J?

58. The track of a cable car on Telegraph Hill in San Francisco rises more than 60 m from its lowest point. Suppose that a car is ascending at 13 km/h along the track when it breaks away from its cable at a height of exactly 60 m. It will then coast up the hill some extra distance, stop, and begin to race down the hill. What speed does the car attain at the lowest point of the track? Ignore friction.

59. In pole vaulting, the jumper achieves great height by converting her kinetic energy of running into gravitational potential energy (Fig. 7.32). The pole plays an intermediate role in this process. When the jumper leaves the ground, part of her translational kinetic energy has been converted into kinetic energy of rotation (with the foot of the pole as the center of rotation) and part has been converted into elastic potential energy of deformation of the pole. When the jumper reaches her highest point, all of this energy has been converted into gravitational potential energy. Suppose that a jumper runs at a speed of 10 m/s. If the jumper converts all of the corresponding kinetic energy into gravitational potential energy, how high will her center of mass rise? The actual height reached by pole vaulters is 5.7 m (measured from the ground). Is this consistent with your calculation?

FIGURE 7.32 A pole vaulter.

60. Because of brake failure, a bicycle with its rider careens down a steep hill 45 m high. If the bicycle starts from rest and there is no friction, what is the final speed attained at the bottom of the hill?

61. Under suitable conditions, an avalanche can reach extremely great speeds because the snow rides down the mountain on a cushion of trapped air that makes the sliding motion nearly frictionless. Suppose that a mass of 2.0×10^7 kg of snow breaks loose from a mountain and slides down into a valley 500 m below the starting point. What is the speed of the snow when it hits the valley? What is its kinetic energy? The explo-

sion of 1 short ton (2000 lb) of TNT releases 4.2×10^9 J. How many tons of TNT release the same energy as the avalanche?

62. A parachutist of mass 60 kg jumps out of an airplane at an altitude of 800 m. Her parachute opens and she lands on the ground with a speed of 5.0 m/s. How much energy has been lost to air friction in this jump?

63. A block released from rest slides down to the bottom of a plane of incline 15° from a height of 1.5 m; the block attains a speed of 3.5 m/s at the bottom. By considering the work done by gravity and the frictional force, determine the coefficient of friction.

64. A bobsled run leading down a hill at Lake Placid, New York, descends 148 m from its highest to its lowest point. Suppose that a bobsled slides down this hill without friction. What speed will the bobsled attain at the lowest point?

65. A 2.5-g Ping-Pong ball is dropped from a window and strikes the ground 20 m below with a speed of 9.0 m/s. What fraction of its initial potential energy was lost to air friction?

66. A roller coaster begins at rest from a first peak, descends a vertical distance of 45 m, and then ascends a second peak, cresting the peak with a speed of 15 m/s. How high is the second peak? Ignore friction.

67. A skateboarder starts from rest and descends a ramp through a vertical distance of 5.5 m; he then ascends a hill through a vertical distance of 2.5 m and subsequently coasts on a level surface. What is his coasting speed? Ignore friction.

*68. In some barge canals built in the nineteenth century, barges were slowly lifted from a low level of the canal to a higher level by means of wheeled carriages. In a French canal, barges of 70 metric tons were placed on a carriage of 35 tons that was pulled, by a wire rope, to a height of 12 m along an inclined track 500 m long.

(a) What was the tension in the wire rope?

(b) How much work was done to lift the barge and carriage?

(c) If the cable had broken just as the carriage reached the top, what would have been the final speed of the carriage when it crashed at the bottom?

*69. A wrecking ball of mass 600 kg hangs from a crane by a cable of length 10 m. If this wrecking ball is released from an angle of 35°, what will be its kinetic energy when it swings through the lowest point of its arc?

*70. Consider a stone thrown vertically upward. If we take air friction into account, we see that $\frac{1}{2}mv^2 + mgy$ must decrease as a function of time. From this, prove that the stone will take longer for the downward motion than for the upward motion.

*71. A stone of mass 0.90 kg attached to a string swings around a vertical circle of radius 0.92 m. Assume that during this motion the energy (kinetic plus potential) of the stone is constant. If, at the top of the circle, the tension in the string is (just about) zero, what is the tension in the string at the bottom of the circle?

*72. A center fielder throws a baseball of mass 0.17 kg with an initial speed of 28 m/s and an elevation angle of 30°. What is the kinetic energy and what is the potential energy of the baseball when it reaches the highest point of its trajectory? Ignore friction.

*73. A jet aircraft looping the loop (see Problem 70 in Chapter 4) flies along a vertical circle of diameter 1000 m with a speed of 620 km/h at the bottom of the circle and a speed of 350 km/h at the top of the circle. The change of speed is due mainly to the downward pull of gravity. For the given speed at the bottom of the circle, what speed would you expect at the top of the circle if the thrust of the aircraft's engine exactly balances the friction force of air (as in the case for level flight)?

*74. A pendulum consists of a mass hanging from a string of length 1.0 m attached to the ceiling. Suppose that this pendulum is initially held at an angle of 30° with the vertical (see Fig. 7.33) and then released. What is the speed with which the mass swings through its lowest point? At what angle will the mass have one-half of this speed?

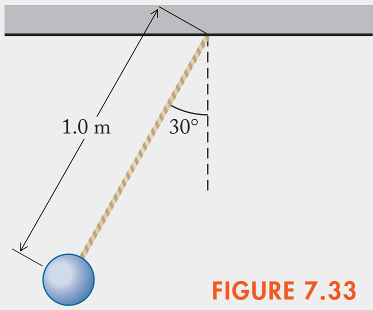


FIGURE 7.33 A pendulum.

**75. A stone is tied to a string of length R. A man whirls this stone in a vertical circle. Assume that the energy of the stone remains constant as it moves around the circle. Show that if the string is to remain taut at the top of the circle, the speed of the stone at the bottom of circle must be at least $\sqrt{5gR}$.

**76. In a loop coaster at an amusement park, cars roll along a track that is bent in a full vertical loop (Fig. 7.34). If the upper portion of the track is an arc of a circle of radius $R = 10$ m, what is the minimum speed that a car must have at the top of the loop if it is not to fall off? If the highest point of the loop has a height $h = 40$ m, what is the minimum speed with which the car must enter the loop at its bottom? Ignore friction.

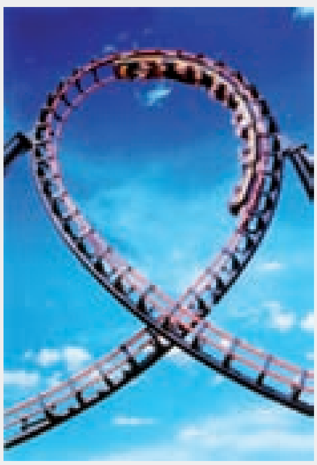


FIGURE 7.34 A roller coaster with a full loop.

**77. You are to design a roller coaster in which cars start from rest at a height $h = 30$ m, roll down into a valley, and then up a mountain (Fig. 7.35).

(a) What is the speed of the cars at the bottom of the valley?

(b) If the passengers are to feel $8g$ at the bottom of the valley, what must be the radius R of the arc of the circle that fits the bottom of the valley?

(c) The top of the next mountain is an arc of a circle of the same radius R. If the passengers are to feel $0g$ at the top of this mountain, what must be its height h'?

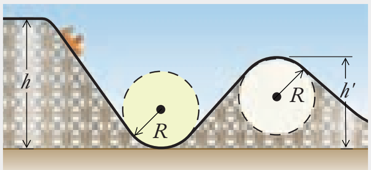


FIGURE 7.35 Profile of a roller coaster.

**78. One portion of the track of a toy roller coaster is bent into a full vertical circle of radius R. A small cart rolling on the track enters the bottom of the circle with a speed $2\sqrt{gR}$. Show that this cart will fall off the track before it reaches the top of the circle, and find the (angular) position at which the cart loses contact with the track.

**79. A particle initially sits on top of a large, smooth sphere of radius R (Fig. 7.36). The particle begins to slide down the sphere, without friction. At what angular position θ will the particle lose contact with the surface of the sphere? Where will the particle land on the ground?

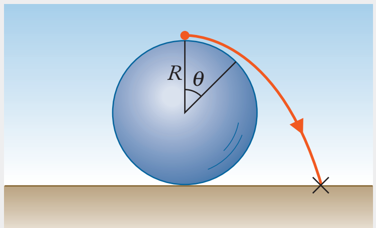


FIGURE 7.36 Particle sliding down a sphere.

REVIEW PROBLEMS

80. An apple falls down 35 m from the fifth floor of an apartment building to the street. The mass of the apple is 0.20 kg. How much work does gravity do on the apple during this fall?

81. A woman pulls a sled by a rope. The rope makes an upward angle of 45° with the ground, and the woman exerts a pull of 150 N on the rope. How much work does the woman do if she pulls this sled 20 m?

82. A man pushes a crate along a flat concrete floor. The mass of the crate is 120 kg, and the coefficient of friction between the crate and the floor is $\mu_k = 0.50$. How much work does the man do if, pushing horizontally, he moves the crate 15 m at constant speed?

83. A 1500-kg automobile is traveling at 20 m/s on a level road. How much work must be done on the automobile to accelerate it from 20 m/s to 25 m/s? From 25 m/s to 30 m/s?

84. A woman slowly lifts a 20-kg package of books from the floor to a height of 1.8 m, and then slowly returns it to the floor. How much work does she do on the package while lifting? How much work does she do on the package while lowering? What is the total work she does on the package? For the information given, can you tell how much work she expends internally in her muscles, that is, how many calories she expends?

85. An automobile of 1200 kg is traveling at 25 m/s when the driver suddenly applies the brakes so as to lock the wheels and cause the automobile to skid to a stop. The coefficient of sliding friction between the tires and the road is 0.90.

(a) What is the deceleration of the automobile, and what is the stopping distance?

(b) What is the friction force of the road on the wheels, and what is the amount of work that this friction force does during the stopping process?

*86. A golf ball of mass 50 g released from a a height of 1.5 m above a concrete floor bounces back to a height of 1.0 m.

(a) What is the kinetic energy of the ball just before contact with the floor begins? Ignore air friction.

(b) What is the kinetic energy of the ball just after contact with the floor ends?

(c) What is the loss of energy during contact with the floor?

87. A small aircraft of mass 1200 kg is cruising at 250 km/h at an altitude of 2000 m.

(a) What is the gravitational potential energy (relative to the ground), and what is the kinetic energy of the aircraft?

(b) If the pilot puts the aircraft into a dive, what will be the gravitational potential energy, what will be the kinetic energy, and what will be the speed when the aircraft reaches an altitude of 1500 m? Assume that the engine of the aircraft compensates the friction force of air, so the aircraft is effectively in free fall.

88. In a roller coaster, a car starts from rest on the top of a 30-m-high mountain. It rolls down into a valley, and then up a 20-m-high mountain. What is the speed of the car at the bottom of the valley, at ground level? What is the speed of the car at the top of the second mountain?

*89. In a compound bow (see Fig. 7.37), the pull of the limbs of the bow is communicated to the arrow by an arrangement of strings and pulleys that ensures that the force of the string against the arrow remains roughly constant while you pull the arrow back in preparation for letting it fly (in an ordinary bow, the force of the string increases as you pull back, which makes it difficult to continue pulling). A typical compound bow provides a steady force of 300 N. Suppose you pull an arrow of 0.020 kg back 0.50 m against this force.

(a) What is the work you do?

(b) When you release the arrow, what is the kinetic energy with which it leaves the bow?

(c) What is the speed of the arrow?

(d) How far will this arrow fly when launched with an elevation angle of 45°? Ignore friction and assume that the heights of the launch and impact points are the same.

(e) With what speed will it hit the target?

FIGURE 7.37 A compound bow.

*90. A large stone-throwing engine designed by Archimedes could throw a 77-kg stone over a range of 180 m. Assume that the stone is thrown at an initial angle of 45° with the horizontal.

(a) Calculate the initial kinetic energy of this stone.

(b) Calculate the kinetic energy of the stone at the highest point of its trajectory.

*91. The luge track at Lillehammer, the site of the 1994 Olympics, starts at a height of 350 m and finishes at 240 m. Suppose that a luger of 95 kg, including the sled starts from rest and reaches the finish at 130 km/h. How much energy has been lost to friction against the ice and the air?

*92. A pendulum consists of a mass m tied to one end of a string of length l. The other end of the string is attached to a fixed point on the ceiling. Suppose that the pendulum is initially held at an angle of 90° with the vertical. If the pendulum is released from this point, what will be the speed of the pendulum at the instant it passes through its lowest point? What will be the tension in the string at this instant?

*93. A roller coaster near St. Louis is 34 m high at its highest point.

(a) What is the maximum speed that the car can attain by rolling down from the highest point if initially at rest? Ignore friction.

(b) Some people claim that cars reach a maximum speed of 100 km/h. If this is true, what must be the initial speed of the car at the highest point?

*94. At a swimming pool, a water slide starts at a height of 6.0 m and ends at a height of 1.0 m above the water level with a short horizontal segment (see Fig. 7.38). A girl slides down the water slide.

(a) What is her speed at the bottom of the slide?

(b) How far from the slide does she land in the water?

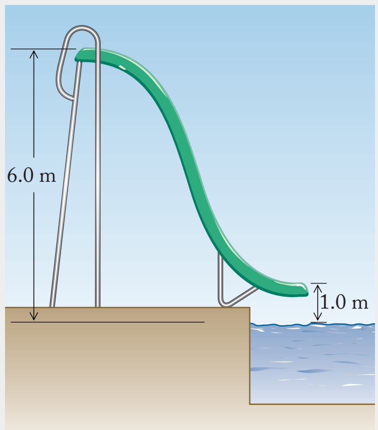


FIGURE 7.38 A water slide.

Answers to Checkups

Checkup 7.1

1. The normal force, which is perpendicular to the motion, does no work. The weight of the roller-coaster car does negative work as the car travels upward, and positive work as the car moves downward, since it has a component against or along the motion, respectively. At the peak, the work done by the weight is zero, since the weight is then perpendicular to the displacement.

2. For both the pushing and the pulling, the force is in the same direction as the displacement (you push when the saw moves forward and pull when the saw moves backward); thus, the work is positive in both cases.

3. The dog's pull is in the same direction as the displacement, and thus does positive work on the woman. The woman's pull is in the opposite direction to the displacement, and thus does negative work on the dog.

4. In each case, the force is opposite to the displacement (whether pushing against the front or pulling on the rear, the force is rearward), and so negative work is done on the cart in both cases.
5. No. The tension provides a centripetal acceleration, which is perpendicular to the (tangential) motion, and thus does no work.
6. The work is positive in (b) and (c), where the angle between the force and displacement is less than 90°; the work is negative in (a), where the angle is greater than 90°. The work is zero in (d), where the force is perpendicular to the displacement. The work is largest when the force is most nearly parallel to the displacement; for force vectors (and displacement vectors) of equal magnitude, this occurs in (c).
7. (E) 4 and 5. To calculate the work done by a constant force, $W = Fs\cos\theta$, you do not need to know the mass, acceleration, or speed. You do need to know the force, the displacement, and the angle between the two.

Checkup 7.2

1. The work done by a variable force is equal to the area under the $F(x)$ vs. x curve. Assuming the two plots are drawn to the same vertical scale, for a displacement from a to b, the upper plot clearly has a greater area between the $F(x)$ curve and the x axis.
2. If we consider a plot such as Fig. 7.13 and imagine extending the curve to the left to $x = -b$ [where $F(x) = +kb$], then we see that *positive* work is done on the particle as it moves from $x = -b$ to $x = 0$ [where the area between the $F(x)$ curve and the x axis is *above* the x axis]. *Negative* work is done on the particle as it moves from $x = 0$ to $x = +b$ [where the area between the $F(x)$ curve and the x axis is *below* the origin]. Thus the net work is zero.
3. The work you must do on the spring is the opposite of what the spring does on you, since the forces involved are an action–reaction pair. Thus the work you do is the negative of the result of Example 4, or $W = +\frac{1}{2}k(b^2 - a^2)$.
4. (D) $3W$. The work to stretch from equilibrium is $\frac{1}{2}kx^2$, so the first stretch requires $W = \frac{1}{2}kd^2$. The second stretch requires work $W' = \frac{1}{2}kx^2\Big|_d^{2d} = \frac{1}{2}k(2d)^2 - \frac{1}{2}kd^2 = 4W - W = 3W$.

Checkup 7.3

1. Yes—the kinetic energy, $K = \frac{1}{2}mv^2$, depends only on the square of the speed, and not on the direction of the velocity. Thus if the two equal masses have the same speed, they have the same kinetic energy.
2. Yes, the kinetic energies can be equal. Since the kinetic energy is proportional to mass and proportional to the square of the speed ($K = \frac{1}{2}mv^2$), if the car has twice the speed of the truck (a factor of 4 contribution to the kinetic energy), then the kinetic energies can be equal if the truck has 4 times the mass of the car.
3. The kinetic energy of the golf ball is largest at the beginning (and end, if we neglect air resistance) of the trajectory; at higher points, the force of gravity has slowed the ball down. The kinetic energy is smallest at the top of the trajectory, where there is only a horizontal contribution to the speed ($v = \sqrt{v_x^2 + v_y^2}$). The kinetic energy is not zero while the ball is in the air (unless the ball was accidentally launched vertically; in that case, the kinetic energy would be zero at the top of the trajectory).
4. No. For the work–energy theorem to apply, one must consider the *net* external force on the sled. If traveling at constant velocity (zero acceleration), the total force must be zero (the horse's pull does positive work and is canceled by the friction force, which does negative work), and so the total work done on the sled is zero. Thus there is no change in kinetic energy.
5. (E) 9. The kinetic energy, $K = \frac{1}{2}mv^2$, is proportional to the square of the speed; thus increasing the speed by a factor of 3 increases the kinetic energy by a factor of 9.

Checkup 7.4

1. As in Example 8, the velocity at the bottom depends only on the height of release (the cars do not even have to have the same mass!); thus, the upper roller coaster will provide the larger speed at the bottom, since Δy is greater.
2. The gravitational potential energy U decreases as the piano is brought to street level from the first house; U remains constant during the trip to the nearby house (assuming travel over flat ground); then, the gravitational potential energy increases back to its original value as the piano is brought up to the second floor of the second house (assuming similar houses).
3. No. At constant speed, K is constant; since U decreases as the truck moves down, $E = K + U$ decreases also, and so is not conserved.
4. Since both the girl and the boy change height by the same amount, they both reach the pool with the same speed (at *any* vertical height, they have the same speed, but the boy's velocity has a horizontal component, so his vertical velocity is slower than that of the girl). Since the girl's velocity is all vertical, a larger vertical velocity implies that she reaches the pool first.
5. (A) $\sqrt{2}$. As in Example 8, the speed at the bottom (starting from rest) is proportional to the square root of the initial height. Thus, for twice the height, the speed of the first bicyclist will be $\sqrt{2}$ times as large at the bottom.

CHAPTER

8

Conservation of Energy

CONCEPTS IN CONTEXT

Concepts in Context

The two orange areas in the middle of this satellite image are the reservoirs of the hydroelectric pumped-storage plant on Brown Mountain in New York State. When full, the upper reservoir (at right) holds 19 million cubic meters of water. This reservoir is linked to the lower reservoir at the base, part of the Schoharie Creek, by a 320-m vertical shaft bored through the mountain. The water flowing out of this shaft drives four large turbines that generate electric power. During periods of low demand, the turbines are operated in reverse, so they pump water back into the upper reservoir.

With the concepts developed in this chapter we can address questions such as:

? How do pumped-storage power plants complement other power plants? (Physics in Practice: Hydroelectric Pumped Storage, page 242)

? What is the speed of water spurting out of the shaft at the bottom? (Example 3, page 242)

? How much gravitational potential energy is stored in the upper reservoir, and how much available electric energy does this represent? (Example 5, page 249)

? When generating power at its maximum capacity, at what rate does the power plant remove water from the upper reservoir? How many hours can it run? (Example 10, page 257)

In the preceding chapter we found how to formulate a law of conservation of mechanical energy for a particle moving under the influence of the Earth's gravity. Now we will seek to formulate the law of conservation of mechanical energy when other forces act on the particle—such as the force exerted by a spring—and we will state the general law of conservation of energy. As in the case of motion under the influence of gravity, the conservation law permits us to deduce some features of the motion without having to deal with Newton's Second Law.

Online Concept Tutorial 10

8.1 POTENTIAL ENERGY OF A CONSERVATIVE FORCE

JOSEPH LOUIS, COMTE LAGRANGE (1736–1813) *French mathematician and theoretical astronomer. In his elegant mathematical treatise* Analytical Mechanics, *Lagrange formulated Newtonian mechanics in the language of advanced mathematics and introduced the general definition of the potential-energy function. Lagrange is also known for his calculations of the motion of planets and for his influential role in securing the adoption of the metric system of units.*

To formulate the law of conservation of energy for a particle moving under the influence of gravity, we began with the work–energy theorem [see Eq. (7.24)],

$$K_2 - K_1 = W \tag{8.1}$$

We then expressed the work W as a difference of two potential energies [see Eq. (7.30)],

$$W = -U_2 + U_1 \tag{8.2}$$

This gave us

$$K_2 - K_1 = -U_2 + U_1$$

from which we immediately found the conservation law for the sum of the kinetic and potential energies, $K_2 + U_2 = K_1 + U_1$, or

$$E = K + U = [\text{constant}] \tag{8.3}$$

As an illustration of this general procedure for the construction of the conservation law for mechanical energy, let us deal with the case of a particle moving under the influence of the elastic force exerted by a spring attached to the particle. If the particle moves along the x axis and the spring lies along this axis, the force has only an x component F_x, which is a function of position:

$$F_x(x) = -kx \tag{8.4}$$

Here, as in Section 6.2, the displacement x is measured from the relaxed position of the spring. The crucial step in the construction of the conservation law is to express the work W as a difference of two potential energies. For this purpose, we take advantage of the result established in Section 7.2 [see Eq. (7.17)], according to which the work done by the spring force during a displacement from x_1 to x_2 is

$$W = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \tag{8.5}$$

This shows that if we identify the **elastic potential energy of the spring** as

potential energy of spring

$$U = \tfrac{1}{2}kx^2 \tag{8.6}$$

then the work is, indeed, the difference between two potential energies $U_1 = \frac{1}{2}kx_1^2$ and $U_2 = \frac{1}{2}kx_2^2$. According to Eq. (8.6), the potential energy of the spring is proportional to the square of the displacement. Figure 8.1 gives a plot of this elastic potential energy.

The potential energy of the spring represents the capacity of the spring to do work by virtue of its deformation. When we compress a spring, we store latent work in it, which we can recover at a later time by letting the spring push against something. An old-fashioned watch, operated by a wound spring, illustrates this storage of energy in a spring (however, the springs in watches are not coil springs, but spiral springs, which are compressed by turning the knob of the watch).

As in the case of the particle moving under the influence of gravity, we conclude that for the particle moving under the influence of the spring force, the sum of the kinetic and elastic potential energies is constant,

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = [\text{constant}] \tag{8.7}$$

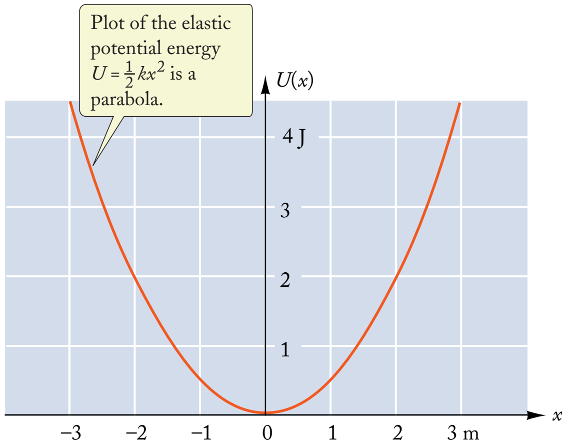


FIGURE 8.1 Plot of the potential energy of a spring as a function of the displacement x. In this plot, the spring constant is $k = 1$ N/m.

This equation gives us some information about the general features of the motion; it shows how the particle trades speed for an increase in the distance from the relaxed position of the spring. For instance, an increase of the magnitude of x requires a decrease of the speed v so as to keep the sum $\frac{1}{2}mv^2 + \frac{1}{2}kx^2$ constant.

EXAMPLE 1 A child's toy gun shoots a dart by means of a compressed spring. The constant of the spring is $k = 320$ N/m, and the mass of the dart is 8.0 g. Before shooting, the spring is compressed by 6.0 cm, and the dart is placed in contact with the spring (see Fig. 8.2). The spring is then released. What will be the speed of the dart when the spring reaches its relaxed position?

SOLUTION: The dart can be regarded as a particle moving under the influence of a force $F_x = -kx$, with a potential energy $U = \frac{1}{2}kx^2$. Taking the positive x axis along the direction of motion, the initial value of x is negative ($x_1 = -6.0$ cm); also, the initial speed is zero. According to Eq. (8.7), the initial energy is

$$E = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}kx_1^2 = 0 + \tfrac{1}{2}kx_1^2 \tag{8.8}$$

When the spring reaches its relaxed position ($x_2 = 0$), the energy will be

$$E = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}kx_2^2 = \tfrac{1}{2}mv_2^2 + 0 \tag{8.9}$$

Conservation of energy demands that the right sides of Eqs. (8.8) and (8.9) be equal:

$$\tfrac{1}{2}mv_2^2 = \tfrac{1}{2}kx_1^2 \tag{8.10}$$

If we cancel the factors of $\frac{1}{2}$ in this equation, divide both sides by m, and take the square root of both sides, we find that the speed of the dart as it leaves the spring at $x_2 = 0$ is

$$v_2 = \sqrt{\frac{k}{m}x_1^2} = \sqrt{\frac{320\ \text{N/m}}{0.0080\ \text{kg}} \times (-0.060\ \text{m})^2} = 12\ \text{m/s} \tag{8.11}$$

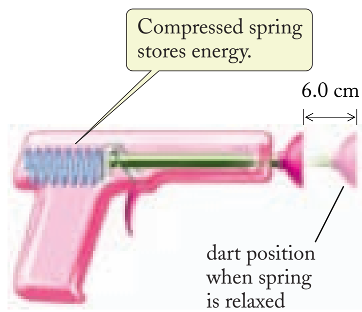


FIGURE 8.2 A toy gun. The spring is initially compressed 6.0 cm.

PROBLEM-SOLVING TECHNIQUES ENERGY CONSERVATION

To obtain an expression for the total mechanical energy, you must include terms for the different kinds of energy that are present:

1. Begin with an expression for the energy at one point [Eq. (8.8)].
2. And an expression for the energy at another point [Eq. (8.9)].
3. Then use energy conservation to equate these expressions [Eq. (8.10)].

With the appropriate expression for the mechanical energy, you can apply energy conservation to solve some problems of motion. As illustrated in the preceding example, this involves the three steps outlined in Section 7.4 and 8.1.

CONTRIBUTIONS TO THE MECHANICAL ENERGY

KIND OF ENERGY	APPLICABLE IF	CONTRIBUTION TO TOTAL MECHANICAL ENERGY
Kinetic energy	Particle is in motion	$K = \frac{1}{2}mv^2$
Gravitational potential energy	Particle is moving up or down near the Earth's surface	$U = mgy$
Elastic potential energy	Particle is subject to a spring force	$U = \frac{1}{2}kx^2$

To formulate the law of conservation of mechanical energy for a particle moving under the influence of some other force, we want to imitate the above construction. We will be able to do this if, and only if, the work performed by this force can be expressed as a difference between two potential energies, that is,

$$W = -U_2 + U_1 \tag{8.12}$$

If the force meets this requirement (and therefore permits the construction of a conservation law), the force is called **conservative.** Thus, the force of gravity and the force of a spring are conservative forces. Note that for any such force, the work done when the particle starts at the point x_1 and *returns* to the same point is necessarily zero, since, with $x_2 = x_1$, Eq. (8.12) implies

$$W = -U_1 + U_1 = 0 \tag{8.13}$$

This simply means that for a round trip that starts and ends at x_1, the work the force does during the outward portion of the trip is exactly the negative of the work the force does during the return portion of the trip, and therefore the net work for the round trip is zero (see Fig. 8.3). Thus, the energy supplied by the force is recoverable: the energy supplied by the force during motion in one direction is restored during the return motion in the opposite direction. For instance, when a particle moves downward from some starting point, gravity performs positive work; and when the particle moves upward, returning to its starting point, gravity performs negative work of a magnitude exactly equal to that of the positive work.

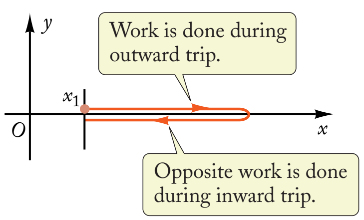


FIGURE 8.3 A particle starts at a point x_1 and returns to the point x_1 after completing some round trip. If the force is conservative, the work done is zero, because the work for the outward portion of the trip is opposite to that for the inward portion.

The requirement of zero work for a round trip can be used to discriminate between conservative and nonconservative forces. Friction is an example of a nonconservative force. If we slide a metal block through some distance along a table and then slide the block back to its starting point, the net work is not zero. The work performed by the friction force during the outward portion of the motion is negative, and the work performed by the friction force during the return portion of the trip is also negative—the friction force always opposes the motion, and the work done by the friction force is always negative. Thus, the work done by the friction force cannot be expressed as a difference between two potential energies, and we cannot formulate a law of conservation of mechanical energy if friction forces are acting. However, as we will see in Section 8.3, we can formulate a more general law of conservation of energy, involving kinds of energy other than mechanical, which remains valid even when there is friction.

In the case of one-dimensional motion, a force is conservative whenever it can be expressed as an explicit function of position, $F_x = F_x(x)$. (Note that the friction force does *not* fit this criterion; the sign of the friction force depends on the direction of motion, and therefore the friction force is not uniquely determined by the position x.) For any such force $F_x(x)$, we can construct the potential energy function by integration. We take a point x_0 as reference point at which the potential energy is zero. The potential energy at any other point x is constructed by evaluating an integral (in the following equations, the integration variables are indicated by primes to distinguish them from the upper limits of integration):

$$U(x) = -\int_{x_0}^{x} F_x(x')\,dx' \tag{8.14}$$

potential energy as integral of force

To check that this construction agrees with Eq. (8.12), we examine $U_1 - U_2$:

$$U_1 - U_2 = U(x_1) - U(x_2) = -\int_{x_0}^{x_1} F_x(x')\,dx' + \int_{x_0}^{x_2} F_x(x')\,dx'$$

By one of the basic rules for integrals (see Appendix 4), the integral changes sign when we reverse the limits of integration. Hence

$$U_1 - U_2 = \int_{x_1}^{x_0} F_x(x')\,dx' + \int_{x_0}^{x_2} F_x(x')\,dx'$$

And by another basic rule, the sum of an integral from x_1 to x_0 and an integral from x_0 to x_2 is equal to a single integral from x_1 to x_2. Thus

$$U_1 - U_2 = \int_{x_1}^{x_2} F_x(x')\,dx' \tag{8.15}$$

Here the right side is exactly the work done by the force as the particle moves from x_1 to x_2, in agreement with Eq. (8.12). This confirms that our construction of the potential energy is correct.

In the special case of the spring force $F_x(x) = -kx$, our general construction (8.14) of the potential energy immediately yields the result (8.6), provided we take $x_0 = 0$.

For a particle moving under the influence of any conservative force, the total mechanical energy is the sum of the kinetic energy and the potential energy; as before, this total mechanical energy is conserved:

$$E = K + U = [\text{constant}] \tag{8.16}$$

or

$$E = \tfrac{1}{2}mv^2 + U = [\text{constant}] \tag{8.17}$$

conservation of mechanical energy

EXAMPLE 2

As we will see in later chapters, the **inverse-square force** plays a large role in physics—gravitational forces are inverse square, and electric forces are inverse square. If we consider a particle that can move in only one dimension along the positive x axis, this force has the form

$$F_x(x) = \frac{A}{x^2} \tag{8.18}$$

where A is a constant. The point $x = 0$ is called the **center of force**. If A is positive, the force is repulsive (F_x is positive, and the force therefore pushes a particle on the positive x axis away from the center of force); if A is negative, the force is attractive (F_x is negative, and the force pulls the particle toward the center of force). The magnitude of the force is very large near $x = 0$, and it decreases as the distance from this point increases (Figs. 8.4a and b). What is the potential energy for this force?

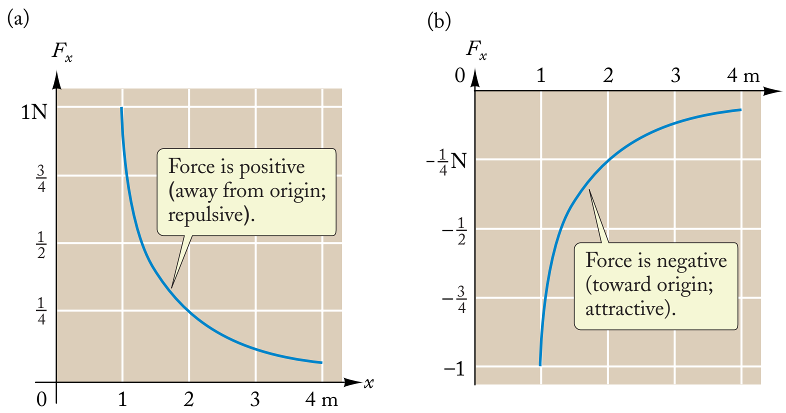


FIGURE 8.4 The inverse-square force A/x^2 as a function of x, (a) for a positive value of A (repulsive force; $A = 1\ \text{N·m}^2$) and (b) for a negative value of A (attractive force; $A = -1\ \text{N·m}^2$).

SOLUTION: According to Eq. (8.14),

$$U(x) = -\int_{x_0}^{x} \frac{A}{x'^2}\,dx'$$

In the compact notation of tables of integrals, $\int (1/x'^2)\,dx' = -1/x'$. Hence

$$U(x) = -\left[-\frac{A}{x'}\right]_{x_0}^{x} = -\left[-\frac{A}{x} - \left(-\frac{A}{x_0}\right)\right] = \frac{A}{x} - \frac{A}{x_0}$$

It is usually convenient to take $x_0 = \infty$ as the reference point, with $U_0 = 0$ at $x = \infty$. With this choice,

$$U(x) = \frac{A}{x} \tag{8.19}$$

COMMENT: Note that for a repulsive force ($A > 0$), the potential energy decreases with x (see Fig. 8.5a), and for an attractive force ($A < 0$), the potential energy increases with x (the potential energy is large and negative near $x = 0$, and it increases toward zero as x increases; see Fig. 8.5b).

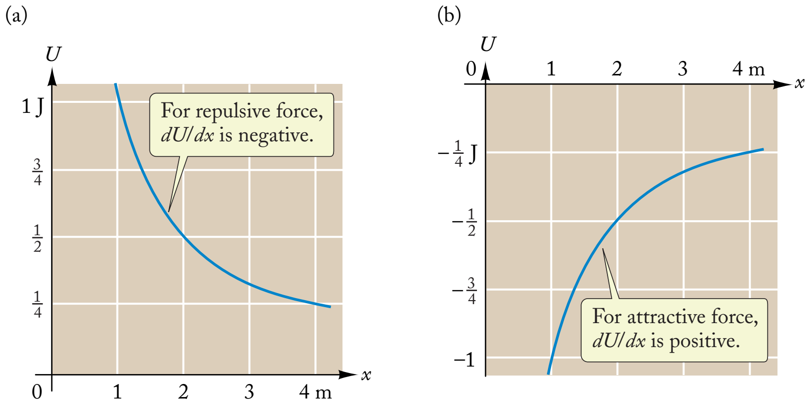


FIGURE 8.5 The potential A/x as a function of x, (a) for a positive value of A and (b) for a negative value of A.

For both the spring force and the inverse-square force, the force can be expressed in terms of the potential energy as $F_x(x) = -dU/dx$; that is, the force is the negative of the derivative of the potential energy. This relationship holds generally, for any kind of conservative force. We can see this by examining the change in potential energy produced by a small displacement dx. From Eq. (8.12) we see that if the points x_1 and x_2 are separated by a small distance $dx = x_2 - x_1$, then [see also Eq. (7.15)]

$$dU = U_2 - U_1 = -dW = -F_x\,dx \tag{8.20}$$

and if we divide this by dx, we obtain

$$F_x = -\frac{dU}{dx} \tag{8.21}$$

force as derivative of potential

This relation gives us a quick way to calculate the force if the potential energy is known.

From Eq. (8.21) we see that the force F_x is positive wherever the potential is a decreasing function of x, that is, wherever the derivative dU/dx is negative. Conversely, the force F_x is negative wherever the potential is an increasing function of x, that is, wherever the derivative dU/dx is positive. This is in agreement with the result we found for repulsive and attractive forces in Example 2.

Although in this section we have focused on one-dimensional motion, the criterion of zero work for a round trip is also valid for conservative forces in two or three dimensions. In one dimension, the path for a round trip is necessarily back and forth along a straight line; in two or three dimensions, the path can be of any shape, provided it forms a closed loop that starts and ends at the same point.

Furthermore, the law of conservation of mechanical energy is valid not only for the motion of a single particle, but also for the motion of more general systems, such as systems consisting of solids, liquids, or gases. When applying the conservation law to the kinetic and potential energies of such bodies, it may be necessary to take into account other forms of energy, such as the heat produced by friction and stored in the bodies (see Section 8.3). However, if such other forms of energy stored in the bodies are constant, then we can ignore them in our examination of the motion, as illustrated in the following example of the motion of water in a pipe.

PHYSICS IN PRACTICE — HYDROELECTRIC PUMPED STORAGE

Concepts in Context

The demand for electric energy by industrial and commercial users is high during working hours, but low during nights and on weekends. For maximum efficiency, electric power companies prefer to run their large nuclear or coal-fired power plants at a steady, full output for 24 hours a day, 7 days a week. Thus electric power companies often have a surplus of electric energy available at night and on weekends, and they often have a deficit of energy during peak-demand times, which requires them to purchase energy from neighboring power companies. Hydroelectric pumped-storage plants help to deal with this mismatch between a fluctuating demand and a steady supply. A hydroelectric pumped-storage plant is similar to an ordinary hydroelectric power plant. It consists of an upper water reservoir and a lower water reservoir, typically separated by a few hundred meters in height. Large pipes (penstocks) connect the upper reservoir to turbines placed at the level of the lower reservoir. The water spurting out of the pipes drives the turbines, which drive electric generators. However, in contrast to an ordinary hydroelectric plant, the pumped-storage plant can be operated in reverse. The electric generators then act as electric motors which drive the turbines in reverse, and thereby pump water from the lower reservoir into the upper reservoir. At peak-demand times the hydroelectric storage plant is used for the generation of electric energy—it converts the gravitational potential energy of the water into electric energy. At low-demand times, the hydroelectric storage plant is used to absorb electric energy—it converts surplus electric energy into gravitational potential energy of the water. This gravitational potential energy can then be held in storage until needed.

The chapter photo shows the reservoirs of a large hydroelectric pumped-storage plant on Brown Mountain in New York State. The upper reservoir on top of the mountain is linked to the lower reservoir at the base by a vertical shaft of more than 320 m bored through the mountain. Each of the four reversible pump/turbines (see Fig. 8.17) and motor/generators in the powerhouse at the base (see Fig. 1) is capable of generating 260 MW of electric power. The upper reservoir holds 1.9×10^7 m^3 of water, which is enough to run the generators at full power for about half a day.

FIGURE 1 Powerhouse at the lower reservoir of the Brown Mountain hydroelectric pumped-storage plant.

Concepts in Context

EXAMPLE 3 At the Brown Mountain hydroelectric storage plant, water from the upper reservoir flows down a pipe in a long vertical shaft (Fig. 8.6). The pipe ends 330 m below the water level of the (full) upper reservoir. Calculate the speed with which the water emerges from the bottom of the pipe. Consider two cases: (a) the bottom of the pipe is wide open, so the pipe does not impede the downward motion of the water; and (b) the bottom of the pipe is closed except for a small hole through which water spurts out. Ignore frictional losses in the motion of the water.

SOLUTION: (a) If the pipe is wide open at the bottom, any parcel of water simply falls freely along the full length of the pipe. Thus, the pipe plays no role at all in the motion of the water, and the speed attained by the water is the same as for a reservoir suspended in midair with water spilling out and falling freely through a height $h = 330$ m. For such free-fall motion, the final speed v can be obtained either from the equations for uniformly accelerated motion [from Eq. (2.29)] or from energy conservation [see Eq. (7.41)]. The result is

$$v = \sqrt{2gh} = \sqrt{2 \times 9.81\ \text{m/s}^2 \times 330\ \text{m}} = 80\ \text{m/s}$$

(b) For a closed pipe with a small hole, the motion of a parcel of water from the top of the upper reservoir to the hole at the bottom of the pipe is complicated and unknown. However, we can find the final speed of the water by relying on the law of energy conservation as applied to the system consisting of the entire volume of water in the reservoir and the pipe. For this purpose, we must examine the kinetic and the potential energy of the water. The water spurting out at the bottom has a large kinetic energy but a low potential energy. In contrast, the water at the top of the upper reservoir has a high potential energy, but next to no kinetic energy (while the water spurts out at the bottom, the water level in the reservoir gradually decreases; but the speed of this downward motion of the water level is very small if the reservoir is large, and this speed can be ignored compared with the large speed of the spurting water).

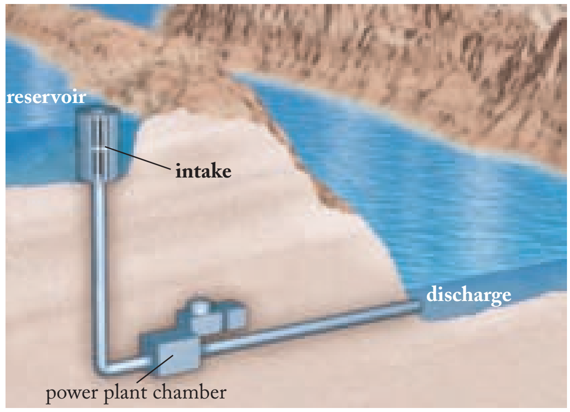


FIGURE 8.6 Cross-sectional view of hydroelectric pumped-storage power plant.

Consider, then, the energy changes that occur when a mass m of water, say, 1 kg of water, spurts out at the bottom of the pipe while, simultaneously, the water level of the upper reservoir decreases slightly. As concerns the energy balance, this effectively amounts to the removal of the potential energy of 1 kg from the top of the reservoir and the addition of the kinetic energy of 1 kg at the bottom of the pipe. All the water at intermediate locations, in the pipe and the reservoir, has the same energy it had before. Thus, energy conservation demands that the kinetic energy of the mass m of water emerging at the bottom be equal to the potential energy of a mass m at the top:

$$\tfrac{1}{2}mv^2 = mgh$$

This again gives

$$v = \sqrt{2gh} = 80\ \text{m/s}$$

that is, the same result as in part (a).

COMMENT: Note that the way the water acquires the final speed of 80 m/s in the cases (a) and (b) is quite different. In case (a), the water accelerates down the pipe with the uniform free-fall acceleration g. In case (b), the water flows down the pipe at a slow and nearly constant speed, and accelerates (strongly) only at the last moment, as it approaches the hole at the bottom. However, energy conservation demands that the result for the final speed of the emerging water be the same in both cases.

✔ Checkup 8.1

QUESTION 1: The potential energy corresponding to the spring force $F = -kx$ is $U = \frac{1}{2}kx^2$. Suppose that some new kind of force has a potential energy $U = -\frac{1}{2}kx^2$. How does this new kind of force differ from the spring force?

QUESTION 2: A particle moves along the positive x axis under the influence of a conservative force. Suppose that the potential energy of this force is as shown in Fig. 8.5a. Is the force directed along the positive x direction or the negative x direction?

QUESTION 3: Suppose that the force acting on a particle is given by the function $F_x = ax^3 + bx^2$, where a and b are constants. How do we know that the work done by this force during a round trip from, say, $x = 1$ back to $x = 1$ is zero?

QUESTION 4: Is the equation $W = U_1 - U_2$ valid for the work done by every kind of force? Is the equation $W = K_2 - K_1$ valid for the work done by each individual force acting on a particle?

(A) Yes; yes (B) Yes; no (C) No; yes (D) No; no

8.2 THE CURVE OF POTENTIAL ENERGY

If a particle of some given energy is moving in one dimension under the influence of a conservative force, then Eq. (8.17) permits us to calculate the speed of the particle as a function of position. Suppose that the potential energy is some known function $U = U(x)$; then Eq. (8.17) states

$$E = \tfrac{1}{2}mv^2 + U(x) \tag{8.22}$$

or, rearranging,

$$v^2 = \frac{2}{m}[E - U(x)] \tag{8.23}$$

Since the left side of this equation is never negative, we can immediately conclude that the particle must always remain within a range of values of x for which $U(x) \leq E$. If $U(x)$ is increasing and the particle reaches a point at which $U(x) = E$, then $v = 0$; that is, the particle will stop at this point, and its motion will reverse. Such a point is called a **turning point** of the motion.

According to Eq. (8.23), v^2 is directly proportional to $E - U(x)$; thus, v^2 is large wherever the difference between E and $U(x)$ is large. We can therefore gain some insights into the qualitative features of the motion by drawing a graph of potential energy as a function of x on which it is possible to display the difference between E and $U(x)$. Such a graph of $U(x)$ vs. x is called the **curve of potential energy**. For example, Fig. 8.7 shows the curve of potential energy for an atom in a diatomic molecule. Treating the atom as a particle, we can indicate the value of the energy of the particle by a horizontal line in the graph (the red line in Fig. 8.7). We call this horizontal line the **energy level** of the particle. At any point x, we can then see the difference between E and $U(x)$ at a glance; according to Eq. (8.23), this tells us v^2. For instance, suppose that a particle has an energy $E = E_1$. Figure 8.7 shows this energy level. The particle has maximum speed at the point $x = x_0$, where the separation between the energy level and the potential-energy curve is maximum. The speed gradually decreases as the particle moves, say, toward the right. The potential-energy curve intersects the energy level at $x = a$; at this point the speed of the particle will reach zero, so this point is a turning point of the motion. The particle then moves toward the left, again attaining the same greatest speed at $x = x_0$. The speed gradually decreases as the particle continues to move toward the left, and the speed reaches zero at $x = a'$, the second turning point of the motion. Here the particle begins to move toward the right, and so on. Thus the particle continues to move back and forth between the two turning points—the particle is confined between the two turning points. The regions $x > a$ and $x < a'$ are forbidden regions;

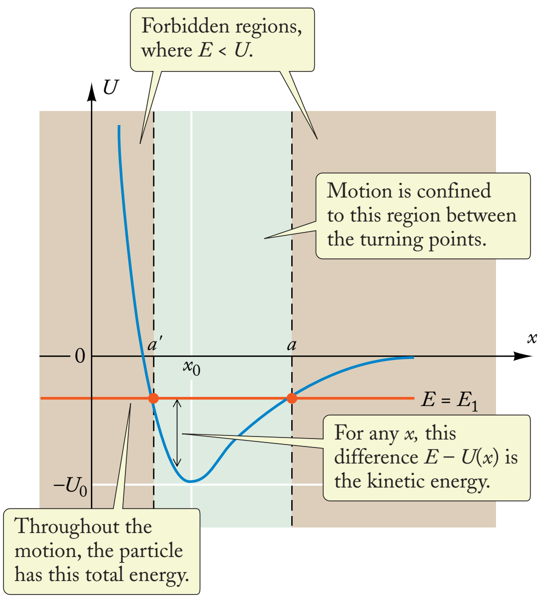


FIGURE 8.7 Potential-energy curve for an atom in a diatomic molecule. The horizontal line (red) is the energy level. The turning points are at $x = a$ and at $x = a'$.

only the region $a' \le x \le a$ is permitted. The particle is said to be in a **bound orbit.** The motion is periodic, that is, repeats again and again whenever the particle returns to its starting point.

The location of the turning points depends on the energy. For a particle with a lower energy level, the turning points are closer together. The lowest possible energy level intersects the potential-energy curve at its minimum (see $E = -U_0$ in Fig. 8.8); the two turning points then merge into the single point $x = x_0$. A particle with this lowest possible energy cannot move at all—it remains stationary at $x = x_0$. Note that the potential-energy curve has zero slope at $x = x_0$; this corresponds to zero force, $F_x = -dU/dx = 0$. A point such as $x = x_0$, where the force is zero, is called an **equilibrium point**. The point $x = x_0$ in Fig. 8.8 is a **stable** equilibrium point, since, after a small displacement, the force pushes the particle back toward that point. In contrast, at an **unstable** equilibrium point, after a small displacement, the force pushes the particle away from the point (see the point x_1 for the potential-energy curve shown in Fig. 8.9); and at a **neutral** equilibrium point no force acts nearby (see the point x_2 in Fig. 8.9). Equivalently, since the force is zero at an equilibrium point, the stable, unstable, and neutral equilibrium points correspond to negative, positive, or zero changes in the force with increasing x, that is, to negative, positive, or zero values of dF_x/dx. But $dF_x/dx = -d^2U/dx^2$, so the stable and unstable equilibrium points respectively correspond to positive and negative second derivatives of the function $U(x)$; in the former case the plot of $U(x)$ curves upward, and in the latter, downward (see Fig. 8.9).

In Fig. 8.7, the right side of the potential-energy curve never rises above $U = 0$. Consequently, if the energy level is above this value (for instance, $E = E_2$; see Fig. 8.10), then there is only one single turning point on the left, and no turning point on the right. A particle with energy E_2 will continue to move toward the right forever; it is not confined. Such a particle is said to be in an **unbound orbit**.

The above qualitative analysis based on the curve of potential energy cannot tell us the details of the motion such as, say, the travel time from one point to another. But the qualitative analysis is useful because it gives us a quick survey of the types of motion that are possible for different values of the energy.

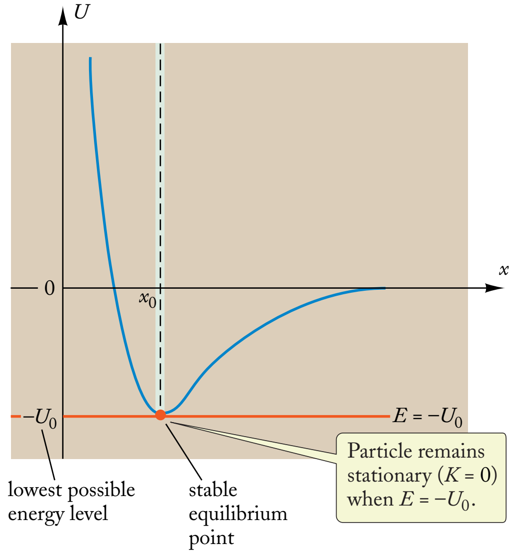


FIGURE 8.8 The energy level (red) coincides with the minimum of the potential-energy curve.

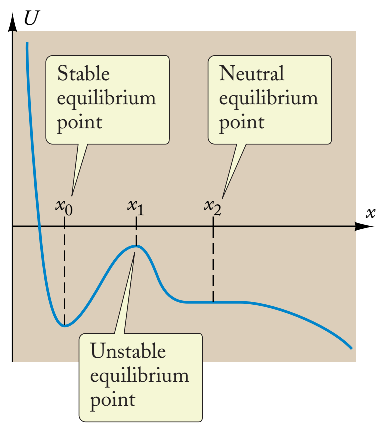


FIGURE 8.9 Types of equilibrium points. At the stable, unstable, and neutral equilibrium points, respectively, the potential-energy curve has a minimum, has a maximum, or is flat.

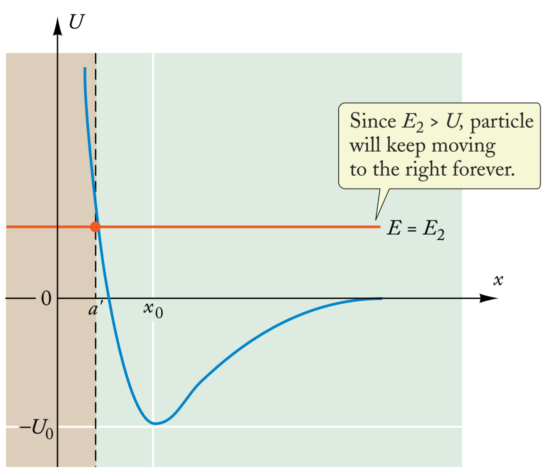


FIGURE 8.10 The energy level (red) is above the maximum height the potential-energy curve attains at its right. There is only one turning point, at $x = a'$.

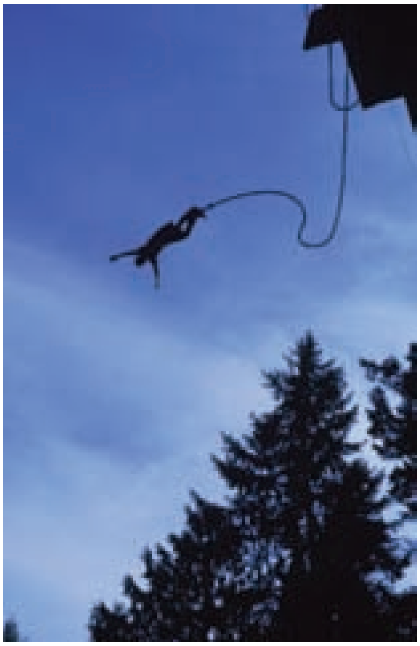


FIGURE 8.11 Bungee jumping.

EXAMPLE 4 Some fanatics, in search of dangerous thrills, jump off high bridges or towers with bungee cords (long rubber cords) tied to their ankles (Fig. 8.11). Consider a jumper of mass 70 kg, with a 9.0-m cord tied to his ankles. When stretched, this cord may be treated as a spring, of spring constant 150 N/m. Plot the potential-energy curve for the jumper, and from this curve estimate the turning point of the motion, that is, the point at which the stretched cord stops the downward motion of the jumper.

SOLUTION: It is convenient to arrange the x axis vertically upward, with the origin at the point where the rubber cord becomes taut, that is, 9.0 m below the jump-off point (see Fig. 8.12a). The potential-energy function then consists of two pieces. For $x > 0$, the rubber cord is slack, and the potential energy is purely gravitational:

$$U = mgx \qquad \text{for} \quad x > 0$$

For $x < 0$, the rubber cord is stretched, and the potential energy is a sum of gravitational and elastic potential energies:

$$U = mgx + \tfrac{1}{2}kx^2 \quad \text{for} \quad x < 0$$

With the numbers specified for this problem,

$$\begin{aligned} U &= 70\text{ kg} \times 9.81\text{ m/s}^2 \times x \\ &= 687x \qquad \text{for} \quad x > 0 \end{aligned} \tag{8.24}$$

and

$$\begin{aligned} U &= 70\text{ kg} \times 9.81\text{ m/s}^2 \times x + \tfrac{1}{2} \times 150\text{ N/m} \times x^2 \\ &= 687x + 75x^2 \qquad \text{for} \quad x < 0 \end{aligned} \tag{8.25}$$

where x is in meters and U in joules. Figure 8.12b gives the plot of the curve of potential energy, according to Eqs. (8.24) and (8.25).

At the jump-off point $x = +9.0$ m, the potential energy is $U = 687x = 687 \times 9.0\text{ J} = 6180\text{ J}$. The red line in Fig. 8.12b indicates this energy level. The left intersection of the red line with the curve indicates the turning point at the lower end of the motion. By inspection of the plot, we see that this turning point is at $x \approx -15$ m. Thus, the jumper falls a total distance of 9.0 m + 15 m = 24 m before his downward motion is arrested.

We can accurately calculate the position of the lower turning point ($x < 0$) by equating the potential energy at that point with the initial potential energy:

$$687x + 75x^2 = 6180\text{ J}$$

This provides a quadratic equation of the form $ax^2 + bx + c = 0$:

$$75x^2 + 687x - 6180 = 0$$

This has the standard solution $x = (-b \pm \sqrt{b^2 - 4ac}\,)/2a$, or

$$\begin{aligned} x &= \frac{-687 \pm \sqrt{(687)^2 + 4 \times 75 \times 6180}}{2 \times 75} \\ &= -14.7\text{ m} \approx -15\text{ m} \end{aligned}$$

in agreement with our graphical result. Here we have chosen the negative solution, since we are solving for x at the lower turning point using the form (8.25), which is valid only for $x < 0$.

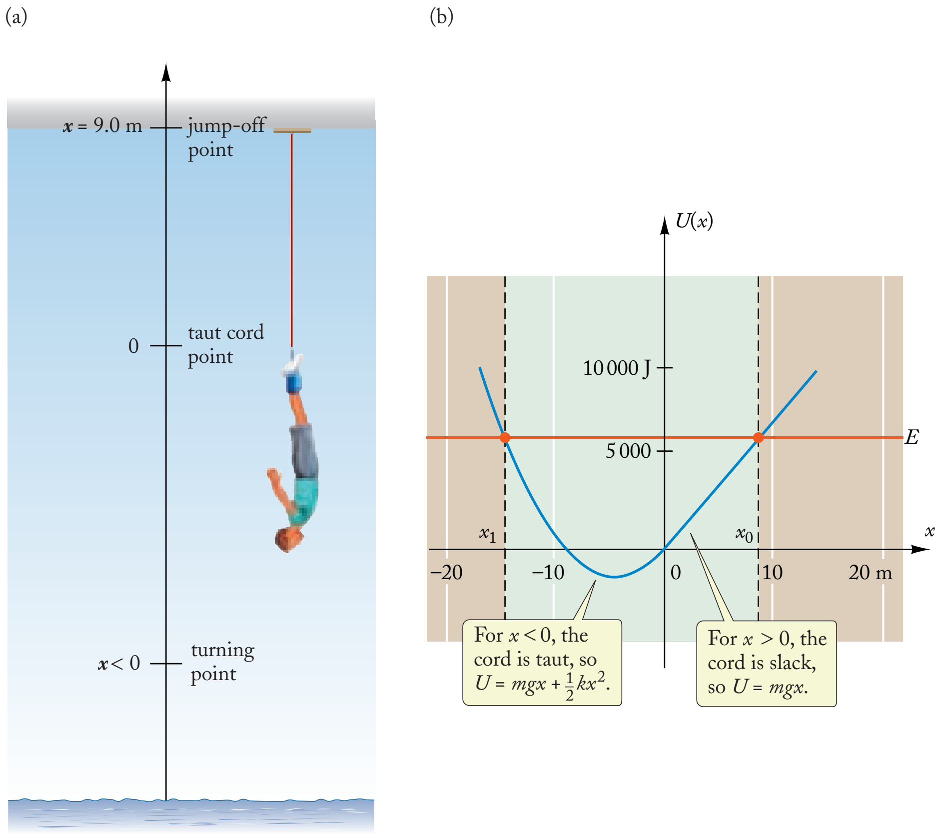


FIGURE 8.12 (a) The origin for the x coordinate is at the point where the rubber cord becomes taut. The jump-off point is at $x = 9.0$ m, and the turning point is at some negative value of x. (b) Curve of potential energy for the bungee jumper. The red line indicates the energy level. This line intersects the curve at approximately $x = -15$ m. This is the turning point for the jumper.

COMMENTS: If there were no friction, the motion would reverse, and the jumper would ascend to the bridge and bang against it. However, like a bouncing ball, the rubber cord has some energy loss due to friction within the material, and the jumper will not bounce back as high as the starting point.

Bungee jumping is a dangerous stunt. The human body has poor tolerance to deceleration in the head-down position. The pooling of blood in the head can lead to loss of consciousness ("redout"), rupture of blood vessels, eye damage, and temporary blindness. And in several instances, jumpers were killed by smashing their heads into the ground or by becoming entangled in their cords during the fall.

✔ Checkup 8.2

QUESTION 1: A particle moving in one dimension under the influence of a given conservative force has either no turning point, one turning point, or two turning points, depending on the energy. Does the number of turning points increase or decrease with the energy? Is there any conceivable value of the energy that will result in three turning points?

QUESTION 2: By examining the curve of potential energy in Fig. 8.12, estimate at what points the bungee jumper attains his maximum downward speed and his maximum acceleration.

QUESTION 3: A particle moving under the influence of the spring force has a positive energy $E = 50$ J. How many turning points are there for this particle?

(A) 1 (B) 2 (C) 3 (D) 0

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8.3 OTHER FORMS OF ENERGY

If the forces acting on a particle are conservative, then the mechanical energy of the particle is conserved. But if some of the forces acting on the particle are not conservative, then the mechanical energy of the particle—consisting of the sum of the kinetic energy and the net potential energy of all the conservative forces acting on the particle—will not remain constant. For instance, if friction forces are acting, they do negative work and thereby decrease the mechanical energy of the particle.

However, it is a remarkable fact about our physical universe that *whenever mechanical energy is lost by a particle or some other body, this energy never disappears—it is merely changed into other forms of energy.* Thus, in the case of friction, the mechanical energy lost by the body is transformed into kinetic and potential energy of the atoms in the body and in the surface against which it is rubbing. The energy that the atoms acquire in the rubbing process is disorderly kinetic and potential energy—it is spread out among the atoms in an irregular, random fashion. At the macroscopic level, we perceive the increase of the disorderly kinetic and potential energy of the rubbed surfaces as an increase of temperature. Thus, friction produces **heat** or **thermal energy.** (You can easily convince yourself of this by vigorously rubbing your hands against each other.)

Heat is a form of energy, but whether it is to be regarded as a new form of energy or not depends on what point of view we adopt. Taking a macroscopic point of view, we ignore the atomic motions; then heat is to be regarded as distinct from mechanical energy. Taking a microscopic point of view, we recognize heat as kinetic and potential energy of the atoms; then heat is to be regarded as mechanical energy. (We will further discuss heat in Chapter 20.)

Chemical energy and nuclear energy are two other forms of energy. The former is kinetic and potential energy of the electrons within the atoms; the latter is kinetic and potential energy of the protons and neutrons within the nuclei of atoms. As in the case of heat, whether these are to be regarded as new forms of energy depends on the point of view.

Electric and magnetic energy are forms of energy associated with electric charges and with light and radio waves. (We will examine these forms of energy in Chapters 25 and 31.)

Table 8.1 lists some examples of different forms of energy. All the energies in Table 8.1 are expressed in joules, the SI unit of energy. However, for reasons of tradition and convenience, some other energy units are often used in specialized areas of physics and engineering.

The energy of atomic and subatomic particles is usually measured in **electron-volts** (eV), where

$$1 \text{ electron-volt} = 1 \text{ eV} = 1.60 \times 10^{-19} \text{ J} \tag{8.26}$$

Electrons in atoms typically have kinetic and potential energies of a few eV.

The energy supplied by electric power plants is usually measured in **kilowatt-hours** (kW·h), where

$$1 \text{ kilowatt-hour} = 1 \text{ kW·h} = 3.60 \times 10^{6} \text{ J} \tag{8.27}$$

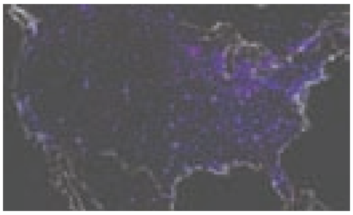
The electric energy used by appliances such as vacuum cleaners, hair dryers, or toasters during one hour of operation is typically 1 kilowatt-hour.

And the thermal energy supplied by the combustion of fuels is often expressed in **kilocalories** (kcal):

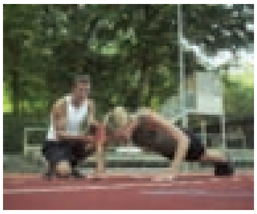
HERMANN VON HELMHOLTZ (1821–1894) *Prussian surgeon, biologist, mathematician, and physicist. His scientific contributions ranged from the invention of the ophthalmoscope and studies of the physiology and physics of vision and hearing to the measurement of the speed of light and studies in theoretical mechanics. Helmholtz formulated the general Law of Conservation of Energy, treating it as a consequence of the basic laws of mechanics and electricity.*

TABLE 8.1 SOME ENERGIES

Nuclear fuel in the Sun	1×10^{45} J
Explosion of a supernova	1×10^{44} J
Fossil fuel available on Earth	2.0×10^{23} J
Yearly energy expenditure of the United States (a)	8×10^{19} J
Volcanic explosion (Krakatoa)	6×10^{18} J
Annihilation of 1 kg of matter–antimatter	9.0×10^{16} J
Explosion of thermonuclear bomb (1 megaton)	4.2×10^{15} J
Gravitational potential energy of airliner (Boeing 747 at 10 000 m)	2×10^{10} J
Combustion of 1 gal of gasoline (b)	1.3×10^{8} J
Daily food intake of man (3000 kcal)	1.3×10^{7} J
Explosion of 1 kg of TNT	4.6×10^{6} J
Metabolization of one apple (110 kcal)	4.6×10^{5} J
One push-up (c)	3×10^{2} J
Fission of one uranium nucleus	3.2×10^{-11} J
Energy of ionization of hydrogen atom	2.2×10^{-18} J


(a)


(b)


(c)

$$1 \text{ kilocalorie} = 1 \text{ kcal} = 4.187 \times 10^3 \text{ J} \tag{8.28}$$

or in **British thermal units** (Btu):

$$1 \text{ Btu} = 1.055 \times 10^3 \text{ J} \tag{8.29}$$

We will learn more about these units in later chapters.

All these forms of energy can be transformed into one another. For example, in an internal combustion engine, chemical energy of the fuel is transformed into heat and kinetic energy; in a hydroelectric power station, gravitational potential energy of the water is transformed into electric energy; in a nuclear reactor, nuclear energy is transformed into heat, light, kinetic energy, etc. However, in any such transformation process, the sum of all the energies of all the pieces of matter involved in the process remains constant: *the form of the energy changes, but the total amount of energy does not change.* This is the general **Law of Conservation of Energy**.

law of conservation of energy

EXAMPLE 5 At the Brown Mountain hydroelectric pumped-storage plant, the average height of the water in the upper reservoir is 320 m above the lower reservoir, and the upper reservoir holds 1.9×10^7 m^3 of water. Expressed in kW·h, what is the gravitational potential energy available for conversion into electric energy?

Concepts in Context

SOLUTION: A cubic meter of water has a mass of 1000 kg. Hence the total mass of water is 1.9×10^{10} kg, and the gravitational potential energy is

$$U = mgh = 1.9 \times 10^{10} \text{ kg} \times 9.81 \text{ m/s}^2 \times 320 \text{ m} = 6.0 \times 10^{13} \text{ J}$$

Expressed in kW·h, this amounts to

$$6.0 \times 10^{13}\ \text{J} \times \frac{1\ \text{kW·h}}{3.6 \times 10^{6}\ \text{J}} = 1.7 \times 10^{7}\ \text{kW·h}$$

(The actual electric energy that can be generated is about 30% less than that, because of frictional losses during the conversion from one form of energy to the other. These frictional losses result in the generation of heat.)

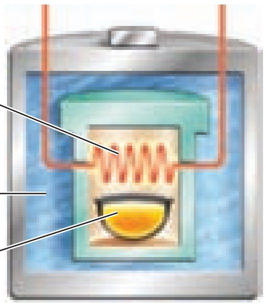


FIGURE 8.13 A bomb calorimeter. The sample is ignited electrically, by a glowing wire.

EXAMPLE 6 The "calorie" used by dietitians to express the energy equivalents of different foods is actually a kilocalorie, or a "large" calorie. To measure the energy equivalent of some kind of food—for instance, sugar—a sample is placed in a bomb calorimeter, a closed vessel filled with oxygen at high pressure (see Fig. 8.13). The sample is ignited and burned completely (complete oxidation). The number of calories released in this chemical reaction—for instance, 4.1 kcal for 1.0 g of sugar—tells us the maximum amount of energy that can be extracted from this food. The human body does not necessarily "burn" food quite as completely, and the muscles do not convert all of the available chemical energy into mechanical energy. However, energy conservation tells us that from one gram of sugar the body cannot produce more than 4.1 kcal of mechanical work.

If you eat one spoonful (4.0 g) of sugar, what is the maximum height to which this permits you to climb stairs? Assume your mass is 70 kg.

SOLUTION: Since 1.0 g of sugar releases 4.1 kcal of energy, the energy equivalent of 4.0 g of sugar is

$$4.0 \times 4.1\ \text{kcal} = 16.4\ \text{kcal} = 16.4\ \text{kcal} \times 4.18 \times 10^{3}\ \text{J/kcal} = 6.9 \times 10^{4}\ \text{J}$$

When you climb the stairs to a height y, this energy becomes gravitational potential energy:

$$mgy = 6.9 \times 10^{4}\ \text{J}$$

from which

$$y = \frac{6.9 \times 10^{4}\ \text{J}}{mg} = \frac{6.9 \times 10^{4}\ \text{J}}{70\ \text{kg} \times 9.81\ \text{m/s}^2} = 100\ \text{m}$$

In practice, because of the limited efficiency of your body, only about 20% of the chemical energy of food is converted into mechanical energy; thus, the actual height you can climb is only about 20 m. (Because of the strong musculature of the human leg, stair climbing is one of your most efficient activities; other physical activities are considerably less efficient in converting chemical energy into mechanical energy.)

✔ Checkup 8.3

QUESTION 1: A parachutist descends at uniform speed. Is the mechanical energy conserved? What happens to the lost mechanical energy?

QUESTION 2: You fire a bullet from a rifle. The increase of kinetic energy of the bullet upon firing must be accompanied by a decrease of some other kind of energy. What energy decreases?

QUESTION 3: A truck travels at constant speed down a road leading from a mountain peak to a valley. What happens to the gravitational potential energy of the truck? How is it dissipated?

QUESTION 4: When you apply the brakes and stop a moving automobile, what happens to the kinetic energy?

(A) Kinetic energy is converted to gravitational potential energy.
(B) Kinetic energy is converted to elastic potential energy.
(C) Kinetic energy is converted to heat due to frictional forces.
(D) Kinetic energy is converted to chemical energy.

8.4 MASS AND ENERGY

One of the great discoveries made by Albert Einstein early in the twentieth century is that energy can be transformed into mass, and mass can be transformed into energy. Thus, *mass is a form of energy*. The amount of energy contained in an amount m of mass is given by Einstein's famous formula

$$E = mc^2 \tag{8.30}$$

energy–mass relation

where c is the speed of light, $c = 3.00 \times 10^8$ m/s. This formula is a consequence of Einstein's relativistic physics. It cannot be obtained from Newton's physics, and its theoretical justification will have to wait until we study the theory of relativity in Chapter 36.

The most spectacular demonstration of Einstein's mass–energy formula is found in the annihilation of matter and antimatter (as we will see in Chapter 41, particles of antimatter are similar to the particles of ordinary matter, except that they have opposite electric charge). If a proton collides with an antiproton, or an electron with an antielectron, the two colliding particles react violently, and they annihilate each other in an explosion that generates an intense flash of very energetic light. According to Eq. (8.30), the annihilation of just 1000 kg of matter and antimatter (500 kg of each) would release an amount of energy

$$E = mc^2 = 1000 \text{ kg} \times (3.00 \times 10^8 \text{ m/s})^2 = 9.0 \times 10^{19} \text{ J} \tag{8.31}$$

This is enough energy to satisfy the requirements of the United States for a full year. Unfortunately, antimatter is not readily available in large amounts. On Earth, antiparticles can be obtained only from reactions induced by the impact of beams of high-energy particles on a target. These collisions occasionally result in the creation of a particle–antiparticle pair. Such pair creation is the reverse of pair annihilation. The creation process transforms some of the kinetic energy of the collision into mass, and a subsequent annihilation merely gives back the original energy.

But the relationship between energy and mass in Eq. (8.30) also has another aspect. *Energy has mass*. Whenever the energy of a body is changed, its mass (and weight) are changed. The change in mass that accompanies a given change of energy is

mass–energy relation

$$\Delta m = \frac{\Delta E}{c^2} \tag{8.32}$$

For instance, if the kinetic energy of a body increases, its mass (and weight) increase. At speeds small compared with the speed of light, the mass increment is not noticeable. But when a body approaches the speed of light, the mass increase becomes very large. The high-energy electrons produced at the Stanford Linear Accelerator provide an extreme example of this effect: these electrons have a speed of 99.999 999 97% of the speed of light, and their mass is 44 000 times the mass of electrons at rest!

The fact that energy has mass indicates that energy is a form of mass. Conversely, as we have seen above, mass is a form of energy. Hence mass and energy must be regarded as two aspects of the same thing. The laws of conservation of mass and conservation of energy are therefore not two independent laws—each implies the other. For example, consider the fission reaction of uranium inside the reactor vessel of a nuclear power plant. The complete fission of 1.0 kg of uranium yields an energy of 8.2×10^{13} J. The reaction conserves energy—it merely transforms nuclear energy into heat, light, and kinetic energy, but does not change the total amount of energy. The reaction also conserves mass—if the reactor vessel is hermetically sealed and thermally insulated from its environment, then the reaction does not change the mass of the contents of the vessel. However, if we open the vessel during or after the reaction and let some of the heat and light escape, then the mass of the residue will not match the mass of the original amount of uranium. The mass of the residues will be about 0.1% smaller than the original mass of the uranium. This mass defect represents the mass carried away by the energy that escapes. Thus, the nuclear fission reactions merely transform energy into new forms of energy and mass into new forms of mass. In this regard, a nuclear reaction is not fundamentally different from a chemical reaction. The mass of the residues in a chemical reaction that releases heat (exothermic reaction) is slightly less than the original mass. The heat released in such a chemical reaction carries away some mass, but, in contrast to a nuclear reaction, this amount of mass is so small as to be quite immeasurable.

EXAMPLE 7 As an example of the small mass loss in a chemical reaction, consider the binding energy of the electron in the hydrogen atom (one proton and one electron), which is 13.6 eV. What is the fractional mass loss when an electron is captured by a proton and the binding energy is allowed to escape?

SOLUTION: In joules, the binding energy is $13.6 \text{ eV} \times 1.60 \times 10^{-19}$ J/eV = 2.18×10^{-18} J. The mass loss corresponding to this binding energy is

$$\Delta m = \frac{\Delta E}{c^2} = \frac{2.18 \times 10^{-18}\text{ J}}{(3.00 \times 10^8 \text{ m/s})^2} = 2.42 \times 10^{-35}\text{ kg}$$

Since the mass of a proton and electron together is 1.67×10^{-27} kg (see Table 5.2), the fractional mass loss is

$$\frac{\Delta m}{m} = \frac{2.42 \times 10^{-35}\text{ kg}}{1.67 \times 10^{-27}\text{ kg}} = 1.45 \times 10^{-8}$$

This is about a millionth of one percent.

Checkup 8.4

QUESTION 1: The Sun radiates heat and light. Does the Sun consequently suffer a loss of mass?

QUESTION 2: In the annihilation of matter and antimatter, a particle and an antiparticle—such as a proton and an antiproton, or an electron and an antielectron—disappear explosively upon contact, giving rise to an intense flash of light. Is energy conserved in this reaction? Is mass conserved?

QUESTION 3: You heat a potful of water to the boiling point. If the pot is sealed so no water molecules can escape, then, compared with the cold water, the mass of the boiling water will:

(A) Increase (B) Decrease (C) Remain the same

8.5 POWER

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When we use an automobile engine to move a car up a hill or when we use an electric motor to lift an elevator cage, the important characteristic of the engine is not how much force it can exert, but rather how much work it can perform in a given amount of time. The force is only of secondary importance, because by shifting to a low gear we can make sure that even a "weak" engine exerts enough force on the wheels to propel the automobile uphill. But the work performed in a given amount of time, or the rate of work, is crucial, since it determines how fast the engine can propel the car up the hill. While the car moves uphill, the gravitational force takes energy from the car; that is, it performs negative work on the car. To keep the car moving, the engine must perform an equal amount of positive work. If the engine is able to perform this work at a fast rate, it can propel the car uphill at a fast speed.

The rate at which a force does work on a body is called the **power** *delivered by the force.* If the force does an amount of work W in an interval of time Δt, then the **average power** is the ratio of W and Δt:

$$\overline{P} = \frac{W}{\Delta t} \tag{8.33}$$

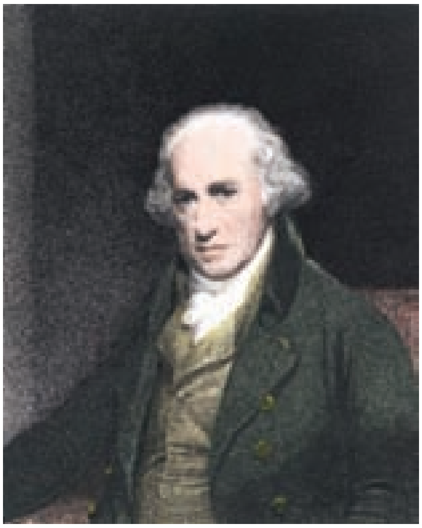
average power

The **instantaneous power** is defined by a procedure analogous to that involved in the definition of the instantaneous velocity. We consider the small amount of work dW done in the small interval of time dt and take the ratio of these small quantities:

$$P = \frac{dW}{dt} \tag{8.34}$$

instantaneous power

According to these definitions, the engine of your automobile delivers high power if it performs a large amount of work on the wheels (or, rather, the driveshaft) in a short time. The maximum power delivered by the engine determines the maximum speed of which this automobile is capable, since at high speed the automobile loses energy to air resistance at a prodigious rate, and this loss has to be made good by the engine. You might also expect that the power of the engine determines the maximum acceleration of which the automobile is capable. But the acceleration is determined



JAMES WATT (1736–1819) *Scottish inventor and engineer. He modified and improved an earlier steam engine and founded the first factory constructing steam engines. Watt introduced the* horsepower *as a unit of mechanical power.*

by the maximum force exerted by the engine on the wheels, and this is not directly related to the power as defined above.

The SI unit of power is the **watt** (W), which is the rate of work of one joule per second:

$$1 \text{ watt} = 1 \text{ W} = 1 \text{ J/s}$$

In engineering practice, power is often measured in **horsepower** (hp) units, where

$$1 \text{ horsepower} = 1 \text{ hp} = 746 \text{ W} \tag{8.35}$$

This is roughly the rate at which a (very strong) horse can do work.

Note that multiplication of a unit of power by a unit of time gives a unit of energy. An example of this is the kilowatt-hour (kW·h), already mentioned in Section 8.3:

$$\begin{aligned} 1 \text{ kilowatt-hour} &= 1 \text{ kW·h} = 1 \text{ kW} \times 1 \text{ h} = 1000 \text{ W} \times 3600 \text{ s} \\ &= 3.6 \times 10^6 \text{ J} \end{aligned} \tag{8.36}$$

This unit is commonly used to measure the electric energy delivered to homes and factories.

For a constant (or average) power P delivered to a body during a time Δt, the work ΔW delivered is the rate times the time [see Eq. (8.33)]:

$$W = P\,\Delta t \tag{8.37}$$

If the rate of doing work P varies with time, then the total work W done between a time t_1 and another time t_2 is the sum of the infinitesimal $P\,\Delta t$ contributions; that is, the work done is the integral of the power over time:

$$W = \int dW = \int_{t_1}^{t_2} P\,dt \tag{8.38}$$

EXAMPLE 8 An elevator cage has a mass of 1000 kg. How many horsepower must the motor deliver to the elevator if it is to raise the elevator cage at the rate of 2.0 m/s? The elevator has no counterweight (see Fig. 8.14).

SOLUTION: The weight of the elevator is $w = mg = 1000 \text{ kg} \times 9.81 \text{ m/s}^2 \approx 9800 \text{ N}$. By means of the elevator cable, the motor must exert an upward force equal to the weight to raise the elevator at a steady speed. If the elevator moves up a distance Δy, the work done by the force is

$$W = F\,\Delta y \tag{8.39}$$

To obtain the power, or the rate of work, we must divide this by the time interval Δt:

$$P = \frac{\Delta W}{\Delta t} = \frac{F\,\Delta y}{\Delta t} = F\frac{\Delta y}{\Delta t} = Fv \tag{8.40}$$

where $v = \Delta y/\Delta t$ is the speed of the elevator. With $F = 9800$ N and $v = 2.0$ m/s, we find

$$P = Fv = 9800 \text{ N} \times 2.0 \text{ m/s} = 2.0 \times 10^4 \text{ W}$$

Since 1 hp = 746 W [see Eq. (8.35)], this equals

$$P = 2.0 \times 10^4 \text{ W} \times \frac{1 \text{ hp}}{746 \text{ W}} = 27 \text{ hp}$$

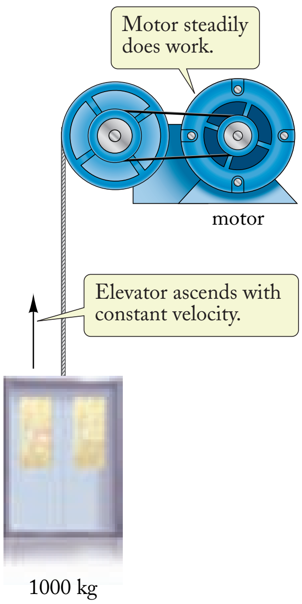


FIGURE 8.14 Elevator cage and motor.

Equation (8.40) is a special instance of a simple formula, which expresses the instantaneous power as the scalar product of force and velocity. To see this, consider that when a body suffers a small displacement $d\mathbf{s}$, the force $\mathbf{F}$ acting on the body will perform an amount of work

$$dW = \mathbf{F}\cdot d\mathbf{s} \tag{8.41}$$

or

$$dW = F\,ds\cos\theta$$

where θ is the angle between the direction of the force and the direction of the displacement (see Fig. 8.15). The instantaneous power delivered by this force is then

$$P = \frac{dW}{dt} = F\frac{ds}{dt}\cos\theta \tag{8.42}$$

Since ds/dt is the speed v, this expression for the power equals

$$P = Fv\cos\theta \tag{8.43}$$

or

$$P = \mathbf{F}\cdot\mathbf{v} \tag{8.44}$$

power delivered by a force

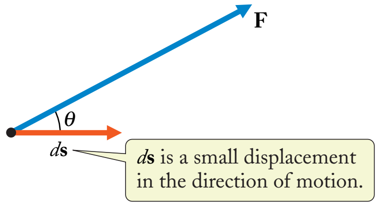


FIGURE 8.15 The force $\mathbf{F}$ makes an angle θ with the displacement $d\mathbf{s}$.

EXAMPLE 9 A horse pulls a sled up a steep snow-covered street of slope 1:7 (see Fig. 8.16a). The sled has a mass of 300 kg, and the coefficient of sliding friction between the sled and the snow is 0.12. If the horse pulls parallel to the surface of the street and delivers a power of 1.0 hp, what is the maximum (constant) speed with which the horse can pull the sled uphill? What fraction of the horse's power is expended against gravity? What fraction against friction?

SOLUTION: Figure 8.16b is a "free-body" diagram for the sled, showing the weight ($w = mg$), the normal force ($N = mg\cos\phi$), the friction force ($f_k = \mu_k N$), and the pull of the horse (T). With the x axis along the street and the y axis at right angles to the street, the components of these forces are

$$w_x = -mg\sin\phi \qquad w_y = -mg\cos\phi$$
$$N_x = 0 \qquad N_y = mg\cos\phi$$
$$f_{k,x} = -\mu_k mg\cos\phi \qquad f_{k,y} = 0$$
$$T_x = T \qquad T_y = 0$$

Since the acceleration along the street is zero (constant speed), the sum of the x components of these forces must be zero:

$$-mg\sin\phi + 0 - \mu_k mg\cos\phi + T = 0 \tag{8.45}$$

We can solve this equation for the pull of the horse:

$$T = mg\sin\phi + \mu_k mg\cos\phi \tag{8.46}$$

This simply says that the pull of the horse must balance the component of the weight along the street plus the friction force. The direction of this pull is parallel to the direction of motion of the sled. Hence, in Eq. (8.43), $\theta = 0$, and the power delivered by the horse is

$$P = Tv = (mg\sin\phi + \mu_k mg\cos\phi)v \tag{8.47}$$

Solving this equation for v, we find

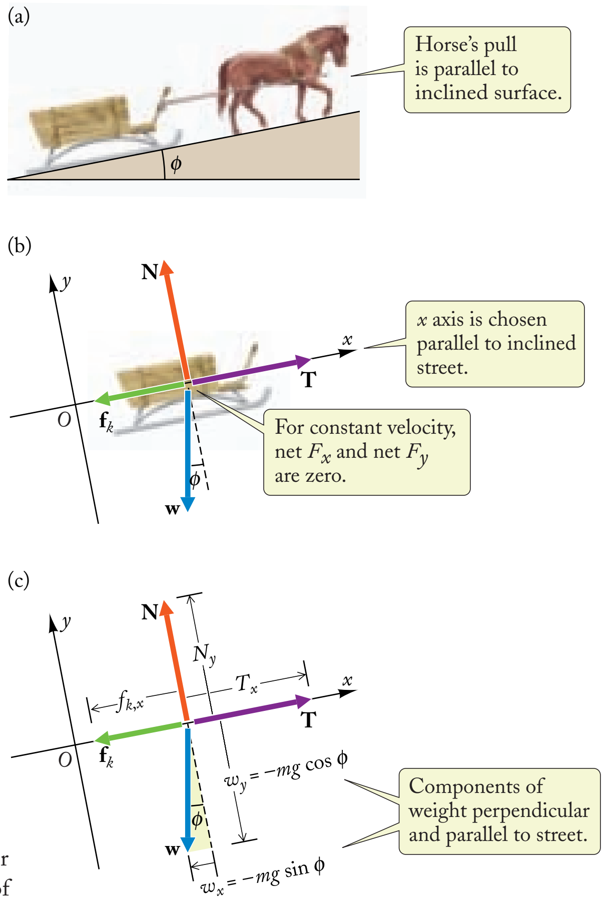


FIGURE 8.16 (a) Horse dragging a sled up a street. (b) "Free-body" diagram for the sled. (c) Components of the forces.

$$v = \frac{P}{mg\sin\phi + \mu_k mg\cos\phi} = \frac{P}{mg(\sin\phi + \mu_k\cos\phi)} \tag{8.48}$$

For a slope of 1:7, the tangent of the angle of inclination is $\tan\phi = 1/7$, and, using a calculator, the inverse tangent of 1/7 gives $\phi = 8.1°$. Hence

$$v = \frac{746\ \text{W}}{300\ \text{kg} \times 9.81\ \text{m/s}^2 \times (\sin 8.1° + 0.12\cos 8.1°)}$$
$$= 0.98\ \text{m/s}$$

The weight of the sled makes an angle of $90.0° + 8.1° = 98.1°$ with the direction of motion (see Fig. 8.16b). The power exerted by the weight of the sled is given by Eq. (8.43), with $F = mg$ and $\cos\theta = \cos 98.1°$:

$$P_{\text{weight}} = mgv\cos 98.1° = 300\ \text{kg} \times 9.81\ \text{m/s}^2 \times 0.98\ \text{m/s} \times \cos 98.1°$$
$$= -406\ \text{W} = -0.54\ \text{hp}$$

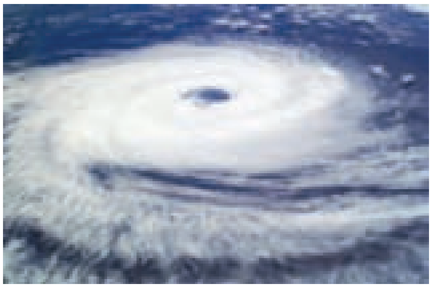
Since the total power is 1.0 hp, this says that 54% of the horse's power is expended against gravity and, consequently, the remaining 46% against friction. The friction portion can also be calculated directly. The friction force acts opposite to the velocity ($\cos\theta = -1$), and so the power exerted is negative:

$$P_{\text{friction}} = -f_k v = -\mu_k mg\cos\phi\, v = -0.12 \times 300\ \text{kg} \times 9.81\ \text{m/s}^2 \times \cos 8.1° \times 0.98\ \text{m/s}$$
$$= -343\ \text{W} = -0.46\ \text{hp}$$

The above equations all refer to *mechanical* power. In general, *power is the rate at which energy is transferred from one form of energy to another or the rate at which energy is transported from one place to another.* For instance, an automobile engine converts chemical energy of fuel into mechanical energy and thermal energy. A nuclear power plant converts nuclear energy into electric energy and thermal energy. And a high-voltage power line transports electric energy from one place to another. Table 8.2 gives some examples of different kinds of power.

TABLE 8.2 SOME POWERS

Light and heat emitted by the Sun		3.9×10^{26} W
Mechanical power generated by hurricane	(a)	2×10^{13} W
Total power used in United States (average)		2×10^{12} W
Large electric power plant		$\approx 10^{9}$ W
Jet airliner engines (Boeing 747)	(b)	2.1×10^{8} W
Automobile engine		1.5×10^{5} W
Solar light and heat per square meter at Earth		1.4×10^{3} W
Electricity used by toaster		1×10^{3} W
Work output of man (athlete at maximum)		2×10^{2} W
Electricity used by light bulb		1×10^{2} W
Basal metabolic rate for man (average)		88 W
Heat and work output of bumblebee (in flight)	(c)	2×10^{-2} W
Atom radiating light		$\approx 10^{-10}$ W



(a)

(b)

(c)

In the previous example, part of the horse's work was converted into heat by the friction between the sled and the snow, and part was converted into gravitational potential energy. In the following example, gravitational potential energy is converted into electric energy.

EXAMPLE 10 Each of the four generators (Fig. 8.17) of the Brown Mountain hydroelectric plant generates 260 MW of electric power. When generating this power, at what rate does the power plant take water from the upper reservoir? How long does a full reservoir last? See the data in Example 5.

Concepts in Context

SOLUTION: We will assume that all of the potential energy of the water in the upper reservoir, at a height of 320 m, is converted into electric energy. The electric

FIGURE 8.17 Turbine generator at Brown Mountain hydroelectric plant, shown during installation.

power $P = 4 \times 260 \times 10^6$ W $= 1.0 \times 10^9$ W must then equal the negative of the rate of change of the potential energy (see Eq. 7.31):

$$P = -\frac{dU}{dt} = -\frac{dm}{dt}gh$$

from which we obtain the rate of change of mass,

$$\frac{dm}{dt} = -\frac{P}{gh} = -\frac{1.0 \times 10^9\ \text{W}}{9.81\ \text{m/s}^2 \times 320\ \text{m}} = -3.3 \times 10^5\ \text{kg/s}$$

Expressed as a volume of water, this amounts to an outflow of 330 m^3 per second. At this rate, the 1.9×10^7 m^3 of water in the reservoir will last for

$$\frac{1.9 \times 10^7 \text{m}^3}{330\ \text{m}^3/\text{s}} = 5.7 \times 10^4\ \text{s} = 16\ \text{h}$$

As mentioned in Example 5, there are also some frictional losses. As a result, the reservoir will actually be depleted about 30% faster than this, that is, in a bit less than half a day.

✔ Checkup 8.5

QUESTION 1: (a) You trot along a flat road carrying a backpack. Do you deliver power to the pack? (b) You trot uphill. Do you deliver power to the pack? (c) You trot downhill. Do you deliver power to the pack? Does the pack deliver power to you?

QUESTION 2: To reach a mountaintop, you have a choice between a short, steep road or a longer, less steep road. Apart from frictional losses, is the energy you have to expend in walking up these two roads the same? Why does the steeper road require more of an effort?

QUESTION 3: In order to keep a 26-m motor yacht moving at 88 km/h, its engines must supply about 5000 hp. What happens to this power?

QUESTION 4: Two cars are traveling up a sloping road, each at a constant speed. The second car has twice the mass and twice the speed of the first car. What is the ratio of the power delivered by the second car engine to that delivered by the first? Ignore friction and other losses.

(A) 1 (B) 2 (C) 4 (D) 8 (E) 16

SUMMARY

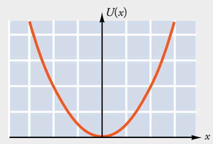
PROBLEM-SOLVING TECHNIQUES Energy Conservation **(page 238)**

PHYSICS IN PRACTICE Hydroelectric Pumped Storage **(page 242)**

CONSERVATIVE FORCE The work done by the force is zero for any round trip.

WORK DONE BY A CONSERVATIVE FORCE $W = -U_2 + U_1 = -\Delta U$ **(8.2)**

POTENTIAL ENERGY OF A SPRING $U = \frac{1}{2}kx^2$ **(8.6)**



CONTRIBUTIONS TO THE MECHANICAL ENERGY

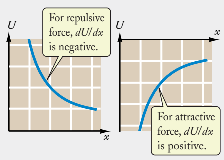
- Kinetic energy: $K = \frac{1}{2}mv^2$ (for motion)
- Gravitational potential energy: $U = mgy$ (near Earth's surface)
- Elastic potential energy: $U = \frac{1}{2}kx^2$ (for a spring)

POTENTIAL ENERGY AS INTEGRAL OF FORCE $U(x) = -\int_{x_0}^{x} F_x(x')\,dx'$ **(8.14)**

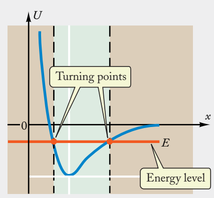
POTENTIAL OF INVERSE-SQUARE FORCE If $F(x) = \frac{A}{x^2}$ then $U(x) = \frac{A}{x}$ **(8.19)**

(for $x > 0$; attractive for $A < 0$, repulsive for $A > 0$.)

FORCE AS DERIVATIVE OF POTENTIAL ENERGY $F_x = -\frac{dU}{dx}$ **(8.21)**



CONSERVATION OF MECHANICAL ENERGY $E = \frac{1}{2}mv^2 + U = [\text{constant}]$ **(8.17)**



MASS IS A FORM OF ENERGY $E = mc^2$ **(8.30)**

ENERGY HAS MASS $\Delta m = \frac{\Delta E}{c^2}$ **(8.32)**

SI UNIT OF POWER	1 watt = 1 W = 1 J/s	
AVERAGE POWER	$\overline{P} = \dfrac{\Delta W}{\Delta t}$	**(8.33)**
INSTANTANEOUS POWER	$P = \dfrac{dW}{dt}$	**(8.34)**
MECHANICAL POWER DELIVERED BY A FORCE	$P = \mathbf{F} \cdot \mathbf{v} = Fv \cos\theta$	**(8.43; 8.44)**
WORK DONE AT CONSTANT POWER	$W = P\,\Delta t$	**(8.37)**
WORK DONE WITH TIME-DEPENDENT POWER	$W = \int_{t_1}^{t_2} P\,dt$	**(8.38)**

QUESTIONS FOR DISCUSSION

1. A body slides on a smooth horizontal plane. Is the normal force of the plane on the body a conservative force? Can we define a potential energy for this force according to the recipe in Section 8.1?
2. If you stretch a spring so far that it suffers a permanent deformation, is the force exerted by the spring during this operation conservative?
3. Is there any frictional dissipation of mechanical energy in the motion of the planets of the Solar System or in the motion of their satellites? (Hint: Consider the tides.)
4. What happens to the kinetic energy of an automobile during braking without skidding? With skidding?
5. An automobile travels down a road leading from a mountain peak to a valley. What happens to the gravitational potential energy of the automobile? How is it dissipated?
6. Suppose you wind up a watch and then place it into a beaker full of nitric acid and let it dissolve. What happens to the potential energy stored in the spring of the watch?
7. News reporters commonly speak of "energy consumption." Is it accurate to say that energy is *consumed*? Would it be more accurate to say that energy is *dissipated*?
8. The explosive yield of thermonuclear bombs (Fig. 8.18) is usually reported in kilotons or megatons of TNT. Would the explosion of a 1-megaton hydrogen bomb really produce the same effects as the explosion of 1 megaton of TNT (a mountain of TNT more than a hundred meters high)?
9. When you heat a potful of water, does its mass increase?
10. Since mass is a form of energy, why don't we measure mass in the same units as energy? How could we do this?
11. In order to travel at 130 km/h, an automobile of average size needs an engine delivering about 40 hp to overcome the effects of air friction, road friction, and internal friction (in the transmission and drive train). Why do most drivers think they need an engine of 150 or 200 hp?

FIGURE 8.18 A thermonuclear explosion.

PROBLEMS

8.1 Potential Energy of a Conservative Force†

1. The spring used in the front suspension of a Triumph sports car has a spring constant $k = 3.5 \times 10^4$ N/m. How far must we compress this spring to store a potential energy of 100 J?

2. A particle moves along the x axis under the influence of a variable force $F_x = 2x^3 + 1$ (where force is measured in newtons and distance in meters). Show that this force is conservative; that is, show that for any back-and-forth motion that starts and ends at the same place (round trip), the work done by the force is zero.

3. Consider a force that is a function of the velocity of the particle (and is not perpendicular to the velocity). Show that the work for a round trip along a closed path can then be different from zero.

4. The force acting on a particle moving along the x axis is given by the formula $F_x = K/x^4$, where K is a constant. Find the corresponding potential-energy function. Assume that $U(x) = 0$ for $x = \infty$.

5. A 50-g particle moving along the x axis experiences a force $F_x = -Ax^3$, where $A = 50$ N/m^3. Find the corresponding potential-energy function. If the particle is released from rest at $x = 0.50$ m, what is its speed as it passes the origin?

6. The force on a particle confined to move along the positive x axis is constant, $F_x = -F_0$, where $F_0 = 25$ N. Find the corresponding potential-energy function. Assume $U(x) = 0$ at $x = 0$.

7. A particular spring is not ideal; for a distance x from equilibrium, the spring exerts a force $F_x = -2x - x^3$, where x is in meters and F_x is in newtons. What is the potential-energy function for this spring? How much energy is stored in the spring when it is stretched 1.0 m? 2.0 m? 3.0 m?

8. The force on a particle moving along the x axis is given by

$$F_x = \begin{cases} F_0 & x \leq -a \\ 0 & -a < x < a \\ -F_0 & x \geq a \end{cases}$$

where F_0 is a constant. What is the potential-energy function for this force? Assume $U(x) = 0$ for $x = 0$.

9. Consider a particle moving in a region where the potential energy is given by $U = 2x^2 + x^4$, where U is in joules and x is in meters. What is the position-dependent force on this particle?

10. The force on an electron in a particular region of space is given by $\mathbf{F} = F_0 \sin(ax)\,\mathbf{i}$, where F_0 and a are constants (this force is achieved with two oppositely directed laser beams). What is the corresponding potential-energy function?

*11. A bow may be regarded mathematically as a spring. The archer stretches this "spring" and then suddenly releases it so that the bowstring pushes against the arrow. Suppose that when the archer stretches the "spring" 0.52 m, he must exert a force of 160 N to hold the arrow in this position. If he now releases the arrow, what will be the speed of the arrow when the "spring" reaches its equilibrium position? The mass of the arrow is 0.020 kg. Pretend that the "spring" is massless.

*12. A mass m hangs on a vertical spring of a spring constant k.
   (a) How far will this hanging mass have stretched the spring from its relaxed length?
   (b) If you now push up on the mass and lift it until the spring reaches its relaxed length, how much work will you have done against gravity? Against the spring?

*13. A particle moving in the x–y plane experiences a conservative force

$$\mathbf{F} = by\mathbf{i} + bx\mathbf{j}$$

where b is a constant.
   (a) What is the work done by this force as the particle moves from $x_1 = 0, y_1 = 0$ to $x_2 = x, y_2 = y$?
   (Hint: Use a path from the origin to the point x_2, y_2 consisting of a segment parallel to the x axis and a segment parallel to the y axis.)
   (b) What is the potential energy associated with this force? Assume that the potential energy is zero when the particle is at the origin.

*14. The four wheels of an automobile of mass 1200 kg are suspended below the body by vertical springs of constant $k = 7.0 \times 10^4$ N/m. If the forces on all wheels are the same, what will be the maximum instantaneous deformation of the springs if the automobile is lifted by a crane and dropped on the street from a height of 0.80 m?

*15. A rope can be regarded as a long spring; when under tension, it stretches and stores elastic potential energy. Consider a nylon rope similar to that which snapped during a giant tug-of-war at a school in Harrisburg, Pennsylvania (see Problem 23 of Chapter 5). Under a tension of 58000 N (applied at its ends), the rope of initial length 300 m stretches to 390 m. What is the elastic energy stored in the rope at this tension? What happens to this energy when the rope breaks?

*16. Among the safety features on elevator cages are spring-loaded brake pads which grip the guide rail if the elevator cable should break. Suppose that an elevator cage of 2000 kg has two such brake pads, arranged to press against opposite sides of the guide rail, each with a force of 1.0×10^5 N. The friction coefficient for the brake pads sliding on the guide rail is 0.15. Assume that the elevator cage is falling freely with an initial speed of 10 m/s when the brake pads come into action. How long will the elevator cage take to stop? How far will it travel? How much energy is dissipated by friction?

† For help, see Online Concept Tutorial 10 at www.ww norton.com/physics

*17. The force between two inert-gas atoms is often described by a function of the form

$$F_x = Ax^{-13} - Bx^{-7}$$

where A and B are positive constants and x is the distance between the atoms. What is the corresponding potential-energy function, called the **Lennard–Jones potential**?

*18. A particle moving in three dimensions is confined by a force $\mathbf{F} = -k(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, where k is a constant. What is the work required to move the particle from the origin to a point $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$? What is the potential-energy function?

**19. Mountain climbers use nylon safety rope whose elasticity plays an important role in cushioning the sharp jerk if a climber falls and is suddenly stopped by the rope.

(a) Suppose that a climber of 80 kg attached to a 10-m rope falls freely from a height of 10 m above to a height of 10 m below the point at which the rope is anchored to a vertical wall of rock. Treating the rope as a spring with $k = 4.9 \times 10^3$ N/m (which is the appropriate value for a braided nylon rope of 9.2 mm diameter), calculate the maximum force that the rope exerts on the climber during stopping.

(b) Repeat the calculations for a rope of 5.0 m and an initial height of 5.0 m. Assume that this second rope is made of the same material as the first, and remember to take into account the change in the spring constant due to the change in length. Compare your results for (a) and (b) and comment on the advantages and disadvantages of long ropes vs. short ropes.

**20. A package is dropped on a horizontal conveyor belt (Fig. 8.19). The mass of the package is m, the speed of the conveyor belt is v, and the coefficient of kinetic friction for the package on the belt is μ_k. For what length of time will the package slide on the belt? How far will it move in this time? How much energy is dissipated by friction? How much energy does the belt supply to the package (including the energy dissipated by friction)?

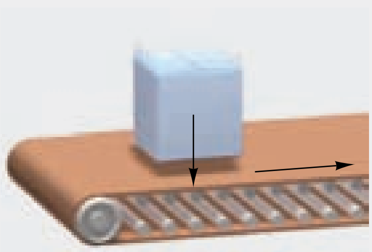


FIGURE 8.19 Package dropped on a conveyor belt

21. The potential energy of a particle moving in the x–y plane is $U = a/(x^2 + y^2)^{1/2}$, where a is a constant. What is the force on the particle? Draw a diagram showing the particle at the position x, y and the force vector.

22. The potential energy of a particle moving along the x axis is $U(x) = K/x^2$, where K is a constant. What is the corresponding force acting on the particle?

23. According to theoretical calculations, the potential energy of two quarks (see the Prelude) separated by a distance r is $U = \eta r$, where $\eta = 1.18 \times 10^{24}$ eV/m. What is the force between the two quarks? Express your answer in newtons.

8.2 The Curve of Potential Energy

24. The potential energy of a particle moving along the x axis is $U(x) = 2x^4 - x^2$, where x is measured in meters and the energy is measured in joules.

(a) Plot the potential energy as a function of x.

(b) Where are the possible equilibrium points?

(c) Suppose that $E = -0.050$ J. What are the turning points of the motion?

(d) Suppose that $E = 1.0$ J. What are the turning points of the motion?

25. In Example 4, we determined the turning point for a bungee jump graphically and numerically. Use the data given in this example for the following calculations.

(a) At what point does the jumper attain maximum speed? Calculate this maximum speed.

(b) At what point does the jumper attain maximum acceleration? Calculate this maximum acceleration.

26. The potential energy of one of the atoms in the hydrogen molecule is

$$U(x) = U_0\left[e^{-2(x-x_0)/b} - 2e^{-(x-x_0)/b}\right]$$

with $U_0 = 2.36$ eV, $x_0 = 0.037$ nm, and b = 0.034 nm.[2] Under the influence of the force corresponding to this potential, the atom moves back and forth along the x axis within certain limits. If the energy of the atom is $E = -1.15$ eV, what will be the turning points of the motion; i.e., at what positions x will the kinetic energy be zero? [Hint: Solve this problem graphically by making a careful plot of $U(x)$; from your plot find the values of x that yield $U(x) = -1.15$ eV.]

27. Suppose that the potential energy of a particle moving along the x axis is

$$U(x) = \frac{b}{x^2} - \frac{2c}{x}$$

where b and c are positive constants.

(a) Plot $U(x)$ as a function of x; assume $b = c = 1$ for this purpose. Where is the equilibrium point?

(b) Suppose the energy of the particle is $E = -\frac{1}{2}c^2/b$. Find the turning points of the motion.

(c) Suppose that the energy of the particle is $E = \frac{1}{2}c^2/b$. Find the turning points of the motion. How many turning points are there in this case?

[2] These values of U_0, x_0, and b are half as large as those usually quoted, because we are looking at the motion of *one* atom relative to the center of the molecule.

28. A particle moves along the x axis under the influence of a conservative force with a potential energy $U(x)$. Figure 8.20 shows the plot of $U(x)$ vs. x. Figure 8.20 shows several alternative energy levels for the particle: $E = E_1$, $E = E_2$, and $E = E_3$. Assume that the particle is initially at $x = 1$ m. For each of the three alternative energies, describe the motion qualitatively, answering the following questions:
   (a) Roughly, where are the turning points (right and left)?
   (b) Where is the speed of the particle maximum? Where is the speed minimum?
   (c) Is the orbit bound or unbound?

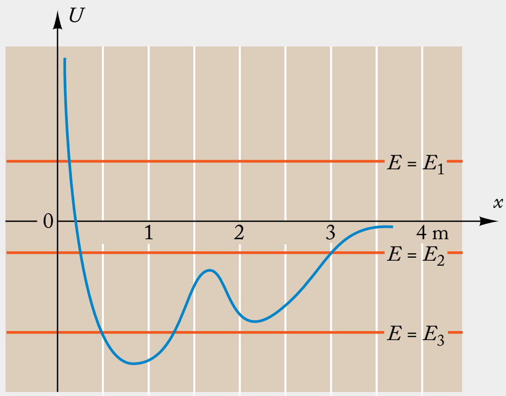


FIGURE 8.20 Plot of $U(x)$ vs. x.

29. A particle moving along the x axis experiences a potential of the form $U(x) = A|x|$, where A is a constant. A particle of mass m has speed v at the origin. Where are the turning points of its motion?
30. A particle initially at the origin moves in a potential of the form $U(x) = -U_0 \cos(ax)$, where U_0 and a are constants. What is the lowest energy the particle may have? If the energy of the particle is $E = 0$ and the particle is initially at $x = 0$, what are the turning points of the motion? For what energies is the particle motion unbound?
31. The potential energy of a particle moving along the x axis is $U(x) = -U_0/[1 + (x/a)^2]$, where $U_0 = 2.0$ J and $a = 1.0$ m. Sketch this function for $-3\text{ m} \leq x \leq 3$ m. What are the turning points for a particle with energy $E = -1.0$ J? For what energies is the particle unbound?

*32. Consider a particle moving in a region where the potential energy is given by $U = 2x^2 + x^4$, where U is in joules and x is in meters. Where are the turning points for a particle with total mechanical energy $E = 1.0$ J? with $E = 2.0$ J?

*33. The potential-energy function (Lennard–Jones potential) for two argon atoms as a function of their separation x is given by $U(x) = Cx^{-12} - Dx^{-6}$, where $C = 1.59 \times 10^{-24}\text{ J}\cdot(\text{nm})^{12}$ and $D = 1.03 \times 10^{-21}\text{ J}\cdot(\text{nm})^6$. (Recall that 1 nm $= 10^{-9}$ m.)
   (a) What is their equilibrium separation in nanometers (nm)?
   (b) What is the lowest possible energy?
   (c) What are the turning points for a particle with energy $E = -2.0 \times 10^{-21}$ J?

8.3 Other Forms of Energy†

34. Express the last two entries in Table 8.1 in electron-volts.
35. The chemical formula for TNT is $CH_3C_5H_2(NO_2)_3$. The explosion of 1 kg of TNT releases 4.6×10^6 J. Calculate the energy released per molecule of TNT. Express your answer in electron-volts.
36. Using the data of Table 8.1, calculate the amount of gasoline that would be required if all the energy requirements of the United States were to be met by the consumption of gasoline. How many gallons per day would have to be consumed?
37. The following table lists the fuel consumption and the passenger capacity of several vehicles. Assume that the energy content of the fuel is that of gasoline (see Table 8.1). Calculate the amount of energy used by each vehicle per passenger per mile. Which is the most energy-efficient vehicle? The least energy-efficient?

VEHICLE	PASSENGER CAPACITY	FUEL CONSUMPTION
Motorcycle	1	60 mi/gal
Snowmobile	1	12
Automobile	4	12
Intercity bus	45	5
Concorde SST	110	0.12
Jetliner	360	0.1

38. The energy released by the metabolization of fat is about 9000 kcal per kg of fat. While jogging on a level road, you use 750 kcal/h. How long do you need to jog to eliminate 1.0 kg of fat?
39. A 12-ounce can of soda typically contains 150 kcal of food energy (150 food "calories"). If your body uses one-fifth of this to climb stairs, how high does one soda enable you to climb?
40. A large household may use as much as 3000 kilowatt-hours of energy during a hot summer month. Express this amount of energy in joules.
41. On food labels in Europe, energy content is typically listed in kilojoules (kJ) instead of kcal (food "calories"). Express a daily intake of 2500 kcal in kJ.

*42. When a humpback whale breaches, or jumps out of the water (see Fig. 8.21), it typically leaves the water at an angle of about 70° at high speed and sometimes attains a height of 3 m,

† For help, see Online Concept Tutorial 10 at www.ww norton.com/physics

FIGURE 8.21 A whale breaching.

measured from the water surface to the center of the whale. For a rough estimate of the energy requirements for such a breach, we can treat the translational motion of the whale as that of a particle moving from the surface of the water upward to a height of 3.0 m (for a more accurate calculation, we would have to take into account the buoyancy of the whale, which assists it in getting out of the water, but let us ignore this). What is the initial speed of the whale when it emerges from the water? What is the initial kinetic energy of a whale of 33 metric tons? Express the energy in kilocalories.

*43. The following table gives the rate of energy dissipation by a man engaged in diverse activities; the energies are given per kilogram of body mass:

RATE OF ENERGY DISSIPATION OF A MAN (PER kg OF BODY MASS)

Standing	1.3 kcal/(kg·h)
Walking (5 km/h)	3.3
Running (8 km/h)	8.2
Running (16 km/h)	15.2

Suppose the man wants to travel a distance of 2.5 km in one-half hour. He can walk this distance in exactly half an hour, or run slow and then stand still until the half hour is up, or run fast and then stand still until the half hour is up. What is the energy per kg of body mass dissipated in each case? Which program uses the most energy? Which the least?

8.4 Mass and Energy

44. The atomic bomb dropped on Hiroshima had an explosive energy equivalent to that of 20 000 tons of TNT, or 8.4×10^{13} J. How many kilograms of mass must have been converted into energy in this explosion?

45. How much energy is released by the annihilation of one proton and one antiproton (both initially at rest)? Express your answer in electron-volts.

46. How much energy is released by the annihilation of one electron and one antielectron (both initially at rest)? Express your answer in electron-volts.

47. The mass of the Sun is 2×10^{30} kg. The thermal energy in the Sun is about 2×10^{41} J. How much does the thermal energy contribute to the mass of the Sun?

48. The masses of the proton, electron, and neutron are 1.672623×10^{-27} kg, 9.11×10^{-31} kg, and 1.674929×10^{-27} kg, respectively. If a neutron decays into a proton and an electron, how much energy is released (other than the energy of the mass of the proton and electron)? Compare this extra energy with the energy of the mass of the electron.

49. Express the mass energy of the electron in keV. Express the mass energy of the proton in MeV.

50. A typical household may use approximately 1000 kilowatt-hours of energy per month. What is the equivalent amount of rest mass?

51. Combustion of one gallon of gasoline releases 1.3×10^8 J of energy. How much mass is converted to energy? Compare this with 2.8 kg, the mass of one gallon of gasoline.

52. A small silicon particle of diameter 0.20 micrometers has a mass of 9.8×10^{-18} kg. What is the mass energy of such a "nanoparticle" (in J)?

*53. In a high-speed collision between an electron and an antielectron, the two particles can annihilate and create a proton and an antiproton. The reaction

$$e + \bar{e} \rightarrow p + \bar{p}$$

converts the mass energy and kinetic energy of the electron and antielectron into the mass energy of the proton and the antiproton. Assume that the electron and the antielectron collide head-on with opposite velocities of equal magnitudes and that the proton and the antiproton are at rest immediately after the reaction. Calculate the kinetic energy of the electron required for this reaction; express your answer in electron-volts.

8.5 Power†

54. For an automobile traveling at a steady speed of 65 km/h, the friction of the air and the rolling friction of the ground on the wheels provide a total external friction force of 500 N. What power must the engine supply to keep the automobile moving? At what rate does the friction force remove momentum from the automobile?

55. In 1979, B. Allen flew a very lightweight propeller airplane across the English Channel. His legs, pushing bicycle pedals, supplied the power to turn the propeller. To keep the airplane flying, he had to supply about 0.30 hp. How much energy did he supply for the full flight lasting 2 h 49 min? Express your answer in kilocalories.

† For help, see Online Concept Tutorial 10 at www.ww norton.com/physics

56. The ancient Egyptians and Romans relied on slaves as a source of mechanical power. One slave, working desperately by turning a crank, could deliver about 200 W of mechanical power (at this power the slave would not last long). How many slaves would be needed to match the output of a modern automobile engine (150 hp)? How many slaves would an ancient Egyptian have to own in order to command the same amount of power as the average per capita power used by residents of the United States (14 kW)?

57. An electric clock uses 2.0 W of electric power. How much electric energy (in kilowatt-hours) does this clock use in 1 year? What happens to this electric energy?

58. While an automobile is cruising at a steady speed of 65 km/h, its engine delivers a mechanical power of 20 hp. How much energy does the engine deliver per hour?

59. A large windmill delivers 10 kW of mechanical power. How much energy does the windmill deliver in a working day of 8 hours?

60. The heating unit of a medium-sized house produces 170 000 Btu/h. Is this larger or smaller than the power produced by a typical automobile engine of 150 hp?

61. The heart of a resting person delivers a mechanical power of about 1.1 W for pumping blood. Express this power in hp. How much work does the heart do on the blood per day? Express this work in kcal.

62. The lasers to be used for controlled fusion experiments at the National Ignition Facility at the Lawrence Livermore Laboratory will deliver a power of 2.0×10^{15} W, a thousand times the output of all the power stations in the United States, in a brief pulse lasting 1.0×10^{-9} s. What is the energy in this laser pulse? How does it compare with the energy output of all the power stations in the United States in one day?

63. During the seven months of the cold season in the Northeastern United States, a medium-sized house requires about 1.0×10^8 Btu of heat to keep warm. A typical furnace delivers 1.3×10^5 Btu of heat per gallon of fuel oil.

   (a) How many gallons of fuel oil does the house consume during the cold season?

   (b) What is the average power delivered by the furnace?

64. Experiments on animal muscle tissue indicate that it can produce up to 100 watts of power per kilogram. A 600-kg horse has about 180 kg of muscle tissue attached to the legs in such a way that it contributes to the external work the horse performs while pulling a load. Accordingly, what is the theoretical prediction for the maximum power delivered by a horse? In trials, the actual maximum power that a horse can deliver in a short spurt was found to be about 12 hp. How does this compare with the theoretical prediction?

65. If a 60-W light bulb is left on for 24 hours each day, how many kilowatt-hours of electric power does it use in one year? If the electric energy costs you 15 cents per kilowatt-hour, what is your cost for one year?

66. Nineteenth-century English engineers reckoned that a laborer turning a crank can do steady work at the rate of 5000 ft·lbf/min. Suppose that four laborers working a manual crane attempt to lift a load of 9.0 short tons (1 short ton = 2000 lb). If there is no friction, what is the rate at which they can lift this load? How long will it take them to lift the load 15 ft?

67. The driver of an automobile traveling on a straight road at 80 km/h pushes forward with his hands on the steering wheel with a force of 50 N. What is the rate at which his hands do work on the steering wheel in the reference frame of the ground? In the reference frame of the automobile?

68. An automobile with a 100-hp engine has a top speed of 160 km/h. When at this top speed, what is the friction force (from air and road) acting on the automobile?

69. A horse walks along the bank of a canal and pulls a barge by means of a long horizontal towrope making an angle of 35° with the bank. The horse walks at the rate of 5.0 km/h, and the tension in the rope is 400 N. What horsepower does the horse deliver?

70. A 900-kg automobile accelerates from 0 to 80 km/h in 7.6 s. What are the initial and the final translational kinetic energies of the automobile? What is the average power delivered by the engine in this time interval? Express your answer in horsepower.

71. A six-cylinder internal combustion engine, such as used in an automobile, delivers an average power of 150 hp while running at 3000 rev/min. Each of the cylinders fires once every two revolutions. How much energy does each cylinder deliver each time it fires?

72. In Chapter 6, we saw that an automobile must overcome the force of air resistance, $f_{air} = \frac{1}{2}\rho C A v^2$. For the automobile of Example 6 of Chapter 6 ($C = 0.30$, $A = 2.8$ m^2, and $\rho =$ 1.3 kg/m^3), calculate the power dissipation due to air resistance when traveling at 30 km/h and when traveling at 90 km/h. What is the difference in the total energy supplied to overcome air friction for a 300-km trip at 30km/h? For a 300-km trip at 90 km/h?

73. A constant force of 40 N is applied to a body as the body moves uniformly at a speed of 3.5 m/s. The force does work on the body at a rate of 90 W. What is the angle between the force and the direction of motion of the body?

74. An electric motor takes 1.0 s to get up to speed; during this time, the power supplied by the motor varies with time according to $P = P_1 + (P_0 - P_1)(t - 1)^2$, where t is in seconds, $P_0 = 1.50$ kW, and $P_1 = 0.75$ kW. What is the total energy supplied for the time period $0 \leq t \leq 1$ s?

75. A constant force $\mathbf{F} = (6.0\text{ N})\mathbf{i} + (8.0\text{ N})\mathbf{j}$ acts on a particle. At what instantaneous rate is this force doing work on a particle with velocity $\mathbf{v} = (3.0\text{ m/s})\mathbf{i} - (2.5\text{ m/s})\mathbf{j}$?

76. An automobile engine typically has an efficiency of about 25%; i.e., it converts about 25% of the chemical energy available in gasoline into mechanical energy. Suppose that an automobile engine has a mechanical output of 110 hp. At what

rate (in gallons per hour) will this engine consume gasoline? See Table 8.1 for the energy content in gasoline.

77. The takeoff speed of a DC-3 airplane is 100 km/h. Starting from rest, the airplane takes 10 s to reach this speed. The mass of the (loaded) airplane is 11 000 kg. What is the average power delivered by the engines to the airplane during takeoff?

78. The Sun emits energy in the form of radiant heat and light at the rate of 3.9×10^{26} W. At what rate does this energy carry away mass from the Sun? How much mass does this amount to in 1 year?

79. The energy of sunlight arriving at the surface of the Earth amounts to about 1.0 kW per square meter of surface (facing the Sun). If all of the energy incident on a collector of sunlight could be converted into useful energy, how many square meters of collector area would we need to satisfy all of the energy demands in the United States? See Table 8.1 for the energy expenditure of the United States.

80. Equations (2.11) and (2.16) give the velocity and the acceleration of an accelerating Maserati sports car as a function of time. The mass of this automobile is 1770 kg. What is the instantaneous power delivered by the engine to the automobile? Plot the instantaneous power as a function of time in the time interval from 0 to 10 s. At what time is the power maximum?

81. The ship *Globtik Tokyo*, a supertanker, has a mass of 650 000 metric tons when fully loaded.
   (a) What is the kinetic energy of the ship when her speed is 26 km/h?
   (b) The engines of the ship deliver a power of 44 000 hp. According to the energy requirements, how long a time does it take the ship to reach a speed of 26 km/h, starting from rest? Make the assumption that 50% of the engine power goes into friction or into stirring up the water and 50% remains available for the translational motion of the ship.
   (c) How long a time does it take the ship to stop from an initial speed of 26 km/h if her engines are put in reverse? Estimate roughly how far the ship will travel during this time.

82. At Niagara Falls, 6200 m^3 per second of water falls down a height of 49 m.
   (a) What is the rate (in watts) at which gravitational potential energy is dissipated by the falling water?
   (b) What is the amount of energy (in kilowatt-hours) wasted in 1 year?
   (c) Power companies get paid about 5 cents per kilowatt-hour of electric energy. If all the gravitational potential energy wasted in Niagara Falls could be converted into electric energy, how much money would this be worth?

83. The movement of a grandfather clock is driven by a 5.0-kg weight which drops a distance of 1.5 m in the course of a week. What is the power delivered by the weight to the movement?

84. A 27 000-kg truck has a 550-hp engine. What is the maximum speed with which this truck can move up a 10° slope?

*85. Consider a "windmill ship," which extracts mechanical energy from the wind by means of a large windmill mounted on the deck (see Fig. 8.22). The windmill generates electric power, which is fed into a large electric motor, which propels the ship. The mechanical efficiency of the windmill is 70% (that is, it removes 70% of the kinetic energy of the wind and transforms it into rotational energy of its blades). The efficiency of the electric generator attached to the windmill is 90%, and the efficiency of the electric motor connected to the generator is also 90%. We want the electric motor to deliver 20 000 hp in a (relative) wind of 40 km/h. What size windmill do we need? The density of air is 1.29 kg/m^3.

FIGURE 8.22
A "windmill ship."

*86. An electric water pump is rated at 15 hp. If this water pump is to lift water to a height of 30 m, how many kilograms of water can it lift per second? How many liters? Neglect the kinetic energy of the water.

*87. The engines of the Sikorski Blackhawk helicopter generate 3080 hp of mechanical power, and the maximum takeoff mass of this helicopter is 7400 kg. Suppose that this helicopter is climbing vertically at a steady rate of 5.0 m/s.
   (a) What is the power that the engines deliver to the body of the helicopter?
   (b) What is the power that the engines deliver to the air (by friction and by the work that the rotors of the helicopter perform on the air)?

*88. In order to overcome air friction and other mechanical friction, an automobile of mass 1500 kg requires a power of 20 hp from its engine to travel at 64 km/h on a level road. Assuming the friction remains the same, what power does the same automobile require to travel uphill on an incline of slope 1:10 at the same speed? Downhill on the same incline at the same speed?

*89. With the gears in neutral, an automobile rolling down a long incline of slope 1:10 reaches a terminal speed of 95 km/h. At this speed the rate of decrease of the gravitational potential energy matches the power required to overcome air friction and other mechanical friction. What power (in horsepower) must the engine of this automobile deliver to drive it at 95 km/h on a level road? The mass of the automobile is 1500 kg.

*90. The power supplied to an electric circuit decreases exponentially with time according to $P = P_0 e^{-t/\tau}$, where $P_0 =$ 2.0 W and $\tau = 5.0$ s are constants. What is the total energy supplied to the circuit during the time interval $0 \le t \le 5.0$ s? During $0 \le t \le \infty$?

*91. Each of the two Wright Cyclone engines on a DC-3 airplane generates a power of 850 hp. The mass of the loaded plane is 10 900 kg. The plane can climb at the rate of 260 m/min. When the plane is climbing at this rate, what percentage of the engine power is used to do work against gravity?

*92. A fountain shoots a stream of water 10 m up in the air. The base of the stream is 10 cm across. What power is expended to send the water to this height?

*93. The record of 203.1 km/h for speed skiing set by Franz Weber at Velocity Peak in Colorado was achieved on a mountain slope inclined downward at 51°. At this speed, the force of friction (air and sliding friction) balances the pull of gravity along the slope, so the motion proceeds at constant velocity.

(a) What is the rate at which gravity does work on the skier? Assume that the mass of the skier is 75 kg.

(b) What is the rate at which sliding friction does work? Assume that the coefficient of friction is $\mu_k = 0.03$.

(c) What is the rate at which air friction does work?

*94. A windmill for the generation of electric power has a propeller of diameter 1.8 m. In a wind of 40 km/h, this windmill delivers 200 W of electric power.

(a) At this wind speed, what is the rate at which the air carries kinetic energy through the circular area swept out by the propeller? The density of air is 1.29 kg/m^3.

(b) What percentage of the kinetic energy of the air passing through this area is converted into electric energy?

*95. A small electric kitchen fan blows 8.5 m^3/min of air at a speed of 5.0 m/s out of the kitchen. The density of air is 1.3 kg/m^3. What electric power must the fan consume to give the ejected air the required kinetic energy?

*96. The final portion of the Tennessee River has a downward slope of 0.074 m per kilometer. The rate of flow of water in the river is 280 m^3/s. Assume that the speed of the water is constant along the river. How much power is dissipated by friction of the water against the riverbed per kilometer?

*97. Off the coast of Florida, the Gulf Stream has a speed of 4.6 km/h and a rate of flow of 2.2×10^3 km^3/day. At what rate is kinetic energy flowing past the coast? If all this kinetic energy could be converted into electric power, how many kilowatts would it amount to?

*98. Figure 8.23 shows an overshot waterwheel, in which water flowing onto the top of the wheel fills buckets whose weight causes the wheel to turn. The water descends in the buckets to the bottom, and there it is spilled out, so the ascending buckets are always empty. If in a waterwheel of diameter 10 m the amount of water carried down by the wheel is 20 liters per second (or 20 kg per second), what is the mechanical power that the descending water delivers to the wheel? Assume that the water flowing onto the top of the wheel has roughly the same speed as the wheel and exerts no horizontal push on the wheel. [Hint: The kinetic energy of the water is the same when the water enters the bucket and when it spills out (since the speed of the bucket is constant); hence the kinetic energy of the water does not affect the answer.]

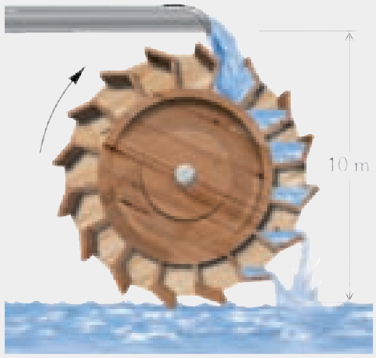


FIGURE 8.23 An overshot waterwheel.

*99. Suppose that in the undershot waterwheel shown in Fig. 8.24, the stream of water against the blades of the wheel has a speed of 15 m/s, and the amount of water is 30 liters per second (or 30 kg per second). If the water gives all of its kinetic energy to the blades (and then drips away with zero horizontal speed), how much mechanical power does the water deliver to the wheel?

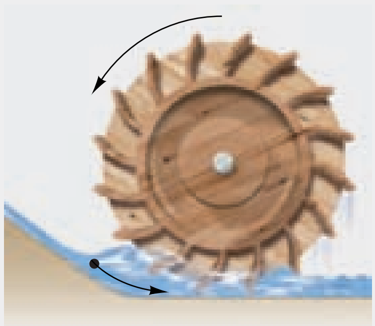


FIGURE 8.24 An undershot waterwheel.

*100. (a) With its engines switched off, a small two-engine airplane of mass 1100 kg glides downward at an angle of 13° at a speed of 90 knots. Under these conditions, the weight of the plane, the lift force (perpendicular to the direction of motion) generated by air flowing over the wings, and the frictional force (opposite to the direction of motion) exerted by the air are in balance. Draw a "free-body" diagram for these forces, and calculate their magnitudes.

(b) Suppose that with its engine switched on, the plane climbs at an upward angle of 13° at a speed of 90 knots. Draw a "free-body" diagram for the forces acting on the airplane under these conditions; include the push that the air exerts on the propeller. Calculate the magnitudes of all the forces.

(c) Calculate the power that the engine must deliver to compensate for the rate of increase of the potential energy of the plane and the power lost to friction. For a typical small plane of 1100 kg, the actual engine power required for such a climb of 13° is about 400 hp. Explain the discrepancy between your result and the actual engine power. (Hint: What does the propeller do to the air?)

*101. The reaction that supplies the Sun with energy is

$$\mathrm{H + H + H + H \rightarrow He + [energy]}$$

(The reaction involves several intermediate steps, but this need not concern us now.) The mass of the hydrogen (H) atom is 1.008 13 u, and that of the helium (He) atom is 4.003 88 u.

(a) How much energy is released in the reaction of four hydrogen atoms (by the conversion of mass into energy)?

(b) How much energy is released in the reaction of 1.0 kg of hydrogen atoms?

(c) The Sun releases energy at the rate of 3.9×10^{26} W. At what rate (in kg/s) does the Sun consume hydrogen?

(d) The Sun contains about 1.5×10^{30} kg of hydrogen. If it continues to consume hydrogen at the same rate, how long will the hydrogen last?

REVIEW PROBLEMS

102. A particle moves along the x axis under the influence of a variable force $F_x = 5x^2 + 3x$ (where force is measured in newtons and distance in meters).

(a) What is the potential energy associated with this force? Assume that $U(x) = 0$ at $x = 0$.

(b) How much work does the force do on a particle that moves from $x = 0$ to $x = 2.0$ m?

*103. A particle is subjected to a force that depends on position as follows:

$$\mathbf{F} = 4\mathbf{i} + 2x\mathbf{j}$$

where the force is measured in newtons and the distance in meters.

(a) Calculate the work done by this force as the particle moves from the origin to the point $x = 1.0$ m, $y = 1.0$ m along the straight path I shown in Fig. 8.25.

(b) Calculate the work done by this force if the particle returns from the point $x = 1.0$ m, $y = 1.0$ m to the origin along the path II consisting of a horizontal and a vertical segment (see Fig. 8.25). Is the force conservative?

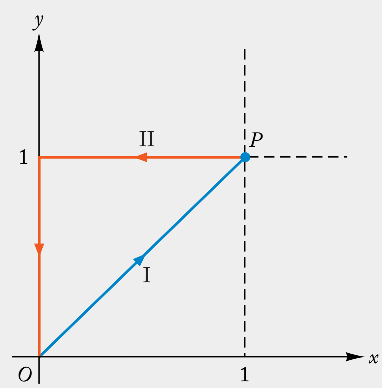


FIGURE 8.25 Outward and return paths of a particle.

*104. A 3.0-kg block sliding on a horizontal surface is accelerated by a compressed spring. At first, the block slides without friction. But after leaving the spring, the block travels over a new portion of the surface, with a coefficient of friction 0.20, for a distance of 8.0 m before coming to rest (see Fig. 8.26). The force constant of the spring is 120 N/m.

(a) What was the maximum kinetic energy of the block?

(b) How far was the spring compressed before being released?

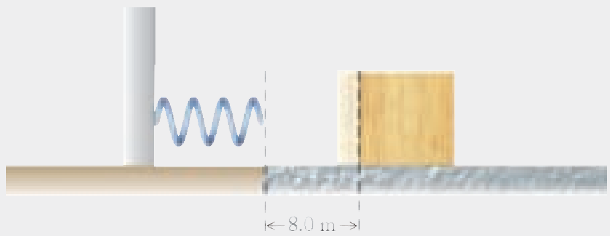


FIGURE 8.26 Block released from a spring.

105. The ancient Egyptians moved large stones by dragging them across the sand in sleds (Fig. 8.27). Suppose that 6000 Egyptians are dragging a sled with a coefficient of sliding friction $\mu_k = 0.30$ along a level surface of sand. Each Egyptian can exert a force of 360 N, and each can deliver a mechanical power of 0.20 hp.

(a) What is the maximum weight they can move at constant speed?

(b) What is the maximum speed with which they can move this weight?

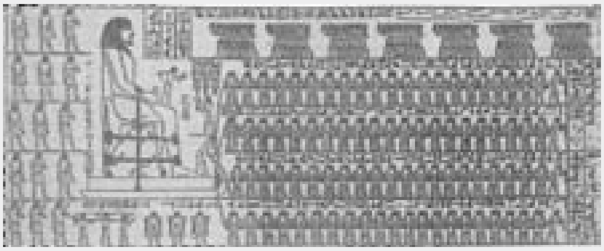


FIGURE 8.27 Ancient Egyptian wall mural from 1900 B.C.

106. In a braking test, a 990-kg automobile takes 2.1 s to come to a full stop from an initial speed of 60 km/h. What is the amount of energy dissipated in the brakes? What is the average power dissipated in the brakes? Ignore external friction in your calculation and express the power in horsepower.

107. In a waterfall on the Alto Paraná river (between Brazil and Paraguay), the height of fall is 33 m and the average rate of flow is 13 000 m^3 of water per second. What is the power dissipated by this waterfall?

108. When jogging at 12 km/h on a level road, a 70-kg man uses 750 kcal/h. How many kilocalories per hour does he require when jogging up a 1:10 incline at the same speed? Assume that the frictional losses are independent of the value of the slope.

109. Consider a projectile traveling horizontally and slowing down under the influence of air resistance, as described in Problems 47 and 48 of Chapter 2. The mass of this projectile is 45.36 kg, and the speed as a function of time is

$$v = 655.9 - 61.1t + 3.26t^2$$

where speed is in m/s and time in seconds.

(a) What is the instantaneous power removed from the projectile by air resistance?

(b) What is the kinetic energy at time $t = 0$? At time $t = 3.00$ s?

(c) What is the average power for the time interval from 0 to 3.00 s?

110. A woman exercising on a rowing machine pulls the oars back once per second. During such a pull, each hand moves 0.50 m while exerting an average force of 100 N.

(a) What is the work the woman does during each stroke (with both hands)?

(b) What is the average power the woman delivers to the oars?

111. The world's tallest staircase, of 2570 steps, is located in the CN tower in Toronto. It reaches a height of 457 m. Estimate how long it would take an athlete to climb this staircase. The athlete has a mass of 75 kg, and his leg muscle can deliver a power of 200 W.

112. A pump placed on the shore of a pond takes in 0.80 kg of water per second and squirts it out of a nozzle at 50 m/s. What mechanical power does this pump supply to the water?

113. The hydroelectric pumped-storage plant in Northfield, Massachusetts, has a reservoir holding 2.2×10^7 m^3 of water on top of a mountain. The water flows 270 m vertically down the mountain in pipes and drives turbines connected to electric generators.

(a) How much electric energy, in kW·h, can this storage plant generate with the water available in the reservoir?

(b) In order to generate 1000 MW of electric energy, at what rate, in m^3/s, must this storage plant withdraw water from the reservoir?

114. A 50-kg circus clown is launched vertically from a spring-loaded cannon using a spring with spring constant 3500 N/m. The clown attains a height of 4.0 m above the initial position (when the spring was compressed).

(a) How far was the spring compressed before launch?

(b) What was the maximum acceleration of the clown during launch?

(c) What was the maximum speed of the clown?

Answers to Checkups

Checkup 8.1

1. The force can be obtained from $F_x = -dU/dx = +kx$. Thus, the force is positive for positive x and negative for negative x; that is, the new force is repulsive (it pushes a particle away from $x = 0$), whereas the spring force is attractive.

2. The potential energy shown in Fig. 8.5a has a negative slope as a function of x. By Eq. (8.21), the force is the negative of the slope of the potential; the negative of a negative is positive, and so the force is directed along the positive x direction.

3. A force is always conservative if the force is an explicit function of position x. In that case, a potential-energy function can always be constructed by integration of the force according to Eq. (8.14).

4. (D) No; no. The work done is equal to the negative of the change in potential energy only for conservative forces. The work done is equal to the change in kinetic energy only for the *net* force acting on a particle.

Checkup 8.2

1. The number of turning points must decrease with increasing energy [we do not consider a stationary point of stable equilibrium (Fig. 8.8), since the particle is moving]. Consider the potential of Fig. 8.7: for small energies, the particle will move back and forth (two turning points); for somewhat higher energy, the particle will move back from the left end but escape from the right end (one turning point). Unless $U = \infty$, for sufficiently high energy the particle could escape from the left end also (no turning point). In one dimension, there cannot be more than two turning points, although the two turning points will of course be different for different energies.
2. The maximum speed corresponds to the deepest part of the curve (maximum kinetic energy, $K = E - U$); from the figure, this occurs at $x \approx -6$ m. The maximum acceleration and force ($F = -dU/dx$) occurs where the slope is largest; for the bungee jumper, this is at $x \approx -15$ m.
3. (B) 2. The potential-energy curve of the spring force is a simple parabola (Fig. 8.1), so there are two turning points for any positive energy.

Checkup 8.3

1. No. Gravitational potential energy is lost as the parachutist descends (at uniform speed, there is no change in kinetic energy). From a macroscopic viewpoint, the energy lost due to friction with the air is converted into heat.
2. The energy comes from a decrease in the chemical energy of the exploding gunpowder; microscopically, such chemical energy comes from changes in the kinetic and potential energy of electrons in the atoms and bonds of the elements involved.
3. The energy is converted to heat due to frictional forces; these may include friction in the engine, brakes, tires, and road, as well as air friction.
4. (C) The kinetic energy is converted into heat due to frictional forces, mostly in the brakes (brake pads rub against drums or disks), partly where the tires contact the road, and some from air friction. All the heat is eventually transferred to the air as the brakes, tires, and road cool.

Checkup 8.4

1. Yes; the Sun continually loses mass in the form of heat and light, as well as by emitting particles with mass.
2. Energy and mass are both conserved; the original rest mass is converted to the energy of the light (electromagnetic radiation), and this light carries away mass as well as energy.
3. (A) Increase. The mass of the water will increase by the usual $\Delta m = \Delta E/c^2$, where ΔE is the increase in thermal energy of the water.

Checkup 8.5

1. (a) No; there is no force parallel to the motion, so there is no work done and no power expended. (b) Trotting uphill, you deliver power at a rate $P = Fv = mg \sin \phi\, v$, where m is the mass of the pack, ϕ is the angle of the incline, and v is the speed along the incline. (c) Trotting downhill, the component of $\mathbf{F}$ along $\mathbf{v}$ is negative, so you do negative work on the backpack; that is, the backpack delivers power to you.
2. Yes, the energy you have to expend is mgh, whichever slope of road you take. The steeper road requires more of an effort, since, for example, for the same walking speed, the force is more nearly parallel to the velocity, and so the power expended, $P = \mathbf{F} \cdot \mathbf{v}$, is greater.
3. Some of the power is lost as heat, due to the friction force between the boat and the water; some of the energy is converted into a more macroscopic kinetic energy of the water, by the generation of water waves.
4. (C) 4. The power is equal to the force times the speed. At the same speed, a car with twice the mass will require twice the power to move against gravity; if that car is also traveling at twice the speed, it will then require four times as much power (ignoring other losses).

CHAPTER

9

Gravitation

Concepts in Context

CONCEPTS IN CONTEXT

Hundreds of artificial satellites have been placed in orbit around the Earth, such as this Syncom communications satellite shown just after launch from the Space Shuttle.

With the concepts we will develop in this chapter, we can answer various questions about artificial satellites:

? Communications satellites and weather satellites are placed in high-altitude "geosynchronous" orbits that permit them to keep in step with the rotation of the Earth, so the satellite always remains at a fixed point above the equator. What is the radius of such a geosynchronous orbit? (Example 6, page 279; and Physics in Practice: Communications Satellites and Weather Satellites, page 281)

? Surveillance satellites and spacecraft such as the Space Shuttle usually operate in low-altitude orbits, just above the Earth's atmosphere. How quickly does such a satellite circle the Earth? (Example 7, page 280)

? The Syncom satellite was carried by the Space Shuttle to a low-altitude orbit, and then it used its own booster rocket to lift itself to the high-altitude geosynchronous orbit. What is the increase of mechanical energy (kinetic and gravitational) of the satellite during this transfer from one orbit to another? (Example 9, page 290)

Within the Solar System, planets orbit around the Sun, and satellites orbit around the planets. These circular, or nearly circular, motions require a centripetal force pulling the planets toward the Sun and the satellites toward the planets. It was Newton's great discovery that this interplanetary force holding the celestial bodies in their orbits is of the same kind as the force of gravity that causes apples, and other things, to fall downward near the surface of the Earth. Newton found that a single formula, his Law of Universal Gravitation, encompasses both the gravitational forces acting between celestial bodies and the gravitational force acting on bodies near the surface of the Earth.

By the nineteenth century, Newton's theory of gravitation had proved itself so trustworthy that when astronomers noticed an irregularity in the motion of Uranus, they could not bring themselves to believe that the theory was at fault. Instead, they suspected that a new, unknown planet caused these irregularities by its gravitational pull on Uranus. The astronomers J. C. Adams and U. J. J. Leverrier proceeded to calculate the expected position of this hypothetical planet—and the new planet was immediately found at just about the expected position. This discovery of a new planet, later named Neptune, was a spectacular success of Newton's theory of gravitation. Newton's theory remains one of the most accurate and successful theories in all of physics, and in all of science.

In this chapter, we will examine Newton's Law of Universal Gravitation; we will see how it includes the familiar gravitational force near the Earth's surface. We will also examine circular and elliptical orbits of planets and satellites, and we will become familiar with Kepler's laws describing these orbits. Finally, we will discuss gravitational potential energy and apply energy conservation to orbital motion.

Online Concept Tutorial 11

9.1 NEWTON'S LAW OF UNIVERSAL GRAVITATION

Newton proposed that just as the Earth gravitationally attracts bodies placed near its surface and causes them to fall downward, the Earth also attracts more distant bodies, such as the Moon, or the Sun, or other planets. In turn, the Earth is gravitationally attracted by all these bodies. More generally, *every* body in the Universe attracts *every* other body with a gravitational force that depends on their masses and on their distances. The gravitational force that two bodies exert on each other is large if their masses are large, and small if their masses are small. The gravitational force decreases if we increase the distance between the bodies. The **Law of Universal Gravitation** formulated by Newton can be stated most easily for the case of particles:

> *Every particle attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them.*

Expressed mathematically, the magnitude of the gravitational force that two particles of masses M and m separated by a distance r exert on each other is

Law of Universal Gravitation

$$F = \frac{GMm}{r^2} \tag{9.1}$$

where G is a universal constant of proportionality, the same for all pairs of particles.

The direction of the force on each particle is directly toward the other particle. Figure 9.1 shows the directions of the forces on each particle. Note that the two forces are of equal magnitudes and opposite directions; they form an action–reaction pair, as required by Newton's Third Law.

The constant G is known as the **gravitational constant**. In SI units its value is approximately given by

$$G = 6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2 \tag{9.2}$$

gravitational constant

The gravitational force of Eq. (9.1) is an inverse-square force: it decreases by a factor of 4 when the distance increases by a factor of 2, it decreases by a factor of 9 when the distance increases by a factor of 3, and so on. Figure 9.2 is a plot of the magnitude of the gravitational force as a function of the distance. Although the force decreases with distance, it never quite reaches zero. Thus, every particle in the universe continually attracts every other particle at least a little bit, even if the distance between the particles is very, very large.

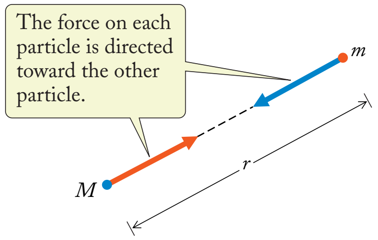


FIGURE 9.1 Two particles attract each other gravitationally. The forces are of equal magnitudes and of opposite directions.

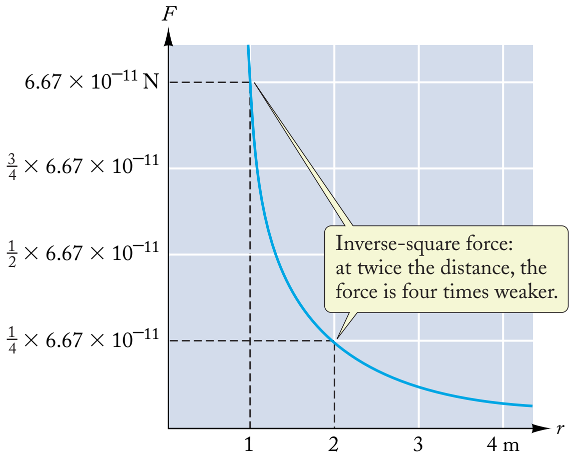


FIGURE 9.2 Magnitude of the gravitational force exerted by a particle of mass 1 kg on another particle of mass 1 kg.

EXAMPLE 1 What is the gravitational force between a 70-kg man and a 70-kg woman separated by a distance of 10 m? Treat both masses as particles.

SOLUTION: From Eq. (9.1),

$$F = \frac{GMm}{r^2}$$

$$= \frac{6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2 \times 70\ \text{kg} \times 70\ \text{kg}}{(10\ \text{m})^2}$$

$$= 3.3 \times 10^{-9}\ \text{N}$$

This is a very small force, but as we will see in the next section, the measurement of such small forces is not beyond the reach of sensitive instruments.

The gravitational force does not require any contact between the interacting particles. In reaching from one remote particle to another, the gravitational force somehow bridges the empty space between the particles. This is called **action-at-a-distance**.

It is also quite remarkable that the gravitational force between two particles is unaffected by the presence of intervening masses. For example, a particle in Washington attracts a particle in Beijing with exactly the force given by Eq. (9.1), even though all of the bulk of the Earth lies between Washington and Beijing. This means that it is impossible to shield a particle from the gravitational attraction of another particle.

Newton's theorem

Since the gravitational attraction between two particles is completely independent of the presence of other particles, it follows that the net gravitational force between two bodies (e.g., the Earth and the Moon or the Earth and an apple) is merely the vector sum of the individual forces between all the particles making up the bodies—that is, the gravitational force obeys the principle of linear superposition of forces (see Section 5.3). As a consequence of this simple vector summation of the gravitational forces of the individual particles in a body, it can be shown that *the net gravitational force between two spherical bodies acts just as though each body were concentrated at the center of its respective sphere*. This result is known as **Newton's theorem**. The proof of Newton's theorem involves a somewhat tedious summation. Later, in the context of electrostatic force, we provide a much simpler derivation of Newton's theorem using Gauss' Law (see Chapter 24). Since the Sun, the planets, and most of their satellites are almost exactly spherical, this important theorem permits us to treat all these celestial bodies as point-like particles in all calculations concerning their gravitational attractions. For instance, since the Earth is (nearly) spherical, the gravitational force exerted by the Earth on a particle above its surface is as though the mass of the Earth were concentrated at its center; thus, this force has a magnitude

$$F = \frac{GM_E m}{r^2} \tag{9.3}$$

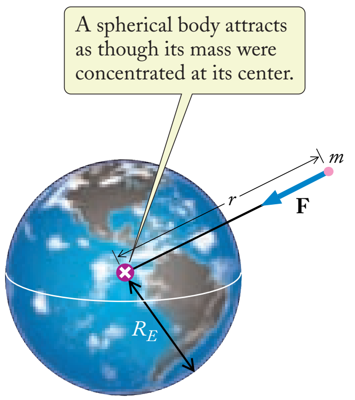


FIGURE 9.3 The gravitational force exerted by the Earth on a particle is directed toward the center of the Earth.

where m is the mass of the particle, M_E is the mass of the Earth, and r is the distance from the center of the Earth (see Fig. 9.3).

If the particle is at the surface of the Earth, at a radius $r = R_E$, then Eq. (9.3) gives a force

$$F = \frac{GM_E m}{R_E^2} \tag{9.4}$$

The corresponding acceleration of the mass m is

$$a = \frac{F}{m} = \frac{GM_E}{R_E^2} \tag{9.5}$$

But this acceleration is what we usually call the acceleration of free fall; and usually designate by g. Thus, g is related to the mass and the radius of the Earth,

$$g = \frac{GM_E}{R_E^2} \tag{9.6}$$

This equation establishes the connection between the ordinary force of gravity we experience at the surface of the Earth and Newton's Law of Universal Gravitation. Notice that g is only approximately constant. Small changes in height near the Earth's surface have little effect on the value given by Eq. (9.6), since $R_E \approx 6.4 \times 10^6$ m is so large. But for a large altitude h above the Earth's surface, we must replace R_E with $R_E + h$ in Eq. (9.6), and appreciable changes in g can occur.

Note that an equation analogous to Eq. (9.6) relates the acceleration of free fall at the surface of any (spherical) celestial body to the mass and the radius of that body.

For example, we can calculate the acceleration of free fall on the surface of the Moon from its mass and radius.

EXAMPLE 2 The mass of the Moon is 7.35×10^{22} kg, and its radius is 1.74×10^6 m. Calculate the acceleration of free fall on the Moon and compare with acceleration of free fall on the Earth.

SOLUTION: For the Moon, the formula analogous to Eq. (9.6) is

$$g_{\text{Moon}} = \frac{GM_{\text{Moon}}}{R_{\text{Moon}}^2} = \frac{6.67 \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2 \times 7.35 \times 10^{22}\,\text{kg}}{(1.74 \times 10^6\,\text{m})^2}$$
$$= 1.62\ \text{m/s}^2$$

This is about 1/6 the acceleration of free fall on the surface of the Earth ($g = 9.81\ \text{m/s}^2$). If you can jump upward to a height of one-half meter on the Earth, then this same jump will take you to a height of 3 meters on the Moon!

EXAMPLE 3 The masses of the Sun, Earth, and Moon are 1.99×10^{30} kg, 5.98×10^{24} kg, and 7.35×10^{22} kg, respectively. Assume that the location of the Moon is such that the angle subtended by the lines from the Moon to the Sun and from the Moon to the Earth is 45.0°, as shown in Fig. 9.4a. What is the net force on the Moon due to the gravitational forces of the Sun and Earth? The Moon is 1.50×10^{11} m from the Sun and 3.84×10^8 m from the Earth.

SOLUTION: Before finding the resultant force, we first find the magnitudes of the individual forces. The magnitude of the force due to the Sun on the Moon is

$$F_{SM} = \frac{GM_S M_M}{R_{SM}^2}$$
$$= \frac{6.67 \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2 \times 1.99 \times 10^{30}\,\text{kg} \times 7.35 \times 10^{22}\,\text{kg}}{(1.50 \times 10^{11}\,\text{m})^2}$$
$$= 4.34 \times 10^{20}\ \text{N}$$

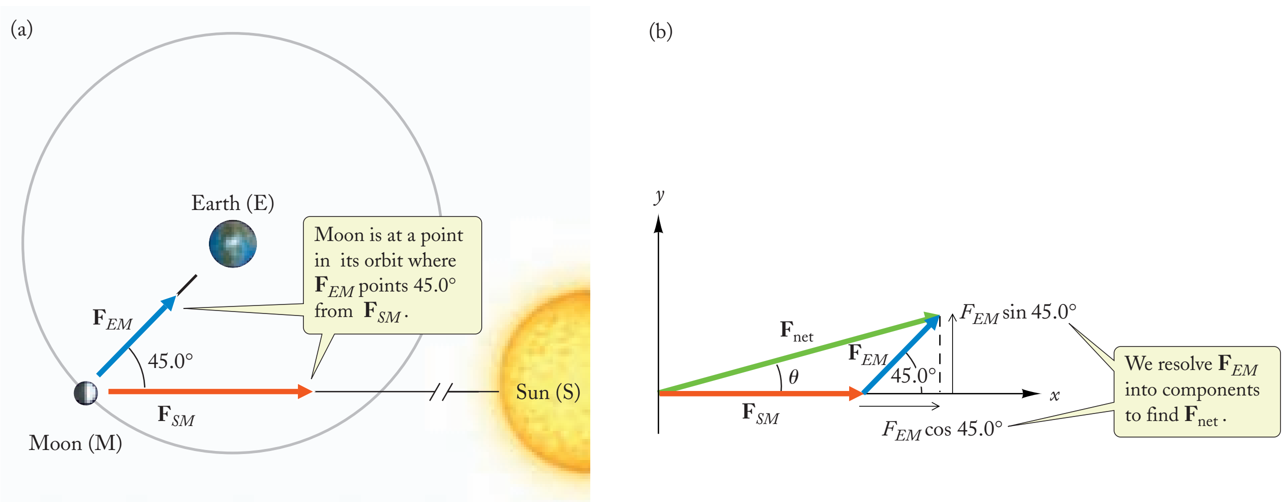


FIGURE 9.4 (a) Each of the gravitational forces on the Moon is directed toward the body producing the force. (b) Vector addition of the two forces.

The magnitude of the force due to the Earth on the Moon is

$$F_{EM} = \frac{GM_E M_M}{R_{EM}^2}$$

$$= \frac{6.67 \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2 \times 5.98 \times 10^{24}\,\text{kg} \times 7.35 \times 10^{22}\,\text{kg}}{(3.84 \times 10^8\,\text{m})^2}$$

$$= 1.99 \times 10^{20}\,\text{N}$$

The direction of each force on the Moon is toward the body producing that force, as indicated in Fig. 9.4a. We choose the x axis along the Moon–Sun direction and add the two forces vectorially as shown in Fig. 9.4b. By resolving $\mathbf{F}_{EM}$ into components, we see that the resultant force $\mathbf{F}_{\text{net}}$ has x component

$$F_{\text{net},x} = F_{SM} + F_{EM} \cos 45.0°$$

$$= 4.34 \times 10^{20}\,\text{N} + 1.99 \times 10^{20}\,\text{N} \times \cos 45.0° = 5.75 \times 10^{20}\,\text{N}$$

and y component

$$F_{\text{net},y} = F_{EM} \sin 45.0° = 1.99 \times 10^{20}\,\text{N} \times \sin 45.0° = 1.41 \times 10^{20}\,\text{N}$$

Thus the resultant force has magnitude

$$F_{\text{net}} = \sqrt{F_{\text{net},x}^2 + F_{\text{net},y}^2}$$

$$= \sqrt{(5.75 \times 10^{20}\,\text{N})^2 + (1.41 \times 10^{20}\,\text{N})^2} = 5.92 \times 10^{20}\,\text{N}$$

The direction of $\mathbf{F}_{\text{net}}$ is given by $\tan\theta = \dfrac{F_{\text{net},y}}{F_{\text{net},x}} = \dfrac{1.41 \times 10^{20}\,\text{N}}{5.75 \times 10^{20}\,\text{N}} = 0.245.$

With a calculator, we find that the inverse tangent of 0.238 is

$$\theta = 13.8°$$

✔ Checkup 9.1

QUESTION 1: Neptune is about 30 times as far away from the Sun as the Earth. Compare the gravitational force that the Sun exerts on a 1-kg piece of Neptune with the force it exerts on a 1-kg piece of the Earth. By what factor do these forces differ?

QUESTION 2: Saturn is about 10 times as far away from the Sun as the Earth, and its mass is about 100 times as large as the mass of the Earth. Is the force that the Sun exerts on Saturn larger, smaller, or about equal to the force it exerts on the Earth? Is the acceleration of Saturn toward the Sun larger, smaller, or about equal to the acceleration of the Earth?

QUESTION 3: Equation (9.6) gives the gravitational acceleration at the surface of the Earth, that is, at a radial distance of $r = R_E$ from the center. What is the gravitational acceleration at a radial distance of $r = 2R_E$? At $r = 3R_E$?

QUESTION 4: Uranus has a larger mass than the Earth, but a smaller gravitational acceleration at its surface. How could this be possible?

QUESTION 5: Consider a particle located at the exact center of the Earth. What is the gravitational force that the Earth exerts on this particle?

QUESTION 6: If the radius of the Earth were twice as large as it is but the mass remained unchanged, what would be the gravitational acceleration at its surface?

(A) $\frac{1}{8}g$ (B) $\frac{1}{4}g$ (C) g (D) $4g$ (E) $8g$

9.2 THE MEASUREMENT OF G

The gravitational constant G is rather difficult to measure with precision. The trouble is that gravitational forces between masses of laboratory size are extremely small, and thus a very delicate apparatus is needed to detect these forces. Measurements of G are usually done with a **Cavendish torsion balance** (see Fig. 9.5). Two equal, small spherical masses m and m' are attached to a lightweight horizontal beam which is suspended at its middle by a thin vertical fiber. When the beam is left undisturbed, it will settle into an equilibrium position such that the fiber is completely untwisted. If two equal, large masses M and M' are brought near the small masses, the gravitational attraction between each small mass and the neighboring large mass tends to rotate the beam clockwise (as seen from above). The twist of the fiber opposes this rotation, and the net result is that the beam settles into a new equilibrium position in which the force on the beam generated by the gravitational attraction between the masses is exactly balanced by the force exerted by the twisted fiber. The gravitational constant can then be calculated from the measured values of the angular displacement between the two equilibrium positions, the values of the masses, their distances, and the characteristics of the fiber.

HENRY CAVENDISH (1731–1810)
English experimental physicist and chemist. His torsion balance for the absolute measurement of the gravitational force was based on an earlier design used by Coulomb for the measurement of the electric force.

Note that the mass of the Earth can be calculated from Eq. (9.6) using the known values of G, R_E, and g:

$$M_E = \frac{R_E^2 g}{G} = \frac{(6.38 \times 10^6\ \text{m})^2 \times 9.81\ \text{m/s}^2}{6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2}$$
$$= 5.98 \times 10^{24}\ \text{kg}$$

This calculation would seem to be a rather roundabout way to arrive at the mass of the Earth, but there is no direct route, since we cannot place the Earth on a balance. Because the calculation requires a prior measurement of the value of G, the Cavendish experiment has often been described figuratively as "weighing the Earth."

FIGURE 9.5 Model of large torsion balance used by Cavendish. The small masses m, m' hang from the ends of a horizontal beam which is suspended at its middle by a thin vertical fiber.

Checkup 9.2

QUESTION 1: Why don't we determine G by measuring the (fairly large) force between the Earth and a mass of, say, 1 kg?

QUESTION 2: Large mountains produce a (small) deflection of a plumb bob suspended nearby. Could we use this effect to determine G?

(A) Yes (B) No

Online Concept Tutorial 11

9.3 CIRCULAR ORBITS

The gravitational force is responsible for holding the Solar System together; it makes the planets orbit around the Sun, and it makes the satellites orbit around the planets. Although the mutual gravitational forces of the Sun on a planet and of the planet on the Sun are of equal magnitudes, the mass of the Sun is more than a thousand times as large as the mass of even the largest planet, and hence its acceleration is much smaller. It is therefore an excellent approximation to regard the Sun as fixed and immovable, and it then only remains to investigate the motion of the planet. If we designate the masses of the Sun and the planet by M_S and m, respectively, and their center-to-center separation by r, then the magnitude of the gravitational force on the planet is

$$F = \frac{GM_S m}{r^2} \tag{9.7}$$

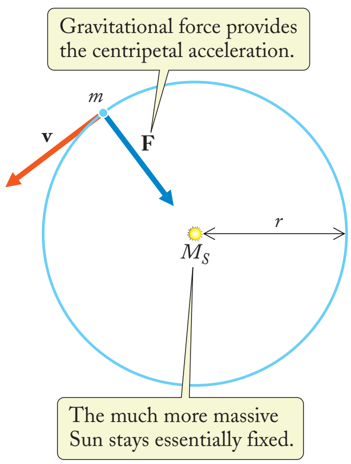


FIGURE 9.6 Circular orbit of a planet around the Sun.

This force points toward the center of the Sun; that is, the center of the Sun is the center of force (see Fig. 9.6). For a particle moving under the influence of such a central force, the simplest orbital motion is uniform circular motion, with the gravitational force acting as centripetal force. The motion of the planets in our Solar System is somewhat more complicated than that—as we will see in the next section, the planets move along ellipses, instead of circles. However, none of these planetary ellipses deviates very much from a circle, and as a first approximation we can pretend that the planetary orbits are circles.

By combining the expression (9.7) for the centripetal force with Newton's Second Law we can find a relation between the radius of the circular orbit and the speed. If the speed of the planet is v, then the centripetal acceleration is v^2/r [see Eq. (4.49)], and the equation of motion, $ma = F$, becomes

$$\frac{mv^2}{r} = F \tag{9.8}$$

Consequently,

$$\frac{mv^2}{r} = \frac{GM_S m}{r^2} \tag{9.9}$$

We can cancel a factor of m and a factor of $1/r$, in this equation, and we obtain

$$v^2 = \frac{GM_S}{r}$$

or

speed for circular orbit

$$v = \sqrt{\frac{GM_S}{r}} \tag{9.10}$$

EXAMPLE 4 The mass of the Sun is 1.99×10^{30} kg, and the radius of the Earth's orbit around the Sun is 1.5×10^{11} m. From this, calculate the orbital speed of the Earth.

SOLUTION: According to Eq. (9.10), the orbital speed is

$$v = \sqrt{\frac{GM_S}{r}} = \sqrt{\frac{6.67 \times 10^{-11}\,\text{N·m}^2/\text{kg}^2 \times 1.99 \times 10^{30}\,\text{kg}}{1.5 \times 10^{11}\,\text{m}}}$$
$$= 3.0 \times 10^4\,\text{m/s} = 30\,\text{km/s}$$

NICHOLAS COPERNICUS (1473–1543) *Polish astronomer. In his book* De Revolutionibus Orbium Coelestium *he formulated the heliocentric system for the description of the motion of the planets, according to which the Sun is immovable and the planets orbit around it.*

The time a planet takes to travel once around the Sun, or the time for one revolution, is called the **period** *of the planet*. We will designate the period by T. The speed of the planet is equal to the circumference $2\pi r$ of the orbit divided by the time T:

$$v = \frac{2\pi r}{T} \tag{9.11}$$

With this expression for the speed, the square of Eq. (9.10) becomes

$$\frac{4\pi^2 r^2}{T^2} = \frac{GM_S}{r} \tag{9.12}$$

which can be rearranged to read

period for circular orbit

$$T^2 = \frac{4\pi^2}{GM_S} r^3 \tag{9.13}$$

This says that *the square of the period is proportional to the cube of the radius of the orbit*, with a constant of proportionality depending on the mass of the central body.

EXAMPLE 5 Both Venus and the Earth have approximately circular orbits around the Sun. The period of the orbital motion of Venus is 0.615 year, and the period of the Earth is 1 year. According to Eq. (9.13), by what factor do the sizes of the two orbits differ?

SOLUTION: If we take the cube root of both sides of Eq. (9.13), we see that the orbital radius is proportional to the 2/3 power of the period. Hence we can set up the following proportion for the orbital radii of the Earth and Venus:

$$\frac{r_E}{r_V} = \frac{T_E^{2/3}}{T_V^{2/3}} = \frac{(1\text{ year})^{2/3}}{(0.615\text{ year})^{2/3}} = 1.38 \tag{9.14}$$

An equation analogous to Eq. (9.13) also applies to the circular motion of a moon or artificial satellite around a planet. In this case, the planet plays the role of the central body and, in Eq. (9.13), its mass replaces the mass of the Sun.

EXAMPLE 6 A communications satellite is in a circular orbit around the Earth, in the equatorial plane. The period of the orbit of such a satellite is exactly 1 day, so that the satellite always hovers in a fixed position relative to the rotating Earth. What must be the radius of such a "geosynchronous," or "geostationary," orbit?

Concepts in Context

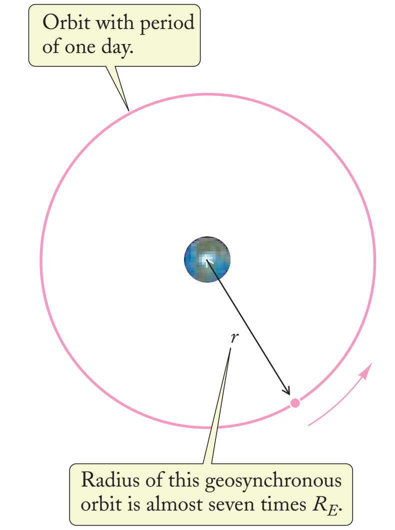


FIGURE 9.7 Orbit of a "geostationary" satellite around the Earth.

SOLUTION: Since the central body is the Earth, the equation analogous to Eq. (9.13) is

$$T^2 = \frac{4\pi^2}{GM_E} r^3 \tag{9.15}$$

or

$$r^3 = \frac{GM_E T^2}{4\pi^2} \tag{9.16}$$

Taking the cube root of both sides of this equation, we find

$$r = \left(\frac{GM_E T^2}{4\pi^2}\right)^{1/3}$$

$$= \left(\frac{6.67 \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2 \times 5.98 \times 10^{24}\,\text{kg} \times (24 \times 60 \times 60\,\text{s})^2}{4\pi^2}\right)^{1/3}$$

$$= 4.23 \times 10^7\,\text{m} \tag{9.17}$$

The orbit is shown in Fig. 9.7, which is drawn to scale. The radius of the orbit is about 6.6 times the radius of the Earth.

Concepts *in* Context

EXAMPLE 7 Surveillance satellites and spacecraft such as the Space Shuttle (Fig. 9.8) often operate in low-altitude orbits quite near the Earth, just above the atmosphere. Such orbits can have a radius as small as $r_{\text{low}} = 6.6 \times 10^6$ m; this is less than one-sixth of the geostationary orbit radius $r_{\text{geo}} = 4.23 \times 10^7$ m. Calculate how often the low-altitude satellites and spacecraft circle the Earth.

SOLUTION: Taking the square root of both sides of Eq. (9.13), we see that the period is proportional to the 3/2 power of the orbital radius. Hence we can set up the following proportion for the orbital periods:

$$\frac{T_{\text{low}}}{T_{\text{geo}}} = \left(\frac{r_{\text{low}}}{r_{\text{geo}}}\right)^{3/2} = \left(\frac{6.6 \times 10^6\,\text{m}}{4.23 \times 10^7\,\text{m}}\right)^{3/2}$$

$$= 0.062$$

or, since the geostationary period T_{geo} is one day, or 24 h,

$$T_{\text{low}} = 0.062 \times 24\,\text{h} = 1.5\,\text{h}$$

Thus such "fly-bys" occur quite frequently: 16 times per day.

FIGURE 9.8 The Space Shuttle in orbit with its cargo bay open.

Checkup 9.3

QUESTION 1: The orbit of the geostationary satellite illustrated in Fig. 9.7 is in the equatorial plane, and the satellite is stationary above a point on the Earth's equator. Why can't we keep a satellite stationary above a point that is not on the equator, say, above San Francisco?

PHYSICS IN PRACTICE — COMMUNICATIONS SATELLITES AND WEATHER SATELLITES

Concepts in Context

More than a hundred communications and weather satellites have been placed in geostationary orbits. The communications satellites use radio signals to relay telephone and TV signals from one point on the Earth to another. The weather satellites capture pictures of the cloud patterns and measure the heights of clouds, wind speeds, atmospheric and ground temperatures, and moisture in the atmosphere. These observations are especially useful for monitoring weather conditions over the oceans, where there are few observation stations at ground level. Data collected by weather satellites permit early detection of dangerous tropical storms (hurricanes, typhoons) and forecasting of the tracks and the strengths of these storms.

The launch vehicle for these satellites usually consists of a two-stage rocket, which carries the satellite to a low-altitude orbit. A small rocket motor attached to the satellite is then used to lift the satellite from the low-altitude orbit to the high-altitude geostationary orbit. Alternatively, the satellite can be ferried to the low-altitude orbit by the Space Shuttle.

At the high altitude of the geostationary orbit there is no atmospheric drag, and a satellite placed in such an orbit will continue to orbit the Earth indefinitely. However, the orbital motion of the satellite is disturbed by the Moon and the Sun, and it is also affected by the nonspherical shape of the Earth, which produces deviations from the ideal uniform centripetal force. These disturbances cause the satellite to drift from its geostationary position. This requires an adjustment of the orbit every few weeks, which is done with small control nozzles on the satellite. Typically, a satellite carries enough propellant to operate its control nozzles for 10 years, by which time other components in the satellite will also have worn out, or will have been superseded by new technology, so it becomes desirable to switch the satellite off, and replace it by a new model.

Communications satellites contain a radio receiver and a transmitter connected to dish antennas aimed at radio stations on the ground. The signal received from one station on the ground is amplified by the satellite, and then this amplified signal is retransmitted to the other station (the satellite acts as a transponder).

FIGURE 1 Astronauts handle an INTELSAT communications satellite.

Figure 1 shows a recent model of the INTELSAT series of communications satellites. This satellite has a length of 5.2 m, a diameter of 3.6 m, and a mass of 2240 kg. It is powered by solar panels that convert the energy of sunlight into electricity, delivering 2300 watts of power. It contains 50 transponders and is capable of handling 40 000 telephone circuits simultaneously.

For intercontinental communications, three groups of INTELSAT satellites are deployed at geostationary positions over the Atlantic, Pacific, and Indian Oceans. But communications satellites are also cost-effective for communications over shorter ranges, when there is a shortage of telephone cables. Many countries have launched communications satellites to handle telephone traffic within their borders. Communications satellites also relay TV transmissions. A small dish antenna connected to an amplifier permits home television sets to pick up a multitude of TV channels from these satellites.

QUESTION 2: The period of the orbital motion of the Moon around the Earth is 27 days. If the orbit of the Moon were twice as large as it is, what would be the period of its motion?

QUESTION 3: The mass of a planet can be determined by observing the period of a moon in a circular orbit around the planet. For such a mass determination, which of the following do we need: the period, the radius of the moon's orbit, the mass of the moon, the radius of the planet?

QUESTION 4: The radius of the orbit of Saturn around the Sun is about 10 times the radius of the orbit of the Earth. Accordingly, what must be the approximate period of its orbital motion?

(A) 1000 yr (B) 100 yr (C) 30 yr (D) 10 yr (E) 3 yr

Online Concept 12 Tutorial

9.4 ELLIPTICAL ORBITS; KEPLER'S LAWS

Although the orbits of the planets around the Sun are approximately circular, none of these orbits are *exactly* circular. We will not attempt the general solution of the equation of motion for such noncircular orbits. A complete calculation shows that with the inverse-square force of Eq. (9.1), the planetary orbits are ellipses. This is **Kepler's First Law**:

Kepler's First Law

The orbits of the planets are ellipses with the Sun at one focus.

Figure 9.9 shows an elliptical planetary orbit (for the sake of clarity, the elongation of this ellipse has been exaggerated; actual planetary orbits have only very small elongations). The point closest to the Sun is called the **perihelion**; the point farthest from the Sun is called the **aphelion**. The sum of the perihelion and the aphelion distances is the **major axis** of the ellipse. The distance from the center of the ellipse to the perihelion (or aphelion) is the **semimajor axis**; this distance equals the average of the perihelion and aphelion distances.

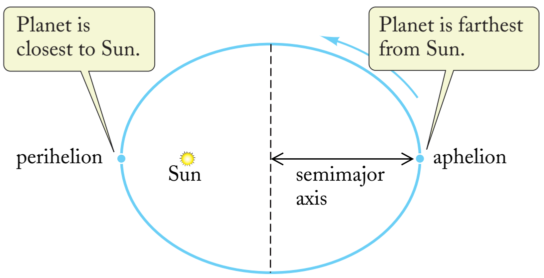


FIGURE 9.9 Orbit of a planet around the Sun. The orbit is an ellipse, with the Sun at one focus.

Kepler originally discovered his First Law and his other two laws (see below) early in the seventeenth century, by direct analysis of the available observational data on planetary motions. Hence, Kepler's laws were originally purely phenomenological statements; that is, they described the phenomenon of planetary motion but did not explain its causes. The explanation came only later, when Newton laid down his laws of motion and his Law of Universal Gravitation and deduced the features of planetary motion from these fundamental laws.

Kepler's Second Law describes the variation in the speed of the motion:

Kepler's Second Law

The radial line segment from the Sun to the planet sweeps out equal areas in equal times.

MATH HELP ELLIPSES

An ellipse is defined geometrically by the condition that the sum of the distance from one focus of the ellipse and the distance from the other focus is the same for all points on the ellipse. This geometrical condition leads to a simple method for the construction of an ellipse: Stick pins into the two foci and tie a length of string to these points. Stretch the string taut to the tip of a pencil, and move this pencil around the foci while keeping the string taut (see Fig. 1a).

An ellipse can also be constructed by slicing a cone obliquely (see Fig. 1b). Because of this, an ellipse is said to be a conic section.

The largest diameter of the ellipse is called the major axis, and the smallest diameter is called the minor axis. The semimajor axis and the semiminor axis are one-half of these diameters, respectively (see Fig. 1c).

If the semimajor axis of length a is along the x axis and the semiminor axis of length b is along the y axis, then the x and y coordinates of an ellipse centered on the origin satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The foci are on the major axis at a distance f from the origin given by

$$f = \sqrt{a^2 - b^2}$$

The separation between a planet and the Sun is $a - f$ at perihelion and is $a + f$ at aphelion.

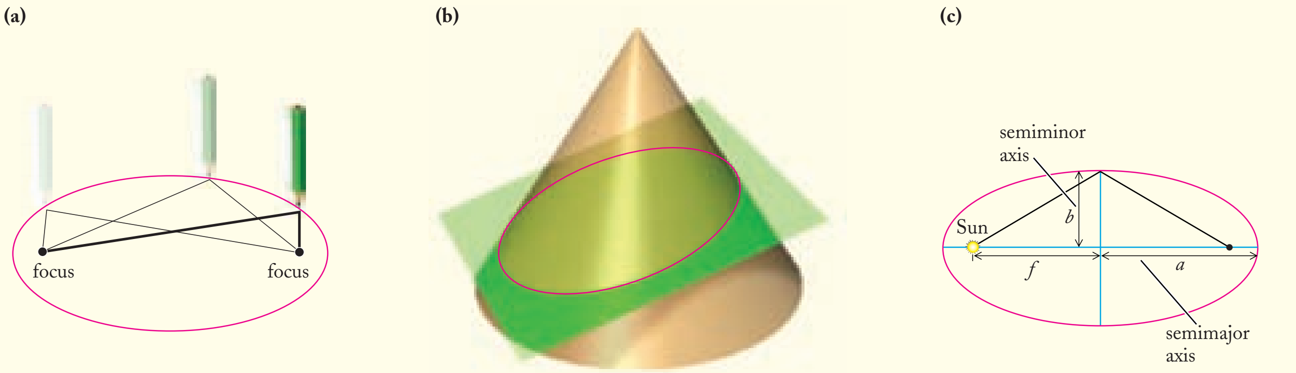


FIGURE 1 (a) Constructing an ellipse. (b) Ellipse as a conic section. (c) Focal distance f, semimajor axis a, and semiminor axis b of an ellipse.

Figure 9.10 illustrates this law. The two colored areas are equal, and the planet takes equal times to move from P to P' and from Q to Q'. According to Fig. 9.10, the speed of the planet is larger when it is near the Sun (at Q) than when it is far from the Sun (at P).

Kepler's Second Law, also called the law of areas, is a direct consequence of the central direction of the gravitational force. We can prove this law by a simple geometrical argument. Consider three successive positions P, P', P'' on the orbit, separated by a relatively small distance. Suppose that the time intervals between P, P' and between P', P'' are equal—say, each of the two intervals is one second. Figure 9.11 shows the positions P, P', P''. Between these positions the curved orbit can be approximated by straight line segments PP' and $P'P''$. Since the time intervals are one unit of time (1 second), the lengths of the segments PP' and $P'P''$ are in proportion to the

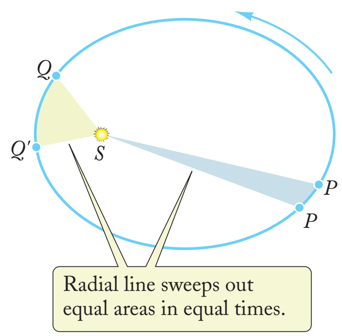


FIGURE 9.10 For equal time intervals, the areas SQQ' and SPP' are equal. The distance QQ' is larger than the distance PP'.

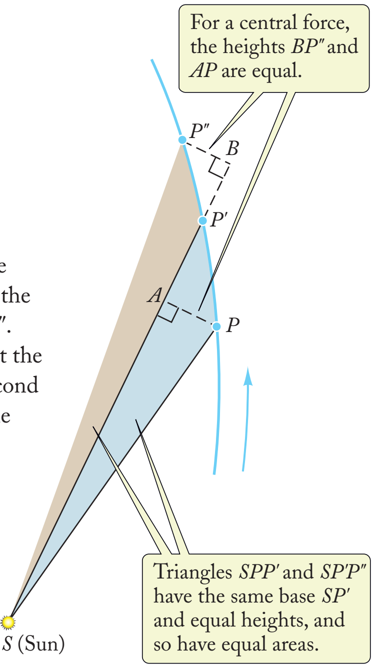


FIGURE 9.11 In one second the planet travels from P to P', and in the next second it travels from P' to P''. The radial line segment sweeps out the triangular area SPP' in the first second and the triangular area $SP'P''$ in the next second.

average velocities in the two time intervals. The velocities differ because the gravitational force causes an acceleration. However, since the direction of the force is toward the center, parallel to the radius, the component of the velocity perpendicular to the radius cannot change. The component of the velocity perpendicular to the radius is represented by the line segment PA for the first time interval, and it is represented by BP'' for the second time interval. These line segments perpendicular to the radius are, respectively, the heights of the triangles SPP' and $SP'P''$ (see Fig. 9.11). Since these heights are equal and since both triangles have the same base SP', their areas must be equal. Thus, the areas swept out by the radial line in the two time intervals must be equal, as asserted by Kepler's Second Law. Note that this geometrical argument depends only on the fact that the force is directed toward a center; it does not depend on the magnitude of the force. This means that Kepler's Second Law is valid not only for planetary motion, but also for motion with any kind of central force.

Let us explore what Kepler's Second Law has to say about the speeds of a planet at aphelion and at perihelion. Figure 9.12 shows the triangular area SPP' swept out by the radial line in a time Δt at, or near, aphelion. The height PP' of this triangle equals the speed v_1 at aphelion times the time Δt; hence the area of the triangle is $\frac{1}{2}r_1 v_1\,\Delta t$. Likewise, the triangular area SQQ' swept out by the radial line in an equal time Δt at, or near, perihelion is $\frac{1}{2}\,r_2 v_2\,\Delta t$. By Kepler's Second Law these two areas must be equal; if we cancel the common factors of $\frac{1}{2}$ and Δt, we obtain

$$\underset{\text{at aphelion}}{r_1 v_1} = \underset{\text{at perihelion}}{r_2 v_2} \tag{9.18}$$

According to this equation, the ratio of the aphelion and perihelion distances is the inverse of the ratio of the speeds.

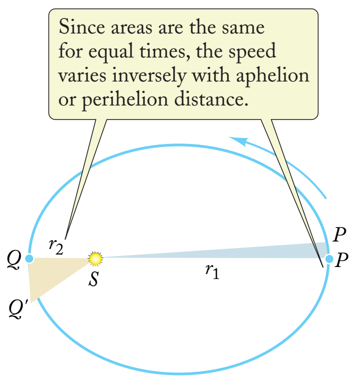


FIGURE 9.12 Triangular area SPP' swept out in one interval Δt after aphelion, and triangular area SQQ' swept out in an identical interval Δt after perihelion.

EXAMPLE 8 The perihelion and aphelion distances for Mercury are 45.9×10^9 m and 69.8×10^9 m, respectively. The speed of Mercury at aphelion is 3.88×10^4 m/s. What is the speed at perihelion?

SOLUTION: From Eq. (9.18),

$$v_2 = \frac{r_1}{r_2}v_1 = \frac{69.8 \times 10^9\ \text{m}}{45.9 \times 10^9\ \text{m}} \times 3.88 \times 10^4\ \text{m/s}$$

$$= 5.90 \times 10^4\ \text{m/s}$$

In Chapter 13 we will become acquainted with the **angular momentum** L, which, for a planet at aphelion or perihelion, is equal to the product rmv. By multiplying both sides of Eq. (9.18) by the mass of the planet m, we see that $r_1 m v_1 = r_2 m v_2$; that is, the angular momentum at aphelion equals the angular momentum at perihelion. Thus, Kepler's Second Law can be regarded as a consequence of a conservation law for angular momentum. We will see that angular momentum is conserved when a particle is under the influence of any central force.

Kepler's Third Law relates the period of the orbit to the size of the orbit:

The square of the period is proportional to the cube of the semimajor axis of the planetary orbit.

Kepler's Third Law

This Third Law, or law of periods, is nothing but a generalization of Eq. (9.13) to elliptical orbits.

Table 9.1 lists the orbital data for the planets of the Solar System. The mean distance listed in this table is defined as the average of the perihelion and aphelion distances; that is, it is the semimajor axis of the ellipse. The difference between the perihelion and aphelion distances gives an indication of the elongation of the ellipse.

TABLE 9.1 THE PLANETS

PLANET (a)	MASS	MEAN DISTANCE FROM SUN (SEMIMAJOR AXIS)	PERIHELION DISTANCE	APHELION DISTANCE	PERIOD
Mercury	3.30×10^{23} kg	57.9×10^{6} km	45.9×10^{6} km	69.8×10^{6} km	0.241 yr
Venus	4.87×10^{24}	108	107	109	0.615
Earth	5.98×10^{24}	150	147	152	1.00
Mars	6.42×10^{23}	228	207	249	1.88
Jupiter	1.90×10^{27}	778	740	816	11.9
Saturn	5.67×10^{26}	1430	1350	1510	29.5
Uranus	8.70×10^{25}	2870	2730	3010	84.0
Neptune	1.03×10^{26}	4500	4460	4540	165
Pluto	1.50×10^{22}	5890	4410	7360	248

(a) A photomontage of the planets in sequence from Mercury (top left, partly hidden) to Pluto (bottom left).

JOHANNES KEPLER (1571–1630) *German astronomer and mathematician. Kepler relied on the theoretical framework of the Copernican system, and he extracted his three laws by a meticulous analysis of the observational data on planetary motions collected by the great Danish astronomer Tycho Brahe.*

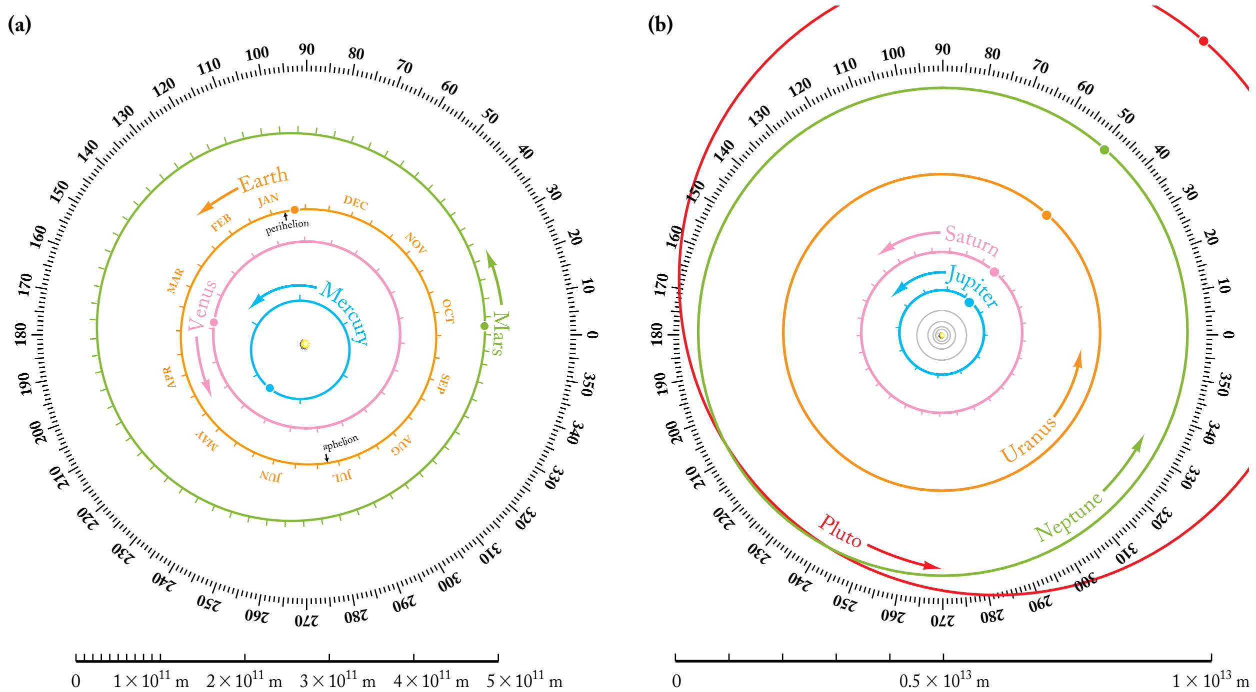


FIGURE 9.13 (a) Orbits of Mercury, Venus, Earth, and Mars. Elliptical orbits can appear quite circular, even when the focus is noticeably off-center, as with Mercury and Mars. The colored dots indicate the positions of these planets on January 1, 2000. The tick marks indicate the positions at intervals of 10 days. (b) Orbits of Jupiter, Saturn, Uranus, and Neptune, and a portion of the orbit of Pluto. The tick marks for Jupiter and Saturn indicate the positions at intervals of 1 year.

Figure 9.13a shows the orbits of the planets Mercury, Venus, Earth, and Mars on scale diagrams. The orbits of Saturn, Jupiter, Uranus, and Neptune and part of the orbit of Pluto are shown in Fig. 9.13b. Inspection of these diagrams reveals that the orbits of Mercury, Mars, and Pluto are noticeably different from circles.[1]

Kepler's three laws apply not only to planets, but also to satellites and to comets. For example, Fig. 9.14 shows the orbits of a few of the many artificial Earth satellites. All these orbits are ellipses. For Earth orbits, the point closest to the Earth is called **perigee**; the point farthest from Earth is called **apogee**. The early artificial satellites were quite small, with masses below 100 kg (see Fig. 9.15). Nowadays, satellites with masses

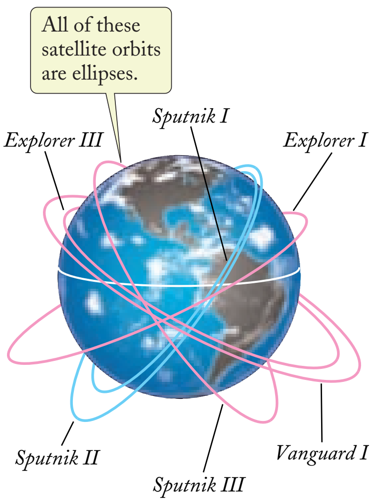


FIGURE 9.14 Orbits of the first artificial Earth satellites. See Table 9.3 for more data.

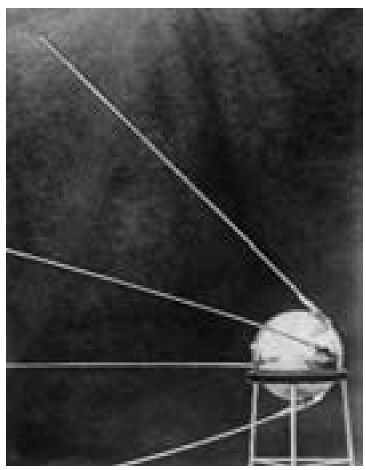


FIGURE 9.15 *Sputnik I*, the first artificial Earth satellite. This satellite had a mass of 83 kg.

[1] Pluto has recently been reclassified by the International Astronomical Union as a dwarf planet, in the same category as Ceres and 2003 UB_{313} (Xena).

of several tons are not unusual. All of the early artificial satellites burned up in the atmosphere after a few months or a few years because they were not sufficiently far from the Earth to avoid the effects of residual atmospheric friction.

Kepler's laws also apply to the motion of a projectile near the Earth. For instance, Fig. 9.16 shows the trajectory of an intercontinental ballistic missile (ICBM). During most of its trajectory, the only force acting on the missile is the gravity of the Earth; the thrust of the engines and the friction of the atmosphere act only during the relatively short initial and final segments of the trajectory (on the scale of Fig. 9.16, these initial and final segments of the trajectory are too small to be noticed). The trajectory is a portion of an elliptical orbit cut short by the impact on the Earth. Likewise, the motion of an ordinary low-altitude projectile, such as a cannonball, is also a portion of an elliptical orbit (if we ignore atmospheric friction). In Chapter 4 we made the near-Earth approximation that gravity was constant in magnitude and direction; with these approximations we found that the orbit of a projectile was a parabola. Although the exact orbit of a projectile is an ellipse, the parabola approximates this ellipse quite well over the relatively short distance involved in ordinary projectile motion; deviations do become noticeable for long-range trajectories (see Fig. 9.17).

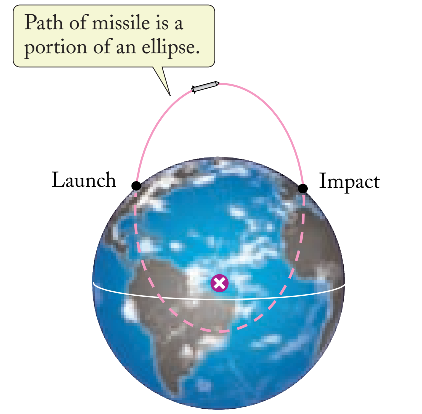


FIGURE 9.16 Orbit of an intercontinental ballistic missile (ICBM). The elongation of the ellipse and the height of the orbit are exaggerated.

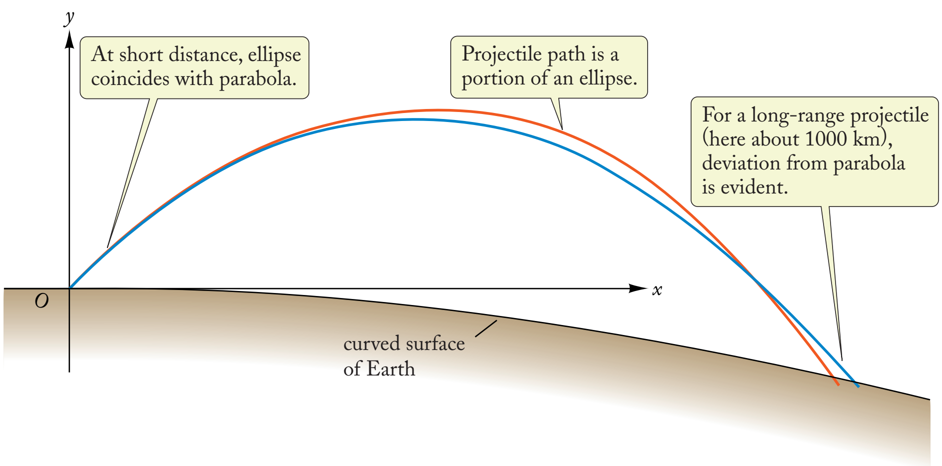


FIGURE 9.17 The parabola (blue curve) approximates the ellipse (red curve) for short distances.

The connection between projectile motion and orbital motion was neatly illustrated by Newton by means of an imaginary experiment, or what today we would call a *Gedankenexperiment*.[1] Newton proposed to fire a cannonball horizontally from a gun emplaced on a high mountain (see Fig. 9.18). If the muzzle velocity is fairly low, the cannonball will arc toward the Earth and strike near the base of the mountain. The trajectory is a segment of a parabola, or, more precisely, a segment of an ellipse. If we increase the muzzle velocity, the cannonball will describe larger and larger arcs. Finally, if the muzzle velocity is just large enough, the rate at which the trajectory curves downward is precisely matched by the curvature of the surface of the Earth—the cannonball never hits the Earth and keeps on falling forever while moving in a circular orbit. This example makes it very clear that orbital motion is free-fall motion.

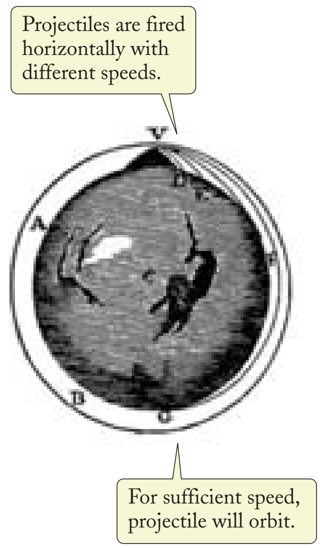


FIGURE 9.18 This drawing from Newton's *Principia* illustrates an imaginary experiment with a cannonball fired from a gun on a high mountain. For a sufficiently large muzzle velocity, the trajectory of the cannonball is a circular orbit.

[1] *Gedankenexperiment* is German for "thought experiment." This word is used by physicists for an imaginary experiment that can be done in principle, but that has never been done in practice, and whose outcome can be discovered by thought.

Finally, we note that in our mathematical description of planetary motion we have neglected the gravitational forces that the planets exert on one another. These forces are much smaller than the force exerted by the Sun, but in a precise calculation the vector sum of all the forces must be taken into account. The net force on any planet then depends on the positions of all the other planets. This means that the motions of all the planets are coupled together, and the calculation of the motion of one planet requires the simultaneous calculation of the motions of all the other planets. This makes the precise mathematical treatment of planetary motion extremely complicated. Kepler's simple laws neglect the complications introduced by the interplanetary forces; these laws therefore do not provide an exact description of planetary motions, but only a very good first approximation.

✓ Checkup 9.4

QUESTION 1: Suppose that the gravitational force were an inverse-cube force, instead of an inverse-square force. Would Kepler's Second Law remain valid? Would Kepler's Third Law remain valid?

QUESTION 2: A comet has an aphelion distance twice as large as its perihelion distance. If the speed of the comet is 40 km/s at perihelion, what is its speed at aphelion?

QUESTION 3: A comet has an elliptical orbit of semimajor axis equal to the Earth–Sun distance. What is the period of such a comet?

QUESTION 4: If you want to place an artificial satellite in an elliptical orbit of period 8 years around the Sun, what must be the semimajor axis of this ellipse? (Answer in units of the Earth–Sun distance.)

(A) 64 (B) $16\sqrt{2}$ (C) 8 (D) 4 (E) 2

9.5 ENERGY IN ORBITAL MOTION

The gravitational force is a conservative force; that is, the work done by this force on a particle moving from some point P_1 to some other point P_2 can be expressed as a difference between two potential energies, and the work done on any round trip starting and ending at some given point is zero. To construct the potential energy, we proceed as in Section 8.1: we calculate the work done by the gravitational force as the particle moves from point P_1 to point P_2, and we seek to express this work as a difference of two terms. In Fig. 9.19, the points P_1 and P_2 are at distances r_1 and r_2, respectively, from the central mass. To calculate the work, we must take into account that the force is a function of the distance; that is, the force is variable. From Section 7.2, we know that for such a variable force, the work is the integral of the force over the distance. If we place the x axis along the line connecting P_1 and P_2 (see Fig. 9.19), then the force can be expressed as

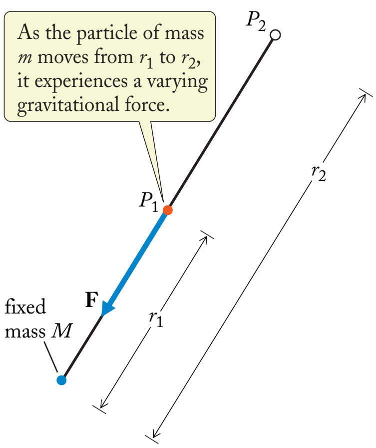


FIGURE 9.19 Two points P_1 and P_2 at distances r_1 and r_2 from the central mass.

$$F_x = -\frac{GMm}{x^2}$$

and the work is

$$W = \int_{P_1}^{P_2} F_x(x)\,dx = \int_{r_1}^{r_2} \left(-\frac{GMm}{x^2}\right) dx$$

We already have evaluated this kind of integral in Example 2 of Chapter 8 (in the case of the gravitational force, the constant A in that example is $A = -GMm$). The result of the integration is

$$W = \frac{GMm}{x}\bigg|_{r_1}^{r_2} = \frac{GMm}{r_2} - \frac{GMm}{r_1} \tag{9.19}$$

As expected, this result shows that the work is the difference between two potential energies. Accordingly, we can identify the **gravitational potential energy** as

$$U = -\frac{GMm}{r} \tag{9.20}$$

gravitational potential energy

Note that in this calculation of the gravitational potential energy we assumed that the points P_1 and P_2 lie on the same radius (see Fig. 9.19). However, Eq. (9.19) is valid in general, even if P_1 and P_2 do not lie on the same radial line. We can see this by introducing an intermediate point Q, which is on the radial line of P_1 but at the radial distance of P_2 (see Fig. 9.20). To move the particle from P_1 to P_2, we first move it from P_1 to Q along the radial line; this takes the amount of work given by Eq. (9.19). We then move the particle from Q to P_2, along the circular arc of radius r_2; this costs no work, since such a displacement is perpendicular to the force. Any more general path can be constructed from small radial segments and small arcs of circles, and so Eq. (9.19) is true in general.

The potential energy (9.20) is always negative, and its magnitude is inversely proportional to r. Figure 9.21 gives a plot of this potential energy as a function of distance. If the distance r is small, the potential energy is low (the potential energy is much below zero); if the distance r is large, the potential energy is higher (the potential energy is still negative, but not so much below zero). Thus, the potential energy *increases* with distance; it increases from a large negative value to a smaller negative value or to zero. Such an increase of potential energy with distance is characteristic of an attractive force. For instance, if we want to lift a communications satellite from a low initial orbit (just above the Earth's atmosphere) into a high final orbit (such as the geostationary orbit described in Example 6), we must do work on this satellite (by means of a rocket engine). The work we do while lifting the satellite increases the gravitational potential energy from a large negative value (much below zero) to a smaller negative value (not so much below zero).

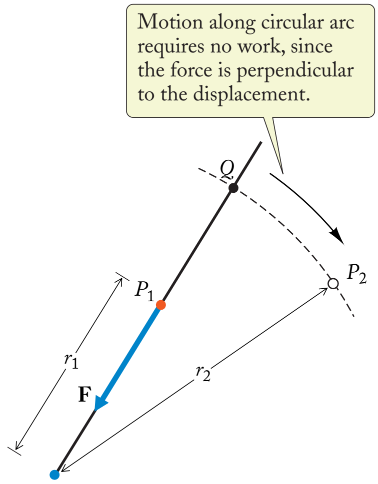


FIGURE 9.20 Two points P_1 and P_2 at distances r_1 and r_2 in different directions. The particle moves from P_1 to Q and then from Q to P_2.

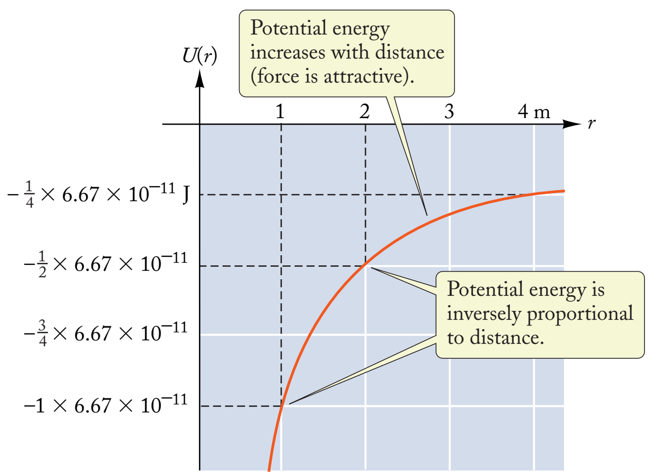


FIGURE 9.21 Gravitational potential energy for a particle of mass 1 kg gravitationally attracted by another particle of mass 1 kg.

The total mechanical energy is the sum of the potential energy and the kinetic energy. Since we are assuming that the mass M is stationary, the kinetic energy is entirely due to the motion of the mass m, and the Law of Conservation of Energy takes the form

Law of Conservation of Energy

$$E = K + U = \tfrac{1}{2}mv^2 - \frac{GMm}{r} = [\text{constant}] \tag{9.21}$$

If the only force acting on the body is the gravitational force (no rocket engine or other external force!), then this total energy remains constant during the motion. For instance, the energy (9.21) is constant for a planet orbiting the Sun, and for a satellite or a spacecraft (with rocket engines shut off) orbiting the Earth. As we saw in Chapter 8, examination of the energy reveals some general features of the motion. Equation (9.21) shows how the orbiting body trades distance ("height") for speed; it implies that if r decreases, v must increase, so that the sum of the two terms $\frac{1}{2}mv^2$ and $-GMm/r$ remains constant. Conversely, if r increases, v must decrease.

Let us now investigate the possible orbits around, say, the Sun from the point of view of their energy. For a circular orbit, we saw in Eq. (9.10) that the orbital speed is

$$v = \sqrt{\frac{GM_S}{r}} \tag{9.22}$$

and so the kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \frac{GM_S m}{2r} \tag{9.23}$$

Hence the total energy is

$$E = K + U = \tfrac{1}{2}mv^2 - \frac{GM_S m}{r} = \frac{GM_S m}{2r} - \frac{GM_S m}{r}$$

or

$$E = -\frac{GM_S m}{2r} \tag{9.24}$$

Consequently, the total energy for a circular orbit is negative and is exactly one-half of the potential energy.

Concepts in Context

EXAMPLE 9 The 1300-kg Syncom communications satellite was placed in its high-altitude geosynchronous orbit of radius 4.23×10^7 m in two steps. First the satellite was carried by the Space Shuttle to a low-altitude circular orbit of radius 6.65×10^6 m; there it was released from the cargo bay of the Space Shuttle, and it used its own booster rocket to lift itself to the high-altitude circular orbit. What is the increase of the total mechanical energy during this change of orbit?

SOLUTION: The total mechanical energy is exactly one-half of the potential energy [Eq. (9.24)]. For an Earth orbit, we replace M_S in Eq. (9.24) by M_E. For the low-altitude circular orbit of radius r_1, the total energy is $E_1 = -GM_E m/2r_1$, and for the high-altitude circular orbit of radius r_2, the total energy is $E_2 = -GM_E m/2r_2$. So the change of the energy is

$$E_2 - E_1 = -\frac{GM_E m}{2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$= -\frac{6.67 \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2 \times 5.98 \times 10^{24}\,\text{kg} \times 1300\,\text{kg}}{2}$$

$$\times \left(\frac{1}{4.23 \times 10^{7}\,\text{m}} - \frac{1}{6.65 \times 10^{6}\,\text{m}}\right)$$

$$= 3.29 \times 10^{10}\,\text{J}$$

This energy was supplied by the booster rocket of the satellite.

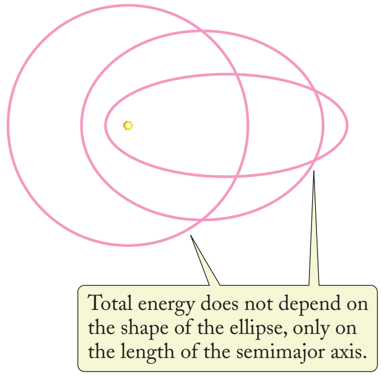


FIGURE 9.22 Orbits of the same total energy. All these orbits have the same semi-major axis.

For an elliptical orbit, the total energy is also negative. It can be demonstrated that the energy can still be written in the form of Eq. (9.24), but the quantity r must be taken equal to the semimajor axis of the ellipse. The total energy of the orbit does not depend on the shape of the ellipse, but only on its larger overall dimension. Figure 9.22 shows several orbits of different shapes but with exactly the same total energy.

From Eq. (9.24) we see that if the energy is nearly zero, then the size of the orbit is very large (note that $E \to 0$ as $r \to \infty$). Such orbits are characteristic of comets, many of which have elliptical orbits that extend far beyond the edge of the Solar System (see Fig. 9.23). If the energy is exactly zero, then the "ellipse" extends all the way to infinity and never closes; such an "open ellipse" is actually a parabola (see Fig. 9.24).

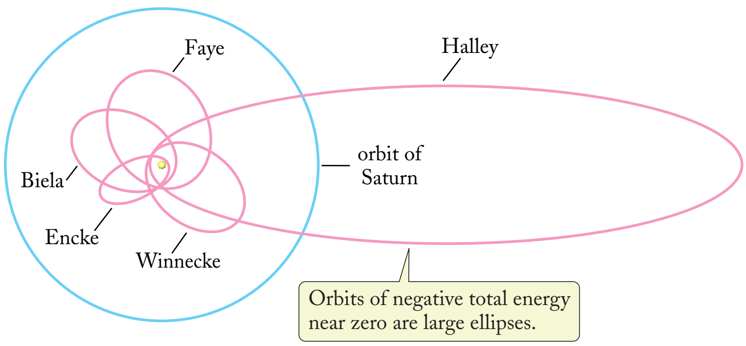


FIGURE 9.23 Orbits of some periodic comets.

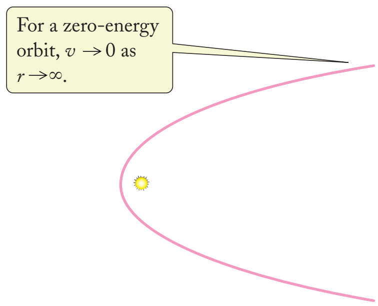


FIGURE 9.24 Orbit of zero energy—a parabola.

Equation (9.21) indicates that if the energy is zero, the comet will reach infinite distance with zero velocity (if $r = \infty$, then $v = 0$). By considering the reverse of this motion, we see that a comet of zero energy, initially at very large distance from the Sun, will fall along this type of parabolic orbit.

If the energy is positive, then the orbit again extends all the way to infinity and again fails to close; such an open orbit is a hyperbola. The comet will then reach infinite distance with some nonzero velocity and continue moving along a straight line (see Fig. 9.25).

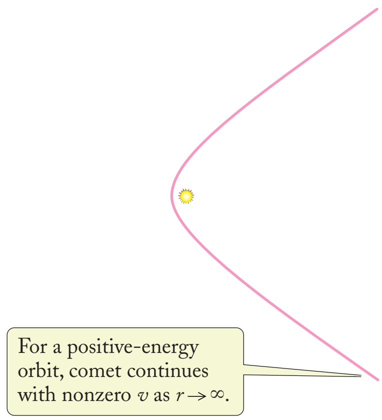


FIGURE 9.25 Orbit of positive energy—a hyperbola.

EXAMPLE 10 A meteoroid (a chunk of rock) is initially at rest in interplanetary space at a large distance from the Sun. Under the influence of gravity, the meteoroid begins to fall toward the Sun along a straight radial line. With what speed does it strike the Sun? The radius of the Sun is 6.96×10^8 m.

SOLUTION: The energy of the meteoroid is

$$E = K + U = \tfrac{1}{2}mv^2 - \frac{GM_S m}{r} = [\text{constant}] \qquad (9.25)$$

Initially, both the kinetic and potential energies are zero ($v = 0$ and $r \approx \infty$). Hence at any later time

$$\frac{1}{2}mv^2 - \frac{GM_S m}{r} = 0$$

or

$$\frac{1}{2}mv^2 = \frac{GM_S m}{r} \tag{9.26}$$

If we cancel a factor of m and multiply by 2 on both sides of this equation, take the square root of both sides, and substitute $r = R_S$ for the impact on the Sun's surface, we find the speed at the moment of impact:

impact speed and escape velocity

$$v = \sqrt{\frac{2GM_S}{R_S}} \tag{9.27}$$

With $M_S = 1.99 \times 10^{30}$ kg (see Example 4) and $R_S = 6.96 \times 10^8$ m, we obtain

$$v = \sqrt{\frac{2 \times 6.67 \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2 \times 1.99 \times 10^{30}\,\text{kg}}{6.96 \times 10^8\,\text{m}}}$$
$$= 6.18 \times 10^5\ \text{m/s} = 618\ \text{km/s}$$

The quantity given by Eq. (9.27) is called the Sun's **escape velocity** because it is the minimum initial velocity with which a body must be launched upward from the surface of the Sun if it is to escape and never fall back. We can recognize this by looking at the motion of the meteoroid in Example 10 in reverse: it starts with a velocity of 618 km/s at the surface of the Sun and gradually slows as it rises, but never quite stops until it reaches a very large distance ($r \approx \infty$).

The escape velocity for a body launched from the surface of the Earth can be calculated from a formula analogous to Eq. (9.27), provided that we ignore atmospheric friction and the pull of the Sun on the body. Atmospheric friction will be absent if we launch the body from just above the atmosphere. The pull of the Sun has only a small effect on the velocity of escape from the Earth if we contemplate a body that "escapes" to a distance of, say, $r = 100R_E$ or $200R_E$ rather than $r = \infty$, where we would also have to consider escape from the Sun. For such a motion, the displacement relative to the Sun can be neglected, and the escape velocity v is approximately $\sqrt{2GM_E/R_E} \approx 11.2$ km/s.

Note that the direction in which the escaping body is launched is immaterial—the body will succeed in its escape whenever the direction of launch is above the horizon. Of course, the path that the body takes will depend on the direction of launch (see Fig. 9.26).

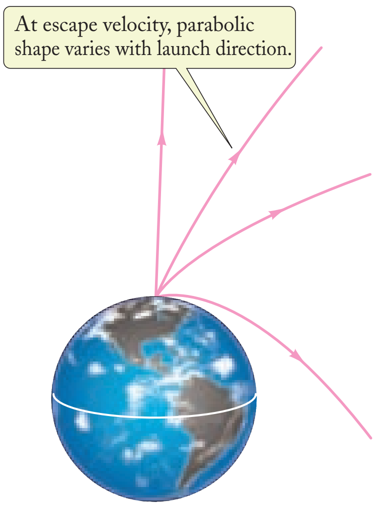


FIGURE 9.26 Different orbits with the same starting point and initial speed. All these orbits are segments of parabolas.

Checkup 9.5

QUESTION 1: An artificial satellite is initially in a circular orbit of fairly low altitude around the Earth. Because of friction with the residual atmosphere, the satellite loses some energy and enters a circular orbit of smaller radius. The speed of the satellite will then be *larger* in the new orbit. How can friction result in an increase of kinetic energy?

QUESTION 2: Does Kepler's Second Law apply to parabolic and hyperbolic orbits?

QUESTION 3: Suppose that we launch a body horizontally from the surface of the Earth, with a velocity exactly equal to the escape velocity of 11.2 km/s. What kind of orbit will this body have? Ignore atmospheric friction.

QUESTION 4: Uranus has a smaller gravitational acceleration at its surface than the Earth. Can you conclude that the escape velocity from its surface is smaller than from the Earth's surface?

QUESTION 5: Suppose that three comets, I, II, and III, approach the Sun. At the instant they cross the Earth's orbit, comet I has a speed of 42 km/s, comet II has a larger speed, and comet III a smaller speed. Given that the orbit of comet I is parabolic, what are the kinds of orbit for comets II and III, respectively?

(A) Elliptical; hyperbolic (B) Elliptical; parabolic (C) Hyperbolic; elliptical
(D) Hyperbolic; parabolic (E) Parabolic; elliptical

SUMMARY

PHYSICS IN PRACTICE Communication Satellites and Weather Satellites **(page 281)**

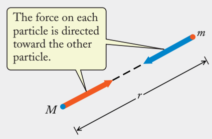
MATH HELP Ellipses **(page 283)**

LAW OF UNIVERSAL GRAVITATION **(9.1)**

Magnitude:

$$F = \frac{GMm}{r^2}$$

Direction: The force on each mass is directed toward the other mass.



GRAVITATIONAL CONSTANT **(9.2)**

$$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$$

ACCELERATION OF FREE FALL ON EARTH **(9.6)**

$$g = \frac{GM_E}{R_E^2}$$

SPEED FOR CIRCULAR ORBIT AROUND SUN **(9.10)**

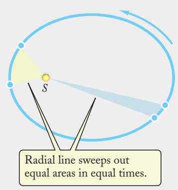
$$v = \sqrt{\frac{GM_S}{r}}$$

PERIOD OF ORBIT AROUND SUN

$$T^2 = \frac{4\pi^2}{GM_S} r^3$$

KEPLER'S FIRST LAW The orbits of the planets are ellipses with the Sun at one focus.

KEPLER'S SECOND LAW The radial line segment from the Sun to a planet sweeps out equal areas in equal times.



KEPLER'S THIRD LAW The square of the period is proportional to the cube of the semimajor axis of a planetary orbit.

GRAVITATIONAL POTENTIAL ENERGY

$$U = -\frac{GMm}{r} \tag{9.20}$$

CONSERVATION OF ENERGY

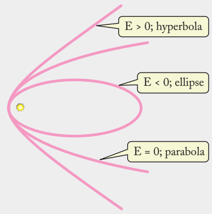
$$E = K + U = \tfrac{1}{2}mv^2 - \frac{GMm}{r} = [\text{constant}] \tag{9.21}$$

ENERGY FOR A CIRCULAR ORBIT AROUND THE SUN (Also the energy for an elliptical orbit with semimajor axis r.)

$$E = -\frac{GM_S m}{2r} \tag{9.24}$$

SHAPES OF ORBITS For total mechanical energy E,

$E < 0$ elliptical orbit
$E = 0$ parabolic trajectory
$E > 0$ hyperbolic trajectory



ESCAPE VELOCITY FROM EARTH

$$v = \sqrt{\frac{2GM_E}{R_E}} \tag{9.27}$$

QUESTIONS FOR DISCUSSION

1. Can you directly feel the gravitational pull of the Earth with your sense organs? (Hint: Would you feel anything if you were in free fall?)
2. According to a tale told by Professor R. Lichtenstein, some apple trees growing in the mountains of Tibet produce apples of negative mass. In what direction would such an apple fall if it fell off its tree? How would such an apple hang on the tree?
3. Eclipses of the Moon can occur only at full Moon. Eclipses of the Sun can occur only at new Moon. Why?
4. Explain why the sidereal day (the time of rotation of the Earth relative to the stars, or 23 h 56 min 4 s) is shorter than the mean solar day (the time between successive passages of the Sun over a given meridian, or 24 h). (Hint: The rotation of the Earth around its axis and the revolution of the Earth around the Sun are in the same direction.)
5. Suppose that an airplane flies around the Earth along the equator. If this airplane flew *very* fast, it would not need wings to support itself. Why not?
6. The mass of Pluto was not known until 1978 when a moon of Pluto was finally discovered. How did the discovery of this moon help?
7. It is easier to launch an Earth satellite into an eastward orbit than into a westward orbit. Why?
8. Would it be advantageous to launch rockets into space from a pad at very high altitude on a mountain? Why has this not been done?
9. Describe how you would play squash on a small, round asteroid (with no enclosing wall). What rules of the game would you want to lay down?
10. According to an NBC news report of April 5, 1983, a communications satellite launched from the Space Shuttle went into an orbit as shown in Fig. 9.27. Is this believable?

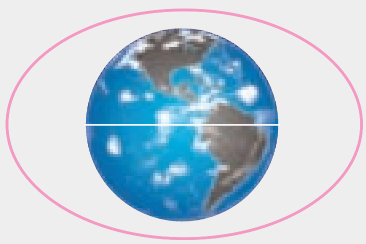


FIGURE 9.27 Proposed orbit for a communications satellite.

11. Does the radial line from the Sun to Mars sweep out area at the same rate as the radial line from the Sun to the Earth?

12. Why were the Apollo astronauts able to jump much higher on the Moon than on the Earth (Fig. 9.28)? If they had landed on a small asteroid, could they have launched themselves into a parabolic or hyperbolic orbit by a jump?

FIGURE 9.28 The jump of the astronaut.

13. The Earth reaches perihelion on January 3 and aphelion on July 6. Why is it not warmer in January than in July?

14. When the Apollo astronauts were orbiting around the Moon at low altitude, they detected several mass concentrations ("mascons") below the lunar surface. What is the effect of a mascon on the orbital motion?

15. An astronaut in a circular orbit above the Earth wants to take his spacecraft into a new circular orbit of larger radius. Give him instructions on how to do this.

16. A Russian and an American astronaut are in two separate spacecraft in the same circular orbit around the Earth. The Russian is slightly behind the American, and he wants to overtake him. The Russian fires his thrusters in the *forward* direction, braking for a brief instant. This changes his orbit into an ellipse. One orbital period later, the astronauts return to the vicinity of their initial positions, but the Russian is now ahead of the American. He then fires his thrusters in the *backward* direction. This restores his orbit to the original circle. Carefully explain the steps of this maneuver, drawing diagrams of the orbits.

17. The gravitational force that a hollow spherical shell of uniformly distributed mass exerts on a particle in its interior is zero. Does this mean that such a shell acts as a gravity shield?

18. Consider an astronaut launched in a rocket from the surface of the Earth and then placed in a circular orbit around the Earth. Describe the astronaut's weight (measured in an inertial reference frame) at different times during this trip. Describe the astronaut's *apparent* weight (measured in his own reference frame) at different times.

19. Several of our astronauts suffered severe motion sickness while under conditions of apparent weightlessness. Since the astronauts were not being tossed about (as in an airplane or a ship in a storm), what caused this motion sickness? What other difficulties does an astronaut face in daily life under conditions of weightlessness?

20. An astronaut on the International Space Station lights a candle. Will the candle burn like a candle on Earth?

21. Astrology is an ancient superstition according to which the planets influence phenomena on the Earth. The only force that can reach over the large distances between the planets and act on pieces of matter on the Earth is gravitation (planets do not have electric charge, and they therefore do not exert electric forces; some planets do have magnetism, but their magnetic forces are too weak to reach the Earth). Given that the Earth is in free fall under the action of the net gravitational force of the planets and the Sun, is there any way that the gravitational forces of the planets can affect what happens on the Earth?

PROBLEMS

9.1 Newton's Law of Universal Gravitation
9.2 The Measurement of $G^\dagger$

1. Two supertankers, each with a mass of 700 000 metric tons, are separated by a distance of 2.0 km. What is the gravitational force that each exerts on the other? Treat them as particles.

2. What is the gravitational force between two protons separated by a distance equal to their diameter, 2.0×10^{-15} m?

3. Somewhere between the Earth and the Moon there is a point where the gravitational pull of the Earth on a particle exactly balances that of the Moon. At what distance from the Earth is this point?

4. Calculate the value of the acceleration of gravity at the surface of Venus, Mercury, and Mars. Use the data on planetary masses and radii given in the table printed inside the book cover.

5. What is the magnitude of the gravitational force that the Sun exerts on you? What is the magnitude of the gravitational force that the Moon exerts on you? The masses of the Sun and the Moon and their distances are given inside the book cover; assume that your mass is 70 kg. Compare these forces with

† For help, see Online Concept Tutorial 11 at www.wwnorton.com/physics

your weight. Why don't you feel these forces? (Hint: You and the Earth are in free fall toward the Sun and the Moon.)

6. Calculate the gravitational force between our Galaxy and the Andromeda galaxy. Their masses are 2.0×10^{11} and 3.0×10^{11} times the mass of the Sun, respectively, and their separation is 2.2×10^6 light-years. Treat both galaxies as point masses.

7. The nearest star is Alpha Centauri, at a distance of 4.4 light-years from us. The mass of this star is 2.0×10^{30} kg. Compare the gravitational force exerted by Alpha Centauri on the Sun with the gravitational force that the Earth exerts on the Sun. Which force is stronger?

8. What is the magnitude of the gravitational attraction the Sun exerts on the Moon? What is the magnitude of the gravitational attraction the Earth exerts on the Moon? Suppose that the three bodies are aligned, with the Earth between the Sun and the Moon (at full moon). What is the direction of the net force acting on the Moon? Suppose that the three bodies are aligned with the Moon between the Earth and the Sun (at new moon). What is the direction of the net force acting on the Moon?

9. Calculate the value of the acceleration due to gravity at the surfaces of Jupiter, Saturn, and Uranus. Use the values of the planetary masses and radii given in the table printed inside the book cover.

10. Somewhere between the Earth and the Sun is a point where the gravitational attraction of the Earth exactly balances that of the Sun. At what fraction of the Earth–Sun distance does this occur?

11. Compare the weight of a 1-kg mass at the Earth's surface with the gravitational force between our Sun and another star of the same mass located at the far end of our galaxy, about 5×10^{20} m away.

12. Each of two adjacent 1.5-kg spheres hangs from a ceiling by a string. The center-to-center distance of the spheres is 8.0 cm. What (small) angle does each string make with the vertical?

13. A 7.0-kg mass is on the x axis at $x = 3.0$ m, and a 4.0-kg mass is on the y axis at $y = 2.0$ m. What is the resultant gravitational force (magnitude and direction) due to these two masses on a third mass of 3.0 kg located at the origin?

14. Three equal masses m are located at the vertices of an equilateral triangle of side a. What is the magnitude of the net gravitational force on each mass due to the other two?

15. Find the acceleration of the Moon due to the pull of the Earth. Express your result in units of the standard g.

16. If a "tower to the sky" of height 2000 km above the Earth's surface could be built, what would be your weight when standing at the top? Assume the tower is located at the South Pole. Express your answer in terms of your weight at the Earth's surface.

17. It has been suggested that strong tidal forces on Io, a moon of Jupiter, could be responsible for the dramatic volcanic activity observed there by Voyager spacecraft. Compare the difference in gravitational accelerations on the near and far surfaces of Io (due to Jupiter) with the difference in accelerations on the near and far side of the Earth (due to the Moon), both as absolute accelerations and as a fraction of the surface g. Io has a mass of 8.9×10^{22} kg and a radius of 1820 km, and is 422×10^3 km from the center of Jupiter.

*18. Suppose that the Earth, Sun, and Moon are located at the vertices of a right triangle, with the Moon located at the right angle (at first or last quarter moon; see Fig. 9.29). Find the magnitude and direction of the sum of the gravitational forces exerted by the Earth and the Sun on the Moon.

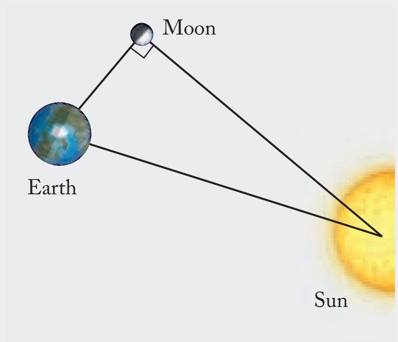


FIGURE 9.29 Earth, Moon, and Sun.

*19. Mimas, a small moon of Saturn, has a mass of 3.8×10^{19} kg and a diameter of 500 km. What is the maximum equatorial velocity with which we can make this moon rotate about its axis if pieces of loose rock sitting on its surface at its equator are not to fly off?

9.3 Circular Orbits†

20. The *Midas II* spy satellite was launched into a circular orbit at a height of 500 km above the surface of the Earth. Calculate the orbital period and the orbital speed of this satellite.

21. Consider the communications satellite described in Example 6. What is the speed of this satellite?

22. Calculate the orbital speed of Venus from the data given in Example 5.

23. The Sun is moving in a circular orbit around the center of our Galaxy. The radius of this orbit is 3×10^4 light-years. Calculate the period of the orbital motion and calculate the orbital speed of the Sun. The mass of our Galaxy is 4×10^{41} kg, and all of this mass can be regarded as concentrated at the center of the Galaxy.

24. Table 9.2 lists some of the moons of Saturn. Their orbits are circular.
   (a) From the information given, calculate the periods and orbital speeds of all these moons.
   (b) Calculate the mass of Saturn.

† For help, see Online Concept Tutorial 11 at www.wwnorton.com/physics

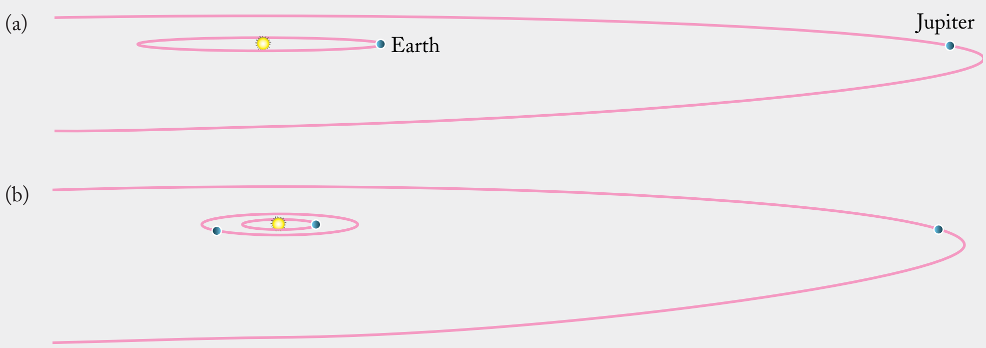
TABLE 9.2 SOME MOONS OF SATURN

MOON	DISTANCE FROM SATURN	PERIOD	ORBITAL SPEED
Tethys (Fig. 9.30)	2.95×10^5 km	1.89 days	—
Dione	3.77	—	—
Rhea	5.27	—	—
Titan	12.22	—	—
Iapetus	35.60	—	—

FIGURE 9.30 Tethys, one of the moons of Saturn.

25. Before clocks with long-term accuracy were constructed, it was proposed that navigators at sea should use the motion of the moons of Jupiter as a clock. The moons Io, Europa, and Ganymede have orbital radii of 422×10^3, 671×10^3, and 1070×10^3 km, respectively. What are the periods of the orbits of these moons? The mass of Jupiter is 1.90×10^{27} kg.
26. A satellite is to be put into an equatorial orbit with an orbital period of 12 hours. What is the radius of the orbit? What is the orbital speed? How many times a day will the satellite be over the same point on the equator if the satellite orbits in the same direction as the Earth's rotation? If it orbits in the opposite direction?
27. An asteroid is in a circular orbit at a distance of two solar diameters from the center of the Sun. What is its orbital period in days?
28. The Sun rotates approximately every 26 days. What is the radius of a "heliosynchronous" orbit, that is, an orbit that stays over the same spot of the Sun?
29. The Apollo command module orbited the Moon while the lunar excursion module visited the surface. If the orbit had a radius of 2.0×10^6 m, how many times per (Earth) day did the command module fly over the excursion module?
30. A Jupiter-sized planet orbits the star 55 Cancri with an orbital radius of 8.2×10^{11} m (see Fig. 9.31). The orbital period of this planet is 13 yr. What is the mass of the star 55 Cancri? How does this compare with the mass of the Sun?

FIGURE 9.31 (a) The Solar System and (b) the 55 Cancri system.



*31. The *Discoverer II* satellite had an approximately circular orbit passing over both poles of the Earth. The radius of the orbit was about 6.67×10^3 km. Taking the rotation of the Earth into account, if the satellite passed over New York City at one instant, over what point of the United States would it pass after completing one more orbit?

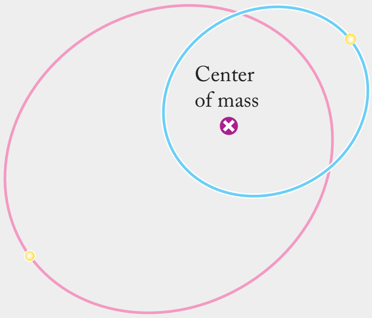
*32. The binary star system PSR 1913+16 consists of two neutron stars orbiting with a period of 7.75 h about their center of mass, which is at the midpoint between the stars. Assume that the stars have equal masses and that their orbits are circular with a radius of 8.67×10^8 m.

(a) What are the masses of the stars?

(b) What are their speeds?

*33. Figure 9.32 shows two stars orbiting about their common center of mass in the binary system Krüger 60. The center of mass is at a point between the stars such that the distances of the stars from this point are in the inverse ratio of their masses. Measure the sizes of their orbits and determine the ratio of their masses.

FIGURE 9.32 The orbits of the two stars in the binary system Krüger 60. Each ellipse has its focus at the center of mass.



**34. A binary star system consists of two stars of masses m_1 and m_2 orbiting about each other. Suppose that the orbits of the stars are circles of radii r_1 and r_2 centered on the center of mass (Fig. 9.33). The center of mass is a point between the stars such that the radii r_1 and r_2 are in the ratio $r_1/r_2 = m_2/m_1$. Show that the period of the orbital motion is given by

$$T^2 = \frac{4\pi^2}{G(m_1 + m_2)}(r_1 + r_2)^3$$

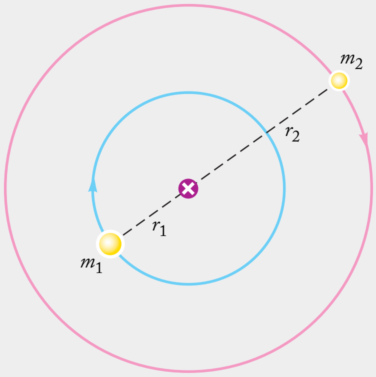


FIGURE 9.33 A binary star system. The orbits are circles about the center of mass.

*35. The binary system Cygnus X-1 consists of two stars orbiting about their center of mass under the influence of their mutual gravitational forces. The orbital period of the motion is 5.6 days. One of the stars is a supergiant with a mass 25 times the mass of the Sun. The other star is believed to be a black hole with a mass about 10 times the mass of the Sun. From the information given, determine the distance between the stars; assume that the orbits of both stars are circular. (Hint: See Problem 34.)

**36. A hypothetical triple star system consists of three stars orbiting about each other. For the sake of simplicity, assume that all three stars have equal masses and that they move along a common circular orbit maintaining an angular separation of 120° (Fig. 9.34). In terms of the mass M of each star and the orbital radius R, what is the period of the motion?

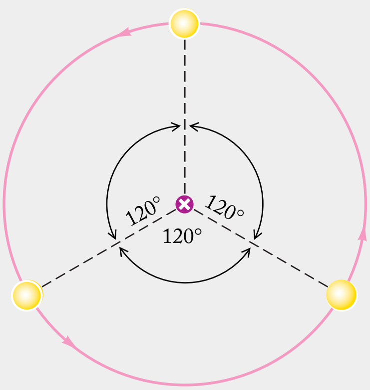


FIGURE 9.34 Three identical stars orbiting about their center of mass.

**37. Take into account the rotation of the Earth in the following problem:

(a) Cape Canaveral is at a latitude of 28° north. What eastward speed (relative to the ground) must a satellite be given if it is to achieve a low-altitude circular orbit (Fig. 9.35)? What westward speed must the satellite be given if it is to travel along the same orbit in the opposite direction? For the purpose of this problem, pretend that "low altitude" means essentially "zero altitude."

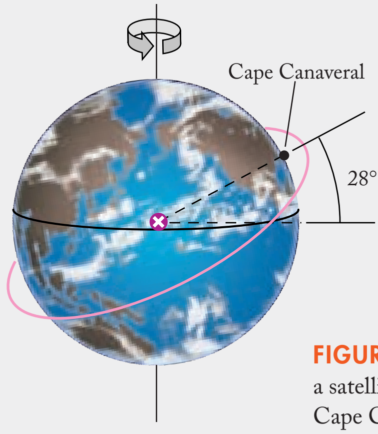


FIGURE 9.35 Orbit of a satellite launched from Cape Canaveral.

(b) Suppose that the satellite has a mass of 14.0 kg. What kinetic energy must the launch vehicle give to the satellite for an eastward orbit? For a westward orbit?

9.4 Elliptical Orbits; Kepler's Laws†

38. Halley's comet (Fig. 9.36) orbits the Sun in an elliptical orbit (the comet reached perihelion in 1986). When the comet is at perihelion, its distance from the Sun is 8.78×10^{10} m, and its speed is 5.45×10^4 m/s. When the comet is at aphelion, its distance is 5.28×10^{12} m. What is the speed at aphelion?

FIGURE 9.36 Halley's comet photographed in 1986.

39. *Explorer I*, the first American artificial satellite, had an elliptical orbit around the Earth with a perigee distance of 6.74×10^6 m and an apogee distance of 8.91×10^6 m. The speed of this satellite was 6.21×10^3 m/s at apogee. Calculate the speed at perigee.

40. The *Explorer X* satellite had an orbit with perigee 175 km and apogee 181,200 km above the surface of the Earth. What was the period of this satellite?

† For help, see Online Concept Tutorial 12 at www.wwnorton.com/physics

TABLE 9.3 THE FIRST ARTIFICIAL EARTH SATELLITES

SATELLITE	MASS	MEAN DISTANCE FROM CENTER OF EARTH (SEMIMAJOR AXIS)	PERIGEE DISTANCE	APOGEE DISTANCE	PERIOD
Sputnik I	83 kg	6.97×10^3 km	6.60×10^3 km	7.33×10^3 km	96.2 min
Sputnik II	3000	7.33	6.61	8.05	104
Explorer I	14	7.83	6.74	8.91	115
Vanguard I	1.5	8.68	7.02	10.3	134
Explorer III	14	7.91	6.65	9.17	116
Sputnik III	1320	7.42	6.59	8.25	106

41. Calculate the orbital periods of *Sputnik I* and *Explorer I* from their apogee and perigee distances in Table 9.3.
42. The aphelion distance for Saturn is 1510×10^6 km; its perihelion distance is 1350×10^6 km. By Kepler's First Law, the Sun is at one focus of this ellipse. How far from the Sun is the other focus? How does this compare with the orbital radius of Mercury?
43. The comet Hale–Bopp was spectacularly visible in the spring of 1997 (see Fig. 9.37) and may be the most viewed comet in history. Its perihelion distance was 137×10^6 km, and its orbital period is 2380 yr. What is its aphelion distance? How does this compare with the mean distance of Pluto from the Sun?

FIGURE 9.37 Comet Hale–Bopp photographed in 1997.

44. The orbit of the Earth deviates slightly from circular: at aphelion, the Earth–Sun distance is 1.52×10^8 km, and at perihelion it is 1.47×10^8 km. By what factor is the speed of the Earth at perihelion greater than the speed at aphelion?

9.5 Energy in Orbital Motion

45. The *Voskhod I* satellite, which carried Yuri Gagarin into space in 1961, had a mass of 4.7×10^3 kg. The radius of the orbit was (approximately) 6.6×10^3 km. What were the orbital speed and the orbital energy of this satellite?
46. What is the kinetic energy and what is the gravitational potential energy for the orbital motion of the Earth around the Sun? What is the total energy?
47. Compare the escape velocity given by Eq. (9.27) with the velocity required for a circular orbit of radius R_S, according to Eq. (9.10). By what factor is the escape velocity larger than the velocity for the circular orbit?
48. In July of 1994, fragments of the comet Shoemaker–Levy struck Jupiter.
    (a) What is the impact speed (equal to the escape speed) for a fragment falling on the surface of Jupiter?
    (b) What is the kinetic energy at impact for a fragment of 1.0×10^{10} kg? Express this energy as an equivalent number of short tons of TNT (the explosion of 1 short ton, or 2000 lb, of TNT releases 4.2×10^9 J).
49. A 1.0-kg mass is in the same orbit around the Earth as the Moon (but far from the Moon). What is the kinetic energy for this orbit? The gravitational potential energy? The total energy?
50. The boosters on a satellite in geosynchronous orbit accidentally fire for a prolonged period. At the instant this "burn" ends, the velocity is parallel to the original tangential direction, but the satellite has been slowed to one-half of its original speed. The satellite is thus at apogee of its new orbit. What is the perigee distance for such an orbit? What happens to the satellite?
51. A black hole is so dense that even light cannot escape its gravitational pull. Assume that all of the mass of the Earth is compressed in a sphere of radius R. How small must R be so the escape speed is the speed of light?

52. The spectacular comet Hale–Bopp (Fig. 9.37), most visible in 1997, entered the Solar System in an elliptical orbit with period 4206 yr. However, after a close encounter with Jupiter on its inbound path, it continues on a new elliptical orbit with a period of 2380 yr. By what fraction did the encounter with Jupiter change the energy of Hale–Bopp's orbit?

53. The typical speed of nitrogen molecules at a temperature of 117°C, the temperature of the Moon's surface at "noon," is 600 m/s; some molecules move slower, others faster. What fraction of the escape velocity from the Moon is this? Can you guess why the Moon has not retained an atmosphere?

*54. The Andromeda galaxy is at a distance of 2.1×10^{22} m from our Galaxy. The mass of the Andromeda is 6.0×10^{41} kg, and the mass of our Galaxy is 4.0×10^{41} kg.

(a) Gravity accelerates the galaxies toward each other. As reckoned in an inertial reference frame, what is the acceleration of the Andromeda galaxy? What is the acceleration of our Galaxy? Treat both galaxies as point particles.

(b) The speed of the Andromeda galaxy *relative to our Galaxy* is 266 km/s. What is the speed of the Andromeda and what is the speed of our Galaxy *relative to the center of mass* of the two galaxies? The center of mass is at a point between the galaxies such that the distances of the galaxies from this point are in the inverse ratios of their masses.

(c) What is the kinetic energy of each galaxy relative to the center of mass? What is the total energy (kinetic and potential) of the system of the two galaxies? Will the two galaxies eventually escape from each other?

*55. Neglect the gravity of the Moon, neglect atmospheric friction, and neglect the rotational velocity of the Earth in the following problem. A long time ago, Jules Verne, in his book *From Earth to the Moon* (1865), suggested sending an expedition to the Moon by means of a projectile fired from a gigantic gun.

(a) With what muzzle speed must a projectile be fired vertically from a gun on the surface of the Earth if it is to (barely) reach the distance of the Moon?

(b) Suppose that the projectile has a mass of 2000 kg. What energy must the gun deliver to the projectile? The explosion of 1 short ton (2000 lb) of TNT releases 4.2×10^9 J. How many tons of TNT are required for firing this gun?

(c) If the gun barrel is 500 m long, what must be the average acceleration of the projectile during firing?

*56. An artificial satellite of 1300 kg made of aluminum is in a circular orbit at a height of 100 km above the surface of the Earth. Atmospheric friction removes energy from the satellite and causes it to spiral downward so that it ultimately crashes into the ground.

(a) What is the initial orbital energy (gravitational plus kinetic) of the satellite? What is the final energy when the satellite comes to rest on the ground? What is the energy change?

(b) Suppose that all of this energy is absorbed in the form of heat by the material of the satellite. Is this enough heat to melt the material of the satellite? To vaporize it? The heats of fusion and of vaporization of aluminum are given in Table 20.4.

*57. According to one theory, glassy meteorites (tektites) found on the surface of the Earth originate in volcanic eruptions on the Moon. With what minimum speed must a volcano on the Moon eject a stone if it is to reach the Earth? With what speed will this stone strike the surface of the Earth? In this problem ignore the orbital motion of the Moon around the Earth; use the data for the Earth–Moon system listed in the tables printed inside the book cover. (Hint: When the rock reaches the intermediate point where the gravitational pulls of the Moon and the Earth cancel out, it must have zero velocity.)

*58. A spacecraft is launched with some initial velocity toward the Moon from 300 km above the surface of the Earth.

(a) What is the minimum initial speed required if the spacecraft is to coast all the way to the Moon without using its rocket motors? For this problem pretend that the Moon does not move relative to the Earth. The masses and radii of the Earth and the Moon and their distance are listed in the tables printed inside the book cover. (Hint: When the spacecraft reaches the point in space where the gravitational pulls of the Earth and the Moon cancel, it must have zero velocity.)

(b) With what speed will the spacecraft strike the Moon?

59. The Pons–Brooks comet had a speed of 47.30 km/s when it reached its perihelion point, 1.160×10^8 km from the Sun. Is the orbit of this comet elliptical, parabolic, or hyperbolic?

*60. At a radial distance of 2.00×10^7 m from the center of the Earth, three artificial satellites (I, II, III) are ejected from a rocket. The three satellites I, II, III are given initial speeds of 5.47 km/s, 4.47 km/s, and 3.47 km/s, respectively; the initial velocities are all in the tangential direction.

(a) Which of the satellites I, II, III will have a circular orbit? Which will have elliptical orbits? Explain your answer.

(b) Draw the circular orbit. Also, superimposed on the same diagram, draw the elliptical orbits of the other satellites; label the orbits with the names of the satellites. (Note: You need not calculate the exact sizes of the ellipses, but your diagram should show where the ellipses are larger or smaller than the circle.)

*61. (a) Since the Moon (*our* Moon) has no atmosphere, it is possible to place an artificial satellite in a circular orbit that skims along the surface of the Moon (provided that the satellite does not hit any mountains!). Suppose that such a satellite is to be launched from the *surface* of the Moon by means of a gun that shoots the satellite in a horizontal direction. With what velocity must the satellite be shot out from the gun? How long does the satellite take to go once around the Moon?

(b) Suppose that a satellite is shot from the gun with a horizontal velocity of 2.00 km/s. Make a rough sketch showing the Moon and the shape of the satellite's orbit; indicate the position of the gun on your sketch.

(c) Suppose that a satellite is shot from the gun with a horizontal velocity of 3.00 km/s. Make a rough sketch showing the Moon and the shape of the satellite's orbit. Is this a closed orbit?

*62. According to an estimate, a large crater on Wilkes Land, Antarctica, was produced by the impact of a 1.2×10^{13}–kg meteoroid incident on the surface of the Earth at 70 000 km/h. What was the speed of this meteoroid relative to the Earth when it was at a "large" distance from the Earth?

*63. An experienced baseball player can throw a ball with a speed of 140 km/h. Suppose that an astronaut standing on Mimas, a small moon of Saturn of mass 3.76×10^{19} kg and radius 195 km, throws a ball with this speed.

(a) If the astronaut throws the ball horizontally, will it orbit around Mimas?

(b) If the astronaut throws the ball vertically, how high will it rise?

*64. An electromagnetic launcher, or rail gun, accelerates a projectile by means of magnetic fields. According to some calculations, it may be possible to attain muzzle speeds as large as 15 km/s with such a device. Suppose that a projectile is launched upward from the surface of the Earth with this speed; ignore air resistance.

(a) Will the projectile escape permanently from the Earth?

(b) Can the projectile escape permanently from the Solar System? (Hint: Take into account the speed of 30 km/s of the Earth around the Sun.)

*65. *Sputnik I,* the first Russian satellite (1957), had a mass of 83.5 kg; its orbit reached perigee at a height of 225 km and apogee at 959 km. *Explorer I,* the first American satellite (1958), had a mass of 14.1 kg; its orbit reached perigee at a height of 368 km and apogee at 2540 km. What was the orbital energy of these satellites?

*66. The orbits of most meteoroids around the Sun are nearly parabolic.

(a) With what speed will a meteoroid reach a distance from the Sun equal to the distance of the Earth from the Sun? (Hint: In a parabolic orbit the speed at any radius equals the escape velocity at the radius. Why?)

(b) Taking into account the Earth's orbital speed, what will be the speed of the meteoroid *relative to the Earth* in a head-on collision with the Earth? In an overtaking collision? Ignore the effect of the gravitational pull of the Earth on the meteoroid.

*67. Calculate the perihelion and the aphelion speeds of Encke's comet. The perihelion and aphelion distances of this comet are 5.06×10^7 km and 61.25×10^7 km. (Hint: Consider the total energy of the orbit.)

*68. The *Explorer XII* satellite was given a tangential velocity of 10.39 km/s when at perigee at a height of 457 km above the Earth. Calculate the height of the apogee. (Hint: Consider the total energy of the orbit.)

**69. Prove that the orbital energy of a planet or a comet in an elliptical orbit around the Sun can be expressed as

$$E = -\frac{GM_S m}{r_1 + r_2}$$

where r_1 and r_2 are, respectively, the perihelion and aphelion distances. [Hint: Use the conservation of energy and the conservation of angular momentum ($r_1 v_1 = r_2 v_2$) at perihelion and at aphelion to solve for v_1^2 and v_2^2 in terms of r_1 and r_2.]

*70. Suppose that a comet is originally at rest at a distance r_1 from the Sun. Under the influence of the gravitational pull, the comet falls radially toward the Sun. Show that the time it takes to reach a radius r_2 is

$$t = -\int_{r_1}^{r_2} \frac{dr}{\sqrt{2GM_S/r - 2GM_S/r_1}}$$

*71. Suppose that a projectile is fired horizontally from the surface of the Moon with an initial speed of 2.0 km/s. Roughly sketch the orbit of the projectile. What maximum height will this projectile reach? What will be its speed when it reaches maximum height?

**72. The Earth has an orbit of radius 1.50×10^8 km around the Sun; Mars has an orbit of radius 2.28×10^8 km. In order to send a spacecraft from the Earth to Mars, it is convenient to launch the spacecraft into an elliptical orbit whose perihelion coincides with the orbit of the Earth and whose aphelion coincides with the orbit of Mars (Fig. 9.38); this orbit requires the least amount of energy for a trip to Mars.

(a) To achieve such an orbit, with what speed (relative to the Earth) must the spacecraft be launched? Ignore the pull of the gravity of the Earth and Mars on the spacecraft.

(b) With what speed (relative to Mars) does the spacecraft approach Mars at the aphelion point? Assume that Mars actually is at the aphelion point when the spacecraft arrives.

(c) How long does the trip from Earth to Mars take?

(d) Where must Mars be (in relation to the Earth) at the instant the spacecraft is launched? Where will the Earth be when the spacecraft arrives at its destination? Draw a diagram showing the relative positions of Earth and Mars at these two times.

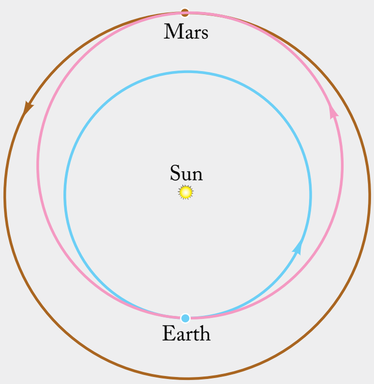


FIGURE 9.38 Orbit for a spacecraft on a trip to Mars.

**73. Repeat the calculations of Problem 72 for the case of a spacecraft launched on a trip to Venus. The orbit of Venus has a radius of 1.08×10^8 km.

**74. If a spacecraft, or some other body, approaches a moving planet on a hyperbolic orbit, it can gain some energy from the motion of the planet and emerge with a larger speed than it had initially. This slingshot effect has been used to boost the speeds of the two Voyager spacecraft as they passed near Jupiter. Suppose that the line of approach of the satellite makes an angle θ with the line of motion of the planet and the line of recession of the spacecraft is parallel to the line of motion of the planet (Fig. 9.39; the planet can be regarded as moving on a straight line during the time interval in question). The speed of the planet is u, and the initial speed of the spacecraft is v (in the reference frame of the Sun).

(a) Show that the final speed of the spacecraft is

$$v' = u + \sqrt{v^2 + u^2 - 2uv\cos\theta}$$

(b) Show that the spacecraft will not gain any speed in this encounter if $\theta = 0$, and show that the spacecraft will gain maximum speed if $\theta = 180°$.

(c) If a spacecraft with $v = 3.0$ km/s approaches Jupiter at an angle of $\theta = 20°$, what will be its final speed?

75. According to one design studied by NASA, a large space colony in orbit around the Earth would consist of a torus of diameter 1.8 km, looking somewhat like a gigantic wheel (see Fig. 9.40). In order to generate artificial gravity of $1g$, how fast must this space colony rotate about its axis?

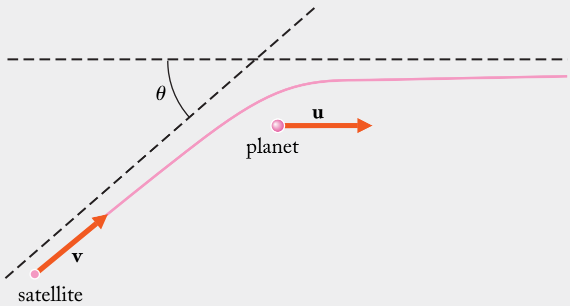


FIGURE 9.39 Trajectory of a spacecraft passing by a planet.

FIGURE 9.40 A rotating space station.

REVIEW PROBLEMS

76. Calculate the gravitational force that the Earth exerts on an astronaut of mass 75 kg in a space capsule at a height of 100 km above the surface of the Earth. Compare with the gravitational force that this astronaut would experience if on the surface of the Earth.

77. The masses used in the Cavendish experiment typically are a few kilograms for the large masses and a few tens of grams for the small masses. Suppose that a "large" spherical mass of 8.0 kg is at a center-to-center distance of 10 cm from a "small" spherical mass of 30 g. What is the magnitude of the gravitational force?

78. The asteroid Ceres has a diameter of 1100 km and a mass of (approximately) 7×10^{20} kg. What is the value of the acceleration of gravity at its surface? On the surface of this asteroid, what would be the weight (in lbf) of a man whose weight on the surface of the Earth is 170 lbf?

79. The asteroid belt of the Solar System consists of chunks of rock orbiting around the Sun in approximately circular orbits. The mean distance of the asteroid belt from the Sun is about 2.9 times the distance of the Earth. What is the mean period of the orbital motion of the asteroids?

80. Imagine that somewhere in interstellar space a small pebble is in a circular orbit around a spherical asteroid of mass 1000 kg. If the radius of the circular orbit is 1.0 km, what is the period of the motion?

81. Europa (Fig. 9.41) is a moon of Jupiter. Astronomical observations show that this moon is in a circular orbit of radius 6.71×10^8 m with a period of 3.55 days. From these data deduce the mass of Jupiter.

82. Observations with the Hubble Space Telescope have revealed that at the center of the galaxy M87, gas orbits around a very massive compact object, believed to be a black hole. The measurements show that gas clouds in a circular orbit of radius 250 light-years have an orbital speed of 530 km/s. From this information, deduce the mass of the black hole.

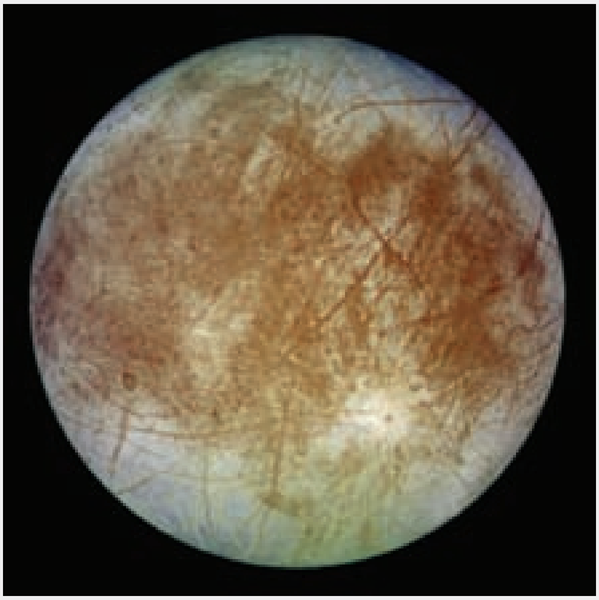


FIGURE 9.41 Europa, one of the moons of Jupiter.

83. Consider a space station in a circular orbit at an altitude of 400 km around the Earth and a piece of debris, left over from, say, the disintegration of a rocket, in an orbit of the same radius but of opposite direction.
    (a) What is the speed of the debris relative to the space station when they pass?
    (b) If the debris hit the spacecraft, it would penetrate the space station with catastrophic consequences for the crew. Penetration depends on the kinetic energy of the debris. What must be the mass of a piece of debris if it is to have an impact energy of 4.6×10^5 J, which corresponds to the explosion of 100 g of TNT?
84. *Vanguard I,* the second American artificial satellite (Fig. 9.42), moved in an elliptical orbit around the Earth with a perigee distance of 7.02×10^6 m and an apogee distance of 10.3×10^6 m. At perigee, the speed of this satellite was 8.22×10^3 m/s. What was the speed at apogee?

*85. The motor of a Scout rocket uses up all its fuel and stops when the rocket is at an altitude of 200 km above the surface of the Earth and is moving vertically at 8.50 km/s. How high will this rocket rise? Neglect any residual atmospheric friction.

*86. An astronaut in a spacecraft in a circular orbit around the Earth wants to get rid of a defective solar panel that he has detached from the spacecraft. He hits the panel with a blast from the steering rocket of the spacecraft, giving it an increment of velocity. This sends the solar panel into an elliptical orbit.
    (a) Sketch the circular orbit of the spacecraft and the elliptical orbit of the solar panel if the velocity increment is parallel to the velocity of the spacecraft and if it is antiparallel.
    (b) If the ratio of the semimajor axis of the ellipse to the radius of the circle has a special value, it is possible for the panel to meet with the spacecraft again after several orbits. What are these special values of the ratio?

*87. A communications satellite of mass 700 kg is placed in a circular orbit of radius 4.23×10^7 m around the Earth.
    (a) What is the total orbital energy of this satellite?
    (b) How much extra energy would we have to give this satellite to put it into a parabolic orbit that permits it to escape to infinite distance from the Earth?

88. What is the escape velocity for a projectile launched from the surface of our Moon?

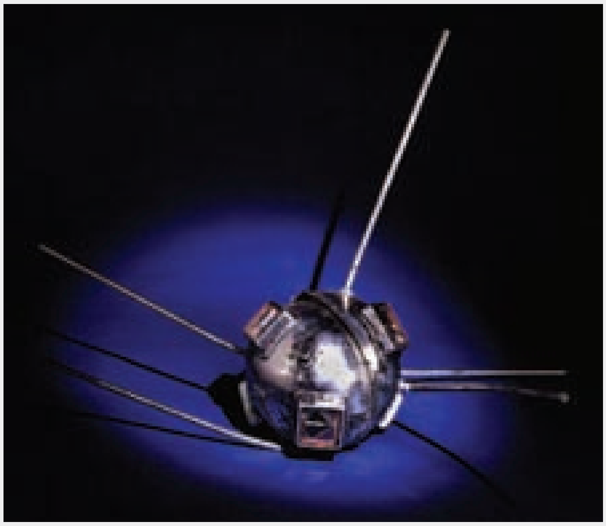


FIGURE 9.42 The *Vanguard I* satellite.

Answers to Checkups

Checkup 9.1

1. The gravitational force varies inversely with the square of the distance, so the force will be $(30)^2 = 900$ times weaker for a 1-kg piece of Neptune than for a 1-kg piece of the Earth.
2. The gravitational force varies in proportion to the mass and in inverse proportion to the square of the distance, so the 100-times-larger mass for Saturn cancels the 10-times-larger distance; thus, the gravitational force that the Sun exerts on Saturn is about equal to that on the Earth. The acceleration is $a = F/m$, and so is about 100 times smaller for Saturn.
3. The acceleration varies inversely with the square of the distance, and so is $\frac{1}{4}g$ at $r = 2R_E$, and is $\frac{1}{9}g$ at $r = 3R_E$.

4. Since the acceleration at a planet's surface is $a = GM/R^2$, a larger mass M and a smaller gravitational acceleration a are possible only because the radius R of Uranus is sufficiently larger than that of the Earth.

5. At the exact center of the Earth, a particle would be equally attracted in all directions, and so would experience zero net force.

6. (B) $\frac{1}{4}g$. The acceleration at the surface is $a = GM_E/R_E^2$, so a doubled radius would result in an acceleration one-fourth as large, or $\frac{1}{4}g$.

Checkup 9.2

1. To determine G by measuring the force between the Earth and some known mass, we would also have to know the mass of the Earth; we have no independent way of determining the mass of the Earth.

2. (A) Yes. If we knew the mass of the mountain (and the spatial distribution of such mass), then we could determine the gravitational force from the plumb bob's deflection, and thus G.

Checkup 9.3

1. An orbit that is a circle at the latitude of San Francisco is impossible, since the center of every orbit must coincide with the center of the Earth.

2. The period is proportional to the 3/2 power of the radius of the orbit, so for a doubled radius, the period of the Moon would become $2^{3/2} \times 27$ days ≈ 76 days.

3. As in Eq. (9.13), we need only know the period and radius of the moon's orbit to determine the mass of the planet.

4. (C) 30 yr. The period is proportional to the 3/2 power of the radius of the orbit, so the period of Saturn's motion is $10^{3/2} \times 1$ yr ≈ 30 yr.

Checkup 9.4

1. Kepler's Second Law would remain valid, since it depends only on the central nature of the force, and otherwise not on any particular form of the force. Kepler's Third Law, however, like the law of periods, Eq. (9.13), depends on the inverse-square nature of the force. If we were to perform a similar derivation to that preceding Eq. (9.13) for an inverse-cube force, we would find that the period was proportional to the square of the radius.

2. As in Eq. (9.18), the speeds vary inversely with the distances, so for an aphelion distance twice as large as the perihelion distance, the speed at aphelion will be half as large as the speed at perihelion, or will be 20 km/s.

3. According to Kepler's Third Law, the period must be exactly one year. This is so because both the Earth's orbit (nearly circular; the semimajor axis of a circle is its radius) and the comet's orbit have the same semimajor axis, and both orbit the same central body, the Sun.

4. (D) 4. Kepler's Third Law states that the square of the period is proportional to the cube of the semimajor axis of the orbit, so to make the period 8 times as large as the Earth's period would make the cube of the semimajor axis 64 times as large; thus the semimajor axis would be $64^{1/3} = 4$ times as large as the Earth–Sun distance.

Checkup 9.5

1. For a circular orbit, we found that the magnitude of the (negative) potential energy is twice the size of the kinetic energy. Thus the potential energy decreases so much for the lower orbit (it becomes more negative) that the kinetic energy can increase and energy can be lost to friction.

2. Yes—our derivation of the law depended only on the central nature of the force, not on any particular type of orbit (or even any particular form of the central force).

3. If we ignore air friction (and the body does not encounter any obstacles), then the body will escape the Earth's influence in a parabolic "orbit," since the escape velocity provides for zero net energy. The orbit would be similarly parabolic if we launched the body at any angle (except straight up, although that resulting linear path can be considered a special case of the parabola). Ultimately, far from the Earth's influence, the path would be modified by the Sun.

4. No. The gravitational acceleration is $g = GM/R^2$, whereas the escape velocity depends on the gravitational potential energy, which is proportional to M/R. For example, a body with twice the mass and twice the radius of the Earth would have half the gravitational acceleration at the surface, but would have the same escape velocity.

5. (C) Hyperbolic; elliptical. Recall that a parabolic orbit is a zero-energy orbit, where the comet can just barely escape to infinity. The energy of comet II must be positive, since it has a larger speed (a greater kinetic energy, but the same potential energy as it crosses the Earth's orbit); we found that a positive-energy orbit is a hyperbola. Similarly, the energy of comet III must be negative, since it has a smaller speed; negative-energy orbits are ellipses, with a semimajor axis given by Eq. (9.24).

CHAPTER

10

Systems of Particles

Concepts in Context

CONCEPTS IN CONTEXT

While this high jumper is passing over the bar, he bends backward and keeps his extremities below the level of the bar. This means that the average height of his body parts is less than if he were to keep his body straight, and he requires less energy to pass over the bar.

The concepts introduced in this chapter permit us to examine in detail several aspects of the motion of the jumper:

? The body of the jumper is a system of particles. Where is the average position of the mass of this system of particles when the body is in a straight configuration? How does this change when the jumper reconfigures his extremities? (Example 8, part (a), page 322)

? What is the gravitational potential energy of a system of particles, and how much does the jumper reduce his potential energy by bending his body? (Page 321 in Section 10.2 and Example 8, part (b), page 322)

? What is equation of motion of a system of particles, and to what extent does the translational motion of a jumper resemble projectile motion? (Page 324 in Section 10.3)

So far we have dealt almost exclusively with the motion of a single particle. Now we will begin to study systems of particles interacting with each other via some forces. This means we must examine, and solve, the equations of motion of all these particles simultaneously.

Since chunks of ordinary matter are made of particles—electrons, protons, and neutrons—all the macroscopic bodies that we encounter in our everyday environment are in fact many-particle systems containing a very large number of particles. However, for most practical purposes, it is not desirable to adopt such an extreme microscopic point of view, and in the preceding chapters we treated the motion of a macroscopic body, such as an automobile, as motion of a particle. Likewise, in dealing with a system consisting of several macroscopic bodies, we will often find it convenient to treat each of these bodies as a particle and ignore the internal structure of the bodies. For example, when investigating a collision between two automobiles, we may find it convenient to pretend that each of the automobiles is a particle—we then regard the colliding automobiles as a system of two particles which exert forces on each other when in contact. And when investigating the Solar System, we may find it convenient to pretend that each planet and each satellite is a particle—we then regard the Solar System as a system of such planet and satellite particles loosely held together by gravitation and orbiting around the Sun and around each other.

The equations of motion of a system of several particles are often hard, and sometimes impossible, to solve. It is therefore necessary to make the most of any information that can be extracted from the general conservation laws. In the following sections we will become familiar with the *momentum* vector, and we will see how the laws of conservation of momentum and of energy apply to a system of particles.

10.1 MOMENTUM

Newton's laws can be expressed very neatly in terms of **momentum,** a vector quantity of great importance in physics. *The momentum of a single particle is defined as the product of the mass and the velocity of the particle:*[1]

momentum of a particle

$$\mathbf{p} = m\mathbf{v} \tag{10.1}$$

Thus, the momentum $\mathbf{p}$ is a vector that has the same direction as the velocity vector, but a magnitude that is m times the magnitude of the velocity. The SI unit of momentum is kg·m/s; this is the momentum of a mass of 1 kg when moving at 1 m/s.

The mathematical definition of momentum is consistent with our intuitive, everyday notion of "momentum." If two cars have equal masses but one has twice the velocity of the other, it has twice the momentum. And if a truck has three times the mass of a car and the same velocity, it has three times the momentum. During the nineteenth century physicists argued whether momentum or kinetic energy was the best measure of the "amount of motion" in a body. They finally decided that the answer

[1] The momentum $\mathbf{p} = m\mathbf{v}$ is sometimes referred to as *linear momentum* to distinguish it from *angular momentum,* discussed in Chapter 13.

depends on the context—as we will see in the examples in this chapter and the next, sometimes momentum is the most relevant quantity, sometimes energy is, and sometimes both are relevant.

Newton's First Law states that, in the absence of external forces, the velocity of a particle remains constant. Expressed in terms of momentum, *the First Law therefore states that the momentum remains constant:*

$$\mathbf{p} = [\text{constant}] \qquad \text{(no external forces)} \tag{10.2}$$

First Law in terms of momentum

Thus, we can say that the momentum of the particle is conserved. Of course, we could equally well say that the velocity of this particle is conserved; but the deeper significance of momentum will emerge when we study the motion of a system of several particles exerting forces on one another. We will find that the total momentum of such a system is conserved—any momentum lost by one particle is compensated by a momentum gain of some other particle or particles.

To express the Second Law in terms of momentum, we note that since the mass is constant, the time derivative of Eq. (10.1) is

$$\frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt}$$

or

$$\frac{d\mathbf{p}}{dt} = m\mathbf{a}$$

But, according to Newton's Second Law, $m\mathbf{a}$ equals the force; hence, *the rate of change of the momentum with respect to time equals the force:*

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{10.3}$$

Second Law in terms of momentum

This equation gives the Second Law a concise and elegant form.

EXAMPLE 1 A tennis player smashes a ball of mass 0.060 kg at a vertical wall. The ball hits the wall perpendicularly with a speed of 40 m/s and bounces straight back with the same speed. What is the change of momentum of the ball during the impact?

SOLUTION: Take the positive x axis along the direction of the initial motion of the ball (see Fig. 10.1a). The momentum of the ball before impact is then in the positive direction, and the x component of the momentum is

$$p_x = mv_x = 0.060 \text{ kg} \times 40 \text{ m/s} = 2.4 \text{ kg}\cdot\text{m/s}$$

The momentum of the ball after impact has the same magnitude but the opposite direction:

$$p'_x = -2.4 \text{ kg}\cdot\text{m/s}$$

(Throughout this chapter, the primes on mathematical quantities indicate that these quantities are evaluated *after* the collision.) The change of momentum is

$$\Delta p_x = p'_x - p_x = -2.4 \text{ kg}\cdot\text{m/s} - 2.4 \text{ kg}\cdot\text{m/s} = -4.8 \text{ kg}\cdot\text{m/s}$$

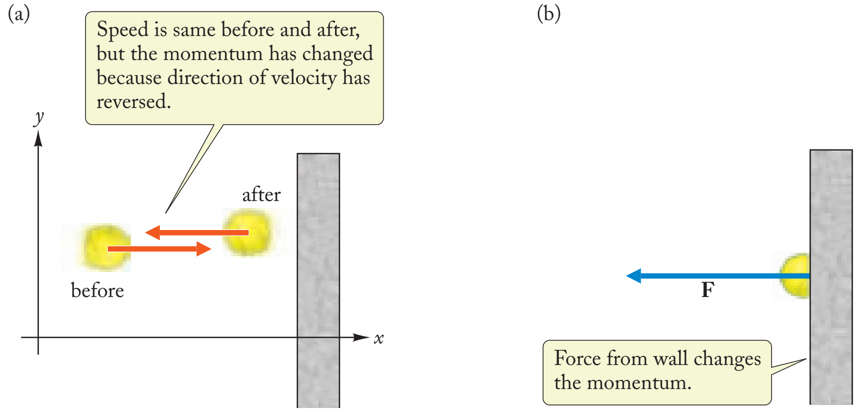


FIGURE 10.1 (a) A tennis ball bounces off a wall. (b) At the instant of impact, the wall exerts a large force on the ball.

This change of momentum is produced by the (large) force that acts on the ball during impact on the wall (see Fig. 10.1b). The change of momentum is negative because the force is negative (the force is in the negative x direction, opposite to the direction of the initial motion).

We can also express Newton's Third Law in terms of momentum. Since the action force is exactly opposite to the reaction force, the rate of change of momentum generated by the action force on one body is exactly opposite to the rate of change of momentum generated by the reaction force on the other body. Hence, we can state the Third Law as follows:

Third Law in terms of momentum

Whenever two bodies exert forces on each other, the resulting changes of momentum are of equal magnitudes and of opposite directions.

This balance in the changes of momentum leads us to a general law of conservation of the total momentum for a system of particles.

The total momentum of a system of n particles is simply the (vector) sum of all the individual momenta of all the particles. Thus, if $\mathbf{p}_1 = m_1\mathbf{v}_1$, $\mathbf{p}_2 = m_2\mathbf{v}_2$, . . . , and $\mathbf{p}_n = m_n\mathbf{v}_n$ are the individual momenta of the particles, then the total momentum is

momentum of a system of particles

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n \tag{10.4}$$

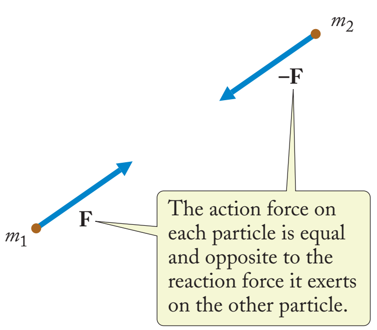


FIGURE 10.2 Two particles exerting mutual forces on each other. The net change of momentum of the isolated particle pair is zero.

The simplest of all many-particle systems consists of just two particles exerting some mutual forces on one another (see Fig. 10.2). Let us assume that the two particles are isolated from the rest of the Universe so that, apart from their mutual forces, they experience no extra forces of any kind. According to the above formulation of the Third Law, the rates of change of $\mathbf{p}_1$ and $\mathbf{p}_2$ are then exactly opposite:

$$\frac{d\mathbf{p}_1}{dt} = -\frac{d\mathbf{p}_2}{dt}$$

The rate of change of the sum $\mathbf{p}_1 + \mathbf{p}_2$ is therefore zero, since the rate of change of the first term in this sum is canceled by the rate of change of the second term:

$$\frac{d(\mathbf{p}_1 + \mathbf{p}_2)}{dt} = 0$$

This means that the sum $\mathbf{p}_1 + \mathbf{p}_2$ is a constant of the motion:

$$\mathbf{p}_1 + \mathbf{p}_2 = [\text{constant}] \qquad (10.5)$$

momentum conservation for two particles

This is the **Law of Conservation of Momentum.** Note that Newton's Third Law is an essential ingredient for establishing the conservation of momentum: the total momentum is constant because the equality of action and reaction keeps the momentum changes of the two particles exactly equal in magnitude but opposite in direction—the particles merely exchange some momentum by means of their mutual forces. Thus, for our particles, the total momentum $\mathbf{P}$ at some instant equals the total momentum $\mathbf{P}'$ at some other instant, so

$$\mathbf{P} = \mathbf{P}'$$

Conservation of momentum is a powerful tool which permits us to calculate some general features of the motion even when we are ignorant of the detailed properties of the interparticle forces. The following examples illustrate how we can use conservation of momentum to solve some problems of motion.

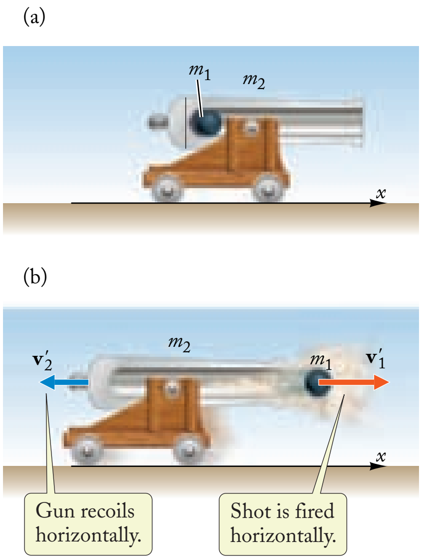


FIGURE 10.3 (a) Initially, the gun and the shot are at rest. (b) After the firing, the gun recoils toward the left (the velocity $\mathbf{v}_2'$ of the gun is negative).

EXAMPLE 2 A gun used onboard an eighteenth-century warship is mounted on a carriage which allows the gun to roll back each time it is fired (Fig. 10.3). The mass of the gun, including the carriage, is 2200 kg. The gun fires a 6.0-kg shot horizontally with a velocity of 500 m/s. What is the recoil velocity of the gun?

SOLUTION: The total momentum of the shot plus the gun must be the same before the firing and just after the firing. Before, the total momentum is zero (Fig. 10.3a):

$$\mathbf{P} = 0$$

After, the (horizontal) velocity of the shot is $\mathbf{v}_1'$, and the velocity of the gun is $\mathbf{v}_2'$ (as above, the primes on mathematical quantities indicate that these are evaluated *after* the firing); hence the total momentum is

$$\mathbf{P}' = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'$$

where $m_1 = 6.0$ kg is the mass of the shot and $m_2 = 2200$ kg is the mass of the gun (including the carriage). Thus, momentum conservation tells us

$$0 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'$$

or

$$\mathbf{v}_2' = -\frac{m_1}{m_2}\,\mathbf{v}_1'$$

The negative sign indicates that $\mathbf{v}_2'$, the recoil velocity of the gun, is opposite to the velocity of the shot and has a magnitude

$$v_2' = \frac{m_1}{m_2}v_1'$$

$$= \frac{6.0\ \text{kg}}{2200\ \text{kg}} \times 500\ \text{m/s} = 1.4\ \text{m/s}$$

COMMENTS: Note that the final velocities are in the inverse ratio of the masses: the shot emerges with a large velocity, and the gun rolls back with a low velocity.

This is a direct consequence of the equality of the magnitudes of the action and reaction forces that act on the shot and the gun during the firing. The force gives the shot (of small mass) a large acceleration, and the reaction force gives the gun (of large mass) a small acceleration.

In this calculation we neglected the mass and momentum of the gases released in the explosion of the gunpowder. This extra momentum increases the recoil velocity somewhat.

EXAMPLE 3 An automobile of mass 1500 kg traveling at 24 m/s crashes into a similar parked automobile. The two automobiles remain joined together after the collision. What is the velocity of the wreck immediately after the collision? Neglect friction against the road, since this force is insignificant compared with the large mutual forces that the automobiles exert on each other.

SOLUTION: Under the assumptions of the problem, the only horizontal forces are the mutual forces of one automobile on the other. Thus, momentum conservation applies to the horizontal component of the momentum: the value of this component must be the same before and after the collision. Before the collision, the (horizontal) velocity of the moving automobile is $v_1 = 24$ m/s and that of the other is $v_2 = 0$. With the x axis along the direction of motion (see Fig. 10.4), the total momentum is therefore

$$P_x = m_1 v_1 + m_2 v_2 = m_1 v_1$$

After the collision, both automobiles have the same velocity (see Fig. 10.4b). We will designate the velocities of the automobiles after the collision by v_1' and v_2', respectively. We can write $v_1' = v_2' = v'$ (the automobiles have a common v', since they remain joined), so the total momentum is

$$P_x' = m_1 v_1' + m_2 v_2' = (m_1 + m_2) v'$$

PROBLEM-SOLVING TECHNIQUES — CONSERVATION OF MOMENTUM

Note that the solution of these examples involves three steps similar to those we used in examples of energy conservation:

1. First write an expression for the total momentum **P** before the firing of the gun or the collision of the automobiles.
2. Then write an expression for the total momentum **P**′ after the firing or the collision.
3. And then use momentum conservation to equate these expressions.

However, in contrast to energy conservation, you must keep in mind that momentum conservation applies to the components of the momentum—*the x, y, and z components of the momentum are conserved separately.* Thus, before writing the expressions for the momentum, you need to select coordinate axes and decide which components of the momentum you want to examine. If the motion is one-dimensional, place one axis along the direction of motion, such as the x axis in the above examples. It then suffices to examine the x component of the momentum. However, sometimes it is necessary to examine two components of the momentum (or, rarely, three); then two (or three) equations result. When writing the components of the momentum, pay attention to the signs; the component is positive if the motion is along the direction of the axis, negative if the motion is opposite to the direction of the axis.

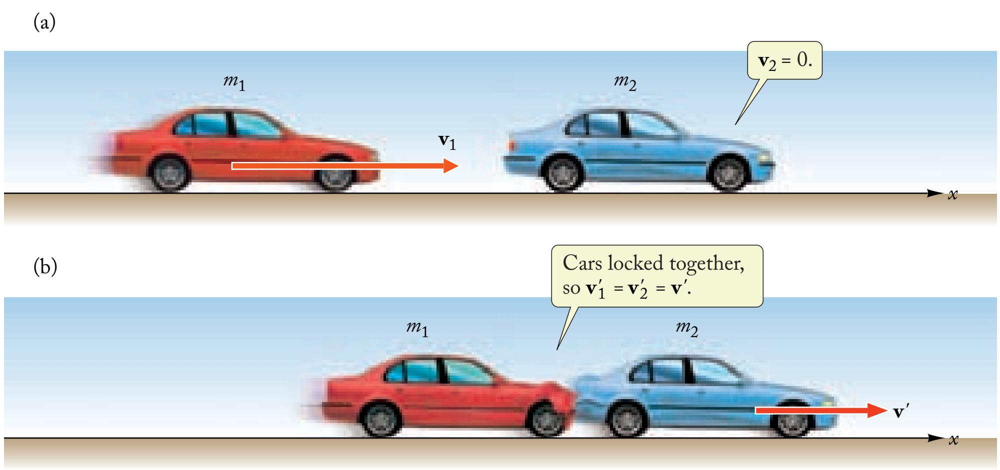


FIGURE 10.4 (a) Initially, the red automobile has a speed of 24 m/s, and the blue automobile is at rest. (b) After the collision, both automobiles are in motion with velocity $\mathbf{v}'$.

By momentum conservation, the momenta P_x and P'_x before and after the collision must be equal:

$$m_1 v_1 = (m_1 + m_2)v' \tag{10.6}$$

When we solve this for the velocity of the wreck v', we find

$$\begin{aligned} v' &= \frac{m_1 v_1}{m_1 + m_2} \\ &= \frac{1500\,\text{kg} \times 24\,\text{m/s}}{1500\,\text{kg} + 1500\,\text{kg}} = 12\,\text{m/s} \end{aligned} \tag{10.7}$$

The forces acting during the firing of the gun or the collision of the automobiles are quite complicated, but momentum conservation permits us to bypass these complications and directly obtain the answer for the final velocities. Incidentally: It is easy to check that kinetic energy is *not* conserved in these examples. During the firing of the gun, kinetic energy is supplied to the shot and the gun by the explosive combustion of the gunpowder, and during the collision of the automobiles, some kinetic energy is used up to produce changes in the shapes of the automobiles.

The conservation law for momentum depends on the absence of "extra" forces. If the particles are not isolated from the rest of the Universe, then besides the mutual forces exerted by one particle on the other, there are also forces exerted by other bodies not belonging to the particle system. The former forces are called **internal forces** of the system and the latter **external forces.** For instance, for the colliding automobiles of Example 3 the gravity of the Earth, the normal force of the road, and the friction of the road are external forces. In Example 3 we ignored these external forces, because gravity and the normal force cancel each other, and the friction force can be neglected in comparison with the much larger impact force that the automobiles exert on each other. But if the external forces are significant, we must take them into account, and we must modify Eq. (10.5). If the internal force on particle 1 is $\mathbf{F}_{1,\text{int}}$ and the external force is $\mathbf{F}_{1,\text{ext}}$, then the total force on particle 1 is $\mathbf{F}_{1,\text{int}} + \mathbf{F}_{1,\text{ext}}$ and its equation of motion will be

internal forces and external forces

$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{1,\text{int}} + \mathbf{F}_{1,\text{ext}} \tag{10.8}$$

Likewise

$$\frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{2,\text{int}} + \mathbf{F}_{2,\text{ext}} \tag{10.9}$$

If we add the left sides of these equations and the right sides, the contributions from the internal forces cancel (that is, $\mathbf{F}_{1,\text{int}} + \mathbf{F}_{2,\text{int}} = 0$), since they are action–reaction pairs. What remains is

$$\frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{1,\text{ext}} + \mathbf{F}_{2,\text{ext}} \tag{10.10}$$

The sum of the rates of change of the momenta is the same as the rate of change of the sum of the momenta; hence,

$$\frac{d(\mathbf{p}_1 + \mathbf{p}_2)}{dt} = \mathbf{F}_{1,\text{ext}} + \mathbf{F}_{2,\text{ext}} \tag{10.11}$$

The sum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ is the total momentum, and the sum $\mathbf{F}_{1,\text{ext}} + \mathbf{F}_{2,\text{ext}}$ is the total external force on the particle system. Thus, Eq. (10.11) states that the rate of change of the total momentum of the two-particle system equals the total *external* force.

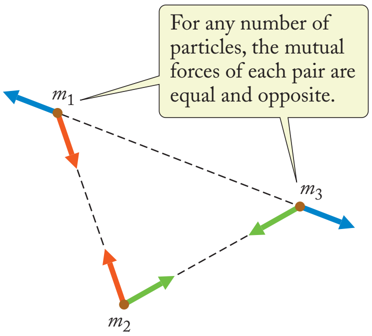


FIGURE 10.5 Three particles exerting forces on each other. As in the case of two particles, the mutual forces between pairs of particles merely exchange momentum between them.

For a system containing more than two particles, we can obtain similar results. If the system is isolated so that there are no external forces, then the mutual interparticle forces acting between pairs of particles merely transfer momentum from one particle of the pair to the other, just as in the case of two particles. Since all the internal forces necessarily arise from such forces between pairs of particles, these internal forces cannot change the total momentum. For example, Fig. 10.5 shows three isolated particles exerting forces on one another. Consider particle 1; the mutual forces between particles 1 and 2 exchange momentum between these two, while the mutual forces between particles 1 and 3 exhange momentum between those two. But none of these momentum transfers will change the total momentum. The same holds for particles 2 and 3. Consequently, the total momentum is constant. More generally, for an isolated system of n particles, the total momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n$ obeys the conservation law

momentum conservation for a system of particles

$$\mathbf{P} = [\text{constant}] \qquad \text{(no external forces)} \tag{10.12}$$

If, besides the internal forces, there are external forces, then the latter will change the momentum. The rate of change can be calculated in essentially the same way as for the two-particle system, and again, the rate of change of the total momentum is equal to the total external force. We can write this as

Second Law for a system of particles

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{\text{ext}} \tag{10.13}$$

where $\mathbf{F}_{\text{ext}} = \mathbf{F}_{1,\text{ext}} + \mathbf{F}_{2,\text{ext}} + \cdots + \mathbf{F}_{n,\text{ext}}$ is the total external force acting on the system.

Equations (10.12) and (10.13) have exactly the same mathematical form as Eqs. (10.2) and (10.3), and they may be regarded as the generalizations for a system of particles of Newton's First and Second Laws. As we will see in Section 10.3, Eq. (10.13) is an equation of motion for the system of particles—it determines the overall translational motion of the system.

✔ Checkup 10.1

QUESTION 1: An automobile and a truck have equal momenta. Which has the larger speed? Which has the larger kinetic energy?

QUESTION 2: An automobile and a truck are traveling along a street in opposite directions. Can they have the same momentum? The same kinetic energy?

QUESTION 3: A rubber ball, dropped on a concrete floor, bounces up with reversed velocity. Is the momentum before the impact the same as after the impact?

QUESTION 4: Is the net momentum of the Sun and all the planets and moons of the Solar System constant? Is the net kinetic energy constant?

QUESTION 5: Consider two automobiles of equal masses m and equal speeds v. (a) If both automobiles are moving southward on a street, what are the total kinetic energy and the total momentum of this system of two automobiles? (b) If one automobile is moving southward and one northward? (c) If one automobile is moving southward and one eastward?

QUESTION 6: An automobile and a truck have equal kinetic energies. Which has the larger speed? Which has the larger momentum? Assume that the truck has the larger mass.

(A) Truck; truck (B) Truck; automobile
(C) Automobile; truck (D) Automobile; automobile

10.2 CENTER OF MASS

In our study of kinematics and dynamics in the preceding chapters we always ignored the size of the bodies; even when analyzing the motion of a large body—an automobile or a ship—we pretended that the motion could be treated as particle motion, position being described by means of some reference point marked on the body. In reality, large bodies are systems of particles, and their motion obeys Eq. (10.13) for a system of particles. This equation can be converted into an equation of motion containing just one acceleration rather than the rate of change of momentum of the entire system, by taking as reference point the **center of mass** of the body. The equation that describes the motion of this special point has the same mathematical form as the equation of motion of a particle; that is, the motion of the center of mass mimics particle motion (see, for example, Fig. 10.6).

center of mass

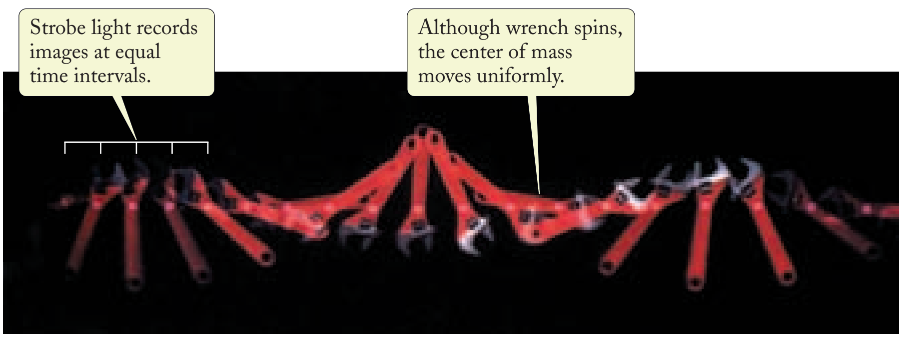


FIGURE 10.6 A wrench moving freely in the absence of external forces. The center of mass, marked with a dot, moves with uniform velocity, along a straight line (you can check this by laying a ruler along the dots).

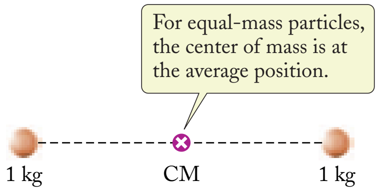


FIGURE 10.7 Two particles of equal masses, and their center of mass.

The position of the center of mass is merely the average position of the mass of the system. For instance, if the system consists of two particles, each of mass 1 kg, then the center of mass is halfway between them (see Fig. 10.7). In any system consisting of n particles of equal masses—such as a piece of pure metal with atoms of only one kind—the x coordinate of the center of mass is simply the sum of the x coordinates of all the particles divided by the number of particles,

$$x_{\text{CM}} = \frac{x_1 + x_2 + \cdots + x_n}{n} \qquad \text{(for equal-mass particles)} \qquad (10.14)$$

Similar equations apply to the y and the z coordinates, if the particles of the system are distributed over a three-dimensional region. The three coordinate equations can be expressed concisely in terms of position vectors:

$$\mathbf{r}_{\text{CM}} = \frac{\mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_n}{n} \qquad \text{(for equal-mass particles)} \qquad (10.15)$$

If the system consists of particles of unequal mass, then the position of the center of mass can be calculated by first subdividing the particles into fragments of equal mass. For instance, if the system consists of two particles, the first of mass 2 kg and the second of 1 kg, then we can pretend that we have *three* particles of equal masses 1 kg, two of which are located at the same position. The coordinate of the center of mass is then

$$x_{\text{CM}} = \frac{x_1 + x_1 + x_2}{3}$$

We can also write this in the equivalent form

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad (10.16)$$

where $m_1 = 2$ kg and $m_2 = 1$ kg. The formula (10.16) is actually valid for any values of the masses m_1 and m_2. The formula simply asserts that in the average position, the position of particle 1 is included m_1 times and the position of particle 2 is included m_2 times—that is, the number of times each particle is included in the average is directly proportional to its mass.

EXAMPLE 4 A 50-kg woman and an 80-kg man sit on the two ends of a seesaw of length 3.00 m (see Fig. 10.8). Treating them as particles, and ignoring the mass of the seesaw, find the center of mass of this system.

SOLUTION: In Fig. 10.8, the origin of coordinates is at the center of the seesaw; hence the woman has a negative x coordinate ($x = -1.50$ m) and the man a positive x coordinate ($x = +1.50$ m). According to Eq. (10.16), the coordinate of the center of mass is

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{50\,\text{kg} \times (-1.50\,\text{m}) + 80\,\text{kg} \times 1.50\,\text{m}}{50\,\text{kg} + 80\,\text{kg}}$$

$$= 0.35\,\text{m}$$

COMMENT: Note that the distance of the woman from the center of mass is 1.50 m + 0.35 m = 1.85 m, and the distance of the man from the center of mass is 1.50 m

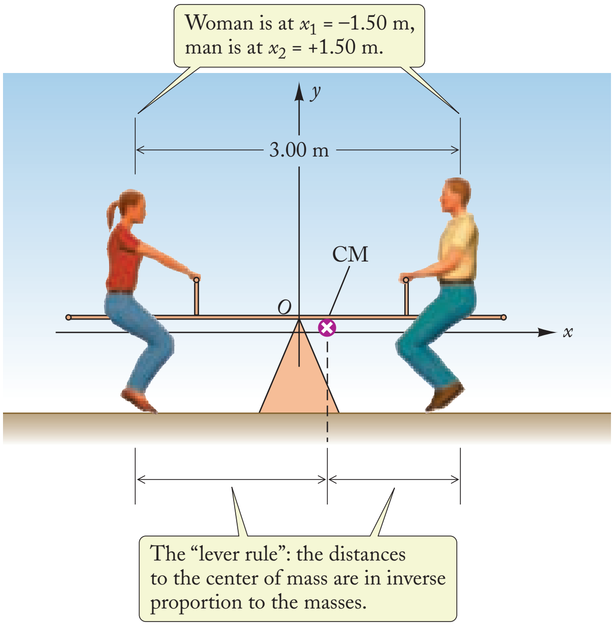


FIGURE 10.8 A woman and a man on a seesaw.

$-$ 0.35 m = 1.15 m. The ratio of these distances is 1.6, which coincides with the inverse of the ratio of the masses, 50/80 = 1/1.6. This "lever rule" is quite general: the position of the center of mass of two particles divides the line segment connecting them in the ratio $m_1 : m_2$, with the smaller length segment nearer to the larger mass.

If the system consists of n particles of different masses $m_1, m_2, \ldots, m_n$, then we apply the same prescription: the number of times each particle is included in the average is in direct proportion to its mass; the exact factor by which each particle's coordinate is multiplied is that particle's fraction of the total mass. This gives the following general expression for the coordinate of the center of mass:

$$x_{\mathrm{CM}} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} \tag{10.17}$$

or

$$x_{\mathrm{CM}} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{M} \tag{10.18}$$

where M is the total mass of the system, $M = m_1 + m_2 + \cdots + m_n$. Similar formulas apply to the y and the z coordinates, if the particles of the system are distributed over a three-dimensional region:

$$y_{\mathrm{CM}} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{M} \tag{10.19}$$

$$z_{\mathrm{CM}} = \frac{m_1 z_1 + m_2 z_2 + \cdots + m_n z_n}{M} \tag{10.20}$$

By introducing the standard notation Σ for a summation of n terms, we can express these formulas more concisely as

coordinates of center of mass

$$x_{\text{CM}} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \tag{10.21}$$

$$y_{\text{CM}} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i \tag{10.22}$$

$$z_{\text{CM}} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i \tag{10.23}$$

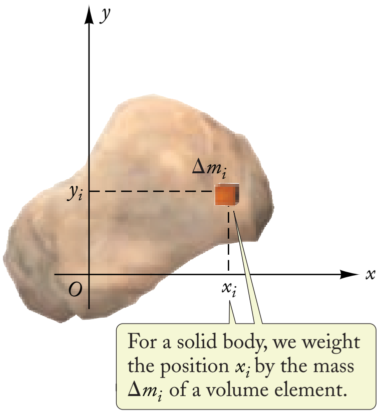


FIGURE 10.9 A small volume element of the body at position x_i has a mass Δm_i.

The position of the center of mass of a solid body can, in principle, be calculated from Eqs. (10.21)–(10.23), since a solid body is a collection of atoms, each of which can be regarded as a particle. However, it would be awkward to deal with the 10^{23} or so atoms that make up a chunk of matter the size of, say, a coin. It is more convenient to pretend that matter in bulk has a smooth and continuous distribution of mass over its entire volume. The mass in some small volume element at position x_i in the body is then Δm_i (see Fig. 10.9), and the x position of the center of mass is

$$x_{\text{CM}} = \frac{1}{M}\sum_{i=1}^{n} x_i \Delta m_i \tag{10.24}$$

In the limiting case of $\Delta m_i \to 0$ (and $n \to \infty$), this sum becomes an integral:

$$x_{\text{CM}} = \frac{1}{M}\int x\, dm \tag{10.25}$$

Similar expressions are valid for the y and z positions of the center of mass:

$$y_{\text{CM}} = \frac{1}{M}\int y\, dm \tag{10.26}$$

$$z_{\text{CM}} = \frac{1}{M}\int z\, dm \tag{10.27}$$

Thus, the position of the center of mass is the average position of all the mass elements making up the body.

For a body of uniform density, the amount of mass dm in any given volume element dV is directly proportional to the amount of volume. *For a uniform-density body, the position of the center of mass is simply the average position of all the volume elements of the body* (in mathematics, this is called the **centroid** of the volume). If the body has a symmetric shape, this average position will often be obvious by inspection. For instance, a sphere of uniform density, or a ring, or a circular plate, or a cylinder, or a parallelepiped will have its center of mass at the geometrical center (see Fig. 10.10). But for a less symmetric body, the center of mass must often be calculated, either by considering parts of the body (as in the next example) or by integrating over the entire body (as in the two subsequent examples).

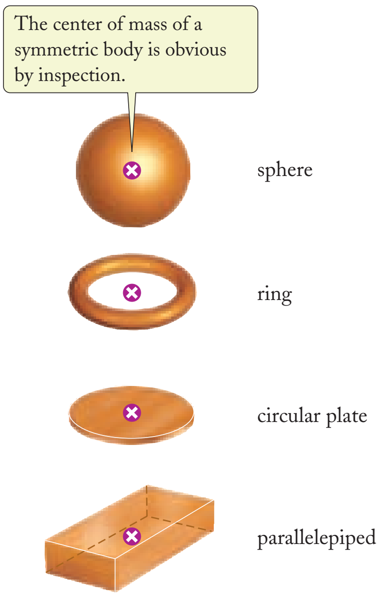


FIGURE 10.10 Several bodies for which the center of mass coincides with the geometrical center.

EXAMPLE 5 A meterstick of aluminum is bent at its midpoint so that the two halves are at right angles (see Fig. 10.11). Where is the center of mass of this bent stick?

SOLUTION: We can regard the bent stick as consisting of two straight pieces, each of 0.500 m. The centers of mass of these straight pieces are at their midpoints,

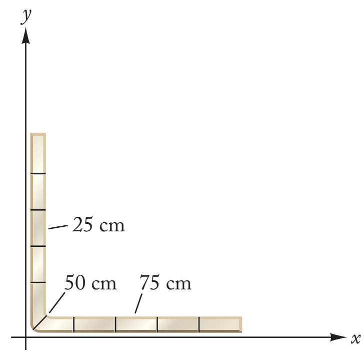


FIGURE 10.11 A meterstick, bent through 90° at its midpoint.

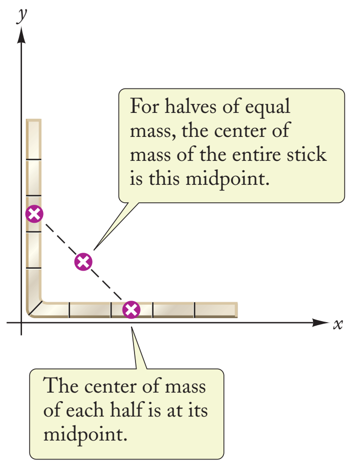


FIGURE 10.12 The center of mass of the bent meterstick is at the midpoint of the line connecting the centers of the halves. The coordinates x_{CM} and y_{CM} of this midpoint are one-half of the distances to the centers of mass of the horizontal and vertical sides—that is, 0.125 m each.

0.250 m from their ends (see Fig. 10.12). The center of mass of the entire stick is the average position of the centers of mass of the two halves. With the coordinate axes arranged as in Fig. 10.12, the x coordinate of the center of mass is, according to Eq. (10.14),

$$x_{CM} = \frac{0.250 \text{ m} + 0}{2} = 0.125 \text{ m} \tag{10.28}$$

Likewise, the y coordinate is

$$y_{CM} = \frac{0.250 \text{ m} + 0}{2} = 0.125 \text{ m}$$

Note that the center of mass of this bent stick is *outside* the stick; that is, it is not in the volume of the stick (see Fig. 10.12).

EXAMPLE 6 Figure 10.13 shows a mobile by Alexander Calder, which contains a uniform sheet of steel, in the shape of a triangle, suspended at its center of mass. Where is the center of mass of a right triangle of perpendicular sides a and b?

SOLUTION: Figure 10.14 shows the triangle positioned with a vertex at the origin and its right angle at a distance b along the x axis. To calculate the x coordinate of the center of mass, we need to sum mass contributions dm at each value of x; one such contribution is the vertical strip in Fig. 10.14, which has a height $y = (a/b)x$ and a width dx. Since the sheet is uniform, the strip has a fraction of the total mass M equal to the strip's area $y\,dx = (a/b)x\,dx$ divided by the total area $\frac{1}{2}ab$:

$$\frac{dm}{M} = \frac{(a/b)x\,dx}{\frac{1}{2}ab}$$

or

$$dm = M\frac{2x}{b^2}dx$$

FIGURE 10.13 This mobile by Alexander Calder contains a triangle suspended above its center of mass.

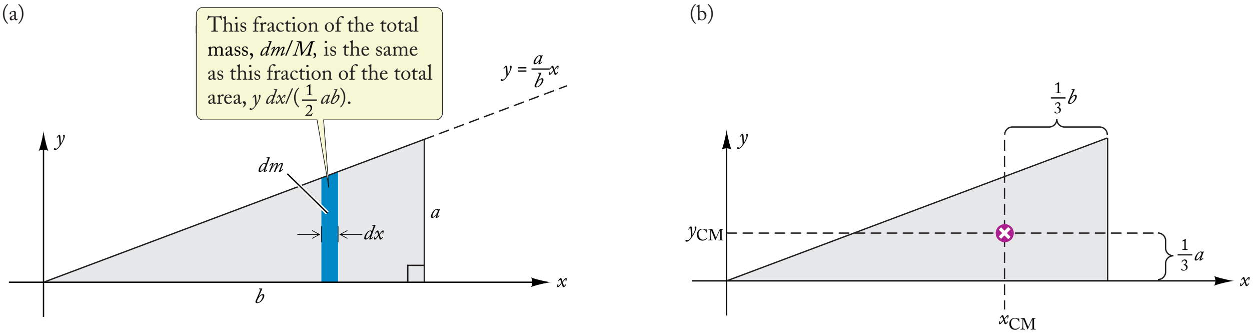


FIGURE 10.14 (a) A right triangle, with mass element dm of height y and width dx. (b) The center of mass is one-third of the distance from the right angle along sides a and b.

We integrate this in Eq. (10.25) for x_{CM} and sum the contributions from $x = 0$ to $x = b$:

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^b xM\frac{2x}{b^2}\,dx$$

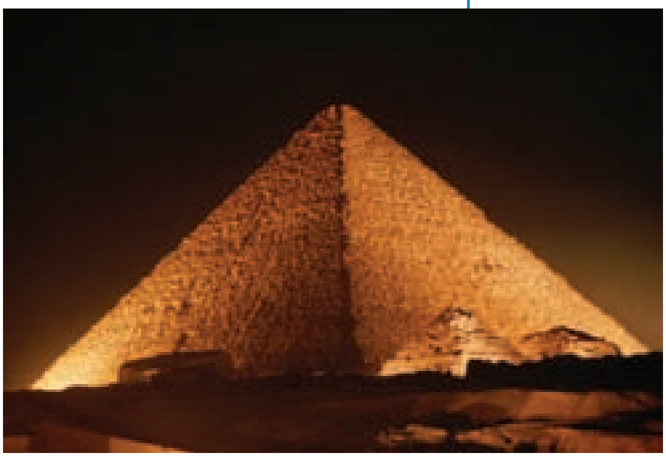
$$= \frac{2}{b^2}\int_0^b x^2\,dx = \frac{2}{b^2}\frac{1}{3}x^3\Big|_0^b$$

$$= \frac{2}{b^2}\frac{1}{3}(b^3 - 0) = \frac{2}{3}b$$

So the center of mass is two-thirds of the distance toward the right angle. Performing a similar calculation for y_{CM} yields $y_{CM} = \frac{1}{3}a$. Thus each of x_{CM} and y_{CM} is a distance *away* from the right angle equal to one-third of the length of the corresponding side (see Fig. 10.14b).

EXAMPLE 7 The Great Pyramid at Giza (see Fig. 10.15) has a height of 147 m and a square base. Assuming that the entire volume is completely filled with stone of uniform density, find its center of mass.

FIGURE 10.15 The Great Pyramid.



SOLUTION: Because of symmetry, the center of mass must be on the vertical line through the apex. For convenience, we place the y axis along this line, and we arrange this axis downward, with origin at the apex. We must then find where the center of mass is on this y axis. Figure 10.16a shows a cross section through the pyramid, looking parallel to two sides. The half-angle at the apex is ϕ. By examination of the colored triangle, we see that at a height y (measured from the apex) the half-width is $x = y\tan\phi$ and the full width is $2x = 2y\tan\phi$. A horizontal slice through the pyramid at this height is a square measuring $2x \times 2x$ (see Fig. 10.16b). The volume of a horizontal slab of thickness dy at this height y is therefore $dV = (2x)^2\,dy = (2y\tan\phi)^2\,dy$. If we represent the uniform density of the stone by ρ (the Greek letter *rho*), the proportionality between mass and volume can be written

$$dm = \rho dV$$

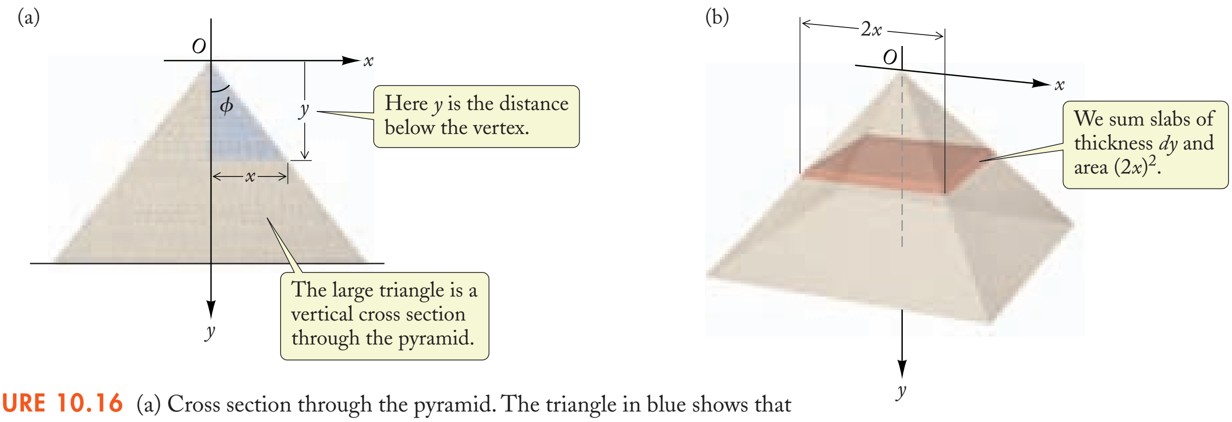


FIGURE 10.16 (a) Cross section through the pyramid. The triangle in blue shows that at a height y measured from the apex, the half-width of the pyramid is $x = y \tan \phi$. (b) The thin horizontal slab indicated in red is a square measuring $2x \times 2x$ with a thickness dy.

Thus the mass of the slab of thickness dy at this height y is

$$dm = \rho dV = \rho(2y \tan\phi)^2 dy = 4\rho(\tan^2\phi)y^2 dy$$

Equation (10.26) then gives us the y coordinate of the center of mass:

$$y_{CM} = \frac{1}{M}\int y\,dm = \frac{1}{M}\int 4\rho(\tan^2\phi)y^3 dy \qquad (10.29)$$

The total mass is

$$M = \int dm = \int 4\rho(\tan^2\phi)y^2 dy \qquad (10.30)$$

When we substitute Eq. (10.30) into Eq. (10.29), the common factor $4\rho \tan^2\phi$ cancels, leaving

$$y_{CM} = \frac{\int y^3 dy}{\int y^2 dy} \qquad (10.31)$$

As we sum the square slabs of thickness dy in both of these integrals, the integration runs from $y = 0$ at the top of the pyramid to $y = h$ at the bottom, where h is the height of the pyramid. Evaluation of these integrals yields

$$\int_0^h y^3 dy = \left.\frac{y^4}{4}\right|_0^h = \frac{h^4}{4}$$

$$\int_0^h y^2 dy = \left.\frac{y^3}{3}\right|_0^h = \frac{h^3}{3}$$

The y coordinate of the center of mass is therefore

$$y_{CM} = \frac{h^4/4}{h^3/3} = \frac{3}{4}h$$

This means that the center of mass is $3/4 \times 147$ m below the apex; that is, it is $1/4 \times 147$ m = 37 m above the ground.

PROBLEM-SOLVING TECHNIQUES — CENTER OF MASS

Calculations of the position of the center of mass of a body can often be simplified by exploiting the shape or the symmetry of the body.

- Sometimes it is profitable to treat the body as consisting of several parts and to begin by calculating the positions of the centers of mass of these parts (as in the example of the bent meterstick). Each part can then be treated as a particle located at its center of mass, and the center of mass of the entire body is then the center of mass of this system of particles, which can be calculated by the sums, Eqs. (10.18)–(10.20).
- If the body or some part of it has symmetry, the position of the center of mass will often be obvious by inspection. For instance, in the example of the bent meterstick, it is obvious that the center of mass of each half is at its center.
- Geometrical arguments can sometimes replace algebraic calculations of the coordinates of the center of mass. For instance, in the example of the bent meterstick, instead of the algebraic calculations of the coordinates [such as for x_{CM} in Eq. (10.28)], the coordinates can be obtained by regarding the stick as consisting of two straight pieces with known centers of mass; then the coordinates of the overall center of mass can be found from the geometry of a diagram, such as Fig. 10.12.

PHYSICS IN PRACTICE — CENTER OF MASS AND STABILITY

In the design of ships, engineers need to ensure that the position of the center of mass is low in the ship, to enhance the stability. If the center of mass is high, the ship is top-heavy and liable to tip over. Ships often carry ballast at the bottom of the hull to lower the center of mass. Many ships have been lost because of insufficient ballast or because of an unexpected shifting of the ballast. For instance, in 1628, the Swedish ship *Vasa* (see Fig. 1), the pride and joy of the Swedish navy and King Gustavus II Adolphus, capsized and sank on its maiden voyage when struck by a gust of wind, just barely out of harbor. It carried an excessive number of heavy guns on its upper decks, which made it top-heavy; and it should have carried more ballast to lower its center of mass.

The position of the center of mass is also crucial in the design of automobiles. A top-heavy automobile, such as an SUV, will tend to roll over when speeding around a sharp curve. High-performance automobiles, such as the Maserati shown in Fig. 2, have a very low profile, with the engine and transmission slung low in the body, so the center of mass is as low as possible and the automobile hugs the ground.

FIG. 1 The Swedish ship *Vasa*.

FIG. 2 A Maserati sports car.

The position of the center of mass enters into the calculation of the gravitational potential energy of an extended body located near the surface of the Earth. According to Eq. (7.29), the potential energy of a single particle of mass m at a height y above the ground is mgy. For a system of particles, the total gravitational potential energy is then

$$\begin{aligned} U &= m_1 g y_1 + m_2 g y_2 + \cdots + m_n g y_n \\ &= (m_1 y_1 + m_2 y_2 + \cdots + m_n y_n)\, g \end{aligned} \tag{10.32}$$

Comparison with Eq. (10.19) shows that the quantity in parentheses is My_{CM}. Hence, Eq. (10.32) becomes

$$U = Mgy_{CM} \tag{10.33}$$

potential energy in terms of height of center of mass

This expression for the gravitational potential energy of a system near the Earth's surface has the same mathematical form as for a single particle—it is as though the entire mass of the system were located at the center of mass.

Concepts in Context

For a human body standing upright, the position of the center of mass is in the middle of the trunk, at about the height of the navel. This is therefore the height to be used in the calculation of the gravitational potential energy of the body. However, if the body adopts any bent position, the center of mass shifts.

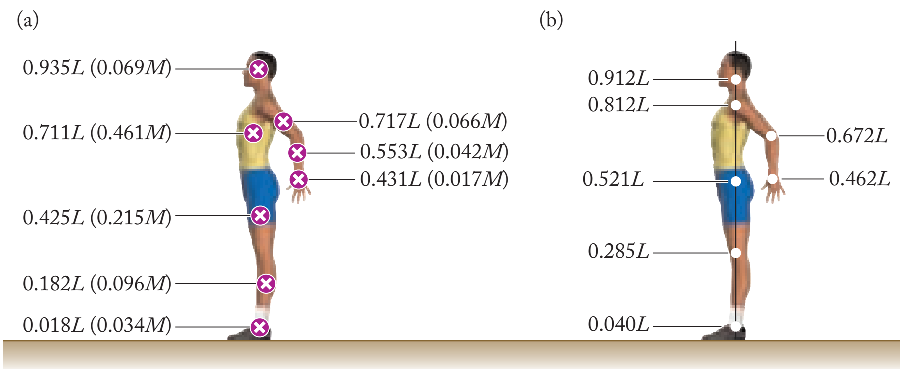


FIGURE 10.17 (a) Centers of mass of the body segments of an average male of mass M and height L standing upright. The numbers give the heights of the centers of mass of the body segments from the floor and (in parentheses) the masses of the body segments; right and left limbs are shown combined. (b) Hinge points of the body. The numbers give the heights of the joints from the floor.

Figure 10.17a gives the centers of mass of the body segments of a man of average proportions standing upright. Figure 10.17b shows the hinge points at which these body segments are joined. From the data in this figure, we can calculate the location of the center of mass when the body adopts any other position, and we can calculate the work done against gravity to change the position of any segment. For instance, if the body is bent in a tight backward arc, the center of mass shifts to a location just outside the body, about 10 cm below the middle of the trunk. Olympic jumpers (see Fig. 10.18) take advantage of this shift of the center of mass to make the most of the gravitational potential energy they can supply for a high jump. By adopting a bent position as they pass over the bar, they raise their trunk above the center of mass, so the trunk passes over the bar while the center of mass can pass *below* the bar. By this trick, the jumper raises the center of her trunk by about 10 cm relative to the center of mass, and she gains extra height without expending extra energy.

FIGURE 10.18 High jumper passing over the bar.

Concepts in Context

EXAMPLE 8 Suppose a man of average proportions performs a high jump, while arching his back (see the chapter opening photo). At the peak of his jump, his torso is approximately horizontal; his thighs, arms, and head make an angle of 45° with the horizontal; and his lower legs are vertical, as shown in Fig. 10.19b. (a) How much is his center of mass shifted downward compared with a man who goes over the pole horizontally (Fig. 10.19a)? (b) How much is his potential energy reduced? Assume the mass of the jumper is $M = 73$ kg and his height $L = 1.75$ m.

SOLUTION: (a) In Fig. 10.19a, the center of mass of the horizontal body is at $y = 0$, since each segment is essentially at $y = 0$. In Fig. 10.20, we have used the relative locations of the hinge points and centers of mass from Fig. 10.17 to determine the vertical position of each body segment in the arched-back position. For example, the center of mass of the thigh is at a distance $0.521L - 0.425L = 0.096L$ from the hip joint, and so is at a vertical distance $0.096L \times \sin 45° = 0.068L$ below $y = 0$. Similarly, we can determine that the centers of mass of the lower legs, the feet, the head, the upper arms, the forearms, and the hands are at $y = -0.270L$, $-0.434L$, $-0.016L$, $-0.067L$, $-0.183L$, and $-0.269L$, respectively. From Fig. 10.17, the masses of all seven segments are $0.215M$, $0.096M$, $0.034M$, $0.069M$, $0.066M$, $0.042M$, and $0.017M$, respectively. The torso, of mass $0.461M$, is again at $y = 0$. Thus, using Eq. (10.19) or (10.22), the arched-back center of mass is at

$$\begin{aligned} y_{CM} &= \frac{1}{M}\sum_{i=1}^{n} m_i y_i \\ &= -\frac{1}{M}(0.215 \times 0.068 + 0.096 \times 0.270 + 0.034 \times 0.434 + 0.069 \\ &\quad \times 0.016 + 0.066 \times 0.067 + 0.042 \times 0.183 + 0.017 \times 0.269 \\ &\quad + 0 \times 0.461)ML \\ &= -0.073L = -0.073 \times 1.75 \text{ m} = -0.13 \text{ m} \end{aligned}$$

Thus a height advantage of 13 cm is gained in this arched position.

(b) According to Eq. (10.33), the potential energy is changed by

$$\begin{aligned} \Delta U &= Mg\,\Delta y_{CM} \\ &= 73 \text{ kg} \times 9.81 \text{ m/s}^2 \times (-0.13 \text{ m}) \qquad (10.34) \\ &= -93 \text{ J} \end{aligned}$$

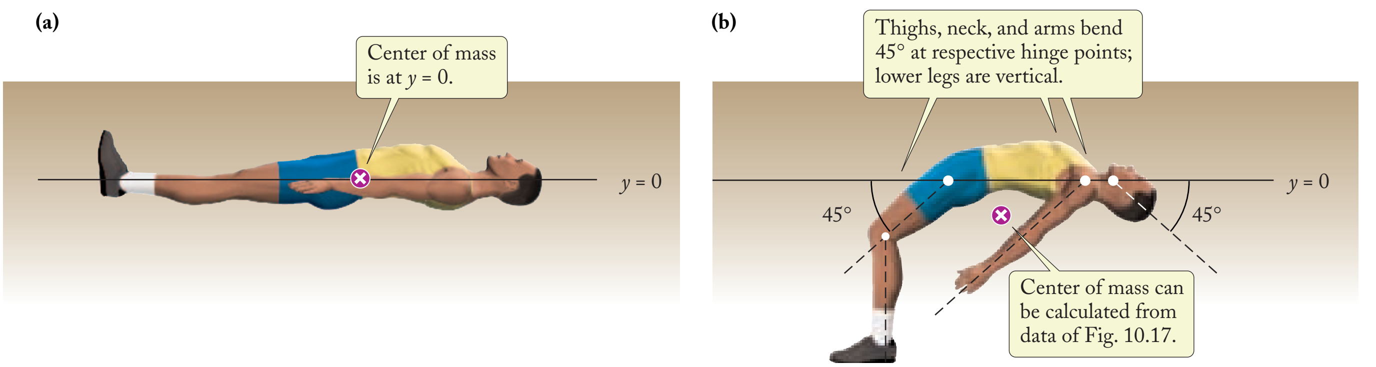


FIGURE 10.19 (a) Horizontal position. (b) High jumper in arched-back position.

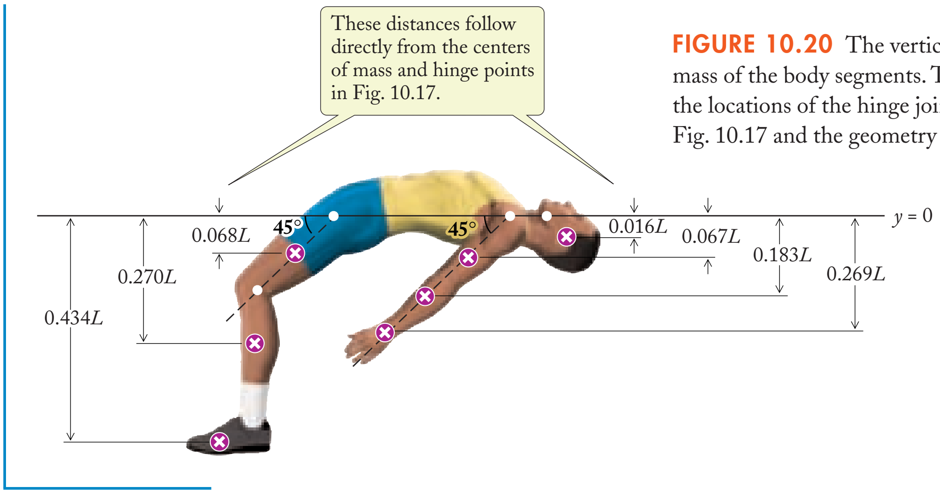


FIGURE 10.20 The vertical positions of the centers of mass of the body segments. These are determined from the locations of the hinge joints and centers of mass in Fig. 10.17 and the geometry of the arched-back position.

Checkup 10.2

QUESTION 1: Roughly where is the center of mass of the snake shown in Fig. 10.21a?

QUESTION 2: Roughly where is the center of mass of the horseshoe shown in Fig. 10.21b?

QUESTION 3: Is it possible for the center of mass of a body to be above the highest part of the body?

QUESTION 4: A sailboat has a keel with a heavy lead bulb at the bottom. If the bulb falls off, the center of mass of the sailboat:

(A) Remains at the same position (B) Shifts downward
(C) Shifts upward

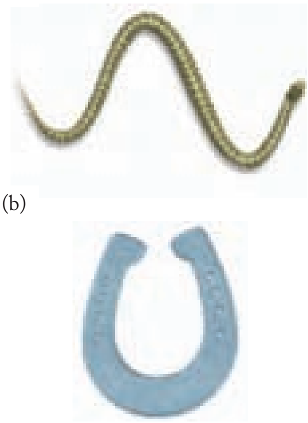
(a)

(b)

FIGURE 10.21 (a) A snake. (b) A horseshoe.

10.3 THE MOTION OF THE CENTER OF MASS

When the particles in a system move, often so does the center of mass. We will now obtain an equation for the motion of the center of mass, an equation which relates the acceleration of the center of mass to the external force. This equation will permit us to calculate the overall translational motion of a system of particles.

According to Eq. (10.18), if the x components of positions of the respective particles change by $dx_1, dx_2, \ldots, dx_n$, then the position of the center of mass changes by

$$dx_{\text{CM}} = \frac{1}{M}(m_1 dx_1 + m_2 dx_2 + \cdots + m_n dx_n) \tag{10.35}$$

Dividing this by the time dt taken for these changes of position, we obtain

$$\frac{dx_{\text{CM}}}{dt} = \frac{1}{M}\left(m_1 \frac{dx_1}{dt} + m_2 \frac{dx_2}{dt} + \cdots + m_n \frac{dx_n}{dt}\right) \tag{10.36}$$

The left side of this equation is the x component of the velocity of the center of mass, and the rates of change on the right side are the x components of the velocities of the individual particles; thus

$$v_{x,\mathrm{CM}} = \frac{m_1 v_{x,1} + m_2 v_{x,2} + \cdots + m_n v_{x,n}}{M}$$

Note that this equation has the same mathematical form as Eq. (10.18); that is, the velocity of the center of mass is an average over the particle velocities, and the number of times each particle velocity is included is directly proportional to its mass.

Since similar equations apply to the y and z components of the velocity, we can write a vector equation for the velocity of the center of mass:

velocity of the center of mass

$$\mathbf{v}_{\mathrm{CM}} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 + \cdots + m_n \mathbf{v}_n}{M} \tag{10.37}$$

The quantity in the numerator is simply the total momentum [compare Eq. (10.1)]; hence Eq. (10.37) says

$$\mathbf{v}_{\mathrm{CM}} = \frac{\mathbf{P}}{M} \tag{10.38}$$

or

momentum in terms of velocity of CM

$$\mathbf{P} = M\mathbf{v}_{\mathrm{CM}} \tag{10.39}$$

This equation expresses the total momentum of a system of particles as the product of the total mass and the velocity of the center of mass. Obviously, this equation is analogous to the familiar equation $\mathbf{p} = m\mathbf{v}$ for the momentum of a single particle.

We know, from Eq. (10.13), that the rate of change of the total momentum equals the net external force on the system,

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{\mathrm{ext}}$$

If we substitute $\mathbf{P} = M\mathbf{v}_{\mathrm{CM}}$ and take into account that the mass is constant, we find

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt}(M\mathbf{v}_{\mathrm{CM}}) = M\frac{d\mathbf{v}_{\mathrm{CM}}}{dt} = M\mathbf{a}_{\mathrm{CM}}$$

and consequently

motion of center of mass

$$M\mathbf{a}_{\mathrm{CM}} = \mathbf{F}_{\mathrm{ext}} \tag{10.40}$$

This equation for a system of particles is the analog of Newton's equation for motion for a single particle. The equation asserts that the center of mass moves as though it were a particle of mass M under the influence of a force $\mathbf{F}_{\mathrm{ext}}$.

Concepts in Context

This result justifies some of the approximations we made in previous chapters. For instance, in Example 9 of Chapter 2 we treated a diver falling from a cliff as a particle. Equation (10.40) shows that this treatment is legitimate: the center of mass of the diver, under the influence of the external force (gravity), moves with a downward acceleration g, just as though it were a freely falling particle. Likewise, after a high jumper leaves the ground, his center of mass moves along a parabolic trajectory, as though it were a projectile, and the shape and height of this parabolic trajectory is unaffected by any contortions the high jumper might perform while in flight. From Chapter 4, we know that the initial vertical velocity v_y determines the maximum height h of the center of mass; that is, $v_y = \sqrt{2gh}$. The contortions of the jumper enable his body to pass over a bar roughly 10 cm above the maximum height of the center of mass.

If the net external force vanishes, then the acceleration of the center of mass vanishes; hence the center of mass remains at rest or it moves with uniform velocity.

FIGURE 10.22 Astronaut on a "space walk" during the *Gemini 4* mission.

EXAMPLE 9 During a "space walk," an astronaut floats in space 8.0 m from his spacecraft orbiting the Earth. He is tethered to the spacecraft by a long umbilical cord (see Fig. 10.22); to return, he pulls himself in by this cord. How far does the spacecraft move toward him? The mass of the spacecraft is 3500 kg, and the mass of the astronaut, including his space suit, is 110 kg.

SOLUTION: In the reference frame of the orbiting (freely falling) astronaut and spacecraft, each is effectively weightless; that is, the external force on the system is effectively zero. The only forces in the system are the forces exerted when the astronaut pulls on the cord; these forces are internal. The forces exerted by the cord on the spacecraft and on the astronaut during the pulling in are of equal magnitudes and opposite directions; the astronaut is pulled toward the spacecraft, and the spacecraft is pulled toward the astronaut. In the absence of external forces, the center of mass of the astronaut–spacecraft system remains at rest. Thus, the spacecraft and the astronaut both move toward the center of mass, and there they meet.

With the x axis as in Fig. 10.23, the x coordinate of the center of mass is

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{10.41}$$

where $m_1 = 3500$ kg is the mass of the spacecraft and $m_2 = 110$ kg is the mass of the astronaut. Strictly, the coordinates x_1 and x_2 of the spacecraft and of the astronaut should correspond to the centers of mass of these bodies, but, for the sake of simplicity, we neglect their size and treat both as particles. The initial values of the coordinates are $x_1 = 0$ and $x_2 = 8.0$ m; hence

$$x_{CM} = \frac{0 + 110\text{ kg} \times 8.0\text{ m}}{3500\text{ kg} + 110\text{ kg}} = 0.24\text{ m}$$

During the pulling in, the spacecraft will move from $x_1 = 0$ to $x_1 = 0.24$ m; simultaneously, the astronaut will move from $x_2 = 8.0$ m to $x_2 = 0.24$ m.

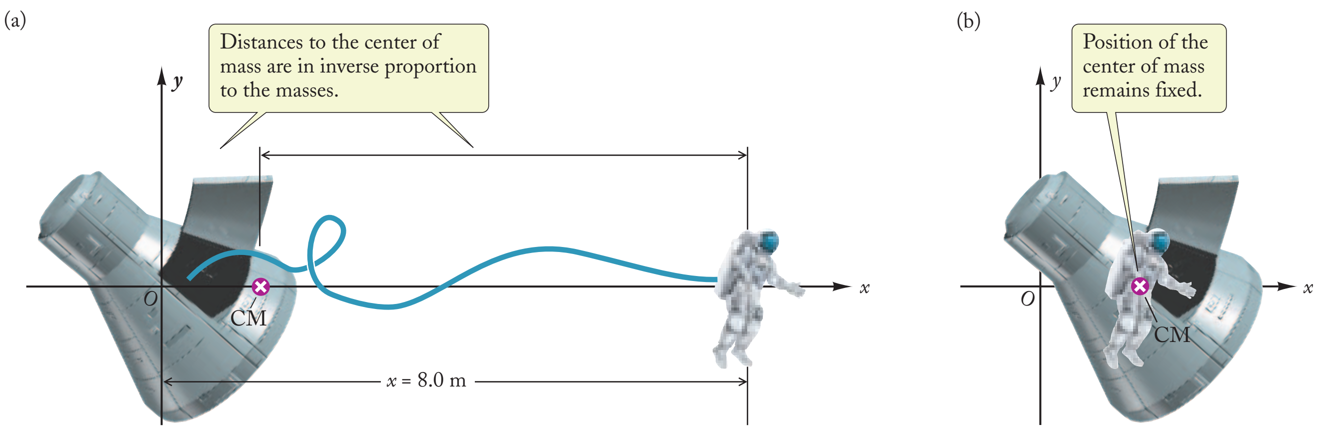


FIGURE 10.23 (a) Initial position of the astronaut and the spacecraft. The center of mass is between them. (b) Final position of the astronaut and the spacecraft. They are both at the center of mass.

COMMENT: The distances moved by the astronaut and by the spacecraft are in the inverse ratio of their masses. The astronaut (of small mass) moves a large distance, and the spacecraft (of large mass) moves a smaller distance. This is the result of the accelerations that the pull of the cord gives to these bodies: with forces of equal magnitudes, the accelerations of the astronaut and spacecraft are in the inverse ratio of their masses. However, our method of calculation based on the fixed position of the center of mass gives us the final positions directly, without any need to examine accelerations.

EXAMPLE 10 A projectile is launched at some angle θ with respect to the horizontal, $0° < \theta < 90°$. Just as it reaches its peak, it explodes into two pieces. The explosion causes a first, rear piece to come to a momentary stop, and it simply drops, striking the ground directly below the peak position. The explosion also causes the speed of the second piece to increase, and it hits the ground a distance five times further from the launch point than the first piece (see Fig. 10.24). If the original projectile had a mass of 12.0 kg, what are the masses of the pieces?

SOLUTION: Because the explosion does not produce external forces, the center of mass continues on its original path, a parabolic trajectory which strikes the ground at the range x_{max}, given by Eq. (4.43). The peak of the parabolic trajectory occurs at half this distance; thus the first piece, of some mass m_1, hits the ground a distance $\frac{1}{2}x_{max}$ from the launch point. We are also told that the second piece, of mass m_2, hits the ground a distance $5 \times \frac{1}{2}x_{max}$ from the launch point. The two pieces will reach the ground at the same instant, since this explosion affected only each piece's horizontal momentum. If we take our origin at the launch point, the x component of the center of mass is thus

$$x_{CM} = x_{max} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_{max}/2 + 5m_2 x_{max}/2}{m_1 + m_2}$$

We can divide both sides of this equation by x_{max} and rearrange to obtain

$$m_1 = 3m_2$$

Since we know the total mass is $m_1 + m_2 = 12.0$ kg, or $4m_2 = 12.0$ kg, we obtain

$$m_1 = 9.0 \text{ kg} \qquad \text{and} \qquad m_2 = 3.0 \text{ kg}$$

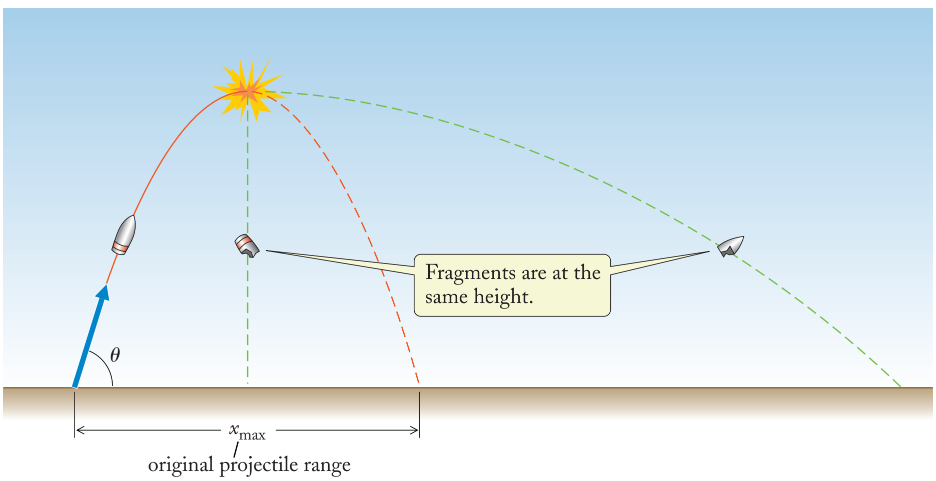


FIGURE 10.24 A projectile explodes at its apex. The rear fragment simply drops, and the forward piece lands five times further from the launch point.

COMMENT: Note that to relate both points of impact to the center of mass, we had to know that the impacts occurred at the *same instant*; we must always use the coordinates of a system of particles at a particular instant when calculating the center of mass.

Checkup 10.3

QUESTION 1: When you crawl from the rear end of a canoe to the front end, the boat moves backward relative to the water. Explain.

QUESTION 2: You are locked inside a boxcar placed on frictionless wheels on railroad tracks. If you walk from the rear end of the boxcar to the front end, the boxcar rolls backward. Is it possible for you to make the boxcar roll a distance longer than its length?

QUESTION 3: You drop a handful of marbles on a smooth floor, and they bang into each other and roll away in all directions. What can you say about the motion of the center of mass of the marbles after the impact on the floor?

QUESTION 4: An automobile is traveling north at 25 m/s. A truck with twice the mass of the automobile is heading south at 20 m/s. What is the velocity of the center of mass of the two vehicles?

(A) 0 (B) 5 m/s south (C) 5 m/s north
(D) 10 m/s south (E) 10 m/s north

10.4 ENERGY OF A SYSTEM OF PARTICLES

The total kinetic energy of a system of particles is simply the sum of the individual kinetic energies of all the particles,

$$K = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \cdots + \tfrac{1}{2}m_nv_n^2 \qquad (10.42)$$

kinetic energy of a system of particles

Since Eq. (10.39) for the momentum of a system of particles resembles the expression for the momentum of a single particle, we might be tempted to guess that the equation for the kinetic energy for a system of particles also can be expressed in the form of the translational kinetic energy of the center of mass $\frac{1}{2}Mv_{\mathrm{CM}}^2$, resembling the kinetic energy of a single particle. But this is wrong! The total kinetic energy of a system of particles is usually larger than $\frac{1}{2}Mv_{\mathrm{CM}}^2$. We can see this in the following simple example: Consider two automobiles of equal masses moving toward each other at equal speeds. The velocity of the center of mass is then zero, and consequently $\frac{1}{2}Mv_{\mathrm{CM}}^2 = 0$. However, since each automobile has a positive kinetic energy, the total kinetic energy is *not* zero.

If the internal and external forces acting on a system of particles are conservative, then the system will have a potential energy. We saw above that for the specific example of the gravitational potential energy near the Earth's surface, the potential energy of the system took the same form as for a single particle, $U = Mgy_{\mathrm{CM}}$ [see Eq. (10.33)]. But this form is a result of the particular force (uniform and proportional to mass); in general, the potential energy for a system does not have the same form as for a single particle. Unless we specify all of the forces, we cannot write down an explicit formula for the potential energy; but in any case, this potential energy will be some function of the positions of all the particles. The total mechanical energy is the sum of the total kinetic energy [Eq. (10.42)] and the total potential energy. This total energy will be conserved during the motion of the system of particles. Note that in reckoning the total potential energy of the system, we must include the potential energy of both the external forces and the internal forces. We know that the internal forces do not contribute to the changes of total momentum of the system, but these internal forces, and their potential energies, contribute to the total energy. For instance, if two particles are falling toward each other under the influence of their mutual gravitational attraction, the momentum gained by one particle is balanced by momentum lost by the other, but the kinetic energy gained by one particle is *not* balanced by kinetic energy lost by the other—both particles gain kinetic energy. In this example the gravitational attraction plays the role of an internal force in the system, and the gain of kinetic energy is due to a loss of mutual gravitational potential energy.

Checkup 10.4

QUESTION 1: Consider a system consisting of two automobiles of equal mass. Initially, the automobiles have velocities of equal magnitudes in opposite directions. Suppose the automobiles collide head-on. Is the kinetic energy conserved?

QUESTION 2: The Solar System consists of the Sun, nine planets, and their moons. Is the total energy of this system conserved? Is the kinetic energy conserved? Is the potential energy conserved?

QUESTION 3: Two equal masses on a frictionless horizontal surface are connected by a spring. Each is given a brief push in a different direction. During the subsequent motion, which of the following remain(s) constant? (**P** = total momentum; K = total kinetic energy; U = total potential energy.)

(A) **P** only (B) **P** and K (C) **P** and U
(D) K and U (E) **P**, K, and U

SUMMARY

PROBLEM-SOLVING TECHNIQUES Conservation of Momentum **(page 310)**

PROBLEM-SOLVING TECHNIQUES Center of Mass **(page 320)**

PHYSICS IN PRACTICE Center of Mass and Stability **(page 320)**

MOMENTUM OF A PARTICLE

$$\mathbf{p} = m\mathbf{v} \tag{10.1}$$

MOMENTUM OF A SYSTEM OF PARTICLES

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n \tag{10.4}$$

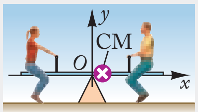
RATE OF CHANGE OF MOMENTUM

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{\text{ext}} \tag{10.13}$$

CONSERVATION OF MOMENTUM (in the absence of external forces)

$$\mathbf{P} = [\text{constant}] \tag{10.12}$$

CENTER OF MASS
(Using $M = m_1 + m_2 + \cdots + m_n$)



$$x_{CM} = \frac{1}{M}(m_1x_1 + m_2x_2 + \cdots + m_nx_n) \tag{10.18}$$

$$y_{CM} = \frac{1}{M}(m_1y_1 + m_2y_2 + \cdots + m_ny_n) \tag{10.19}$$

$$z_{CM} = \frac{1}{M}(m_1z_1 + m_2z_2 + \cdots + m_nz_n) \tag{10.20}$$

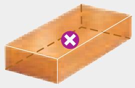
CENTER OF MASS OF CONTINUOUS DISTRIBUTION OF MASS

where $dm = \rho dV$ (ρ is density and dV is a volume element).


sphere


circular plate


ring


parallelepiped

$$x_{CM} = \frac{1}{M}\int x\,dm \tag{10.25}$$

$$y_{CM} = \frac{1}{M}\int y\,dm \tag{10.26}$$

$$z_{CM} = \frac{1}{M}\int z\,dm \tag{10.27}$$

VELOCITY OF THE CENTER OF MASS

$$\mathbf{v}_{CM} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \cdots + m_n\mathbf{v}_n}{M} \tag{10.37}$$

MOMENTUM OF A SYSTEM OF PARTICLES

$$\mathbf{P} = M\mathbf{v}_{CM} \tag{10.39}$$

MOTION OF THE CENTER OF MASS

$$M\mathbf{a}_{CM} = \mathbf{F}_{ext} \tag{10.40}$$

GRAVITATIONAL POTENTIAL ENERGY OF A SYSTEM OF PARTICLES (near the Earth's surface)

$$U = Mgy_{CM} \tag{10.33}$$

KINETIC ENERGY OF A SYSTEM OF PARTICLES

$$K = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \cdots + \tfrac{1}{2}m_nv_n^2 \tag{10.42}$$

QUESTIONS FOR DISCUSSION

1. When the nozzle of a fire hose discharges a large amount of water at high speed, several strong firefighters are needed to hold the nozzle steady. Explain.
2. When firing a shotgun, a hunter always presses it tightly against his shoulder. Why?
3. As described in Example 2, guns onboard eighteenth-century warships were often mounted on carriages (see Fig. 10.3). What was the advantage of this arrangement?
4. Hollywood movies often show a man being knocked over by the impact of a bullet while the man who shot the bullet remains standing, quite undisturbed. Is this reasonable?
5. Where is the center of mass of this book when it is closed? Mark the center of mass with a cross.
6. Roughly, where is the center of mass of this book when it is open, as it is at this moment?

7. A fountain shoots a stream of water up into the air (Fig. 10.25). Roughly, where is the center of mass of the water that is in the air at one instant? Is the center of mass higher or lower than the middle height?

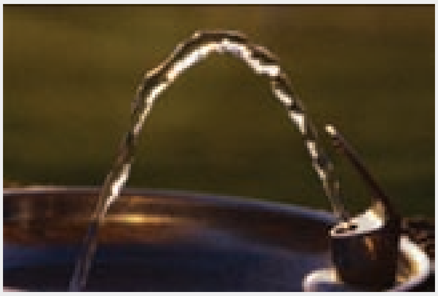


FIGURE 10.25 Stream of water from a fountain.

8. Consider the moving wrench shown in Fig. 10.6. If the center of mass on this wrench had not been marked, how could you have found it by inspection of this photograph?
9. Is it possible to propel a sailboat by mounting a fan on the deck and blowing air on the sail? Is it better to mount the fan on the stern and blow air toward the rear?
10. Cyrano de Bergerac's sixth method for propelling himself to the Moon was as follows: "Seated on an iron plate, to hurl a magnet in the air—the iron follows—I catch the magnet—throw again—and so proceed indefinitely." What is wrong with this method (other than the magnet's insufficient pull)?
11. Within the Mexican jumping bean, a small insect larva jumps up and down. How does this lift the bean off the table?
12. Answer the following question, sent by a reader to the *New York Times:*

> A state trooper pulls a truck driver into the weigh station to see if he's overloaded. As the vehicle rolls onto the scales, the driver jumps out and starts beating on the truck box with a club. A bystander asks what he's doing. The trucker says: "I've got five tons of canaries in here. I know I'm overloaded. But if I can keep them flying I'll be OK." If the canaries are flying in that enclosed box, will the truck really weigh any less than if they're on the perch?

13. An elephant jumps off a cliff. Does the Earth move upward while the elephant falls?
14. A juggler stands on a balance, juggling five balls (Fig. 10.26). On the average, will the balance register the weight of the juggler plus the weight of the five balls? More than that? Less?

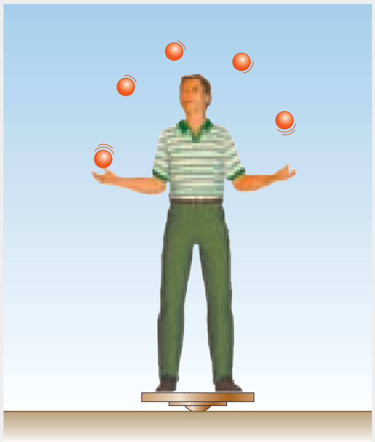


FIGURE 10.26 Juggler on a balance.

15. Suppose you fill a rubber balloon with air and then release it so that the air spurts out of the nozzle. The balloon will fly across the room. Explain.
16. The combustion chamber of a rocket engine is closed at the front and at the sides, but it is open at the rear (Fig. 10.27). Explain how the pressure of the gas on the walls of this combustion chamber gives a net forward force that propels the rocket.

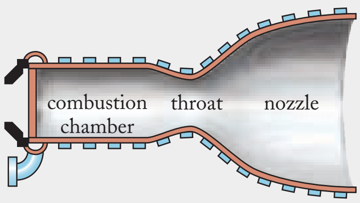


FIGURE 10.27 Combustion chamber of a rocket engine.

PROBLEMS

10.1 Momentum

1. What is the momentum of a rifle bullet of mass 15 g and speed 600 m/s? An arrow of mass 40 g and speed 80 m/s?
2. What is the momentum of an automobile of mass 900 kg moving at 65 km/h? If a truck of mass 7200 kg is to have the same momentum as the automobile, what must be its speed?
3. Using the entries listed in Tables 1.7 and 2.1, find the magnitude of the momentum for each of the following: Earth moving around the Sun, jet airliner at maximum airspeed, automobile at 55 mi/h, man walking, electron moving around a nucleus.
4. The push that a bullet exerts during impact on a target depends on the momentum of the bullet. A Remington .244

rifle, used for hunting deer, fires a bullet of 90 grains (1 grain is $\frac{1}{7000}$ lb) with a speed of 975 m/s. A Remington .35 rifle fires a bullet of 200 grains with a speed of 674 m/s. What is the momentum of each bullet?

5. An electron, of mass 9.1×10^{-31} kg, is moving in the x–y plane; its speed is 2.0×10^5 m/s, and its direction of motion makes an angle of 25° with the x axis. What are the components of the momentum of the electron?

6. A skydiver of mass 75 kg is in free fall. What is the rate of change of his momentum? Ignore friction.

7. A soccer player kicks a ball and sends it flying with an initial speed of 26 m/s at an upward angle of 30°. The mass of the ball is 0.43 kg. Ignore friction.

   (a) What is the initial momentum of the ball?

   (b) What is the momentum when the ball reaches maximum height on its trajectory?

   (c) What is the momentum when the ball returns to the ground? Is this final momentum the same as the initial momentum?

8. The Earth moves around the Sun in a circle of radius 1.5×10^{11} m at a speed of 3.0×10^4 m/s. The mass of the Earth is 6.0×10^{24} kg. Calculate the magnitude of the rate of change of the momentum of the Earth from these data. (Hint: The magnitude of the momentum does not change, but the direction does.)

9. A 1.0-kg mass is released from rest and falls freely. How much momentum does it acquire after one second? After ten seconds?

10. A 55-kg woman in a 20-kg rowboat throws a 3.0-kg life preserver with a horizontal velocity of 5.0 m/s. What is the recoil velocity of the woman and rowboat?

11. A 90-kg man dives from a 20-kg boat with an initial horizontal velocity of 2.0 m/s (relative to the water). What is the initial recoil velocity of the boat? (Neglect water friction.)

12. A hydrogen atom (mass 1.67×10^{-27} kg) at rest can emit a photon (a particle of light) with maximum momentum 7.25×10^{-27} kg·m/s. What is the maximum recoil velocity of the hydrogen atom?

13. Calculate the change of the kinetic energy in the collision between the two automobiles described in Example 3.

14. A rifle of 10 kg lying on a smooth table discharges accidentally and fires a bullet of mass 15 g with a muzzle speed of 650 m/s. What is the recoil velocity of the rifle? What is the kinetic energy of the bullet, and what is the recoil kinetic energy of the rifle?

15. A typical warship built around 1800 (such as the USS *Constitution*) carried 15 long guns on each side. The guns fired a shot of 11 kg with a muzzle speed of about 490 m/s. The mass of the ship was about 4000 metric tons. Suppose that all of the 15 guns on one side of the ship are fired (almost) simultaneously in a horizontal direction at right angle to the ship. What is the recoil velocity of the ship? Ignore the resistance offered by the water.

16. Two automobiles, moving at 65 km/h in opposite directions, collide head-on. One automobile has a mass of 700 kg; the other, a mass of 1500 kg. After the collision, both remain joined together. What is the velocity of the wreck? What is the change of the velocity of each automobile during the collision?

17. The nucleus of an atom of radium (mass 3.77×10^{-25} kg) suddenly ejects an alpha particle (mass 6.68×10^{-27} kg) of an energy of 7.26×10^{-16} J. What is the velocity of the recoil of the nucleus? What is the kinetic energy of the recoil?

18. A lion of mass 120 kg leaps at a hunter with a horizontal velocity of 12 m/s. The hunter has an automatic rifle firing bullets of mass 15 g with a muzzle speed of 630 m/s, and he attempts to stop the lion in midair. How many bullets would the hunter have to fire into the lion to stop its horizontal motion? Assume the bullets stick inside the lion.

*19. Find the recoil velocity for the gun described in Example 2 if the gun is fired with an elevation angle of 20°.

*20. Consider the collision between the moving and the initially stationary automobiles described in Example 3. In this example we neglected effects of the friction force exerted by the road during the collision. Suppose that the collision lasts for 0.020 s, and suppose that during this time interval the joined automobiles are sliding with locked wheels on the pavement with a coefficient of friction $\mu_k = 0.90$. What change of momentum and what change of speed does the friction force produce in the joined automobiles in the interval of 0.020 s? Is this change of speed significant?

*21. A Maxim machine gun fires 450 bullets per minute. Each bullet has a mass of 14 g and a velocity of 630 m/s.

   (a) What is the average force that the impact of these bullets exerts on a target? Assume that the bullets penetrate the target and remain embedded in it.

   (b) What is the average rate at which the bullets deliver their energy to the target?

*22. An owl flies parallel to the ground and grabs a stationary mouse with its talons. The mass of the owl is 250 g, and that of the mouse is 50 g. If the owl's speed was 4.0 m/s before grabbing the mouse, what is its speed just after the capture?

*23. A particle moves along the x axis under the influence of a time-dependent force of the form $F_x = 2.0t + 3.0t^2$, where F_x is in newtons and t is in seconds. What is the change in momentum of the particle between $t = 0$ and $t = 5.0$ s? [Hint: Rewrite Eq. (10.3) as $dp_x = F_x\,dt$ and integrate.]

*24. A vase falls off a table and hits a smooth floor, shattering into three fragments of equal mass which move away horizontally along the floor. Two of the fragments leave the point of impact with velocities of equal magnitudes v at right angles. What are the magnitude and direction of the horizontal velocity of the third fragment? (Hint: The x and y components of the momentum are conserved separately.)

*25. The nucleus of an atom of radioactive copper undergoing beta decay simultaneously emits an electron and a neutrino. The momentum of the electron is 2.64×10^{-22} kg·m/s, that of the neutrino is 1.97×10^{-22} kg·m/s, and the angle between their directions of motion is 30.0°. The mass of the residual nucleus is 63.9 u. What is the recoil velocity of the nucleus? (Hint: The x and y components of the momentum are conserved separately.)

*26. The solar wind sweeping past the Earth consists of a stream of particles, mainly hydrogen ions of mass 1.7×10^{-27} kg. There are about 1.0×10^{7} ions per cubic meter, and their speed is 4.0×10^{5} m/s. What force does the impact of the solar wind exert on an artificial Earth satellite that has an area of 1.0 m^2 facing the wind? Assume that upon impact the ions at first stick to the surface of the satellite.

*27. The record for the heaviest rainfall is held by Unionville, Maryland, where 3.12 cm of rain (1.23 in.) fell in an interval of 1.0 min. Assuming that the impact velocity of the raindrops on the ground was 10 m/s, what must have been the average impact force on each square meter of ground during this rainfall?

*28. An automobile is traveling at a speed of 80 km/h through heavy rain. The raindrops are falling vertically at 10 m/s, and there are 7.0×10^{-4} kg of raindrops in each cubic meter of air. For the following calculation assume that the automobile has the shape of a rectangular box 2.0 m wide, 1.5 m high, and 4.0 m long.

(a) At what rate (in kg/s) do the raindrops strike the front and top of the automobile?

(b) Assume that when a raindrop hits, it initially sticks to the automobile, although it falls off later. At what rate does the automobile give momentum to the raindrops? What is the horizontal drag force that the impact of the raindrops exerts on the automobile?

*29. A spaceship of frontal area 25 m^2 passes through a cloud of interstellar dust at a speed of 1.0×10^{6} m/s. The density of dust is 2.0×10^{-18} kg/m^3. If all the particles of dust that impact on the spaceship stick to it, find the average decelerating force that the impact of the dust exerts on the spaceship.

**30. A basketball player jumps straight up to launch a long jump shot at an angle of 45° with the horizontal and a speed of 15 m/s. The 75-kg player is momentarily at rest at the top of his jump just before the shot is released, with his feet 0.80 m above the floor. (a) What is the player's velocity immediately after the shot is released? (b) How far from his original position does he land? Treat the player as a point particle. The mass of a basketball is 0.62 kg.

**31. A gun mounted on a cart fires bullets of mass m in the backward direction with a horizontal muzzle velocity u. The initial mass of the cart, including the mass of the gun and the mass of the ammunition, is M, and the initial velocity of the cart is zero. What is the velocity of the cart after firing n bullets? Assume that the cart moves without friction, and ignore the mass of the gunpowder.

10.2 Center of Mass

32. A penny coin lies on a table at a distance of 20 cm from a stack of three penny coins. Where is the center of mass of the system of four coins?

33. A 59-kg woman and a 73-kg man sit on a seesaw, 3.5 m long. Where is their center of mass? Neglect the mass of the seesaw.

34. Consider the system Earth–Moon; use the data in the table printed inside the book cover. How far from the center of the Earth is the center of mass of this system?

35. Consider the Sun and the planet Jupiter as a two-particle system. How far from the center of the Sun is the center of mass of this system? Express your result as a multiple of the radius of the Sun. (Use the data inside the cover of this book.)

36. Two bricks are adjacent, and a third brick is positioned symmetrically above them, as shown in Fig. 10.28. Where is the center of mass of the three bricks?

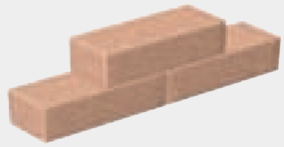


FIGURE 10.28 Three bricks.

*37. Where is the center of mass of a uniform sheet in the shape of an isosceles triangle? Assume that the height of the triangle is h when the unequal side is the base.

*38. Consider a pyramid with height h and a triangular base. Where is its center of mass?

*39. In order to balance the wheel of an automobile, a mechanic attaches a piece of lead alloy to the rim of the wheel. The mechanic finds that if he attaches a piece of 40 g at a distance of 20 cm from the center of the wheel of 30 kg, the wheel is perfectly balanced; that is, the center of the wheel coincides with the center of mass. How far from the center of the wheel was the center of mass before the mechanic balanced the wheel?

*40. The distance between the oxygen and each of the hydrogen atoms in a water (H_2O) molecule is 0.0958 nm; the angle between the two oxygen–hydrogen bonds is 105° (Fig. 10.29). Treating the atoms as particles, find the center of mass.

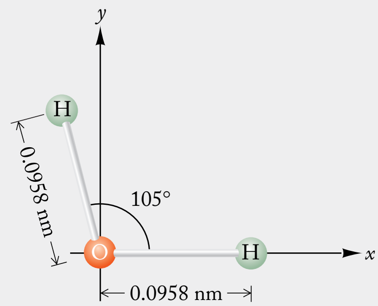


FIGURE 10.29 Atoms in a water molecule.

*41. Figure 10.30 shows the shape of a nitric acid (HNO_3) molecule and its dimensions. Treating the atoms as particles, find the center of mass of this molecule.

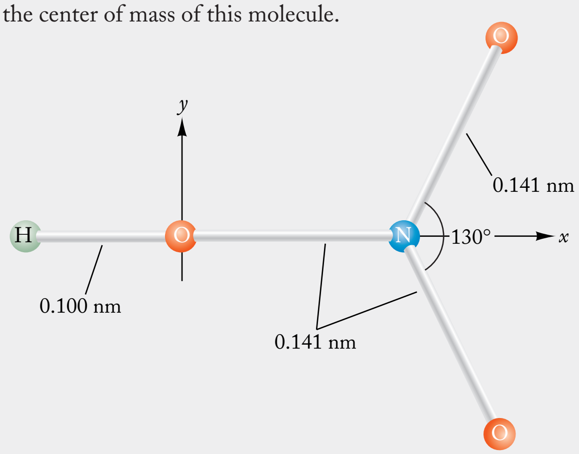


FIGURE 10.30 Atoms in a nitric acid molecule.

*42. Figure 9.13a shows the positions of the three inner planets (Mercury, Venus, and Earth) on January 1, 2000. Measure angles and distances off this figure and find the center of mass of the system of these planets (ignore the Sun). The masses of the planets are listed in Table 9.1.

*43. The Local Group of galaxies consists of our Galaxy and its nearest neighbors. The masses of the most important members of the Local Group are as follows (in multiples of the mass of the Sun): our Galaxy, 2×10^{11}; the Andromeda galaxy, 3×10^{11}; the Large Magellanic Cloud, 2.5×10^{10}; and NGC598, 8×10^{9}. The *x, y, z* coordinates of these galaxies are, respectively, as follows (in thousands of light-years): (0, 0, 0); (1640, 290, 1440), (8.5, 56.7, −149), and (1830, 766, 1170). Find the coordinates of the center of mass of the Local Group. Treat all the galaxies as point masses.

*44. A thin, uniform rod is bent in the shape of a semicircle of radius R (see Fig. 10.31). Where is the center of mass of this rod?

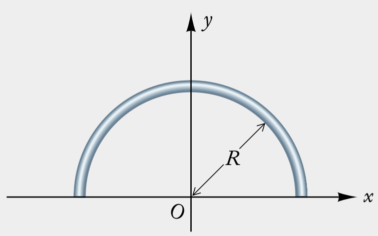


FIGURE 10.31 A rod bent in a semicircle.

*45. Three uniform square pieces of sheet metal are joined along their edges so as to form three of the sides of a cube (Fig. 10.32). The dimensions of the squares are $L \times L$. Where is the center of mass of the joined squares?

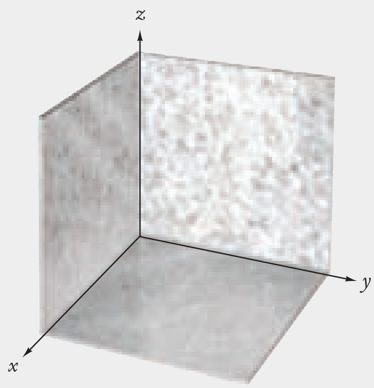


FIGURE 10.32 Three square pieces of sheet metal joined together at their edges.

*46. A box made of plywood has the shape of a cube measuring $L \times L \times L$. The top of the box is missing. Where is the center of mass of the open box?

*47. A cube of iron has dimensions $L \times L \times L$. A hole of radius $\frac{1}{4}L$ has been drilled all the way through the cube so that one side of the hole is tangent to the middle of one face along its entire length (Fig. 10.33). Where is the center of mass of the drilled cube?

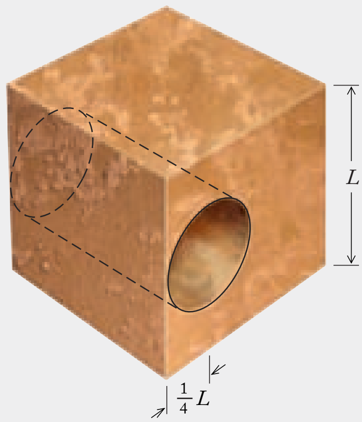


FIGURE 10.33 Iron cube with a hole.

*48. A semicircle of uniform sheet metal has radius R (Fig. 10.34). Find the center of mass.

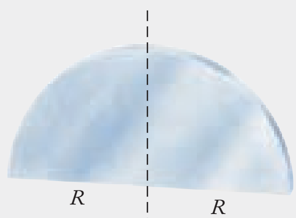


FIGURE 10.34 Semicircle of sheet metal.

*49. Mount Fuji has approximately the shape of a cone. The half-angle at the apex of this cone is 65°, and the height of the apex is 3800 m. At what height is the center of mass? Assume that the material in Mount Fuji has uniform density.

*50. Show that the center of mass of a uniform flat triangular plate is at the point of intersection of the lines drawn from the vertices to the midpoints of the opposite sides.

*51. Consider a man of mass 80 kg and height 1.70 m with the mass distribution described in Fig. 10.17. How much work does this man do to raise his arms from a hanging position to a horizontal position? To a vertically raised position?

*52. Suppose that a man of mass 75 kg and height 1.75 m runs in place, raising his legs high, as in Fig. 10.35. If he runs at the rate of 80 steps per minutes for each leg (160 total per minute), what power does he expend in raising his legs?

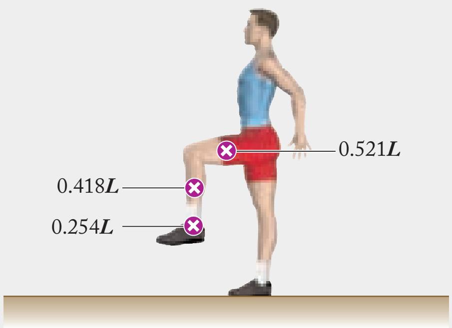


FIGURE 10.35 Man with raised leg.

*53. A lock on the Champlain Canal is 73 m long and 9.2 m wide; the lock has a lift of 3.7 m—that is, the difference between the water levels of the canal on one side of the lock and on the other side is 3.7 m. How much gravitational potential energy is wasted each time the lock goes through one cycle (involving the filling of the lock with water from the high level and then the spilling of this water to the low level)?

*54. The Great Pyramid at Giza has a mass of 6.6×10^6 metric tons and a height of 147 m (see Example 7). Assume that the mass is uniformly distributed over the volume of the pyramid.

(a) How much work must the ancient Egyptian laborers have done against gravity to pile up the stones in the pyramid?

(b) If each laborer delivered work at an average rate of 4.0×10^5 J/h, how many person-hours of work have been stored in this pyramid?

**55. A thin hemispherical shell of uniform thickness is suspended from a point above its center of mass as shown in Fig. 10.36. Where is that center of mass?

**56. Suppose that water drops are released from a point at the edge of a roof with a constant time interval Δt between one water drop and the next. The drops fall a distance l to the ground. If Δt is very short (so the number of drops falling though the air at any given instant is very large), show that the center of mass of the falling drops is at a height of $\frac{2}{3}l$ above the ground.

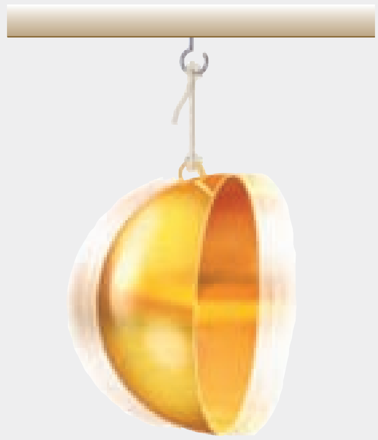


FIGURE 10.36 A hemispherical shell used as a gong.

From this, deduce that the time-average height of a projectile released from the ground and returning to the ground is $\frac{2}{3}$ of its maximum height. (This theorem is useful in the calculation of the average air pressure and air resistance encountered by a projectile.)

10.3 The Motion of the Center of Mass

57. A proton of kinetic energy 1.6×10^{-13} J is moving toward a proton at rest. What is the velocity of the center of mass of the system?

58. In a molecule, the atoms usually execute a rapid vibrational motion about their equilibrium position. Suppose that in an isolated potassium bromide (KBr) molecule the speed of the potassium atom is 5.0×10^3 m/s at one instant (relative to the center of mass). What is the speed of the bromine atom at the same instant?

59. A fisherman in a boat catches a great white shark with a harpoon. The shark struggles for a while and then becomes limp when at a distance of 300 m from the boat. The fisherman pulls the shark by the rope attached to the harpoon. During this operation, the boat (initially at rest) moves 45 m in the direction of the shark. The mass of the boat is 5400 kg. What is the mass of the shark? Pretend that the water exerts no friction.

60. A 75-kg man climbs the stairs from the ground floor to the fourth floor of a building, a height of 15 m. How far does the Earth recoil in the opposite direction as the man climbs?

61. A 6000-kg truck stands on the deck of an 80000-kg ferryboat. Initially the ferry is at rest and the truck is located at its front end. If the truck now drives 15 m along the deck toward the rear of the ferry, how far will the ferry move forward relative to the water? Pretend that the water has no effect on the motion.

62. While moving horizontally at 5.0×10^3 m/s at an altitude of 2.5×10^4 m, a ballistic missile explodes and breaks apart into

two fragments of equal mass which fall freely. One of the fragments has zero speed immediately after the explosion and lands on the ground directly below the point of the explosion. Where does the other fragment land? Ignore the friction of air.

63. A 15-g bullet moving at 260 m/s is fired at a 2.5-kg block of wood. What is the velocity of the center of mass of the bullet–block system?

64. A 60-kg woman and a 90-kg man walk toward each other, each moving with speed v relative to the ground. What is the velocity of their center of mass?

65. A projectile of mass M reaches the peak of its motion a horizontal distance D from the launch point. At its peak, it explodes into three equal fragments. One fragment returns directly to the launch point, and one lands a distance $2D$ from the launch point, at a point in the same plane as the initial motion. Where does the third fragment land?

*66. A projectile is launched with speed v_0 at an angle of θ with respect to the horizontal. At the peak of its motion, it explodes into two pieces of equal mass, which continue to move in the original plane of motion. One piece strikes the ground a horizontal distance D further from the launch point than the point directly below the explosion at a time $t < v_0 \sin\theta/g$ after the explosion. How high does the other piece go? Where does the other piece land? Answer in terms of v_0, θ, D, and t.

**67. Figure 9.13a shows the positions of the three inner planets (Mercury, Venus, Earth) on January 1, 2000. Measuring angles off this figure and using the data on masses, orbital radii, and periods given in Table 9.1, find the velocity of the center of mass of this system of three planets.

10.4 Energy of a System of Particles

68. Two automobiles, each of mass 1500 kg, travel in the same direction along a straight road. The speed of one automobile is 25 m/s, and the speed of the other automobile is 15 m/s. If we regard these automobiles as a system of two particles, what is the translational kinetic energy of the center of mass? What is the total kinetic energy?

69. Repeat the calculation of Problem 68 if the two automobiles travel in *opposite* directions.

70. A projectile of 45 kg fired from a gun has a speed of 640 m/s. The projectile explodes in flight, breaking apart into a fragment of 32 kg and a fragment of 13 kg (we assume that no mass is dispersed in the explosion). Both fragments move along the original direction of motion. The speed of the first fragment is 450 m/s and that of the second is 1050 m/s.
   (a) Calculate the translational kinetic energy of the center of mass motion before the explosion.
   (b) Calculate the translational kinetic energy of the center of mass motion after the explosion. Calculate the total kinetic energy. Where does the extra kinetic energy come from?

71. Consider the automobile collision described in Problem 16. What is the translational kinetic energy of the center of mass motion before the collision? What is the total kinetic energy before and after the collision?

72. Two isolated point masses m_1 and m_2 are connected by a spring. The masses attain their maximum speeds at the same instant. A short time later both masses are stationary. The maximum speed of the first mass is v_1. What is the maximum speed of the second mass? When the masses are stationary, what is the energy stored in the spring?

73. The typical speed of a helium atom in helium gas at room temperature is 1.4 km/s; that of an oxygen molecule (O_2) in oxygen gas is close to 500 m/s. Find the total kinetic energy of one mole of helium atoms and that of one mole of oxygen molecules.

*74. Two automobiles, each of mass $M/2$ and speed v, drive around a one-lane traffic circle. What is the total kinetic energy of the two-car system? What is the quantity $\frac{1}{2}Mv_{CM}^2$ if the automobiles are (a) on opposite sides of the traffic circle, (b) one-quarter of the circle apart, and (c) locked together?

*75. Consider the Sun and Jupiter to be a two-particle system, orbiting around the center of mass. Find the ratio of the kinetic energy of the Sun to that of Jupiter. (Use the data inside the book cover.)

*76. The typical speed of the vibrational motion of the iron atoms in a piece of iron at room temperature is 360 m/s. What is the total kinetic energy of a 1.0-kg chunk of iron?

REVIEW PROBLEMS

77. A hunter on skates on a smooth sheet of ice shoots 10 bullets at a target at the shore. Each bullet has a mass of 15 g and a speed of 600 m/s. The hunter has a mass of 80 kg. What recoil speed does he acquire?

78. Grain is being loaded into an almost full railroad car from an overhead chute (see Fig. 10.37). If 500 kg per second falls freely from a height of 4.0 m to the top of the car, what downward push does the impact of the grain exert on the car?

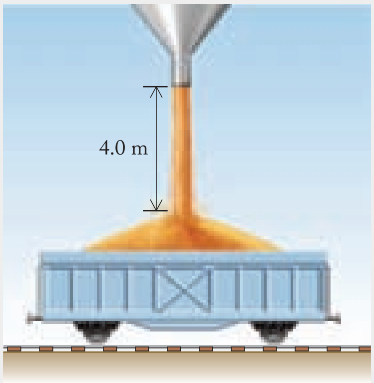


FIGURE 10.37 Grain from a chute falls into a railroad car.

79. A boy and a girl are engaged in a tug-of-war on smooth, frictionless ice. The mass of the boy is 40 kg, and that of the girl is 30 kg; their separation is initially 4.0 m. Each pulls with a force of 200 N on the rope. What is the acceleration of each? If they keep pulling, where will they meet?
80. An automobile of 1200 kg and an automobile of 1500 kg are traveling in the same direction on a straight road. The speeds of the two automobiles are 60 km/h and 80 km/h, respectively. What is the velocity of the center of mass of the two-automobile system?
81. An automobile traveling 40 km/h collides head-on with a truck which has 5 times the mass of the automobile. The wreck remains at rest after the collision. Deduce the speed of the truck.
82. The nozzle of a fire hose ejects 800 liters of water per minute at a speed of 26 m/s. Estimate the recoil force on the nozzle. By yourself, can you hold this nozzle steady in your hands?
83. The distance between the centers of the atoms of potassium and bromine in the potassium bromide (KBr) molecule is 0.282 nm (Fig. 10.38). Treating the atoms as particles, find the center of mass.

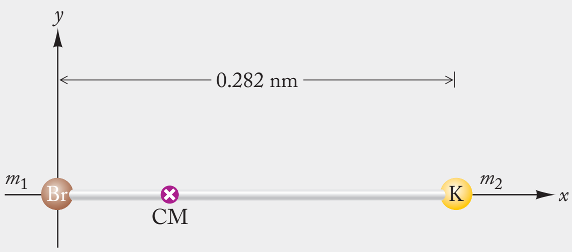


FIGURE 10.38 Atoms in a potassium bromide molecule.

84. A tugboat of mass 400 metric tons and a ship of 28 000 metric tons are joined by a long towrope of 400 m. Both vessels are initially at rest in the water. If the tugboat reels in 200 m of towrope, how far does the ship move relative to the water? The tugboat? Ignore the resistance that the water offers to the motion.
85. A cat stands on a plank of balsa wood floating in water. The mass of the cat is 3.5 kg, and the mass of the balsa is 5.0 kg. If the cat walks 1.0 m along the plank, how far does she move in relation to the water?
86. Three firefighters of equal masses are climbing a long ladder. When the first firefighter is 20 m up the ladder, the second is 15 m up, and the third is 5 m up. Where is the center of mass of the three firefighters?
87. Four identical books are arranged on the vertices of an equilateral triangle of side 1.0 m. Two of the books are together at one vertex of the triangle, and the other two are at the other two vertices. Where is the center of mass of this arrangement?
88. Three identical metersticks are arranged to form the letter U. Where is the center of mass of this system?
89. Two uniform squares of sheet metal of dimensions $L \times L$ are joined at right angles along one edge (see Fig. 10.39). One of the squares has twice the mass of the other. Find the center of mass of the combined squares.

FIGURE 10.39 Two square pieces of sheet metal joined along one edge.

*90. Find the center of mass of a uniform solid hemisphere of radius R.

Answers to Checkups

Checkup 10.1

1. We assume the usual case that the truck has a larger mass than the automobile. Then the equality of their momenta (mv) implies that the automobile has a larger speed (the ratio of the velocities will be the inverse of the ratio of the masses). Since the kinetic energy is $\frac{1}{2}mv^2 = \frac{1}{2} \times mv \times v$ and the momenta (mv) are equal, then the vehicle with the larger speed, the automobile, will also have the larger kinetic energy.
2. They cannot have the same momentum, since the signs of their momenta will be opposite. They can have the same kinetic energy, since that depends only on the speed: $K = \frac{1}{2}mv^2$.
3. No—the momentum, like the velocity, is also reversed, and it has the opposite sign after the impact.
4. Yes—for practical purposes, the Solar System is essentially an isolated system, and so the net momentum is constant. The net kinetic energy is not constant, since during motion, kinetic energy is converted to potential energy and vice versa.
5. (a) For any directions, the total kinetic energy is $\frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$. For parallel motion (both southward), the total momentum is $mv + mv = 2mv$ southward. (b) The total kinetic energy is again mv^2, while for antiparallel motion, the total momentum is $mv - mv = 0$ (no direction). (c) The total kinetic energy is again mv^2, while for perpendicular motion, the total momentum has magnitude $\sqrt{(mv)^2 + (mv)^2} = \sqrt{2}mv$ and is directed 45° south of east.
6. (C) Automobile; truck. The truck has a larger mass M than the automobile mass m. Let the truck speed be V and the automobile speed be v. The equal kinetic energies $(\frac{1}{2}MV^2 = \frac{1}{2}mv^2)$ then imply that the automobile will have the larger speed $v = (M/m)^{1/2}V$. If we substitute one power of this v into the kinetic energy equality and cancel a factor of $\frac{1}{2}V$, we find $MV = (M/m)^{1/2}mv$; thus, the truck momentum is larger.

Checkup 10.2

1. Consider the average position of the mass distribution. For the curved snake shown in the figure, the center of mass is at a point in the space below the top arc, perhaps slightly below center (because of the *two* bottom arcs) and slightly to the right of center (because of the head).
2. Consider the average position of the mass distribution. For the horseshoe, the center of mass is below center in the space in the middle of the arc, along the vertical line of symmetry, at a point well away from the open end.
3. No. The center of mass is a weighted average of position; such an average can never be greater than all of the positions averaged.
4. (C) Shifts upward. If the heavy mass at the bottom falls off, the center of mass is higher. When the center of mass of the sailboat is too high, it is top-heavy, and prone to tip over.

Checkup 10.3

1. For the (isolated) system of person plus canoe, with no initial motion, the center of mass stays at the same fixed position as you begin and continue your crawl. Thus as your mass moves from the rear to the front, the boat moves backward a sufficient distance to keep the center of mass of the combined system fixed.
2. No. In the extreme case where the boxcar has zero mass, you remain fixed relative to the ground, and the boxcar rolls a distance equal to its length as you walk from the rear to the front. If the boxcar has appreciable mass, you will move toward your common center of mass, which will be a distance less than the length of the boxcar.
3. If the marbles were dropped vertically, then the center of mass remains fixed at the point of impact with the floor, even though the marbles scatter in all directions.
4. (B) 5 m/s south. Using Eq. (10.37) with positive velocity northward, $v_{CM} = (M \times 25\text{ m/s} - 2M \times 20\text{ m/s})/(3M) = (-15\text{ m/s})/3 = -5\text{ m/s}$.

Checkup 10.4

1. No. The initial kinetic energy is large, and the final kinetic energy is small or zero; the energy is transformed into other forms: elastic energy (deformation of automobile parts), friction, sound, and heat.
2. The total energy is conserved, if we consider only gravitational potential energy and kinetic energy (in actuality, some other energy is lost, for example, as the Sun's light is radiated away into space). Neither kinetic nor potential energy is separately conserved; these two are traded back and forth, for example, as the planets move in their elliptical orbits.
3. (A) **P** only. Since there is no net external force, the total momentum of such an isolated system is always simply conserved. However, the spring will stretch and compress during the motion, trading kinetic for potential energy, so K and U will not remain constant.

CHAPTER

11 Collisions

CONCEPTS IN CONTEXT

Concepts in Context

In this crash test, the automobile was towed at a speed of 56 km/h (35 mi/h) and then crashed into a rigid concrete barrier. Anthropomorphic dummies that simulate human bodies are used for evaluation of injuries that would be sustained by driver and passengers. Accelerometers installed on the body of the automobile and the bodies of the dummies permit calculation of impact forces.

With the concepts of this chapter we can answer questions such as:

? What is the average force on the front of an automobile during impact? (Example 1, page 341)

? What is force on the head of a dummy during a collision with the windshield or the steering wheel? (Example 2, page 341)

? How do seat belts and air bags protect occupants of an automobile in a crash? (Physics in Practice: Automobile Collisions, page 343)

? How does the stiffness of the front end of an automobile affect the safety of its occupants in a collision? (Checkup 11.1, question 1, page 344)

? In a two-car collision, how are the initial velocities related to the final direction of motion? (Example 7, page 351)

The collision between two bodies—an automobile and a solid wall, a ship and an iceberg, a molecule of oxygen and a molecule of nitrogen—involves a violent change of the motion, a change brought about by very strong forces that begin to act suddenly when the bodies come into contact, last a short time, and then cease just as suddenly when the bodies separate. The forces that act during a collision are usually rather complicated, so their complete theoretical description is impossible (e.g., in an automobile collision) or at least very difficult (e.g., in a collision between subatomic particles). However, even without exact knowledge of the details of the forces, we can make some predictions about the collision by taking advantage of the general laws of conservation of momentum and energy we studied in the preceding chapters. In the following sections we will see what constraints these laws impose on the motion of the colliding bodies.

The study of collisions is an important tool in engineering and physics. In automobile collision and safety studies, engineers routinely subject vehicles to crash tests. Collisions are also essential for the experimental investigation of atoms, nuclei, and elementary particles. All subatomic bodies are too small to be made visible with any kind of microscope. Just as you might use a stick to feel your way around a dark cave, a physicist who cannot see the interior of an atom uses probes to "feel" for subatomic structures. The probe used by physicists in the exploration of subatomic structures is simply a stream of fast-moving particles—electrons, protons, alpha particles (helium nuclei), or others. These projectiles are aimed at a target containing a sample of the atoms, nuclei, or elementary particles under investigation. From the manner in which the projectiles collide and react with the target, physicists can deduce some of the properties of the subatomic structures in the target. Similarly, materials scientists, chemists, and engineers deduce the structure and composition of solids and liquids by bombarding such materials with particles and examining the results of such collisions.

11.1 IMPULSIVE FORCES

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The force that two colliding bodies exert on one another acts for only a short time, giving a brief but strong push. Such a force that acts for only a short time is called an **impulsive force.** During the collision, the impulsive force is much stronger than any other forces that may be present; consequently the impulsive force produces a large change in the motion while the other forces produce only small and insignificant changes. For instance, during the automobile collision shown in Fig. 11.1, the only important force on the automobile is the push of the wall on its front end; the effects produced by gravity and by the friction force of the road during the collision are insignificant.

Suppose the collision lasts some short time Δt, say, from $t = 0$ to $t = \Delta t$, and that during this time an impulsive force $\mathbf{F}$ acts on one of the colliding bodies. This force is zero before $t = 0$ and it is zero after $t = \Delta t$, but it is large between these times. For example, Fig. 11.2 shows a plot of the force experienced by an automobile in a collision with a solid wall lasting 0.120 s. The force is zero before $t = 0$ and after $t = 0.120$ s, and varies in a complicated way between these times.

*The **impulse** delivered by such a force to the body is defined as the integral of the force over time:*

$$\mathbf{I} = \int_0^{\Delta t} \mathbf{F}\, dt \tag{11.1}$$

impulse

FIGURE 11.1 Crash test of a Mercedes–Benz automobile. The photographs show an impact at 49 km/h into a rigid barrier. The first photograph was taken 5×10^{-3} s after the initial contact; the others were taken at intervals of 20×10^{-3} s. The automobile remains in contact with the barrier for 0.120 s; it then recoils from the barrier with a speed of 4.7 km/h. The checkered bar on the ground has a length of 2 m.

According to this equation, the x component of the impulse for the force plotted in Fig. 11.2 is the area between the curve $F_x(t)$ and the t axis.

The SI units of impulse are N·s, or kg·m/s; these units are the same as those for momentum.

By means of the equation of motion, $\mathbf{F} = d\mathbf{p}/dt$, we can transform Eq. (11.1) into

$$\mathbf{I} = \int_0^{\Delta t} \mathbf{F}\, dt = \int_0^{\Delta t} \frac{d\mathbf{p}}{dt}\, dt = \int d\mathbf{p} = \mathbf{p}' - \mathbf{p} \tag{11.2}$$

where $\mathbf{p}$ is the momentum of the body before the collision (at time 0) and $\mathbf{p}'$ is the momentum after the collision (at time $t = \Delta t$). Thus, the impulse of a force is simply equal to the momentum change produced by this force. This equality of impulse and momentum change is sometimes referred to as the *impulse–momentum relation*. However, since the force acting during a collision is usually not known in detail, Eq. (11.2) is not very helpful for calculating momentum changes. It is often best to apply Eq. (11.2) in reverse, for calculating the time-average force from the known momentum change. This time-average force is defined by

$$\overline{\mathbf{F}} = \frac{1}{\Delta t} \int_0^{\Delta t} \mathbf{F}\, dt \tag{11.3}$$

In a plot of force vs. time, such as shown in Fig. 11.2, the time-average force simply represents the mean height of the function above the t axis; this mean height is shown

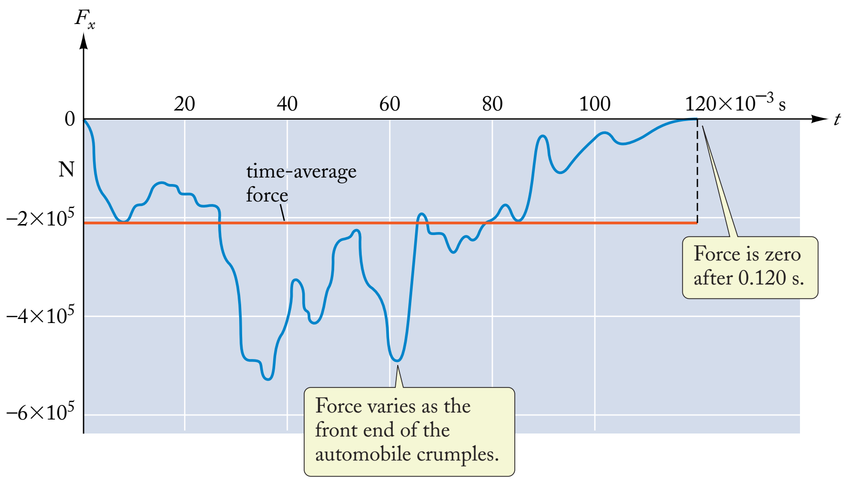

FIGURE 11.2 Force on the automobile as a function of time during the impact shown in Fig. 11.1. The colored horizontal line indicates the time-average force. (Calculated from data supplied by Mercedes–Benz of North America, Inc.)

by the red horizontal line in Fig. 11.2. According to Eq. (11.2), we can write the time-average force as

$$\overline{\mathbf{F}} = \frac{1}{\Delta t}\mathbf{I} = \frac{1}{\Delta t}(\mathbf{p}' - \mathbf{p}) \tag{11.4}$$

average force in collision

This relation gives a quick estimate of the average magnitude of the impulsive force acting on the body if the duration of the collision and the momentum change are known.

EXAMPLE 1 The collision between the automobile and the barrier shown in Fig. 11.1 lasts 0.120 s. The mass of the automobile is 1700 kg, and the initial and final velocities in the horizontal direction are $v_x = 13.6$ m/s and $v'_x = -1.3$ m/s, respectively (the final velocity is negative because the automobile recoils, or bounces back from the barrier). From these data, evaluate the average force that acts on the automobile during the collision. Evaluate the average force that acts on the barrier.

Concepts in Context

SOLUTION: With the x axis along the direction of the initial motion, the change of momentum is

$$\begin{aligned} p'_x - p_x &= mv'_x - mv_x \\ &= 1700 \text{ kg} \times (-1.3 \text{ m/s}) - 1700 \text{ kg} \times 13.6 \text{ m/s} \\ &= -2.53 \times 10^4 \text{ kg·m/s} \end{aligned}$$

According to Eq. (11.4), the average force is then

$$\begin{aligned} \overline{F}_x &= \frac{p'_x - p_x}{\Delta t} \\ &= \frac{-2.53 \times 10^4 \text{ kg·m/s}}{0.120 \text{ s}} = -2.11 \times 10^5 \text{ N} \end{aligned} \tag{11.5}$$

Since the mutual forces on two bodies engaged in a collision are an action–reaction pair, the forces on the automobile and on the barrier are of equal magnitudes and of opposite directions. Thus, the average force on the barrier is $\overline{F}_x = +2.11 \times 10^5$ N. This is quite a large force—it equals the weight of about 2×10^4 kg, or 20 tons.

EXAMPLE 2 When an automobile collides with an obstacle and suddenly stops, a passenger not restrained by a seat belt will not stop simultaneously with the automobile, but instead will continue traveling at nearly constant speed until he or she hits the dashboard and the windshield. The collision of the passenger's head with the windshield often results in severe or fatal injuries. In crash tests, dummies with masses, shapes, and joints simulating human bodies are used to determine likely injuries. Consider a dummy head striking a windshield at 15 m/s (54 km/h) and stopping in a time of 0.015 s (this time is considerably shorter than the time of about 0.12 s for stopping the automobile because the front end of the automobile crumples gradually and cushions the collision to some extent; there is no such cushioning for the head striking the windshield). What is the average force on the head during impact on the windshield? What is the average deceleration? Treat the head as a body of mass 5.0 kg, moving independently of the neck and trunk.

Concepts in Context

SOLUTION: The initial momentum of the head is

$$p_x = mv_x = 5.0 \text{ kg} \times 15 \text{ m/s} = 75 \text{ kg·m/s}$$

When the head stops, the final momentum is zero. Hence the average force is

$$\overline{F}_x = \frac{p'_x - p_x}{\Delta t} = -\frac{p_x}{\Delta t}$$

$$= -\frac{75 \text{ kg·m/s}}{0.015 \text{ s}} = -5.0 \times 10^3 \text{ N}$$

The average acceleration is

$$\overline{a}_x = \frac{\overline{F}_x}{m} = -\frac{5.0 \times 10^3 \text{ N}}{5.0 \text{ kg}} = -1.0 \times 10^3 \text{ m/s}^2$$

which is about 100 standard *g*'s!

Often it is not possible to calculate the motion of the colliding bodies by direct solution of Newton's equation of motion because the impulsive forces that act during the collision are not known in sufficient detail. We must then glean whatever information we can from the general laws of conservation of momentum and energy, which do not depend on the details of these forces. In some simple instances, these general laws permit the deduction of the motion after the collision from what is known about the motion before the collision.

In all collisions between two or more particles, the total momentum of the system is conserved. Whether or not the mechanical energy is conserved depends on the character of the forces that act between the particles. *A collision in which the total kinetic energy before and after the collision is the same is called* **elastic.** (This usage of the word *elastic* is consistent with the usage we encountered previously when discussing the restoring force of a deformable body in Section 6.2. For example, if the colliding bodies exert a force on each other by means of a massless elastic spring placed between them, then the kinetic energy before and after the collision will indeed be the same—that is, the collision will be elastic.) Collisions between macroscopic bodies are usually not elastic—during the collision some of the kinetic energy is transformed into heat by the internal friction forces and some is used up in doing work to change the internal configuration of the bodies. For example, the automobile collision shown in Fig. 11.1 is highly *inelastic;* almost the entire initial kinetic energy is used up in doing work on the automobile parts, changing their shape. On the other hand, the collision of a "Super Ball" and a hard wall or the collision of two billiard balls comes pretty close to being elastic—that is, the kinetic energies before and after the collision are almost the same.

elastic collision

Collisions between "elementary" particles—such as electrons, protons, and neutrons—are often elastic. These particles have no internal friction forces which could dissipate kinetic energy. A collision between such particles can be inelastic only if it involves the creation of new particles; such new particles may arise either by conversion of some of the available kinetic energy into mass or else by transmutation of the old particles by means of a change of their internal structure.

EXAMPLE 3 A Super Ball, made of a rubberlike plastic, is thrown against a hard, smooth wall. The ball strikes the wall from a perpendicular direction with speed v. Assuming that the collision is elastic, find the speed of the ball after the collision.

PHYSICS IN PRACTICE — AUTOMOBILE COLLISIONS

Concepts *in* Context

We can fully appreciate the effects of the secondary impact on the human body if we compare the impact speeds of a human body on the dashboard or the windshield with the speed attained by a body in free fall from some height. The impact of the head on the windshield at 15 m/s is equivalent to falling four floors down from an apartment building and landing headfirst on a hard surface. Our intuition tells us that this is likely to be fatal. Since our intuition about the dangers of heights is much better than our intuition about the dangers of speeds, it is often instructive to compare impact speeds with equivalent heights of fall. The table lists impact speeds and equivalent heights, expressed as the number of floors the body has to fall down to acquire the same speed.

The number of fatalities in automobile collisions has been reduced by the use of air bags. The air bag helps by cushioning the impact over a longer time, reducing the time-average force. To be effective, the air bag must inflate quickly, before the passenger reaches it, typically in about 10 milliseconds. Because of this, a passenger, especially a child, too near an air bag prior to inflation can be injured or killed by the impulse from the inflation. But for a properly seated adult passenger, the inflated air bag cushions the passenger, reducing the severity of injuries.

COMPARISON OF IMPACT SPEEDS AND HEIGHTS OF FALL

SPEED	SPEED	EQUIVALENT HEIGHT (NUMBER OF FLOORS)[a]
15 km/h	9 mi/h	$\frac{1}{3}$
30	19	1
45	28	3
60	37	5
75	47	8
90	56	11
105	65	15

[a]Each floor is 2.9 m.

However, the impact can still be fatal—you wouldn't expect to survive a jump from an 11-floor building onto an air mattress.

For maximum protection, a seat belt should always be worn even in vehicles equipped with air bags. In lateral collisions, in repeated collisions (such as in car pileups), and in rollovers, an air bag is of little help, and a seat belt is essential. The effectiveness of seat belts is well demonstrated by the experiences of race car drivers. Race car drivers wear lap belts and crossed shoulder belts. Even in spectacular crashes at very high speeds (see the figure), the drivers rarely suffer severe injuries.

In a race at the California Speedway in October 2000, a car flips over and breaks in half after a crash, but the driver, Luis Diaz, walks away from the wreck.

SOLUTION: The only horizontal force on the ball is the normal force exerted by the wall; this force reverses the motion of the ball (see Fig. 11.3). Since the wall is very massive, the reaction force of the ball on the wall will not give the wall any appreciable velocity. Hence the kinetic energy of the system, both before and after the collision, is merely the kinetic energy of the ball. Conservation of this

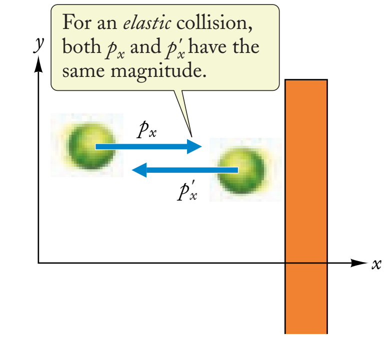

FIGURE 11.3 The initial momentum p_x of the ball is positive; the final momentum p'_x is negative.

kinetic energy then requires that the ball rebound with a speed v equal to the incident speed.

Note that although the kinetic energy of the ball is the same before and after the collision, the momentum is not the same before and after (see also Example 1 of Chapter 10). If the x axis is in the direction of the initial motion, then the momentum of the ball before the collision is $p_x = mv$, and after the collision it is $p'_x = -mv$. Hence the change of momentum is $p'_x - p_x = -2mv$. The wall suffers an equal and opposite momentum change of $+2mv$, so that the total momentum of the system is conserved. The wall can acquire the momentum $2mv$ without acquiring any appreciable velocity because its mass is large and it is attached to a building of even larger mass.

Concepts in Context

Checkup 11.1

QUESTION 1: In order to protect the occupants of an automobile in a collision, is it better to make the front end of the automobile very hard (a solid block of steel) or fairly soft and crushable?

QUESTION 2: If a golf ball and a steel ball of the same mass strike a concrete floor with equal speeds, which will exert the larger average force on the floor?

QUESTION 3: You drop a Super Ball on a hard, smooth floor from a height of 1 m. If the collision is elastic, how high will the ball bounce up?

QUESTION 4: A child throws a wad of chewing gum against a wall, and it sticks. Is this an elastic collision?

QUESTION 5: A 3000-kg truck collides with a 1000-kg car. During this collision the average force exerted by the truck on the car is 3×10^6 N in an eastward direction. What is the magnitude of the average force exerted by the car on the truck?

(A) 0 (B) 1×10^6 N (C) 3×10^6 N (D) 9×10^6 N

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11.2 ELASTIC COLLISIONS IN ONE DIMENSION

The collision of two boxcars on a railroad track is an example of a collision on a straight line. More generally, the collision of any two bodies that approach head-on and recoil along their original line of motion is a collision along a straight line. Such collisions will occur only under exceptional circumstances; nevertheless, we find it instructive to study such collisons because they display in a simple way some of the broad features of more complicated collisions.

In an elastic collision of two particles moving along a straight line, the laws of conservation of momentum and energy completely determine the final velocities in terms of the initial velocities. In the following calculations, we will assume that one particle (the "projectile") is initially in motion and the other (the "target") is initially at rest.

Figure 11.4a shows the particles before the collision, and Fig. 11.4b shows them after; the x axis is along the direction of motion. We will designate the x components of the velocity of particle 1 and particle 2 before the collision by v_1 and v_2, respectively. We will designate the x components of these velocities after the collision by v'_1 and v'_2.

Particle 2 is the target, initially at rest, so $v_2 = 0$. Particle 1 is the projectile. The initial momentum is therefore simply the momentum m_1v_1 of particle 1. The final momentum, after the collision, is $m_1v_1' + m_2v_2'$. Conservation of momentum then tells us that

$$m_1v_1 = m_1v_1' + m_2v_2' \tag{11.6}$$

The initial kinetic energy is $\frac{1}{2}m_1v_1^2$, and the final kinetic energy is $\frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$. Since this collision is *elastic,* conservation of kinetic energy[1] tell us that

$$\tfrac{1}{2}m_1v_1^2 = \tfrac{1}{2}m_1v_1'^2 + \tfrac{1}{2}m_2v_2'^2 \tag{11.7}$$

In these equations, we can regard the initial velocities v_1 and v_2 as known, and the final velocities v_1' and v_2' as unknown. We therefore want to solve these equations for the unknown quantities. For this purpose, it is convenient to rearrange the two equations somewhat. If we subtract m_1v_1' from both sides of Eq. (11.6), we obtain

$$m_1(v_1 - v_1') = m_2v_2' \tag{11.8}$$

If we multiply both sides of Eq. (11.7) by 2 and subtract $m_1v_1'^2$ from both sides, we obtain

$$m_1(v_1^2 - v_1'^2) = m_2v_2'^2 \tag{11.9}$$

With the identity $v_1^2 - v_1'^2 = (v_1 - v_1')(v_1 + v_1')$, this becomes

$$m_1(v_1 - v_1')(v_1 + v_1') = m_2v_2'^2 \tag{11.10}$$

Now divide Eq. (11.10) by Eq. (11.8)—that is, divide the left side of Eq. (11.10) by the left side of Eq. (11.8) and the right side of Eq. (11.10) by the right side of Eq. (11.8). The result is

$$v_1 + v_1' = v_2' \tag{11.11}$$

This trick gets rid of the bothersome squares in Eq. (11.7) and leaves us with two equations—Eqs. (11.8) and (11.11)—without squares. To complete the solution for our unknowns, we take the value $v_2' = v_1 + v_1'$ given by Eq. (11.11) and substitute it into the right side of Eq. (11.8):

$$m_1(v_1 - v_1') = m_2(v_1 + v_1') \tag{11.12}$$

We can solve this immediately for the unknown v_1', with the result

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}v_1 \tag{11.13}$$

final projectile velocity in elastic collision

Finally, we substitute this value of v_1' into the expression from Eq. (11.11), $v_2' = v_1 + v_1'$, and we find

$$v_2' = v_1 + \frac{m_1 - m_2}{m_1 + m_2}v_1 = \frac{(m_1 + m_2)v_1 + (m_1 - m_2)v_1}{m_1 + m_2}$$

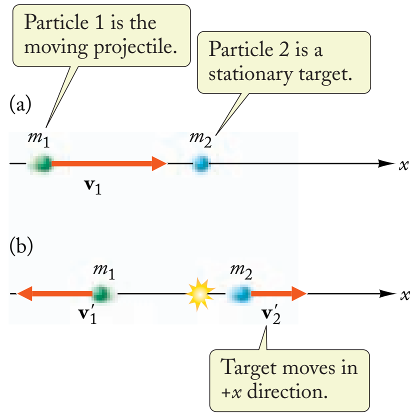


FIGURE 11.4 (a) Before the collision, particle 2 is at rest, and particle 1 has velocity $\mathbf{v}_1$. (b) After the collision, particle 1 has velocity $\mathbf{v}_1'$, and particle 2 has velocity $\mathbf{v}_2'$.

[1] In the context of elastic collisions, "conservation of kinetic energy" is taken to mean that the kinetic energy is the same before and after the collision; duing the collision, when the particles are interacting, what is conserved is not the kinetic energy itself, but the sum of kinetic and potential energies.

or

final target velocity in elastic collision

$$v_2' = \frac{2m_1}{m_1 + m_2} v_1 \tag{11.14}$$

Equations (11.13) and (11.14) give us the final velocities v_1' and v_2' in terms of the initial velocity v_1.

EXAMPLE 4 An empty boxcar of mass $m_1 = 20$ metric tons rolling on a straight track at 5.0 m/s collides with a loaded stationary boxcar of mass $m_2 = 65$ metric tons (see Fig. 11.5). Assuming that the cars bounce off each other elastically, find the velocities after the collision.

SOLUTION: With $m_1 = 20$ tons and $m_2 = 65$ tons, Eqs. (11.13) and (11.14) yield

$$v_1' = \frac{20\text{ tons} - 65\text{ tons}}{20\text{ tons} + 65\text{ tons}} \times 5.0\text{ m/s} = -2.6\text{ m/s}$$

$$v_2' = \frac{2 \times 20\text{ tons}}{20\text{ tons} + 65\text{ tons}} \times 5.0\text{ m/s} = 2.4\text{ m/s}$$

Thus, boxcar 2 acquires a speed of 2.4 m/s, and boxcar 1 recoils with a speed of 2.6 m/s (note the negative sign of v_1').

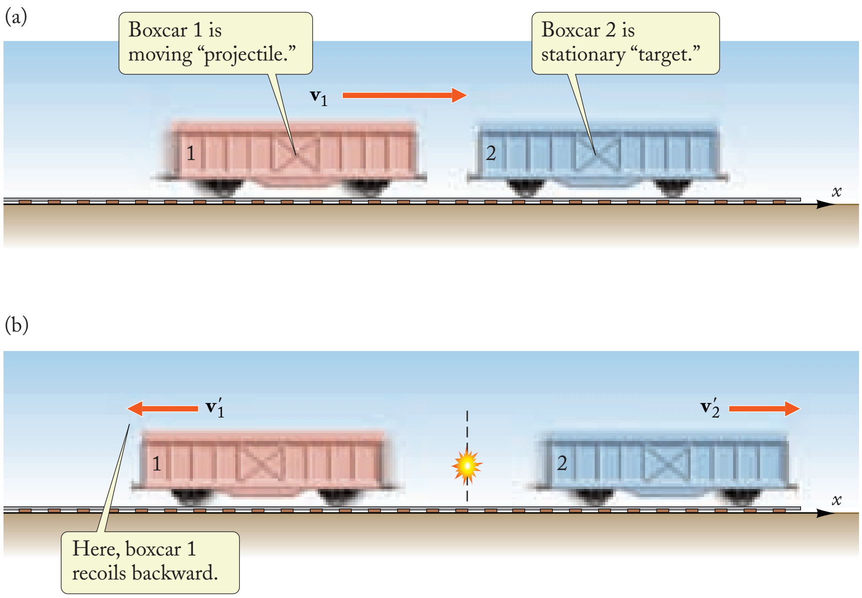


FIGURE 11.5 (a) Initially, boxcar 1 is moving toward the right, and boxcar 2 is stationary. (b) After the collision, boxcar 1 is moving toward the left, and boxcar 2 is moving toward the right.

Note that if the mass of the target is much larger than the mass of the projectile, then m_1 can be neglected compared with m_2. Equation (11.13) then becomes

$$v_1' \approx -\frac{m_2}{m_2} v_1 = -v_1 \tag{11.15}$$

and Eq. (11.14) becomes

$$v_2' \approx \frac{2m_1}{m_2} v_1 \approx 0 \tag{11.16}$$

This means the projectile bounces off the target with a reversed velocity and the target remains nearly stationary (as in the case of the Super Ball bouncing off the wall; see Example 3).

Conversely, if the mass of the projectile is much larger than the mass of the target, then m_2 can be neglected compared with m_1, and Eqs. (11.13) and (11.14) become

$$v_1' \approx \frac{m_1}{m_1} v_1 = v_1 \tag{11.17}$$

and

$$v_2' \approx \frac{2m_1}{m_1} v_1 = 2v_1 \tag{11.18}$$

This means that the projectile plows along with unchanged velocity and the target bounces off with *twice* the speed of the incident projectile. For example, when a (heavy) golf club strikes a golf ball, the ball bounces away at twice the speed of the club (see Fig. 11.6).

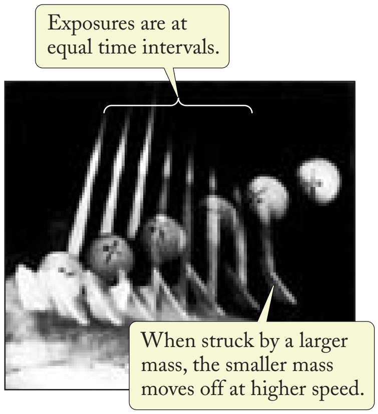


FIGURE 11.6 Impact of club on golf ball. By inspection of this multiple-exposure photograph, we see that the speed of the ball is larger than the initial speed of the club.

Also, if the two masses are equal, Eqs. (11.13) and (11.14) give

$$v_1' = 0 \qquad \text{and} \qquad v_2' = v_1$$

Thus, the projectile stops and the target moves off with the projectile's initial speed. This is common in a head-on collision in billiards, and is also realized in certain pendulum toys (see Discussion Question 9 at the end of the chapter).

Finally, if both particles involved in a one-dimensional elastic collision are initially moving ($v_1 \neq 0$ and $v_2 \neq 0$), conservation of the total momentum and the total kinetic energy can again be applied to uniquely determine the final velocities. The results are more complicated, but they are obtained in the same manner as in the stationary target case above.

✔ Checkup 11.2

In the following questions assume that a projectile traveling in the direction of the positive x axis strikes a stationary target head-on and the collision is elastic.

QUESTION 1: Under what conditions will the velocity of the projectile be positive after the collision? Negative?

QUESTION 2: Can the speed of recoil of the target ever exceed twice the speed of the incident projectile?

QUESTION 3: For an elastic collision, the kinetic energies before and after the collision are the same. Is the kinetic energy *during* the collision also the same?

QUESTION 4: A marble with velocity v_1 strikes a stationary, identical marble elastically and head-on. The final velocities of the shot and struck marbles are, respectively:

(A) $\frac{1}{2}v_1$; $\frac{1}{2}v_1$ (B) v_1; $2v_1$ (C) $-v_1$; 0

(D) $-v_1$; $2v_1$ (E) 0; v_1

Online Concept 13 Tutorial

Online Concept 14 Tutorial

11.3 INELASTIC COLLISIONS IN ONE DIMENSION

If the collision is inelastic, kinetic energy is not conserved, and then the only conservation law that is applicable is the conservation of momentum. This, by itself, is insufficient to calculate the velocities of both particles after the collision. Thus, for most inelastic collisions, one of the final velocities must be measured in order for momentum conservation to provide the other. Alternatively, we must have some independent knowledge of the amount of kinetic energy lost. However, *if the collision is* **totally inelastic,** *so a maximum amount of kinetic energy is lost, then the common velocity of both particles after the collision can be calculated.*

totally inelastic collision

In a totally inelastic collision, the particles do not bounce off each other at all; instead, *the particles stick together,* like two automobiles that form a single mass of interlocking wreckage after a collision, or two railroad boxcars that couple together. Under these conditions, the velocities of both particles must coincide with the velocity of the center of mass. But the velocity of the center of mass *after* the collision is the same as the velocity of the center of mass *before* the collision, because there are no external forces and the acceleration of the center of mass is zero [see Eq. (10.40)]. We again consider a stationary target, so that before the collision the velocity of the target particle is zero ($v_2 = 0$) and the general equation [Eq. (10.37)] for the velocity of the center of mass yields

$$v_{\text{CM}} = \frac{m_1 v_1}{m_1 + m_2} \tag{11.19}$$

This must then be the final velocity of both particles after a totally inelastic collision:

final velocities in totally inelastic collision with stationary target

$$v_1' = v_2' = v_{\text{CM}} = \frac{m_1 v_1}{m_1 + m_2} \tag{11.20}$$

We have already come across an instance of this formula in Example 3 of Chapter 10.

EXAMPLE 5 Suppose that the two boxcars of Example 4 couple during the collision and remain locked together (see Fig. 11.7). What is the velocity of the combination after the collision? How much kinetic energy is dissipated during the collision?

SOLUTION: Since the boxcars remain locked together, this is a totally inelastic collision. With $m_1 = 20$ tons, $m_2 = 65$ tons, and $v_1 = 5.0$ m/s, Eq. (11.19) gives us the velocity of the center of mass:

$$v_{\text{CM}} = \frac{m_1 v_1}{m_1 + m_2} = \frac{20 \text{ tons} \times 5.0 \text{ m/s}}{20 \text{ tons} + 65 \text{ tons}} = 1.2 \text{ m/s}$$

and this must be the velocity of the coupled cars after the collision.

The kinetic energy before the collision is that of the moving boxcar,

$$\tfrac{1}{2} m_1 v_1^2 = \tfrac{1}{2} \times 20\,000 \text{ kg} \times (5.0 \text{ m/s})^2 = 2.5 \times 10^5 \text{ J}$$

and the kinetic energy after the collision is that of the two coupled boxcars,

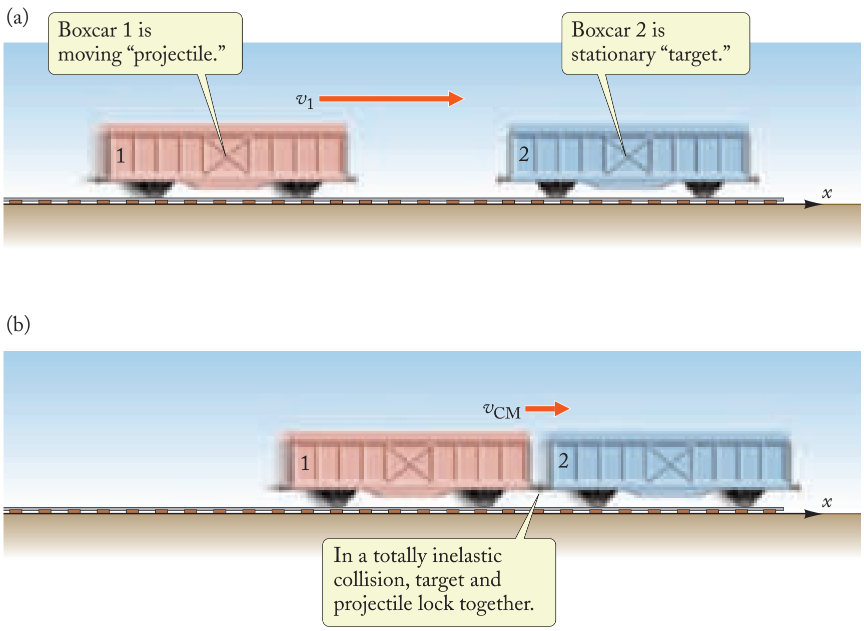


FIGURE 11.7 (a) Initially, boxcar 1 is moving toward the right, and boxcar 2 is stationary, as in Fig. 11.5. (b) After the collision, the boxcars remain locked together. Their common velocity must be the velocity of the center of mass.

$$\tfrac{1}{2}m_1v_{CM}^2 + \tfrac{1}{2}m_2v_{CM}^2 = \tfrac{1}{2}(m_1 + m_2)v_{CM}^2$$
$$= \tfrac{1}{2}(20\,000\text{ kg} + 65\,000\text{ kg}) \times (1.2\text{ m/s})^2 = 0.61 \times 10^5\text{ J}$$

Thus, the loss of kinetic energy is

$$2.5 \times 10^5\text{ J} - 0.61 \times 10^5\text{ J} = 1.9 \times 10^5\text{ J} \qquad (11.21)$$

This energy is absorbed by friction in the bumpers during the coupling of the boxcars.

EXAMPLE 6 Figure 11.8a shows a **ballistic pendulum,** a device once commonly used to measure the speeds of bullets. The pendulum consists of a large block of wood of mass m_2 suspended from thin wires. Initially, the pendulum is at rest. The bullet, of mass m_1, strikes the block horizontally and remains stuck in it. The impact of the bullet puts the block in motion, causing it to swing upward to a height h (see Fig. 11.8b), where it momentarily stops. In a test of a Springfield rifle firing a bullet of 9.7 g, a ballistic pendulum of 4.0 kg swings up to a height of 19 cm. What was the speed of the bullet before impact?

SOLUTION: The collision of the bullet with the wood is totally inelastic. Hence, immediately after the collision, bullet and block move horizontally with the velocity of the center of mass:

$$v_{CM} = \frac{m_1v_1}{m_1 + m_2} \qquad (11.22)$$

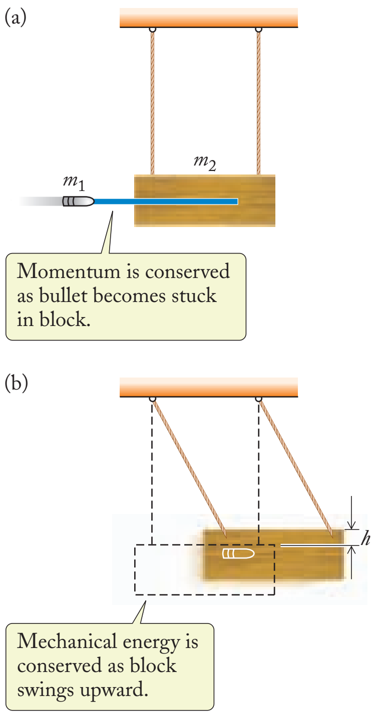


FIGURE 11.8 (a) Before the bullet strikes, the block of wood is at rest. (b) After the bullet strikes, the block, with the embedded bullet, moves toward the right and swings upward to a height h.

After the collision is over, during the subsequent swinging motion of the pendulum, the total mechanical energy (kinetic plus potential) is conserved. At the bottom of the swing, the energy is kinetic, $\frac{1}{2}(m_1 + m_2)v_{CM}^2$; and at the top of the swing at height h, it is potential, $(m_1 + m_2)gh$. Hence, conservation of the total mechanical energy tells us that

$$\frac{1}{2}(m_1 + m_2)v_{CM}^2 = (m_1 + m_2)gh \tag{11.23}$$

If we divide this by $(m_1 + m_2)$ and take the square root of both sides, we find

$$v_{CM} = \sqrt{2gh} \tag{11.24}$$

Substitution of this into Eq. (11.22) yields

$$\sqrt{2gh} = \frac{m_1 v_1}{m_1 + m_2} \tag{11.25}$$

which we can solve for v_1, with the result

$$\begin{aligned} v_1 &= \frac{m_1 + m_2}{m_1}\sqrt{2gh} \\ &= \frac{0.0097\ \text{kg} + 4.0\ \text{kg}}{0.0097\ \text{kg}}\sqrt{2 \times 9.81\ \text{m/s}^2 \times 0.19\ \text{m}} \\ &= 800\ \text{m/s} \end{aligned} \tag{11.26}$$

COMMENT: Note that during the collision, momentum is conserved but not kinetic energy (the collision is totally inelastic); and that during the swinging motion, the total mechanical energy is conserved, but not momentum (the swinging motion proceeds under the influence of the "external" forces of gravity and the tensions in the wires).

✔ Checkup 11.3

QUESTION 1: In a totally inelastic collision, do both particles lose kinetic energy?

QUESTION 2: Consider a collision between two particles of equal masses and of opposite velocities. What is the velocity after this collision if the collision is totally inelastic? If the collision is elastic?

QUESTION 3: Under what conditions is the velocity of the particles after a totally inelastic collision equal to one-half the velocity of the incident projectile? (Assume a stationary target.)

QUESTION 4: Does the length of the suspension wires affect the operation of the ballistic pendulum described in Example 6?

QUESTION 5: A particle is traveling in the positive x direction with speed v. A second particle with one-half the mass of the first is traveling in the opposite direction with the same speed. The two experience a totally inelastic collision. The final x component of the velocity is:

(A) 0 (B) $\frac{1}{3}v$ (C) $\frac{1}{2}v$ (D) $\frac{2}{3}v$ (F) v

11.4 COLLISIONS IN TWO AND THREE DIMENSIONS

In the previous sections, we have focused on collisions on a straight line, in one dimension. Collisions in two or three dimensions are more difficult to analyze, because the conservation laws for momentum and energy do not provide sufficient information to determine the final velocities completely in terms of the initial velocities. Momentum is always conserved during a collision, and this conservation provides one equation for each of the x, y, and z directions. If it is known that the collision is totally elastic, then conservation of the total kinetic energy provides another equation. However, these are not enough to determine the three final velocity components for each and every particle. Some information concerning the final velocities must also be known or measured.

The case of totally inelastic collisions is an exception: in this case, the conservation of momentum determines the outcome completely, even in two or three dimensions. The particles stick together, and their final velocities coincide with the velocity of the center of mass, as illustrated by the following example. The subsequent example explores a case where the solution exploits some knowledge of the final velocities.

EXAMPLE 7 A red automobile of mass 1100 kg and a green automobile of mass 1300 kg collide at an intersection. Just before this collision, the red automobile was traveling due east at 34 m/s, and the green automobile was traveling due north at 15 m/s (see Fig. 11.9). After the collision, the wrecked automobiles remain joined together, and they skid on the pavement with locked wheels. What is the direction of the skid?

Concepts in Context

SOLUTION: The final velocity of the wreck coincides with the final velocity of the center of mass, which is the same as the initial velocity of the center of mass. According to Eq. (10.37), this velocity is

final velocity in totally inelastic collision

$$\mathbf{v}_{\text{CM}} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \tag{11.27}$$

With the x axis eastward and the y axis northward, the initial velocity $\mathbf{v}_1$ of the red automobile has an x component but no y component, and the initial velocity $\mathbf{v}_2$ of the green automobile has a y component but no x component. Hence the x component of $\mathbf{v}_{\text{CM}}$ is

$$v_{\text{CM},x} = \frac{m_1 v_1}{m_1 + m_2} = \frac{1100\text{ kg} \times 34\text{ m/s}}{1100\text{ kg} + 1300\text{ kg}} = 16\text{ m/s}$$

and the y component of $\mathbf{v}_{\text{CM}}$ is

$$v_{\text{CM},y} = \frac{m_2 v_2}{m_1 + m_2} = \frac{1300\text{ kg} \times 15\text{ m/s}}{1100\text{ kg} + 1300\text{ kg}} = 8.1\text{ m/s}$$

The angle between the direction of this velocity and the x axis is given by

$$\tan\theta = \frac{v_{\text{CM},y}}{v_{\text{CM},x}} = \frac{8.1\text{ m/s}}{16\text{ m/s}} = 0.51$$

from which

$$\theta = 27°$$

Since the x axis is eastward, this is 27° north of east.

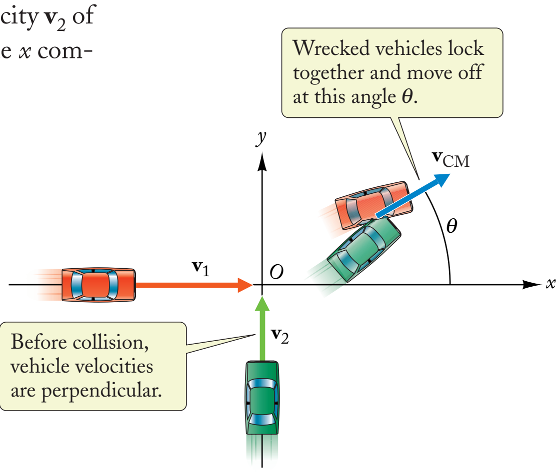


FIGURE 11.9 An automobile collision. Before the collision, the velocities of the automobiles were $\mathbf{v}_1$ and $\mathbf{v}_2$. After the collision, both velocities are $\mathbf{v}_{\text{CM}}$.

EXAMPLE 8 In an atomic collision experiment, or "scattering" experiment, a helium ion of mass $m_1 = 4.0$ u with speed $v_1 = 1200$ m/s strikes an oxygen (O_2) molecule of mass $m_2 = 32$ u which is initially at rest (see Fig. 11.10a). The helium ion exits the collision at 90° from its incident direction with one-fourth of its original kinetic energy. What is the recoil speed of the oxygen molecule? What fraction of the total kinetic energy is lost during the collision? [This energy is lost to the internal (vibrational and rotational) motions of the oxygen molecule.]

SOLUTION: In the absence of external forces, momentum is always conserved. If we choose the direction of incident motion along the x axis, then for 90° scattering, we can choose the direction in which the helium ion exits (the direction of $\mathbf{v}_1'$) to be along the y axis (see Fig. 11.10b). Conservation of momentum in the two directions then requires

$$\text{for } x \text{ direction: } m_1 v_1 = m_2 v_{2x}'$$

$$\text{for } y \text{ direction: } 0 = m_1 v_1' + m_2 v_{2y}'$$

Since the helium ion exits with one-fourth of its initial kinetic energy,

$$\tfrac{1}{2} m_1 v_1'^2 = \tfrac{1}{4} \times \tfrac{1}{2} m_1 v_1^2$$

or

$$v_1' = \tfrac{1}{2} v_1$$

Substituting this v_1' and the given $m_2 = 8m_1$ into the x and y components of the momentum gives for the velocity of the oxygen molecule:

$$v_{2x}' = \frac{m_1}{m_2} v_1 = \frac{1}{8} v_1 = \frac{1}{8} \times 1200 \text{ m/s} = 150 \text{ m/s}$$

$$v_{2y}' = -\frac{m_1}{m_2} v_1' = -\frac{1}{8} \times \frac{1}{2} v_1 = -\frac{1}{16} \times 1200 \text{ m/s} = -75 \text{ m/s}$$

The speed of recoil of the oxygen molecule is thus

$$v_2' = \sqrt{v_{2x}'^2 + v_{2y}'^2} = \sqrt{(150 \text{ m/s})^2 + (-75 \text{ m/s})^2} = 170 \text{ m/s}$$

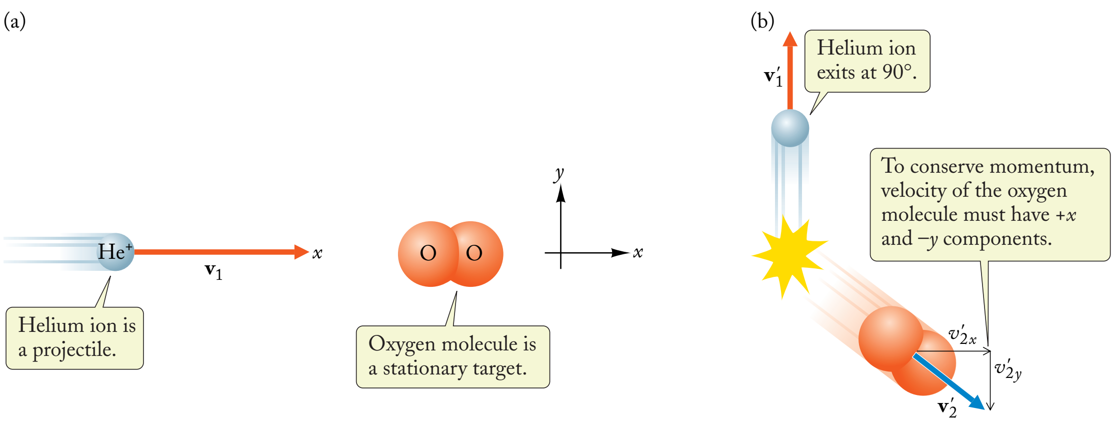


FIGURE 11.10 (a) A helium ion with velocity $\mathbf{v}_1 = v_{1x}\mathbf{i}$ is moving toward a stationary oxygen molecule. (b) After the collision, the helium ion exits perpendicular to its incident direction with velocity $\mathbf{v}_1'$, while the oxygen molecule acquires a velocity $\mathbf{v}_2' = v_{2x}'\mathbf{i} + v_{2y}'\mathbf{j}$.

The fraction of kinetic energy lost is the amount of kinetic energy lost divided by the original kinetic energy:

$$[\text{fraction lost}] = \frac{K - K'}{K} = \frac{\frac{1}{2}m_1 v_1^2 - \left(\frac{1}{4} \times \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2'^2\right)}{\frac{1}{2}m_1 v_1^2}$$

$$= 1 - \frac{1}{4} - \frac{m_2}{m_1}\frac{v_2'^2}{v_1^2} = \frac{3}{4} - \frac{32}{4.0} \times \frac{(170\ \text{m/s})^2}{(1200\ \text{m/s})^2}$$

$$= 0.59$$

Thus, about 59% of the helium ion's initial kinetic energy is lost to internal motions of the oxygen molecule during the collision.

✔ Checkup 11.4

QUESTION 1: A car traveling south collides with and becomes entangled with a car of the same mass and speed heading west. In what direction does the wreckage emerge from the collision?

QUESTION 2: An object at rest explodes into three pieces; one travels due west and another due north. In which quadrant of directions does the third piece travel?

(A) Northeast (B) Southeast (C) Southwest (D) Northwest

PROBLEM-SOLVING TECHNIQUES — CONSERVATION OF ENERGY AND MOMENTUM IN COLLISIONS

For solving problems involving collisions, it is essential to know what conservation laws are applicable. The following table summarizes the conservation laws applicable for different collisions:

TYPE OF COLLISION	CONSERVATION OF KINETIC ENERGY	CONSERVATION OF MOMENTUM	COMMENTS
Elastic	Yes	Yes	For a one-dimensional collision, energy and momentum conservation determine the final velocities in terms of the initial velocities. For a 2- or 3-dimensional collision, there is not enough information in the initial velocities alone to determine the final velocities uniquely.
Totally inelastic	No	Yes	The two colliding bodies stick together, and momentum conservation determines the final velocities (in 1, 2, and 3 dimensions).
Inelastic	No	Yes	If the collision is not totally inelastic, there is not enough information in the initial velocities alone to determine the final velocities. Some information about the energy loss and final velocities must also be known.

SUMMARY

PHYSICS IN PRACTICE Automobile Collisions **(page 343)**

PROBLEM-SOLVING TECHNIQUES **(page 353)**
Conservation of Energy and Momentum in Collisions

IMPULSE

$$\mathbf{I} = \int_0^{\Delta t} \mathbf{F}\,dt = \mathbf{p}' - \mathbf{p} \qquad (11.1)$$

AVERAGE FORCE IN COLLISION

$$\overline{\mathbf{F}} = \frac{\mathbf{p}' - \mathbf{p}}{\Delta t} \qquad (11.4)$$

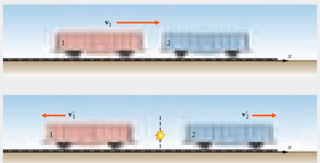
ALL COLLISIONS
The total momentum is conserved.

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \cdots = m_1\mathbf{v}_1' + m_2\mathbf{v}_2' + \cdots \qquad (11.6)$$

ELASTIC COLLISION
The total kinetic energy is conserved.

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \cdots = \tfrac{1}{2}m_1v_1'^2 + \tfrac{1}{2}m_2v_2'^2 + \cdots \qquad (11.7)$$

VELOCITIES IN ONE-DIMENSIONAL ELASTIC COLLISION WITH STATIONARY TARGET



Before: $v_1 \neq 0, \quad v_2 = 0$

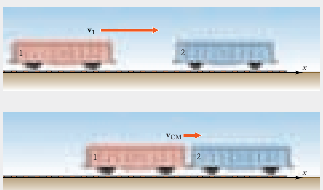
After: $$v_1' = \frac{m_1 - m_2}{m_1 + m_2}v_1 \qquad (11.13)$$

$$v_2' = \frac{2m_1}{m_1 + m_2}v_1 \qquad (11.14)$$

INELASTIC COLLISION
Kinetic energy is not conserved.

TOTALLY INELASTIC COLLISION
The colliding particles stick together.

VELOCITIES IN TOTALLY INELASTIC COLLISION
(1, 2, or 3 dimensions).



Before: $\mathbf{v}_1$ and $\mathbf{v}_2$

After: $$\mathbf{v}_1' = \mathbf{v}_2' = \mathbf{v}_{CM} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \qquad (11.27)$$

QUESTIONS FOR DISCUSSION

1. According to the data given in Example 1, what percentage of the initial kinetic energy does the automobile have after the collision?
2. A (foolish) stuntman wants to jump out of an airplane at high altitude without a parachute. He plans to jump while tightly encased in a strong safe which can withstand the impact on the ground. How would you convince the stuntman to abandon this project?
3. In the crash test shown in the photographs of Fig. 11.1, anthropomorphic dummies were riding in the automobile. These dummies were (partially) restrained by seat belts, which limited their motion relative to the automobile. How would the motion of the dummies have differed from that shown in these photographs if they had not been restrained by seat belts?
4. For the sake of safety, would it be desirable to design automobiles so that their collisions are elastic or inelastic?
5. Two automobiles have collided at a north–south east–west intersection. The skid marks their tires made after the collision point roughly northwest. One driver claims he was traveling west; the other driver claims he was traveling south. Who is lying?
6. Statistics show that, on the average, the occupants of a heavy ("full-size") automobile are more likely to survive a crash than those of a light ("compact") automobile. Why would you expect this to be true?
7. In Joseph Conrad's tale "Gaspar Ruiz", the hero ties a cannon to his back and, hugging the ground on all fours, fires several shots at the gate of a fort. How does the momentum absorbed by Ruiz compare with that absorbed by the gate? How does the energy absorbed by Ruiz compare with that absorbed by the gate?
8. Give an example of a collision between two bodies in which *all* of the kinetic energy is lost to inelastic processes.
9. Explain the operation of the five-pendulum toy, called Newton's cradle, shown in Fig. 11.11.
10. In order to split a log with a small ax, you need a greater impact speed than you would need with a large ax. Why? If the energy required to split the log is the same in both cases, why is it more tiring to use the small ax? (Hint: Think about the kinetic energy of your arms.)
11. If you throw an (elastic) baseball at an approaching train, the ball will bounce back at you with an increased speed. Explain.
12. You are investigating the collision of two automobiles at an intersection. The automobiles remained joined together after this collision, and their wheels made measurable skid marks on the pavement before they came to rest. Assume that during skidding all the wheels remained locked so that the deceleration was entirely due to sliding friction. You know the direction of motion of the automobiles before the collision (drivers are likely to be honest about this), but you do not know the speeds (drivers are likely to be dishonest about this). What do you have to measure at the scene of the accident to calculate the speeds of both the automobiles before the collision?
13. You are sitting in your car, stopped at an intersection. You notice another car approaching from behind, and you notice this car is not slowing down and is going to ram you. Because the time to impact is short, you have only two choices: push hard on your brake, or take your foot off the brake and give your car freedom to roll. Which of these tactics will minimize damage to yourself? Which will minimize damage to your car? Which will minimize damage to the other car?

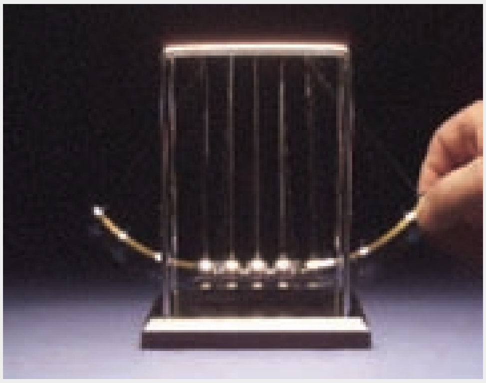


FIGURE 11.11 Newton's cradle.

PROBLEMS

11.1 Impulsive Forces†

1. A stuntman of mass 77 kg "belly-flops" on a shallow pool of water from a height of 11 m. When he hits the pool, he comes to rest in about 0.050 s. What is the impulse that the water and the bottom of the pool deliver to his body during this time interval? What is the time-average force?
2. A large ship of 7.0×10^5 metric tons steaming at 20 km/h runs aground on a reef, which brings it to a halt in 5.0 s. What is the impulse delivered to the ship? What is the average force on the ship? What is the average deceleration?

† For help, see Online Concept Tutorial 13 at www.wwnorton.com/physics

3. The photographs in Fig. 11.1 show the impact of an automobile on a rigid wall.
   (a) Measure the positions of the automobile on these photographs and calculate the average velocity for each of the 20×10^{-3}-s intervals between one photograph and the next; calculate the average acceleration for each time interval from the change between one average velocity and the next.
   (b) The mass of this automobile is 1700 kg. Calculate the average force for each time interval.
   (c) Make a plot of this force as a function of time and find the impulse by estimating the area under this curve.
4. The "land divers" of Pentecost Island (Vanuatu) jump from platforms 21 m high. Long liana vines tied to their ankles jerk them to a halt just short of the ground. If the pull of the liana takes 0.020 s to halt the diver, what is the average acceleration of the diver during this time interval? If the mass of the diver is 64 kg, what is the corresponding average force on his ankles?
5. A shotgun fires a slug of lead of mass 28 g with a muzzle velocity of 450 m/s. The slug acquires this velocity while it accelerates along the barrel of the shotgun, which is 70 cm long.
   (a) What is the impulse the shotgun gives the slug?
   (b) Estimate the average impulsive force; assume constant acceleration of the slug along the barrel.
6. A rule of thumb for automobile collisions against a rigid barrier is that the collision lasts about 0.11 s, for any initial speed and for any model of automobile (for instance, the collision illustrated in Fig. 11.1 lasted 0.120 s, in rough agreement with this rule of thumb). Accordingly, the deceleration experienced by an automobile during a collision is directly proportional to the change of velocity Δv (with a constant factor of proportionality), and therefore Δv can be regarded as a measure of the severity of the collision.
   (a) If the collision lasts 0.11 s, what is the average deceleration experienced by an automobile in an impact on a rigid barrier at 55 km/h? 65 km/h? 75 km/h?
   (b) For each of these speeds, what is the crush distance of the front end of the automobile? Assume constant deceleration for this calculation.
   (c) For each of these speeds, what is the average force the seat belt must exert to hold a driver of 75 kg in his seat during the impact?
7. Suppose that a seat-belted mother riding in an automobile holds a 10-kg baby in her arms. The automobile crashes and decelerates from 50 km/h to 0 in 0.10 s. What average force would the mother have to exert on the baby to hold it? Do you think she can do this?
8. In a test, an air force volunteer belted in a chair placed on a rocket sled was decelerated from 143 km/h to 0 in a distance of 5.5 m. Assume that the mass of the volunteer was 75 kg, and assume that the deceleration was uniform. What was the deceleration? What impulse did the seat belt deliver to the volunteer? What time-average force did the seat belt exert?
9. Assume that the Super Ball of Example 3 has a mass of 60 g and is initially traveling with speed 15 m/s. For simplicity, assume that the acceleration is constant while the ball is in contact with the wall. After touching the wall, the center of mass of the Super Ball moves 0.50 cm toward the wall, and then moves the same distance away to complete the bounce. What is the impulse delivered by the wall? What is the time-average force?
10. A 0.50-kg hammerhead moving at 2.0 m/s strikes a board and stops in 0.020 s. What is the impulse delivered to the board? What is the time-average force?
11. A soccer player applies an average force of 180 N during a kick. The kick accelerates a 0.45-kg soccer ball from rest to a speed of 18 m/s. What is the impulse imparted to the ball? What is the collision time?
12. When an egg ($m = 50$ g) strikes a hard surface, the collision lasts about 0.020 s. The egg will break when the average force during impact exceeds 3.0 N. From what minimum height will a dropped egg break?

*13. The net force on a body varies with time according to $F_x = 3.0t + 0.5t^2$, where F_x is in newtons and t is in seconds. What is the impulse imparted to the body during the time interval $0 \leq t \leq 3.0$ s?

*14. Suppose that in a baseball game, the batter succeeds in hitting the baseball thrown toward him by the pitcher. Suppose that just before the bat hits, the ball is moving toward the batter horizontally with a speed of 35 m/s; and that after the bat has hit, the ball is moving away from the batter and upward at an angle of 50° and finally lands on the ground 110 m away. The mass of the ball is 0.15 kg. From this information, calculate the magnitude and direction of the impulse the ball receives in the collision with the bat. Neglect air friction and neglect the initial height of the ball above the ground.

*15. Bobsleds racing down a bobsled run often suffer glancing collisions with the vertical walls enclosing the run. Suppose that a bobsled of 600 kg traveling at 120 km/h approaches a wall at an angle of 3.0° and bounces off at the same angle. Subsequent inspection of the wall shows that the side of the bobsled made a scratch mark of length 2.5 m along the wall. From these data, calculate the time interval the bobsled was in contact with the wall, and calculate the average magnitude of the force that acted on the side of the bobsled during the collision.

11.2 Elastic Collisions in One Dimension†

16. A particle moving at 10 m/s along the x axis collides elastically with another particle moving at 5.0 m/s in the *same* direction along the x axis. The particles have equal masses. What are their speeds after this collision?
17. In a lecture demonstration, two masses collide elastically on a a frictionless air track. The moving mass (projectile) is 60 g,

† For help, see Online Concept Tutorial 13 and 14 at www.wwnorton.com/physics

and the initially stationary mass (target) is 120 g. The initial velocity of the projectile is 0.80 m/s.

(a) What is the velocity of each mass after the collision?

(b) What is the kinetic energy of each mass before the collision? After the collision?

18. A target sometimes used for target shooting with small bullets consists of a steel disk hanging on a rod which is free to swing on a pivot (in essence, a pendulum). The collision of the bullet with the steel disk is not elastic and not totally inelastic, but somewhere between these extremes. Suppose that a .22-caliber bullet of 15 g and initial speed 600 m/s strikes such a target of mass 40 g. With what velocity would this bullet bounce back (ricochet) if the collision were elastic? Assume that the disk acts like a free particle during the collision.

19. The impact of the head of a golf club on a golf ball can be approximately regarded as an elastic collision. The mass of the head of the golf club is 0.15 kg, and that of the ball is 0.045 kg. If the ball is to acquire a speed of 60 m/s in the collision, what must be the speed of the club before impact?

20. Suppose that a neutron in a nuclear reactor initially has an energy of 4.8×10^{-13} J. How many head-on collisions with carbon nuclei at rest must this neutron make before its energy is reduced to 1.6×10^{-19} J? The collisions are elastic.

21. The impact of a hammer on a nail can be regarded as an elastic collision between the head of the hammer and the nail. Suppose that the mass of the head of the hammer is 0.50 kg and it strikes a nail of mass 12 g with an impact speed of 5.0 m/s. How much energy does the nail acquire in this collision?

22. Consider two coins: a quarter of mass 5.6 g and a dime of mass 2.3 g. If one is sliding at 2.0 m/s on a frictionless surface and hits the other head-on, find the final velocities when either (a) the quarter or (b) the dime is the stationary target. Assume the collision is elastic.

23. Using a straw, a child shoots a series of small balls of mass 1.0 g with speed v at a block of mass 40 g on a frictionless surface. If the small balls elastically collide head-on with the block, how fast will the block be moving after five strikes?

24. A projectile of unknown mass and speed strikes a ball of mass $m =$ 0.15 kg initially at rest. The collision is head-on and elastic. The ball moves off at 1.50 m/s, and the projectile continues in its original direction at 0.50 m/s. What is the mass of the projectile? What was its original speed?

25. A marble of unknown mass m is shot at a larger marble of known mass M, initially at rest in the center of a circle. The collision is head-on and elastic. The smaller marble bounces backward and exits the circle in one-third of the time that it takes the larger marble to do so. What is the mass of the smaller marble? Neglect any rolling motion.

26. In materials science, **Rutherford backscattering** is used to determine the composition of materials. In such an experiment, alpha particles (helium nuclei, mass 4.0 u) of typical kinetic energy 1.6×10^{-13} J strike target nuclei at rest. The collisions are elastic and head-on. What is the recoil kinetic energy of the alpha particle when the target is (a) silicon ($m = 28$ u) and (b) copper ($m = 63$ u)?

*27. An automobile traveling at 60 km/h bumps into the rear of another automobile traveling at 55 km/h in the *same* direction. The mass of the first automobile is 1200 kg, and the mass of the second automobile is 1000 kg. If the collision is elastic, find the velocities of both automobiles immediately after this collision. (Hint: Solve this problem in a reference frame moving with a velocity equal to the initial velocity of one of the automobiles.)

*28. A projectile of 45 kg has a muzzle speed of 656.6 m/s when fired horizontally from a gun held in a rigid support (no recoil). What will be the muzzle speed (relative to the ground) of the same projectile when fired from a gun that is free to recoil? The mass of the gun is 6.6×10^3 kg. (Hint: The kinetic energy of the gun–projectile system is the same in both cases.)

*29. On a smooth, frictionless table, a billiard ball of velocity v is moving toward two other aligned billiard balls in contact (Fig. 11.12). What will be the velocity of each ball after impact? Assume that all balls have the same mass and that the collisions are elastic. Ignore any rotation of the balls. (Hint: Treat this as two successive collisions.)

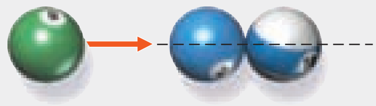


FIGURE 11.12 Three billiard balls along a line.

*30. Repeat Problem 29 but assume that the middle ball has twice the mass of each of the others.

*31. Two small balls are suspended side by side from two strings of length l so that they touch when in their equilibrium position (see Fig. 11.13). Their masses are m and $2m$, respectively. If the left ball (of mass m) is pulled aside and released from a height h, it will swing down and collide with the right ball (of mass $2m$) at the lowest point. Assume the collision is elastic.

(a) How high will each ball swing after the collision?

(b) Both balls again swing down, and they collide once more at the lowest point. How high will each swing after this second collision?

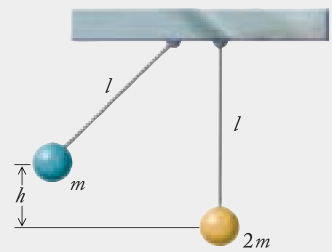


FIGURE 11.13 Two balls suspended from strings.

*32. If a spacecraft, or some other body, approaches a planet at fairly high speed at a suitable angle, it will whip around the planet and recede in a direction almost opposite to the initial direction of motion (Fig. 11.14). This can be regarded approximately as a one-dimensional "collision" between the satellite and the planet; the collision is elastic. In such a collision the satellite will gain kinetic energy from the planet, provided that it approaches the planet along a direction opposite to the direction of the planet's motion. This slingshot effect has been used to boost the speed of both Voyager spacecraft as they passed near Jupiter. Consider the head-on "collision" of a satellite of initial speed 10 km/s with the planet Jupiter, which has a speed of 13 km/s. (The speeds are measured in the reference frame of the Sun.) What is the maximum gain of speed that the satellite can achieve?

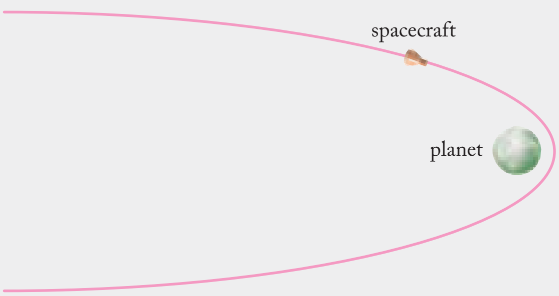


FIGURE 11.14 Spacecraft "colliding" with planet.

**33. A turbine wheel with curved blades is driven by a high-velocity stream of water that impinges on the blades and bounces off (Fig. 11.15). Under ideal conditions the velocity of the water particles after the collision with the blade is exactly zero, so that all of the kinetic energy of the water is transferred to the turbine wheel. If the speed of the water particles is 27 m/s, what is the ideal speed of the turbine blade? (Hint: Treat the collision of a water particle and the blade as a one-dimensional elastic collision.)

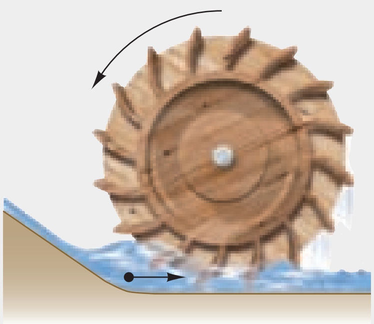


FIGURE 11.15 An undershot turbine wheel.

*34. A nuclear reactor designed and built in Canada (CANDU) contains heavy water (D_2O). In this reactor, the fast neutrons are slowed down by elastic collisions with the deuterium nuclei of the heavy-water molecule.

(a) By what factor will the speed of the neutron be reduced in a head-on collision with a deuterium nucleus? The mass of this nucleus is 2.01 u.

(b) After how many head-on collisions with deuterium nuclei will the speed be reduced by the same factor as in a single head-on collision with a proton?

**35. Because of brake failure, an automobile parked on a hill of slope 1:10 rolls 12 m downhill and strikes a parked automobile. The mass of the first automobile is 1400 kg, and the mass of the second automobile is 800 kg. Assume that the first automobile rolls without friction and that the collision is elastic.

(a) What are the velocities of both automobiles immediately after the collision?

(b) After the collision, the first automobile continues to roll downhill, with acceleration, and the second automobile skids downhill, with deceleration. Assume that the second automobile skids with all its wheels locked, with a coefficient of sliding friction 0.90. At what time after the first collision will the automobiles have another collision, and how far from the initial collision?

**36. (a) Show that for an elastic one-dimensional collision the relative velocity reverses during the collision; that is, show that $v_1' - v_2' = -v_1$ (for $v_2 = 0$).

(b) For a partially inelastic collision the relative velocity after the collision will have a smaller magnitude than the relative velocity before the collision. We can express this mathematically as $v_1' - v_2' = -ev_1$, where $e < 1$ is called the **coefficient of restitution.** For some kinds of bodies, the coefficient e is a constant, independent of v_1 and v_2. Show that in this case the final kinetic energy of the motion relative to the center of mass is less than the initial kinetic energy of this motion by a factor of e^2, that is, that $K' = e^2K$.

(c) Derive formulas analogous to Eqs. (11.13) and (11.14) for the velocities v_1' and v_2' in terms of v_1.

11.3 Inelastic Collisions in One Dimension†

37. In karate, the fighter makes the hand collide at high speed with the target; this collision is inelastic, and a large portion of the kinetic energy of the hand becomes available to do damage in the target. According to a crude estimate, the energy required to break a concrete block (28 cm × 15 cm × 1.9 cm supported only at its short edges) is of the order of 10 J. Suppose the fighter delivers a downward hammer-fist strike with a speed of 12 m/s to such a concrete block. In principle, is there enough energy to break the block? Assume that the fist has a mass of 0.4 kg.

38. According to a tall tale told by Baron Münchhausen, on one occasion, while cannon shots were being exchanged between a

† For help, see Online Concept Tutorial 13 and 14 at www.wwnorton.com/physics

besieged city and the enemy camp, he jumped on a cannonball as it was being fired from the city, rode the cannonball toward the enemy camp, and then, in midair, jumped onto an enemy cannonball and rode back to the city. The collision of Münchhausen and the enemy cannonball must have been inelastic, since he held on to it. Suppose that his speed just before hitting the enemy cannonball was 150 m/s southward and the speed of the enemy cannonball was 300 m/s northward. The mass of Münchhausen was 90 kg, and the mass of the enemy cannonball was 20 kg. What must have been the speed just after the collision? Do you think he made it back to the city?

39. As described in Problem 6, the change of velocity Δv of an automobile during a collision is a measure of the severity of the collision. Suppose that an automobile moving with an initial speed of 15 m/s collides with (a) an automobile of equal mass initially at rest, (b) an automobile of equal mass initially moving in the opposite direction at 15 m/s, or (c) a stationary rigid barrier. Assume that the collision is totally inelastic. What is Δv in each case?

40. A 25-kg boy on a 10-kg sled is coasting at 3.0 m/s on level ice toward his 30-kg sister. The girl jumps vertically and lands on her brother's back. What is the final speed of the siblings and sled? Neglect friction.

41. A 75-kg woman and a 65-kg man face each other on a frictionless ice pond. The woman holds a 5.0-kg "medicine ball." The woman throws the ball to the man with a horizontal velocity of 2.5 m/s relative to the ice. What is her recoil velocity? What is the man's velocity after catching the ball? The man then throws the ball horizontally to the woman at 3.0 m/s relative to himself at the instant before release. What is his final velocity? What is the woman's final velocity after catching it?

42. A 16-u oxygen atom traveling at 600 m/s collides head-on with another oxygen atom at rest. The two join and form an oxygen molecule. With what speed does the molecule move? What fraction of the original translational kinetic energy is transferred to internal energy of the molecule?

*43. A circus clown in a cannon is shot vertically upward with an initial speed of 12 m/s. After ascending 3.5 m, she collides with and grabs a performer sitting still on a trapeze. They ascend together and then fall. What is their speed when they reach the original launch height? The clown and trapeze artist have the same mass.

*44. As described in Problem 6, the change in velocity Δv of an automobile during a collision is a measure of the severity of the collision. For a collision between two automobiles of equal masses, Δv has the same magnitude for each automobile. But for a collision between automobiles of different masses, Δv is larger for the automobile of smaller mass. Suppose that an automobile of 800 kg moving with an initial speed of 15 m/s collides with (a) an automobile of 1400 kg initially at rest, (b) an automobile of 1400 kg initially moving in the opposite direction at 15 m/s, or (c) a stationary rigid barrier. Assume that the collision is totally inelastic. What is Δv in each case for each participating automobile?

*45. Two automobiles of 540 and 1400 kg collide head-on while moving at 80 km/h in opposite directions. After the collision the automobiles remain locked together.

(a) Find the velocity of the wreck immediately after the collision.

(b) Find the kinetic energy of the two-automobile system before and after the collision.

(c) The front end of each automobile crumples by 0.60 m during the collision. Find the acceleration (relative to the ground) of the passenger compartment of each automobile; make the assumption that these accelerations are constant during the collision.

*46. A speeding automobile strikes the rear of a parked automobile. After the impact the two automobiles remain locked together, and they skid along the pavement with all their wheels locked. An investigation of this accident establishes that the length of the skid marks made by the automobiles after the impact was 18 m; the mass of the moving automobile was 2200 kg and that of the parked automobile was 1400 kg, and the coefficient of sliding friction between the wheels and the pavement was 0.95.

(a) What was the speed of the two automobiles immediately after impact?

(b) What was the speed of the moving automobiles before impact?

*47. A proton of energy 8.0×10^{-13} J collides head-on with a proton at rest. How much energy is available for inelastic reactions between these protons?

*48. According to test procedures laid down by the National Highway Traffic Safety Administration, a stationary barrier (of very large mass) and a towed automobile are used for tests of front impacts (Fig. 11.16a), but a moving barrier of 1800 kg and a stationary, unbraked automobile are used for tests of rear impacts (Fig. 11.16b). Explain how this test with the moving barrier and the stationary automobile could be replaced by an equivalent test with a stationary barrier and an automobile towed *backward* at some appropriate speed. If the automobile has a mass of 1400 kg and the moving barrier has a speed of 8 km/h, what is the appropriate equivalent speed of the moving automobile towed backward to the stationary barrier? Assume the collision is inelastic.

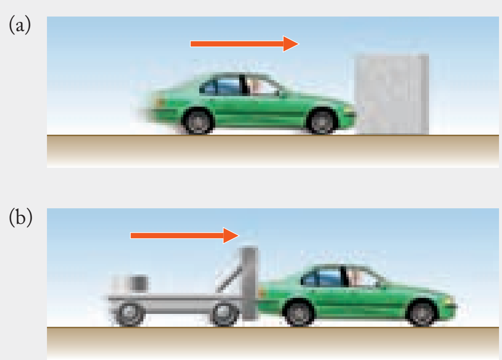


FIGURE 11.16 (a) Test procedure for front impact. (b) Test procedure for rear impact.

*49. Regard the two automobiles described in Example 7 as a system of two particles.

(a) What is the translational kinetic energy of the center of mass before the collision? After the collision?

(b) What is the total kinetic energy before the collision? After the collision?

*50. A cat crouches on the floor, at a distance of 1.2 m from a desk chair of height 0.45 m. The cat jumps onto the chair, landing with zero vertical velocity (this is standard procedure for cat jumps). The desk chair has frictionless coasters and rolls away when the cat lands. The mass of the cat is 4.5 kg, and the mass of the chair is 12 kg. What is the speed of recoil of the chair and cat?

*51. A crude but simple method for measuring the speed of a bullet is to shoot the bullet horizontally into a block of wood resting on a table. The block of wood will then slide until its kinetic energy is expended against the friction of the surface of the table. Suppose that a 3.0-kg block of wood slides a distance of 6.0 cm after it is struck by a bullet of 12 g. If the coefficient of sliding friction for the wood on the table is 0.60, what impact speed can you deduce for the bullet?

*52. Another way (not recommended) to measure the speed of bullets with a ballistic pendulum is to shoot a steady stream of bullets into the pendulum, which will push it aside and hold it in a rough equilibrium position at some angle. The speed can be calculated from this equilibrium angle. Suppose that you shoot .22-caliber bullets of mass 15 g into a 4.0-kg ballistic pendulum at the rate of 2 per second. You find that the equilibrium angle is 24°. What is the speed of the bullets?

*53. You shoot a .22-caliber bullet through a piece of wood sitting on a table. The piece of wood acquires a speed of 8.0 m/s, and the bullet emerges with a reduced speed. The mass of the bullet is 15 g, and its initial speed is 600 m/s; the mass of the piece of wood is 300 g.

(a) What is the change of velocity of the bullet?

(b) What is the change of kinetic energy of the bullet?

(c) What is the change of kinetic energy of the wood?

(d) Account for the missing kinetic energy.

*54. An automobile traveling at 50 km/h strikes the rear of a parked automobile. After the collision, the two automobiles remain joined together. The parked automobile skids with all its wheels locked, but the other automobile rolls with negligible friction. The mass of each automobile is 1300 kg, and the coefficient of sliding friction between the locked wheels and the pavement is 0.90. How far do the joined automobiles move before they stop? How long do they take to stop?

*55. You can make a fairly accurate measurement of the speed of a bullet by shooting it horizontally into a block of wood sitting on a fence. The collision of the bullet and the block is inelastic, and the block will fall off the fence and land on the ground at some distance from the bottom of the fence. The speed of the bullet is proportional to this distance. Suppose that a bullet of mass 15 g fired into a block of 4.0 kg sitting on a 1.8-m fence causes the block to land 1.4 m from the bottom of the fence. Calculate the speed of the bullet.

11.4 Collisions in Two and Three Dimensions

56. A cheetah intercepts a gazelle on the run, and grabs it (a totally inelastic collision). Just before this collision, the gazelle was running due north at 20 m/s, and the cheetah was running on an intercepting course of 45° east of north at 22 m/s. The mass of the cheetah is 60 kg, and the mass of the gazelle is 50 kg. What are the magnitude and the direction of the velocity of the entangled animals at the instant after the collision?

57. Two hydrogen atoms ($m = 1.0$ u) with equal speeds, initially traveling in perpendicular directions, collide and join together to form a hydrogen molecule. If 6.1×10^{-22} J of the initial kinetic energy is transferred to internal energy in the collision, what was the initial speed of the atoms?

*58. Two automobiles of equal masses collide at an intersection. One was traveling eastward and the other northward. After the collision, they remain joined together and skid, with locked wheels, before coming to rest. The length of the skid marks is 18 m, and the coefficient of friction between the locked wheels and the pavement is 0.80. Each driver claims his speed was less than 14 m/s (50 km/h) before the collision. Prove that at least one driver is lying.

*59. Two hockey players (see Fig. 11.17) of mass 80 kg collide while skating at 7.0 m/s. The angle between their initial directions of motion is 130°.

(a) Suppose that the players remain entangled and that the collision is totally inelastic. What is their velocity immediately after collision?

(b) Suppose that the collision lasts 0.080 s. What is the magnitude of the average acceleration of each player during the collision?

FIGURE 11.17 Collision of two hockey players.

*60. On July 27, 1956, the ships *Andrea Doria* (40 000 metric tons) and *Stockholm* (20 000 metric tons) collided in the fog south of Nantucket and remained locked together (for a while). Immediately before the collision the velocity of the *Andrea Doria* was 22 knots at 15° east of south and that of the *Stockholm* was 19 knots at 48° east of south (1 knot = 1 nmi/h = 1.85 km/h).

(a) Calculate the velocity (magnitude and direction) of the combined wreck immediately after the collision.

(b) Find the amount of kinetic energy that was converted into other forms of energy by inelastic processes during the collision.

(c) The large amount of energy absorbed by inelastic processes accounts for the heavy damage to both ships. How many kilograms of TNT would have to be exploded to obtain the same amount of energy as was absorbed by inelastic processes in the collision? The explosion of 1 kg of TNT releases 4.6×10^6 J.

*61. Your automobile of mass $m_1 = 900$ kg collides at a traffic circle with another automobile of mass $m_2 = 1200$ kg. Just before the collision, your automobile was moving due east and the other automobile was moving 40° south of east. After the collision the two automobiles remain entangled while they skid, with locked wheels, until coming to rest. Your speed before the collision was 14 m/s. The length of the skid marks is 17.4 m, and the coefficient of kinetic friction between the tires and the pavement is 0.85. Calculate the speed of the other automobile before the collision.

*62. Two billiard balls are placed in contact on a smooth, frictionless table. A third ball moves toward this pair with velocity v in the direction shown in Fig. 11.18. What will be the velocity (magnitude and direction) of the three balls after the collision? The balls are identical and the collisions are elastic.

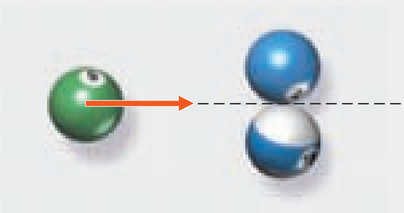


FIGURE 11.18 Three billiard balls.

*63. A billiard ball of mass m and radius R moving with speed v on a smooth, frictionless table collides elastically with an identical stationary billiard ball glued firmly to the surface of the table.

(a) Find a formula for the angular deflection suffered by the moving billiard ball as a function of the impact parameter b (defined in Fig. 11.19). Assume the billiard balls are very smooth so that the force during contact is entirely along the center-to-center line of the balls.

(b) Find a formula for the magnitude of the momentum change suffered by the billiard ball.

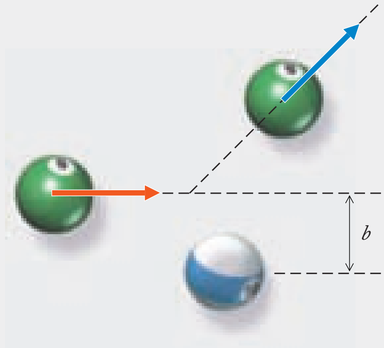


FIGURE 11.19 Two billiard balls.

*64. A coin of mass m slides along a table with speed v and elastically collides with a second, identical coin at rest. The first coin is deflected 60° from its original direction. What are the speeds of each of the two coins after the collision? At what angle does the second coin exit the collision?

*65. In a head-on elastic collision between a projectile and a stationary target of equal mass, we saw that the projectile stops. Show that if such a collision is not head-on, then the projectile and target final velocities are perpendicular (see Fig. 11.20). (Hint: Square the conservation of momentum equation, using $p^2 = \mathbf{p} \cdot \mathbf{p}$, and compare the resulting equation with the energy conservation equation.)

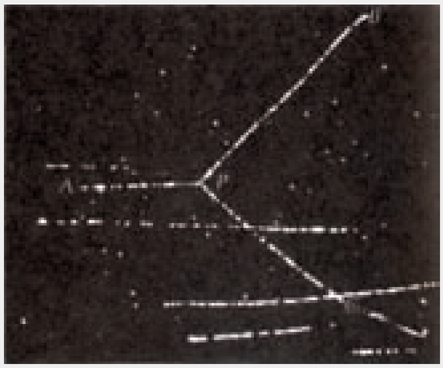


FIGURE 11.20 Elastic collision between two protons. The final velocities of the protons are perpendicular.

*66. In an elastic collision in two dimensions, the projectile has twice the mass of the stationary target. After the collision, the target moves off with three times the final speed of the projectile. Find the angle between the two final directions of motion.

REVIEW PROBLEMS

67. High-speed photography reveals that when a golf club hits a golf ball, the club and the ball typically remain in contact for 1.0×10^{-3} s and the ball acquires a speed of 70 m/s. The mass of the ball is 45 g. What is the impulse the club delivers to the ball? What is the time-average force?

68. In a remarkable incident, a 52-kg woman jumped from the 10th floor of an apartment building, fell 28 m, and landed on her side on soft earth in a freshly dug garden. She fractured her wrist and rib, but remained conscious and fully alert, and recovered completely after some time in a hospital. The earth was depressed 15 cm by her impact.
    (a) What was her impact speed?
    (b) Assuming constant deceleration upon contact with the ground, what was her deceleration?
    (c) What was the force of the ground on her body during deceleration?

69. An automobile approaching an intersection at 10 km/h bumps into the rear of another automobile standing at the intersection with its brakes off and its gears in neutral. The mass of the moving automobile is 1200 kg, and that of the stationary automobile is 700 kg. If the collision is elastic, find the velocities of both automobiles after the collision.

70. It has been reported (fallaciously) that the deer botfly can attain a maximum airspeed of 1318 km/h, that is, 366 m/s. Suppose that such a fly, buzzing along at this speed, strikes a stationary hummingbird and remains stuck in it. What will be the recoil velocity of the hummingbird? The mass of the fly is 2 g; the mass of the hummingbird is 50 g.

*71. A proton of energy 8.0×10^{-13} J collides head-on with a proton of energy 4.0×10^{-13} J moving in the opposite direction. How much energy is available for inelastic reactions between these protons?

*72. When a baseball bat strikes a ball, the impact can be approximately regarded as an elastic collision (the hands of the hitter have little effect on the short time the bat and the ball are in contact). Suppose that a bat of 0.85 kg moving horizontally at 30 m/s encounters a ball of 0.15 kg moving at 40 m/s in the opposite direction. We cannot directly apply the results of Section 11.2 to this collision, since *both* particles are in motion before collision ($\mathbf{v}_1 = 40$ m/s and $\mathbf{v}_2 = -30$ m/s). However, we can apply these results if we use a reference frame that moves at a velocity $\mathbf{V}_0 = -30$ m/s in the direction of the initial motion of the bat; in this reference frame, the initial velocity of the bat is zero ($\mathbf{v}_2 = 0$)
    (a) What is the initial velocity of the ball in this reference frame?
    (b) What are the final velocities of the ball and the bat, just after the collision?
    (c) What are these final velocities in the reference frame of the ground?

*73. A boy throws a baseball at another baseball sitting on a 1.5-m-high fence. The collision of the balls is elastic. The thrown ball moves horizontally at 20 m/s just before the head-on collision.
    (a) What are the velocities of the two balls just after the collision?
    (b) Where do the two balls land on the ground?

74. An automobile of 1200 kg traveling at 45 km/h strikes a moose of 400 kg standing on the road. Assume that the collision is totally inelastic (the moose remains draped over the front end of the automobile). What is the speed of the automobile immediately after this collision?

75. A ship of 3.0×10^4 metric tons steaming at 40 km/h strikes an iceberg of 8.0×10^5 metric tons. If the collision is totally inelastic, what fraction of the initial kinetic energy of the ship is converted into inelastic energy? What fraction remains as kinetic energy of the ship–iceberg system? Ignore the effect of the water on the motion of the ship and iceberg.

*76. When William Tell shot the apple off his son's head, the arrow remained stuck in the apple, which means the collision between the arrow and apple was totally inelastic. Suppose that the velocity of the arrow was horizontal at 80 m/s before it hit, the mass of the arrow was 40 g, and the mass of the apple was 200 g. Suppose Tell's son was 1.40 m high.
    (a) Calculate the velocity of the apple and arrow immediately after the collision.
    (b) Calculate how far behind the son the apple and arrow landed on the ground.

*77. Meteor Crater in Arizona (Fig. 11.21), a hole 180 m deep and 1300 m across, was gouged in the surface of the Earth by the impact of a large meteorite. The mass and speed of this meteorite have been estimated at 2.0×10^9 kg and 10 km/s, respectively, before impact.
    (a) What recoil velocity did the Earth acquire during this (inelastic) collision?

FIGURE 11.21 Meteor Crater in Arizona.

(b) How much kinetic energy was released for inelastic processes during the collision? Express this energy in the equivalent of tons of TNT; 1 ton of TNT releases 4.2×10^9 J upon explosion.

(c) Estimate the magnitude of the impulsive force.

*78. A black automobile smashes into the rear of a white automobile stopped at a stop sign. You investigate this collision and find that before the collision, the black automobile made skid marks 5.0 m long; after the collision the black automobile made skid marks 1.0 m long (in the same direction as the initial direction of motion), and the white automobile made skid marks 2.0 m long. Both automobiles made these skid marks with all their wheels. The mass of the black automobile is 1400 kg, and the mass of the white automobile is 800 kg. The coefficient of sliding friction between the wheels and the pavement is 0.90. From these data, deduce the speed of the black automobile just before the collision, and the speed before it started to brake.

*79. (a) Two identical small steel balls are suspended from strings of length l so they touch when hanging straight down, in their equilibrium position (Fig. 11.22). If we pull one of the balls back until its string makes an angle θ with the vertical and then let it go, it will collide elastically with the other ball. How high will the other ball rise?

(b) Suppose that instead of steel balls we use putty balls. They will then collide inelastically and remain stuck together. How high will the balls rise?

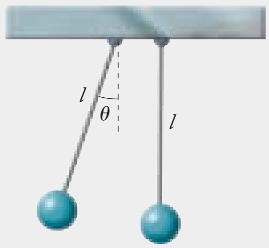


FIGURE 11.22 Two balls suspended from strings.

*80. While in flight, a peregrine falcon spots a pigeon flying 40 m below. The falcon closes its wings and, in free fall, dives on the pigeon and grabs it (a totally inelastic collision). The mass of the falcon is 1.5 kg, and the mass of the pigeon is 0.40 kg. Suppose that the velocity of the pigeon before this collision is horizontal, at 15 m/s, and the velocity of the falcon is vertical, equal to the free-fall velocity. What is the velocity (magnitude and direction) of both birds after the collision?

*81. On a freeway, a truck of 3500 kg collides with an automobile of 1500 kg that is trying to cut diagonally across the path of the truck. Just before the collision, the truck was traveling due north at 70 km/h, and the automobile was traveling at 30° west of north at 100 km/h. After the collision, the vehicles remain joined together.

(a) What is the velocity (magnitude and direction) of the joined vehicles immediately after the collision?

(b) How much kinetic energy is lost during the collision?

*82. Two asteroids of 1.0×10^7 kg and 8.0×10^7 kg, respectively, are initially at rest in interstellar space separated by a large distance. Their mutual gravitational attraction then causes them to fall toward each other on a straight line. Assume the asteroids are spheres of radius 100 m and 200 m, respectively.

(a) What is the velocity of each asteroid just before they hit? What is the kinetic energy of each? What is the total kinetic energy?

(b) The collision is totally inelastic. What is the velocity of the joined asteroids after they hit?

Answers to Checkups

Checkup 11.1

1. The front end should be soft and crushable to protect automobile occupants in a collision; this will spread the momentum change over a longer time, lowering the force experienced by the occupants.

2. The steel ball will exert a larger force, because it is less deformable than the golf ball. Thus, although the change in momentum (the impulse **I**) can be the same, the steel ball is in contact for a shorter time Δt, and so exerts a greater average force during that time ($\overline{\mathbf{F}} = \mathbf{I}/\Delta t$).

3. Because the collision is elastic, the ball will rebound with the same kinetic energy; as this energy gets converted to potential energy, the ball will rise up to the same height, 1 m, from which it was dropped before stopping.

4. No. Since the wad of gum stopped, kinetic energy was lost, and thus the collision was not elastic. We will later refer to such a collision (when the bodies stick together) as totally *in*elastic.

5. (C) 3×10^6 N. In a collision, each vehicle exerts an equal-magnitude, but opposite-direction, force on the other (an action–reaction pair), so the force exerted by the car on the truck is 3×10^6 N westward.

Checkup 11.2

1. As in the cases just discussed, and as in Eq. (11.13), where the projectile's final velocity is proportional to $m_1 - m_2$, the velocity of the projectile will be positive when it is more massive than the target ($m_1 > m_2$), and it will be negative when it is less massive than the target ($m_1 < m_2$).
2. No. As we saw in the cases just discussed, the speed of recoil of a massive target is very small; in the limit of a very light target, the speed approaches twice the speed of the projectile. For any values of m_1 and m_2, the final speed of the target (v_2'), given by Eq. (11.14), cannot exceed twice the projectile speed (v_1).
3. No; for instance, in the collision of the Super Ball and the wall, the ball is instantaneously at rest before it bounces back. The kinetic energy is transformed into elastic energy momentarily, and then converted back into kinetic energy.
4. (E) 0; v_1. As discussed above, when the masses of the target and projectile are identical, the speed of the projectile is zero after the collision [since we have $m_1 - m_2 = 0$ in Eq. (11.13)]. For identical masses, the target speed is equal to the initial speed of the projectile, v_1 [since we have $m_1 = m_2$ in Eq. (11.14)].

Checkup 11.3

1. No; for example, if the target is initially at rest, it gains kinetic energy.
2. Two particles of equal mass and opposite velocities have zero net momentum. Thus, in a totally inelastic collision, the composite particle has zero momentum, and thus zero velocity. In an elastic collision, the total kinetic energy is unchanged; since the net momentum is zero, the particles must again have opposite velocities. If we ignore the possibility that the particles might have passed through each other, then this means that their velocities were reversed by the collision.
3. The velocity of the joined particles after a totally inelastic collision is the velocity of the center of mass, $v_{CM} = m_1 v_1/(m_1 + m_2)$; this is equal to one-half of the velocity of the incident projectile when the masses of the target and projectile are equal, or $m_1 = m_2$.
4. No, assuming the wires are long enough to permit the upward motion of the pendulum to the maximum height h.
5. (B) $\frac{1}{3}v$. Momentum is conserved, so equating the initial and final momenta, we have $mv - (m/2)v = (\frac{3}{2}m)v'$, which implies $v' = \frac{1}{3}v$.

Checkup 11.4

1. Because the cars have equal mass and speed, the total momentum before and after this totally inelastic collision is directed due southwest.
2. (B) Southeast. This explosion is like a three-particle totally inelastic collision in reverse. Since the total momentum before the "collision" (explosion) is zero, so must it be afterward: the third particle must have momentum components which cancel the northward and westward momentum contributions of the other two particles; thus, the third particle travels in the southeast quadrant of directions.

Rotation of a Rigid Body

CHAPTER

12

CONCEPTS IN CONTEXT

Concepts *in* Context

This large centrifuge at the Sandia National Laboratory is used for testing the behavior of components of rockets, satellites, and reentry vehicles when subjected to high accelerations. The components to be tested are placed in a compartment in one arm of this centrifuge; the opposite arm holds a counterweight. The arms rotate at up to 175 revolutions per minute, and they generate a centripetal acceleration of up to 300*g*.

The concepts of this chapter permit us to answer several questions about this centrifuge:

? How are the speed and the centripetal acceleration at the end of an arm related to the rate of rotation? (Example 5, page 373)

? How do we determine the resistance that the centrifuge offers to changes in its rotational motion? (Example 12, part (a), page 383)

? How is the kinetic energy of the centrifuge arms related to the rate of rotation? (Example 12, part (b), page 383)

rigid body

A *body is* **rigid** *if the particles in the body do not move relative to one another.* Thus, the body has a fixed shape, and all its parts have a fixed position relative to one another. A hammer is a rigid body, and so is a baseball bat. A baseball is not rigid—when struck a blow by the bat, the ball suffers a substantial deformation; that is, different parts of the ball move relative to one another. However, the baseball can be regarded as a rigid body while it flies through the air—the air resistance is not sufficiently large to produce an appreciable deformation of the ball. This example indicates that whether a body can be regarded as rigid depends on the circumstances. No body is absolutely rigid; when subjected to a sufficiently large force, any body will suffer some deformation or perhaps even break into several pieces. In this chapter, we will ignore such deformations produced by the forces acting on bodies. We will examine the motion of bodies under the assumption that rigidity is a good approximation.

12.1 MOTION OF A RIGID BODY

A rigid body can simultaneously have two kinds of motion: it can change its position in space, and it can change its orientation in space. Change of position is translational motion; as we saw in Chapter 10, this motion can be conveniently described as motion of the center of mass. Change in orientation is rotational motion; that is, it is rotation about some axis.

As an example, consider the motion of a hammer thrown upward (see Fig. 12.1). The orientation of the hammer changes relative to fixed coordinates attached to the ground. Instantaneously, the hammer rotates about a horizontal axis, say, a horizontal axis that passes through the center of mass. In Fig. 12.1, this horizontal axis sticks out of the plane of the page and moves upward with the center of mass. The complete motion can then be described as a rotation of the hammer about this axis and a simultaneous translation of the axis along a parabolic path.

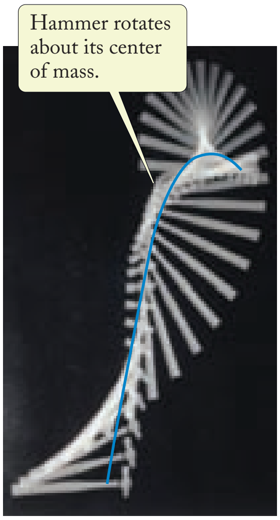


FIGURE 12.1 A hammer in free fall under the influence of gravity. The center of mass of the hammer moves with constant vertical acceleration g, just like a particle in free fall.

In this example of the thrown hammer, the axis of rotation always remains horizontal, out of the plane of the page. In the general case of motion of a rigid body, the axis of rotation can have any direction and can also change its direction. To describe such complicated motion, it is convenient to separate the rotation into three components along three perpendicular axes. The three components of rotation are illustrated by the motion of an aircraft (see Fig. 12.2): the aircraft can turn left or right (yaw), it can tilt to the left or the right (roll), and it can tilt its nose up or down (pitch). However, in the following sections we will usually not deal with this general case of rotation with three components; we will mostly deal only with the simple case of rotation about a fixed axis, such as the rotational motion of a fan, a roulette wheel, a compact disc, a swinging door, or a merry-go-round (see Fig. 12.3).

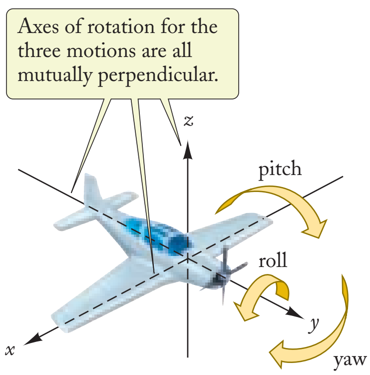


FIGURE 12.2 Pitch, roll, and yaw motions of an aircraft.

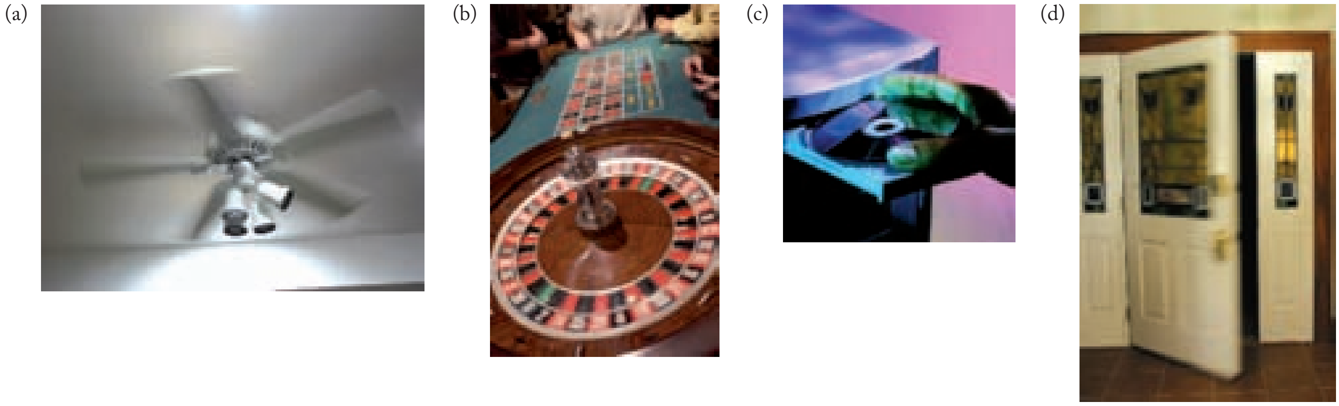


FIGURE 12.3 Some examples of rotational motion with a fixed axis (a) fan, (b) roulette wheel, (c) compact disc, (d) swinging door, (e) merry-go-round).

✔ Checkup 12.1

QUESTION 1: Characterize the following motions as translational, rotational, or both: swinging motion of door, motion of wheel of train, motion of propeller of airplane while in level flight.

QUESTION 2: Suppose that instead of selecting an axis through the center of mass of the hammer in Fig. 12.1, we select a parallel axis through the end of the handle. Can the motion still be described as rotation about this axis and a simultaneous translation of the axis along some path? Is this path parabolic?

QUESTION 3: Under what conditions will the passenger compartment of an automobile exhibit (limited) rolling, pitching, and turning motions?

QUESTION 4: Which of the rotating bodies in Fig. 12.3 does *not* rotate about an axis through its center of mass?

(A) Fan (B) Roulette wheel (C) Compact disc
(D) Swinging door (E) Merry-go-round

12.2 ROTATION ABOUT A FIXED AXIS

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Figure 12.4 shows a rigid body rotating about a fixed axis, which coincides with the z axis. During this rotational motion, each point of the body remains at a given distance from this axis and moves along a circle centered on the axis. To describe the orientation of the body at any instant, we select one particle in the body and use it as a reference point; any particle can serve as reference point, provided that it is not on the axis of rotation. The circular motion of this reference particle (labeled P in Fig. 12.4) is then representative of the rotational motion of the entire body, and the angular position of this particle is representative of the angular orientation of the entire body.

Figure 12.5 shows the rotating rigid body as seen from along the axis of rotation. The coordinates in Fig. 12.5 have been chosen so the z axis coincides with the axis of rotation, whereas the x and y axes are in the plane of the circle traced out by the motion of the reference particle. The angular position of the reference particle—and hence the angular orientation of the entire rigid body—can be described by the position angle ϕ (the Greek

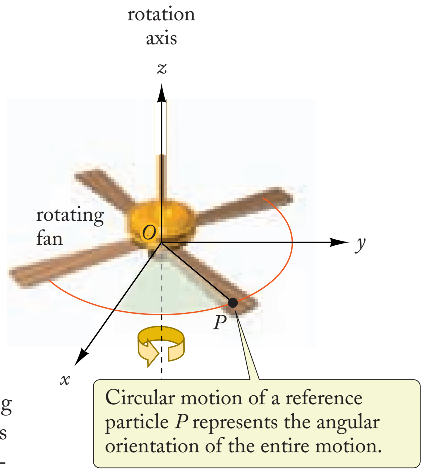


FIGURE 12.4 The four blades of this fan are a rigid body rotating about a fixed axis, which coincides with the z axis. The reference particle P in this rigid body moves along a circle around this axis.

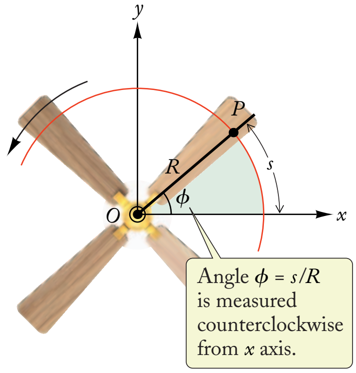


FIGURE 12.5 Motion of a reference particle P in the rigid body rotating about a fixed axis. The axis is indicated by the circled dot O. The radius of the circle traced out by the motion of the reference particle is R.

letter *phi*) between the radial line OP and the x axis. Conventionally, the angle ϕ is taken as positive when reckoned in a counterclockwise direction (as in Fig. 12.5). We will usually measure this position angle in radians, rather than degrees. By definition, *the* **angle ϕ in radians** *is the length s of the circular arc divided by the radius R*, or

angle in radians

$$\phi = \frac{s}{R} \tag{12.1}$$

In Fig. 12.5, the length s is the distance traveled by the reference particle from the x axis to the point P. Note that if the length s is the circumference of a full circle, then $s = 2\pi R$, and $\phi = s/R = 2\pi R/R = 2\pi$. Thus, there are 2π radians in a full circle; that is, there are 2π radians in 360°:

$$2\pi \text{ radians} = 360°$$

Accordingly, 1 radian equals $360°/2\pi$, or

$$1 \text{ radian} = 57.3°$$

EXAMPLE 1 The accuracy of the guidance system of the Hubble Space Telescope is such that if the telescope were sitting in New York, the guidance system could aim at a dime placed on top of the Washington Monument, at a distance of 320 km. The width of a dime is 1.8 cm. What angle does the dime subtend when seen from New York?

SOLUTION: Figure 12.6 shows the circular arc subtended by the dime. The radius of the circle is 320 km. For a small angle, such as in this figure, the length s of the arc from one side of the dime to the other is approximately the same as the length of the straight line from one side to the other, which is the width of the dime. Hence the angle in radians is

$$\phi = \frac{s}{R} = \frac{1.8 \times 10^{-2}\,\text{m}}{3.2 \times 10^{5}\,\text{m}} = 5.6 \times 10^{-8} \text{ radian}$$

Expressed in degrees, this becomes

$$\phi = 5.6 \times 10^{-8} \text{ radian} \times \frac{360°}{2\pi \text{ radians}} = 3.2 \times 10^{-6} \text{ degree}$$

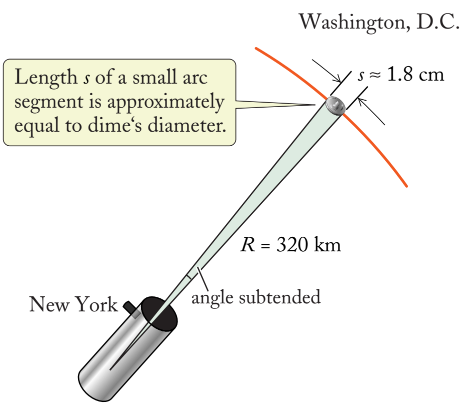


FIGURE 12.6 A dime placed at a distance of 320 km from the telescope. The length $s = 1.8$ cm is the diameter of the dime.

When a rigid body rotates, the position angle ϕ changes in time. The body then has an **angular velocity** ω (the Greek letter *omega*). The definition of the angular velocity for rotational motion is mathematically analogous to the definition of velocity for translational motion (see Sections 2.2 and 2.3). The **average angular velocity** $\overline{\omega}$ is defined as

$$\overline{\omega} = \frac{\Delta\phi}{\Delta t} \tag{12.2}$$

average angular velocity

where $\Delta\phi$ is the change in the angular position and Δt the corresponding change in time. The **instantaneous angular velocity** is defined as

$$\omega = \frac{d\phi}{dt} \tag{12.3}$$

instantaneous angular velocity

According to these definitions, the angular velocity is the rate of change of the angle with time. The unit of angular velocity is the radian per second (1 radian/s). The radian is the ratio of two lengths [compare Eq. (12.1)], and hence it is a pure number; thus, 1 radian/s is the same thing as 1/s. However, to prevent confusion, it is often useful to retain the vacuous label *radian* as a reminder that angular motion is involved. Table 12.1 gives some examples of angular velocities.

If the body rotates with constant angular velocity, then we can also measure the rate of rotation in terms of the ordinary **frequency** f, or the number of revolutions per second. Since each complete revolution involves a change of ϕ by 2π radians, the frequency of revolution is smaller than the angular velocity by a factor of 2π:

$$f = \frac{\omega}{2\pi} \tag{12.4}$$

frequency

This expresses the frequency in terms of the angular velocity. The unit of rotational frequency is the revolution per second (1 rev/s). Like the radian, the revolution is a pure number, and hence 1 rev/s is the same thing as 1/s. But we will keep the label *rev* to prevent confusion between rev/s and radian/s.

As in the case of planetary motion, the time per revolution is called the **period** of the motion. If the number of revolutions per second is f, then the time per revolution is $1/f$, that is,

$$T = \frac{1}{f} \tag{12.5}$$

period of motion

TABLE 12.1 SOME ANGULAR VELOCITIES

Computer hard disk	8×10^2 radians/s	Helicopter rotor	40 radians/s
Circular saw	7×10^2 radians/s	Compact disc (outer track)	22 radians/s
Electric blender blades	5×10^2 radians/s	Phonograph turntable	3.5 radians/s
Jet engine	4×10^2 radians/s	Neutron star (pulsar) rotation	0.1 radian/s
Airplane propeller	3×10^2 radians/s	Earth rotation	7.3×10^{-5} radian/s
Automobile engine	2×10^2 radians/s	Earth revolution about Sun	2.0×10^{-7} radian/s
Small fan	60 radians/s		

EXAMPLE 2 The rotational frequency of machinery is often expressed in revolutions per minute, or rpm. A typical ceiling fan on medium speed rotates at 150 rpm. What is the frequency of revolution? What is the angular velocity? What is the period of the motion?

SOLUTION: Each minute is 60.0 s; hence 150 revolutions per minute amounts to 150 revolutions in 60.0 s; so

$$f = \frac{150\text{ rev}}{60.0\text{ s}} = 2.50\text{ rev/s}$$

Since each revolution comprises 2π radians, the angular velocity is

$$\omega = 2\pi f = 2\pi \times 2.50\text{ rev/s} = 15.7\text{ radians/s}$$

Note that here we have dropped a label *rev* in the third step and inserted a label *radians;* as remarked above, these labels merely serve to prevent confusion, and they can be inserted and dropped at will once they have served their purpose.

The period of the motion is

$$T = \frac{1}{f} = \frac{1}{2.50\text{ rev/s}} = 0.400\text{ s}$$

One complete revolution takes two-fifths of a second.

If the angular velocity of a rigid body is changing, the body has an **angular acceleration** α (the Greek letter *alpha*). The rotational motion of a ceiling fan that is gradually building up speed immediately after being turned on is an example of accelerated rotational motion. The mathematical definition of the **average angular acceleration** is, again, analogous to the definition of acceleration for translational motion. If the angular velocity changes by $\Delta\omega$ in a time Δt, then the average angular acceleration is

average angular acceleration

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} \tag{12.6}$$

and the **instantaneous angular acceleration** is

instantaneous angular acceleration

$$\alpha = \frac{d\omega}{dt} \tag{12.7}$$

Thus, the angular acceleration is the rate of change of the angular velocity. The unit of angular acceleration is the radian per second per second, or radian per second squared (1 radian/s^2).

Since the angular velocity ω is the rate of change of the angular position ϕ [see Eq. (12.3)], the angular acceleration given by Eq. (12.7) can also be written

$$\alpha = \frac{d^2\phi}{dt^2} \tag{12.8}$$

Equations (12.3) and (12.7) give the angular velocity and acceleration of the rigid body; that is, they give the angular velocity and acceleration of every particle in the body. It is interesting to focus on one of the particles and evaluate its *translational* speed and acceleration as it moves along its circular path around the axis of rotation of the rigid body. If the particle is at a distance R from the axis of rotation (see Fig. 12.7), then the length along the circular path of the particle is, according to the definition of angle, Eq. (12.1),

$$s = \phi R \tag{12.9}$$

Since R is a constant, the rate of change of s is entirely due to the rate of change of ϕ, so

$$\frac{ds}{dt} = \frac{d\phi}{dt}R \tag{12.10}$$

Here ds/dt is the translational speed v with which the particle moves along its circular path, and $d\phi/dt$ is the angular velocity ω; hence Eq. (12.10) is equivalent to

$$v = \omega R \tag{12.11}$$

translational speed in circular motion

This shows that the translational speed of the particle along its circular path around the axis is directly proportional to the radius: the farther a particle in the rigid body is from the axis, the faster it moves. We can understand this by comparing the motions of two particles, one on a circle of large radius R_1, and the other on a circle of smaller radius R_2 (see Fig. 12.8). For each revolution of the rigid body, both of these particles complete one trip around their circles. But the particle on the larger circle has to travel a larger distance, and hence must move with a larger speed.

For a particle at a given R, the translational speed is constant if the angular velocity is constant. This speed is the distance around the circular path (the circumference) divided by the time for one revolution (the period), or

$$v = \frac{2\pi R}{T} \quad \text{(constant speed)} \tag{12.12}$$

Since $2\pi/T = 2\pi f = \omega$, Eq. (12.12) can be obtained from Eq. (12.11).

If v is changing, it also follows from Eq. (12.11) that the rate of change of v is proportional to the rate of change of ω:

$$\frac{dv}{dt} = \frac{d\omega}{dt}R$$

A rate of change of the speed along the circle implies that the particle has an acceleration along the circle, called a **tangential acceleration.** According to the last equation, this tangential acceleration is

$$a_{\text{tangential}} = \alpha R \tag{12.13}$$

tangential acceleration

Note that, besides this tangential acceleration directed along the circle, the particle also has a **centripetal acceleration** directed toward the center of the circle. From Section 4.5, we know that the centripetal acceleration for uniform circular motion is

$$a_{\text{centripetal}} = \frac{v^2}{R} \tag{12.14}$$

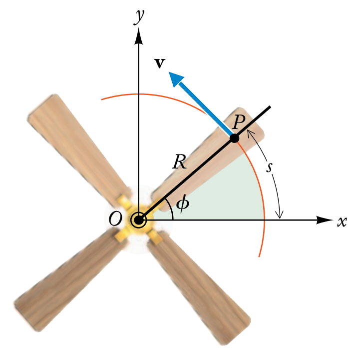


FIGURE 12.7 The instantaneous translational velocity of a particle in a rotating rigid body is tangent to the circular path.

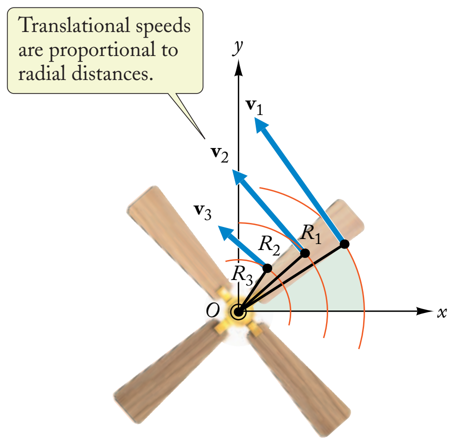


FIGURE 12.8 Several particles in a rigid body rotating about a fixed axis and their velocities.

With $v = \omega R$, this becomes

centripetal acceleration

$$a_{\text{centripetal}} = \omega^2 R \tag{12.15}$$

The net translational acceleration of the particle is the vector sum of the tangential and the centripetal accelerations, which are perpendicular (see Fig. 12.9); thus, the magnitude of the net acceleration is

$$a_{\text{net}} = \sqrt{a_{\text{tangential}}^2 + a_{\text{centripetal}}^2} \tag{12.16}$$

Although we have here introduced the concept of tangential acceleration in the context of the rotational motion of a rigid body, this concept is also applicable to the translational motion of a particle along a circular path or any curved path. For instance, consider an automobile (regarded as a particle) traveling around a curve. If the driver steps on the accelerator (or on the brake), the automobile will suffer a change of speed as it travels around the curve. It will then have both a tangential and a centripetal acceleration.

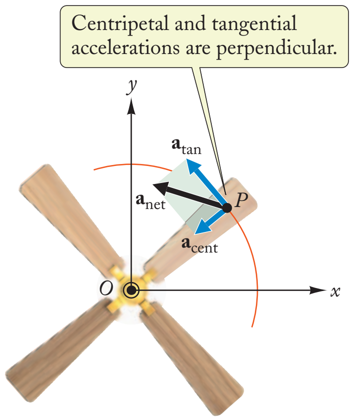


FIGURE 12.9 A particle in a rotating rigid body with an angular acceleration has both a centripetal acceleration $\mathbf{a}_{\text{centripetal}}$ and a tangential acceleration $\mathbf{a}_{\text{tangential}}$. The net instantaneous translational acceleration $\mathbf{a}_{\text{net}}$ is then the vector sum of $\mathbf{a}_{\text{centripetal}}$ and $\mathbf{a}_{\text{tangential}}$.

EXAMPLE 3 The blade of a circular saw is initially rotating at 7000 revolutions per minute. Then the motor is switched off, and the blade coasts to a stop in 8.0 s. What is the average angular acceleration?

SOLUTION: In radians per second, 7000 rev/min corresponds to an initial angular velocity $\omega_1 = 7000 \times 2\pi$ radians/min, or

$$\omega_1 = \frac{7000 \times 2\pi \text{ radians}}{60 \text{ s}} = 7.3 \times 10^2 \text{ radians/s}$$

The final angular velocity is $\omega_2 = 0$. Hence the average angular acceleration is

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{0 - 7.3 \times 10^2 \text{ radians/s}}{8.0 \text{ s} - 0}$$
$$= -91 \text{ radians/s}^2$$

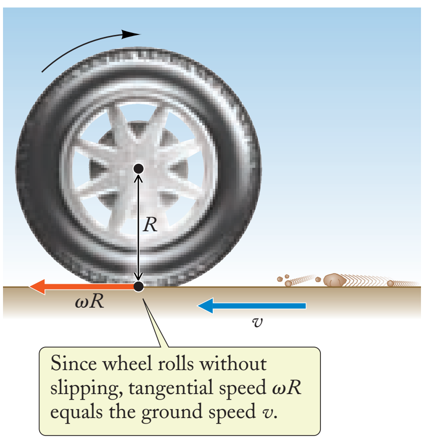


FIGURE 12.10 Rotating wheel of the automobile as viewed in the reference frame of the automobile. The ground moves toward the left at speed v.

EXAMPLE 4 An automobile accelerates uniformly from 0 to 80 km/h in 6.0 s. The wheels of the automobile have a radius of 0.30 m. What is the angular acceleration of the wheels? Assume that the wheels roll without slipping.

SOLUTION: The translational acceleration of the automobile is

$$a = \frac{v - v_0}{t} = \frac{80 \text{ km/h}}{6.0 \text{ s}} = \frac{(80 \text{ km/h}) \times (1000 \text{ m/1 km}) \times (1 \text{ h/3600 s})}{6.0 \text{ s}}$$
$$= 3.7 \text{ m/s}^2$$

The angular acceleration of the wheel is related to this translational acceleration by $a = \alpha R$, the same relation as Eq. (12.13). We can establish this relationship most conveniently by viewing the motion of the wheel in the reference frame of the automobile (see Fig. 12.10). In this reference frame, the ground is moving backward at speed v, and the bottom point of the rotating wheel is moving backward at the tangential speed ωR. Since the wheel is supposed to move without slipping, the speed v of the ground must match the tangential speed of the bottom point of the wheel; that is, $v = \omega R$. This proportionality of v and ω implies the same proportionality of the accelerations a and α, and therefore establishes the relationship $a = \alpha R$.

The angular acceleration of the wheel is then

$$\alpha = \frac{a}{R} = \frac{3.7\ \text{m/s}^2}{0.30\ \text{m}} = 12\ \text{radians/s}^2$$

EXAMPLE 5 The large centrifuge shown in the chapter photo has an arm of length 8.8 m. When rotating at 175 revolutions per minute, what is the speed of the end of this arm, and what is the centripetal acceleration?

Concepts in Context

SOLUTION: 175 rpm amounts to 175/60 = 2.9 revolutions per second. The corresponding angular velocity is

$$\omega = 2\pi f = 2\pi \times 2.9\ \text{rev/s} = 18\ \text{radians/s}$$

According to Eq. (12.11), the speed at a radius $R = 8.8$ m is

$$v = \omega R = 18\ \text{radians/s} \times 8.8\ \text{m} = 1.6 \times 10^2\ \text{m/s}$$

and according to Eq. (12.15), the centripetal acceleration is

$$a_{\text{centripetal}} = \omega^2 R = (18\ \text{radians/s})^2 \times 8.8\ \text{m} = 2.9 \times 10^3\ \text{m/s}^2$$

This is almost 300 standard g's!

✔ Checkup 12.2

QUESTION 1: Consider a point P on the rim of a rotating, accelerating flywheel and a point Q near the center. Which point has the larger instantaneous speed? The larger instantaneous angular velocity? The larger angular acceleration? The larger tangential acceleration? The larger centripetal acceleration?

QUESTION 2: The Earth rotates steadily around its axis once per day. Do all points on the surface of the Earth have the same radius R for their circular motion? Do they all have the same angular velocity ω? The same speed v around the axis? The same centripetal acceleration $a_{\text{centripetal}}$? If not, which points have the largest R, ω, v, and $a_{\text{centripetal}}$?

QUESTION 3: A short segment of the track of a roller coaster can be approximated by a circle of suitable radius. If a (frictionless) roller-coaster car is passing through the highest point of the track, is there a centripetal acceleration? A tangential acceleration? What if the the roller coaster is some distance beyond the highest point?

QUESTION 4: Consider the motion of the hammer shown in Fig. 12.1. Taking into account only the rotational motion, which end of the hammer has the larger speed v around the axis? The larger centripetal acceleration $a_{\text{centripetal}}$?

(A) Head end; head end
(B) Head end; handle end
(C) Handle end; head end
(D) Handle end; handle end

12.3 MOTION WITH CONSTANT ANGULAR ACCELERATION

We will now examine the kinematic equations describing rotational motion for the special case of *constant* angular acceleration; these are mathematically analogous to the equations describing translational motion with constant acceleration (see Section 2.5), and they can be derived by the same methods. In the next section, we will develop an alternative method, based on integration, for obtaining the kinematic equations describing either angular or translational motion for the general case of accelerations with arbitrary time dependence.

If the rigid body rotates with a constant angular acceleration α, then the angular velocity increases at a constant rate, and after a time t has elapsed, the angular velocity will attain the value

constant angular acceleration: ω, α, and t

$$\omega = \omega_0 + \alpha t \qquad (12.17)$$

where ω_0 is the initial value of the angular velocity at $t = 0$.

The angular position can be calculated from this angular velocity by the arguments used in Section 2.5 to calculate x from v [see Eqs. (2.17), (2.22), and (2.25)]. The result is

constant angular acceleration: ϕ, α, and t

$$\phi = \phi_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad (12.18)$$

Furthermore, the arguments of Section 2.5 lead to an identity between acceleration, position, and velocity [see Eqs. (2.20)–(2.22)]:

constant angular acceleration: α, ϕ, and ω

$$\alpha(\phi - \phi_0) = \tfrac{1}{2}(\omega^2 - \omega_0^2) \qquad (12.19)$$

Note that all these equations have exactly the same mathematical form as the equations of Section 2.5, with the angular position ϕ taking the place of the position x, the angular velocity ω taking the place of v, and the angular acceleration α taking the place of a. This analogy between rotational and translational quantities can serve as a useful mnemonic for remembering the equations for rotational motion. Table 12.2 displays analogous equations.

TABLE 12.2 ANALOGIES BETWEEN TRANSLATIONAL AND ROTATIONAL QUANTITIES

$v = \dfrac{dx}{dt}$	$\rightarrow$	$\omega = \dfrac{d\phi}{dt}$
$a = \dfrac{dv}{dt}$	$\rightarrow$	$\alpha = \dfrac{d\omega}{dt}$
$v = v_0 + at$	$\rightarrow$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\rightarrow$	$\phi = \phi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$a(x - x_0) = \frac{1}{2}(v^2 - v_0^2)$	$\rightarrow$	$\alpha(\phi - \phi_0) = \frac{1}{2}(\omega^2 - \omega_0^2)$

EXAMPLE 6 The cable supporting an elevator runs over a wheel of radius 0.36 m (see Fig. 12.11). If the elevator begins from rest and ascends with an upward acceleration of 0.60 m/s^2, what is the angular acceleration of the wheel? How many turns does the wheel make if this accelerated motion lasts 5.0 s? Assume that the cable runs over the wheel without slipping.

SOLUTION: If there is no slipping, the speed of the cable must always coincide with the tangential speed of a point on the rim of the wheel. The acceleration $a = 0.60$ m/s^2 of the cable must then coincide with the tangential acceleration of a point on the rim of the wheel:

$$a = a_{\text{tangential}} = \alpha R \qquad (12.20)$$

where $R = 0.36$ m is the radius of the wheel. Hence

$$\alpha = \frac{a}{R} = \frac{0.60\ \text{m/s}^2}{0.36\ \text{m}} = 1.7\ \text{radians/s}^2$$

According to Eq. (12.18), the angular displacement in 5.0 s is

$$\begin{aligned}\phi - \phi_0 &= \omega_0 t + \tfrac{1}{2}\alpha t^2 \\ &= 0 + \tfrac{1}{2} \times 1.7\ \text{radians/s}^2 \times (5.0\ \text{s})^2 \\ &= 21\ \text{radians}\end{aligned}$$

Each revolution comprises 2π radians; thus, the number of turns the wheel makes is

$$[\text{number of turns}] = \frac{\phi - \phi_0}{2\pi} = \frac{21\ \text{radians}}{2\pi} = 3.3\ \text{revolutions}$$

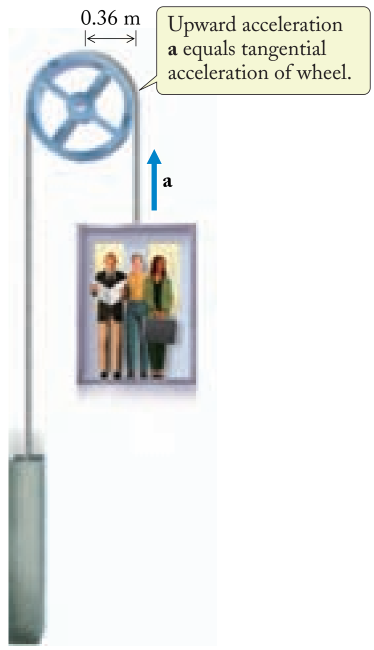


FIGURE 12.11 Elevator supported by a cable that runs over a rotating wheel.

PROBLEM-SOLVING TECHNIQUES — ANGULAR MOTION

The solution of kinematic problems about angular velocity and angular acceleration involves the same techniques as the problems about translational velocity and translational acceleration in Chapter 2. You might find it useful to review the procedures suggested on page 50.

Sometimes a problem contains a link between a rotational motion and a translational motion, such as the link between the rotational and translational motions of the wheels of an automobile (see Example 4) or the link between the translational motion of the elevator cable and the rotational motion of the wheel over which it runs (Example 6). If the body in contact with the rim of the wheel does not slip, the translational speed of this body equals the tangential speed of the contact point at the rim of the wheel; that is, $v = \omega R$ and $a = \alpha R$.

Keep in mind that although some of the equations in this chapter remain valid if the angular quantities are expressed in degrees, any equation that contains both angular quantities and distances (e.g., $v = \alpha R$) is valid only if the angular quantity is expressed in radians. To prevent mistakes, it is safest to express all angular quantities in radians; if degrees are required in the answer, convert from radians to degrees after completing your calculations.

✔ Checkup 12.3

QUESTION 1: Consider a point on the rim of the wheel shown in Fig. 12.11, (instantaneously) at the top of the wheel. What is the direction of the centripetal acceleration of this point? The tangential acceleration?

QUESTION 2: The wheel of a bicycle rolls on a flat road. Is the angular velocity constant if the translational velocity of the bicycle is constant? Is the angular acceleration constant if the translational acceleration of the bicycle is constant?

QUESTION 3: A grinding wheel accelerates uniformly for 3 seconds after being turned on. In the first second of motion, the wheel rotates 5 times. In the first two seconds of motion, the total number of revolutions is:

(A) 6 (B) 10 (C) 15 (D) 20 (E) 25

12.4 MOTION WITH TIME-DEPENDENT ANGULAR ACCELERATION

The equations of angular motion for the general case when the angular acceleration is a function of time are analogous to the corresponding equations of translational motion discussed in Section 2.7. Such equations are solved by integration. Integral calculus was discussed in detail in Chapter 7, and we now revisit the technique of integration of the equations of motion for the case of angular motion. To see how we can obtain kinematic solutions for nonconstant accelerations, consider the angular acceleration $\alpha = d\omega/dt$. We rearrange this relation and obtain

$$d\omega = \alpha\, dt$$

We can integrate this expression directly, for example, from the initial value of the angular velocity ω_0 at time $t = 0$, to some final value ω at time t (the integration variables are indicated by primes to distinguish them from the upper limits of integration):

$$\int_{\omega_0}^{\omega} d\omega' = \int_0^t \alpha\, dt'$$

$$\omega - \omega_0 = \int_0^t \alpha\, dt' \tag{12.21}$$

This gives the angular velocity as a function of time:

angular velocity for time-dependent angular acceleration

$$\omega = \omega_0 + \int_0^t \alpha\, dt' \tag{12.22}$$

Equation (12.22) enables us to calculate the angular velocity as a function of time for any angular acceleration that is a known function of time.

The angular position ϕ can be obtained in a similar manner:

$$d\phi = \omega\, dt$$

$$\int_{\phi_0}^{\phi} d\phi' = \int_0^t \omega\, dt'$$

angular position for time-dependent angular velocity

$$\phi - \phi_0 = \int_0^t \omega\, dt' \tag{12.23}$$

In the special case of constant angular acceleration α, Eq. (12.22) gives us $\omega = \omega_0 + \alpha t$, which agrees with our previous result, Eq. (12.17). If we insert this into Eq. (12.23), we obtain $\phi - \phi_0 = \int_0^t (\omega_0 + \alpha t)dt = \omega_0 t + \frac{1}{2}\alpha t^2$, which agrees with our previous Eq. (12.18).

In the general case of a time-dependent angular acceleration α, we proceed in the same way: first, use Eq. (12.22) to find ω as a function of time, and then insert this function into Eq. (12.23) to find the angular position as a function of time, as in the following example.

EXAMPLE 7 When turned on, a motor rotates a circular saw wheel, beginning from rest, with an angular acceleration that has an initial value $\alpha_0 = 60$ radians/s^2 at $t = 0$ and decreases to zero acceleration during the interval $0 \leq t \leq 3.0$ s according to

$$\alpha = \alpha_0\left(1 - \frac{t}{3.0\,\text{s}}\right)$$

After $t = 3.0$ s, the motor maintains the wheel's angular velocity at a constant value. What is this final angular velocity? In the process of "getting up to speed," how many revolutions occur?

SOLUTION: The angular acceleration α is given as an explicit function of time. Since we are beginning from rest, the initial angular velocity is $\omega_0 = 0$, so Eq. (12.22) gives ω as a function of t:

$$\begin{aligned}\omega &= \omega_0 + \int_0^t \alpha\,dt' = 0 + \int_0^t \alpha_0\left(1 - \frac{t'}{3.0\,\text{s}}\right)dt' \\ &= \alpha_0\left(\int_0^t dt' - \frac{1}{3.0\,\text{s}}\int_0^t t'\,dt'\right) = \alpha_0\left(t'\Big|_0^t - \frac{1}{3.0\,\text{s}}\frac{t'^2}{2}\Big|_0^t\right) \\ &= \alpha_0\left(t - \frac{t^2}{6.0\,\text{s}}\right) \qquad (12.24)\end{aligned}$$

where we have used the property that the integral of the sum is the sum of the integrals, and that $\int t^n\,dt = t^{n+1}/(n + 1)$. At $t = 3.0$ s, this angular velocity reaches its final value of

$$\omega = 60\ \text{radians/s}^2 \times \left(3.0\,\text{s} - \frac{(3.0\,\text{s})^2}{6.0\,\text{s}}\right) = 90\ \text{radians/s}$$

To obtain the number of revolutions during the time of acceleration, we can calculate the change in angular position and divide by 2π. To do so, we must insert the time-dependent angular velocity obtained in Eq. (12.24) into Eq. (12.23):

$$\begin{aligned}\phi - \phi_0 &= \int_0^t \omega\,dt' = \int_0^t \alpha_0\left(t' - \frac{t'^2}{6.0\,\text{s}}\right)dt' \\ &= \alpha_0\left(\int_0^t t'\,dt' - \frac{1}{6.0\,\text{s}}\int_0^t t'^2 dt'\right) = \alpha_0\left(\frac{t'^2}{2}\Big|_0^t - \frac{1}{6.0\,\text{s}}\frac{t'^3}{3}\Big|_0^t\right) \\ &= \alpha_0\left(\frac{t^2}{2} - \frac{t^3}{18\,\text{s}}\right)\end{aligned}$$

Evaluating this expression at $t = 3.0$ s, we find

$$\phi - \phi_0 = 60 \text{ radians/s}^2 \times \left(\frac{(3.0\text{ s})^2}{2} - \frac{(3.0\text{ s})^3}{18\text{ s}} \right) = 180 \text{ radians}$$

Hence the number of revolutions during the acceleration is

$$[\text{number of revolutions}] = \frac{\phi - \phi_0}{2\pi} = \frac{180\text{ radians}}{2\pi} = 29 \text{ revolutions} \quad (12.25)$$

As discussed in Section 2.7, similar integration techniques can be applied to determine any component of the translational velocity and the position when the time-dependent net force and, thus, the time-dependent translational acceleration are known. In Section 2.7 we also examined the case when the acceleration is a known function of the velocity; in that case, integration provides t as a function of v (and v_0), which can sometimes be inverted to find v as a function of t.

We saw in Chapters 7–9 that a conservation-of-energy approach is often the easiest way to determine the motion when the forces are known as a function of position. Now we have seen that direct integration of the equations of motion can be applied when the translational or angular acceleration is known as a function of time or of velocity.

✔ Checkup 12.4

QUESTION 1: Beginning from rest at $t = 0$, the angular velocity of a merry-go-round increases in proportion to the square root of the time t. By what factor is the angular position of the merry-go-round at $t = 4$ s greater than it was at $t = 1$ s?

QUESTION 2: A car on a circular roadway accelerates from rest beginning at $t = 0$, so that its angular acceleration increases in proportion to the time t. With what power of time does its centripetal acceleration increase?

(A) t (B) t^2 (C) t^3 (D) t^4 (E) t^5

12.5 KINETIC ENERGY OF ROTATION; MOMENT OF INERTIA

A rigid body is a system of particles, and as for any system of particles, the total kinetic energy of a rotating rigid body is simply the sum of the individual kinetic energies of all the particles (see Section 10.4). If the particles in the rigid body have masses m_1, m_2, m_3, . . . and speeds v_1, v_2, v_3, . . . , then the kinetic energy is

$$K = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \tfrac{1}{2}m_3v_3^2 + \cdots \quad (12.26)$$

In a rigid body rotating about a given axis, all the particles move with the same angular velocity ω along circular paths. By Eq. (12.11), the speeds of the particles along their paths are proportional to their radial distances:

$$v_1 = R_1\omega, \quad v_2 = R_2\omega, \quad v_3 = R_3\omega, \quad \cdots \quad (12.27)$$

and hence the total kinetic energy is

$$K = \tfrac{1}{2}m_1R_1^2\omega^2 + \tfrac{1}{2}m_2R_2^2\omega^2 + \tfrac{1}{2}m_3R_3^2\omega^2 + \cdots$$

We can write this as

$$K = \tfrac{1}{2} I\omega^2 \qquad (12.28)$$

kinetic energy of rotation

where the quantity

$$I = m_1R_1^2 + m_2R_2^2 + m_3R_3^2 + \cdots \qquad (12.29)$$

moment of inertia

is called the **moment of inertia** of the rotating body about the given axis. The SI unit of moment of inertia is $\text{kg}\cdot\text{m}^2$.

Note that Eq. (12.28) has a mathematical form reminiscent of the familiar expression $\frac{1}{2}mv^2$ for the kinetic energy of a single particle—the moment of inertia replaces the mass, and the angular velocity replaces the translational velocity. As we will see in the next chapter, this analogy between moment of inertia and mass is of general validity. *The moment of inertia is a measure of the resistance that a body offers to changes in its rotational motion,* just as mass is a measure of the resistance that a body offers to changes in its translational motion.

Equation (12.29) shows that the moment of inertia—and consequently the kinetic energy for a given value of ω—is large if most of the mass of the body is at a large distance from the axis of rotation. This is very reasonable: for a given value of ω, particles at large distance from the axis move with high speeds, and therefore have large kinetic energies.

EXAMPLE 8 A 50-kg woman and an 80-kg man sit on a massless seesaw separated by 3.00 m (see Fig. 12.12). The seesaw rotates about a fulcrum (the point of support) placed at the center of mass of the system; the center of mass is 1.85 m from the woman and 1.15 m from the man, as obtained in Example 4 of Chapter 10. If the (instantaneous) angular velocity of the seesaw is 0.40 radian/s, calculate the kinetic energy. Treat both masses as particles.

SOLUTION: The moment of inertia for particles rotating about an axis depends only on the masses and their distances from the axis:

$$\begin{aligned} I &= m_1R_1^2 + m_2R_2^2 \\ &= 50\ \text{kg} \times (1.85\ \text{m})^2 + 80\ \text{kg} \times (1.15\ \text{m})^2 = 280\ \text{kg}\cdot\text{m}^2 \end{aligned} \qquad (12.30)$$

The kinetic energy for the rotational motion is

$$\begin{aligned} K &= \tfrac{1}{2} I\omega^2 \\ &= \tfrac{1}{2} \times 280\ \text{kg}\cdot\text{m}^2 \times (0.40\ \text{radian/s})^2 = 22\ \text{J} \end{aligned} \qquad (12.31)$$

This kinetic energy could equally well have been obtained by first calculating the individual speeds of the woman and the man ($v_1 = R_1\omega$, $v_2 = R_2\omega$) and then adding the corresponding individual kinetic energies.

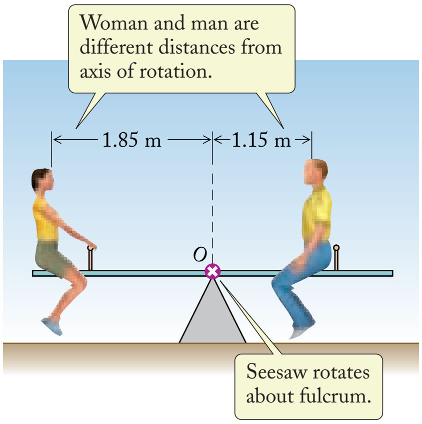


FIGURE 12.12 Woman and man on a seesaw.

If we regard the mass of a solid body as continuously distributed throughout its volume, then we can calculate the moment of inertia by the same method we used for the calculation of the center of mass: we subdivide the body into small mass elements and add the moments of inertia contributed by all these small amounts of mass. This leads to an approximation for the moment of inertia,

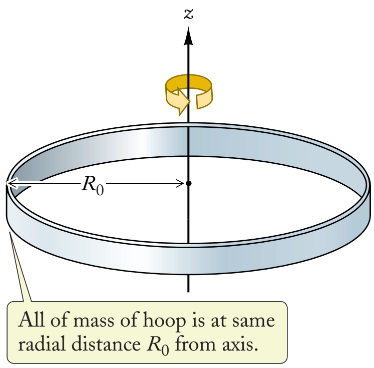


FIGURE 12.13 A thin hoop rotating about its axis of symmetry.

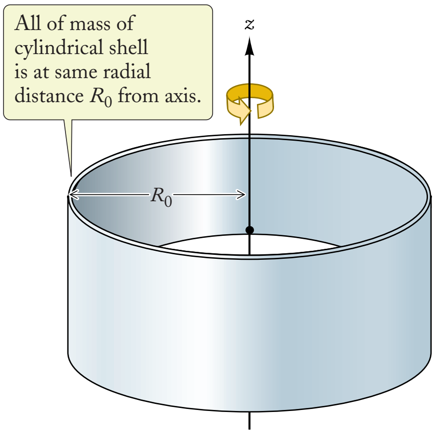


FIGURE 12.14 A thin cylindrical shell rotating about its axis of symmetry.

$$I \approx \sum_i R_i^2 \, \Delta m_i \tag{12.32}$$

where R_i is the radial distance of the mass element Δm_i from the axis of rotation. In the limit $\Delta m_i \to 0$, this approximation becomes exact, and the sum becomes an integral:

$$I = \int R^2 \, dm \tag{12.33}$$

In general, the calculation of the moment of inertia requires the evaluation of the integral (12.33). However, in a few exceptionally simple cases, it is possible to find the moment of inertia without performing this integration. For example, if the rigid body is a thin hoop (see Fig. 12.13) or a thin cylindrical shell (see Fig. 12.14) of radius R_0 rotating about its axis of symmetry, then *all* of the mass of the body is at the same distance from the axis of rotation—the moment of inertia is then simply the total mass M of the hoop or shell multiplied by its radius R_0 squared,

$$I = MR_0^2$$

If all of the mass is *not* at the same distance from the axis of rotation, then we must perform the integration (12.33); when summing the individual contributions, we usually write the small mass contribution as a mass per unit length times a small length, or as a mass per unit area times a small area, as in the following examples.

EXAMPLE 9 Find the moment of inertia of a uniform thin rod of length l and mass M rotating about an axis perpendicular to the rod and through its center.

SOLUTION: Figure 12.15 shows the rod lying along the x axis; the axis of rotation is the z axis. The rod extends from $x = -l/2$ to $x = +l/2$. Consider a small slice dx of the rod. The amount of mass within this slice is proportional to the length dx, and so is equal to the mass per unit length times this length:

$$dm = \frac{M}{l}\,dx$$

The square of the distance of the slice from the axis of rotation is $R^2 = x^2$, so Eq. (12.33) becomes

$$\begin{aligned} I &= \int R^2 \, dm = \int_{-l/2}^{+l/2} x^2 \frac{M}{l}\,dx = \frac{M}{l}\left(\frac{x^3}{3}\right)\Bigg|_{-l/2}^{+l/2} \\ &= \frac{M}{l} \times \frac{2(l/2)^3}{3} = \frac{1}{12}Ml^2 \end{aligned} \tag{12.34}$$

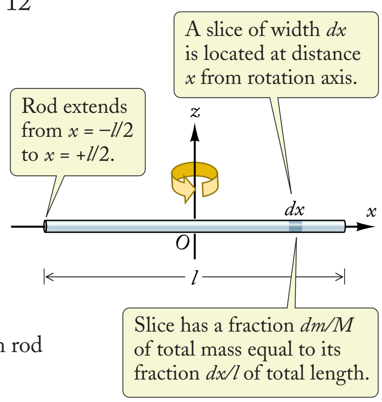


FIGURE 12.15 A thin rod rotating about its center.

EXAMPLE 10 Repeat the calculation of the preceding example for an axis of rotation through one end of the rod.

SOLUTION: Figure 12.16 shows the rod and the axis of rotation. The rod extends from $x = 0$ to $x = l$. Hence, instead of Eq. (12.34) we now obtain

$$I = \int_0^l x^2 \frac{M}{l}\,dx = \frac{M}{l}\left(\frac{x^3}{3}\right)\Bigg|_0^l = \frac{M}{l} \times \frac{l^3}{3} = \frac{1}{3}Ml^2 \qquad (12.35)$$

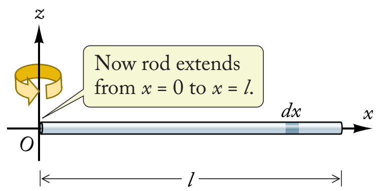


FIGURE 12.16 A thin rod rotating about its end.

EXAMPLE 11 Find the moment of inertia of a wide ring, or annulus, made of sheet metal of inner radius R_1, outer radius R_2, and mass M rotating about its axis of symmetry (see Fig. 12.17).

SOLUTION: The annulus can be regarded as made of a large number of thin concentric hoops fitting around one another. Figure 12.17 shows one such hoop, of radius R and width dR. All of the mass dm of this hoop is at the same radius R from the axis of rotation; hence the moment of inertia of the hoop is

$$dI = R^2\,dm$$

The area dA of the hoop is the product of its length (the perimeter $2\pi R$) and its width dR, so $dA = 2\pi R\,dR$. The mass dm of the hoop equals the product of this area and the mass per unit area of the sheet metal. Since the total area of the annulus is $\pi R_2^2 - \pi R_1^2$, the mass per unit area is $M/\pi(R_2^2 - R_1^2)$. The mass contributed by each hoop is the mass per unit area times its area:

$$dm = \frac{M}{\pi(R_2^2 - R_1^2)} \times 2\pi R\,dR = \frac{2M}{R_2^2 - R_1^2} R\,dR \qquad (12.36)$$

We sum the contributions dI from $R = R_1$ to $R = R_2$; hence

$$I = \int R^2\,dm = \frac{2M}{R_2^2 - R_1^2}\int_{R_1}^{R_2} R^3\,dR$$

$$= \frac{2M}{R_2^2 - R_1^2}\left(\frac{R^4}{4}\right)\Bigg|_{R_1}^{R_2} = \frac{M}{2(R_2^2 - R_1^2)} \times (R_2^4 - R_1^4)$$

$$= \frac{M}{2(R_2^2 - R_1^2)} \times (R_2^2 + R_1^2)(R_2^2 - R_1^2) = \frac{M}{2}(R_2^2 + R_1^2) \qquad (12.37)$$

COMMENT: Note that for $R_1 = 0$, this becomes $I = MR_2^2/2$, which is the moment of inertia of a disk (see Table 12.3). And for $R_1 = R_2$, it becomes $I = MR_1^2$, which is the moment of inertia of a hoop. Note that the result (12.37) for a sheet also applies to a thick annulus or a thick cylindrical shell (rotating about the axis of symmetry).

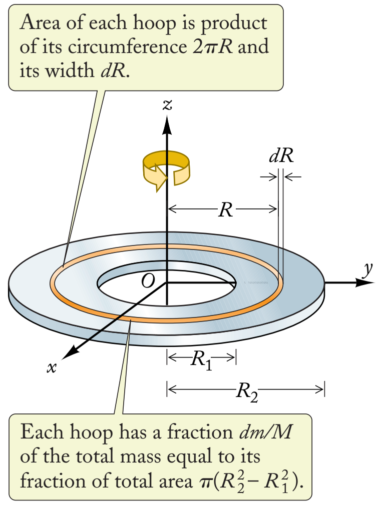
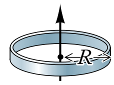
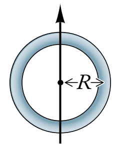
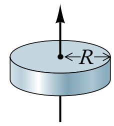
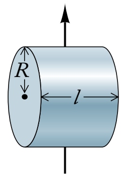
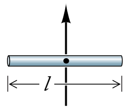
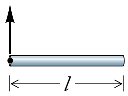
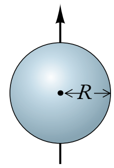
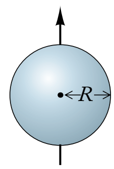


FIGURE 12.17 An annulus of sheet metal rotating about its axis of symmetry. The annulus can be regarded as made of a large number of concentric hoops. The hoop shown in the figure has radius R and width dR.

Comparison of Eqs. (12.34) and (12.35) for the moment of inertia of a rod makes it clear that the value of the moment of inertia depends on the location of the axis of rotation. The moment of inertia is small if the axis passes through the center of mass, and large if it passes through the end of the rod. In the latter case, more of the mass of the rod is at a larger distance from the axis of rotation, which leads to a larger moment of inertia.

TABLE 12.3 SOME MOMENTS OF INERTIA

BODY		MOMENT OF INERTIA
	Thin hoop about symmetry axis	MR^2
	Thin hoop about diameter	$\frac{1}{2}MR^2$
	Disk or cylinder about symmetery axis	$\frac{1}{2}MR^2$
	Cylinder about diameter through center	$\frac{1}{4}MR^2 + \frac{1}{12}Ml^2$
	Thin rod about perpendicular axis through center	$\frac{1}{12}Ml^2$
	Thin rod about perpendicular axis through end	$\frac{1}{3}Ml^2$
	Sphere about diameter	$\frac{2}{5}MR^2$
	Thin spherical shell about diameter	$\frac{2}{3}MR^2$

It is possible to prove a theorem that relates the moment of inertia I_{CM} about an axis through the center of mass to the moment of inertia I about a parallel axis through some other point. This theorem, called the **parallel-axis theorem,** asserts that

parallel-axis theorem

$$I = I_{CM} + Md^2 \quad (12.38)$$

where M is the total mass of the body and d the distance between the two axes. We will not give the proof, but merely check that the theorem is consistent with our results for the moments of inertia of the rod rotating about an axis through the center [$I_{CM} = \frac{1}{12}Ml^2$; see Eq. (12.34)] and an axis through an end [$I = \frac{1}{3}Ml^2$; see Eq. (12.35)]. In this case, $d = l/2$, and the parallel-axis theorem asserts

$$\frac{1}{3}Ml^2 = \frac{1}{12}Ml^2 + M\left(\frac{l}{2}\right)^2 \qquad (12.39)$$

which is identically true.

Note that it is a corollary of Eq. (12.38) that the moment of inertia about an axis passing through the center of mass is always smaller than that about any other parallel axis.

Table 12.3 lists the moments of inertia of a variety of rigid bodies about an axis through their center of mass; all the bodies are assumed to have uniform density.

EXAMPLE 12 The large centrifuge shown in the chapter photo carries the payload in a chamber in one arm and counterweights at the end of the opposite arm. The mass distribution depends on the choice of payload and the choice of counterweights. Figure 12.18 is a schematic diagram of the mass distribution attained with a particular choice of payload and counterweights. The payload arm (including the payload) has a mass of 1.8×10^3 kg uniformly distributed over a length of 8.8 m. The counterweight arm has a mass of 1.1×10^3 kg uniformly distributed over a length of 5.5 m, and it carries a counterweight of 8.6×10^3 kg at its end. (a) What is the moment of inertia of the centrifuge for this mass distribution? (b) What is the rotational kinetic energy when the centrifuge is rotating at 175 revolutions per minute?

Concepts in Context

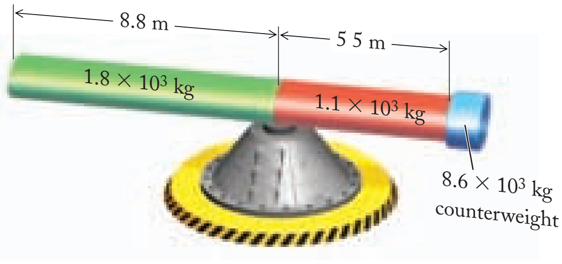


FIGURE 12.18 Centrifuge mass distribution.

SOLUTION: (a) The total moment of inertia is the sum of the moments of inertia of a rod of mass $m_1 = 1.8 \times 10^3$ kg, length $l_1 = 8.8$ m rotating about its end; a second rod of mass $m_2 = 1.1 \times 10^3$ kg, length $l_2 = 5.5$ m also rotating about its end; and a mass of $m = 8.6 \times 10^3$ kg at a radial distance of $R = 5.5$ m. The moments of inertia of the rods are given by Eq. (12.35), and the moment of inertia of the counterweight is mR^2. So the total moment of inertia is

$$\begin{aligned} I &= \tfrac{1}{3}m_1 l_1^2 + \tfrac{1}{3}m_2 l_2^2 + mR^2 \\ &= \tfrac{1}{3} \times 1.8 \times 10^3 \text{ kg} \times (8.8 \text{ m})^2 + \tfrac{1}{3} \times 1.1 \times 10^3 \text{ kg} \times (5.5 \text{ m})^2 \\ &\quad + 8.6 \times 10^3 \text{ kg} \times (5.5 \text{ m})^2 \\ &= 3.2 \times 10^5 \text{ kg}\cdot\text{m}^2 \end{aligned}$$

(b) At 175 revolutions per minute, the angular velocity is $\omega = 18$ radians/s (see Example 5), and the rotational kinetic energy is

$$\begin{aligned} K &= \tfrac{1}{2}I\omega^2 \\ &= \tfrac{1}{2} \times 3.2 \times 10^5 \text{ kg}\cdot\text{m}^2 \times (18 \text{ radians/s})^2 \\ &= 5.2 \times 10^7 \text{ J} \end{aligned}$$

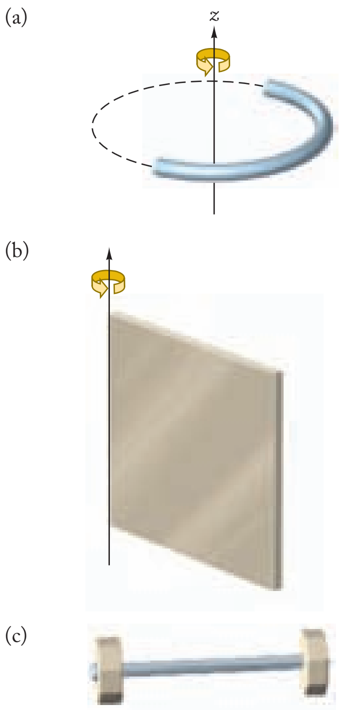


FIGURE 12.19 (a) A rod bent into an arc of a circle of radius R, rotating about its center of curvature. (b) A square plate rotating about an axis along one edge. (c) A dumbbell.

Checkup 12.5

QUESTION 1: What is the moment of inertia of a rod of mass M bent into an arc of a circle of radius R when rotating about an axis through the center and perpendicular to the circle (see Fig. 12.19a)?

QUESTION 2: Consider a rod rotating about (a) an axis along the rod, (b) an axis perpendicular to the rod through its center, and (c) an axis perpendicular to the rod through its end. For which axis is the moment of inertia largest? Smallest?

QUESTION 3: What is the moment of inertia of a square plate of mass M and dimension $L \times L$ rotating about an axis along one of its edges (see Fig. 12.19b)? What is the moment of inertia if this square plate rotates about an axis through its center parallel to an edge?

QUESTION 4: A dumbbell consists of two particles of mass m each attached to the ends of a rigid, massless rod of length l (Fig. 12.19c). Assume the particles are point particles. What is the moment of inertia of this rigid body when rotating about an axis through the center and perpendicular to the rod? When rotating about a parallel axis through one end? Are these moments of inertia consistent with the parallel-axis theorem?

QUESTION 5: According to Table 12.3, the moment of inertia of a hoop about its symmetry axis is $I_{CM} = MR^2$. What is the moment of inertia if you twirl a large hoop around your finger, so that in essence it rotates about a point on the hoop, about an axis parallel to the symmetry axis?

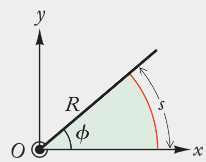
(A) $5MR^2$ (B) $2MR^2$ (C) $\frac{3}{2}MR^2$.
(D) MR^2 (E) $\frac{1}{2}MR^2$.

SUMMARY

PROBLEM-SOLVING TECHNIQUES Angular Motion **(page 375)**

DEFINITION OF ANGLE (in radians)

$$\phi = \frac{[\text{arc length}]}{[\text{radius}]} = \frac{s}{R} \qquad (12.1)$$



ANGLE CONVERSIONS

1 revolution = 2π radians = 360°

AVERAGE ANGULAR VELOCITY

$$\bar{\omega} = \frac{\Delta\phi}{\Delta t} \qquad (12.2)$$

INSTANTANEOUS ANGULAR VELOCITY

$$\omega = \frac{d\phi}{dt} \qquad (12.3)$$

FREQUENCY

$$f = \frac{\omega}{2\pi} \tag{12.4}$$

PERIOD OF MOTION

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{12.5}$$

AVERAGE ANGULAR ACCELERATION

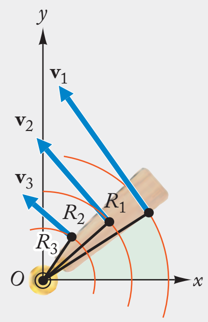
$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} \tag{12.6}$$

INSTANTANEOUS ANGULAR ACCELERATION

$$\alpha = \frac{d\omega}{dt} \tag{12.7}$$

SPEED OF PARTICLE ON ROTATING BODY

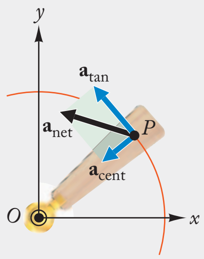
$$v = \omega R \tag{12.11}$$



ACCELERATION OF PARTICLE ON ROTATING BODY

$$a_{\text{tangential}} = \alpha R, \tag{12.13}$$

$$a_{\text{centripetal}} = \omega^2 R \tag{12.15}$$



MOTION WITH CONSTANT ANGULAR ACCELERATION

$$\omega = \omega_0 + \alpha t \tag{12.17}$$

$$\phi = \phi_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \tag{12.18}$$

$$\alpha(\phi - \phi_0) = \tfrac{1}{2}(\omega^2 - \omega_0^2) \tag{12.19}$$

MOTION WITH TIME-DEPENDENT ANGULAR ACCELERATION

$$\omega = \omega_0 + \int_0^t \alpha\, dt' \tag{12.22}$$

$$\phi = \phi_0 + \int_0^t \omega\, dt' \tag{12.23}$$

MOMENT OF INERTIA

where R_i is the radial distance of m_i from the axis of rotation.

$$I = m_1 R_1^2 + m_2 R_2^2 + m_3 R_3^3 + \cdots \tag{12.29}$$

MOMENT OF INERTIA OF RIGID BODY (see also Table 12.3) where R is the radial distance of the mass element dm from the axis of rotation; for uniformly distributed mass, dm is given by	$I = \int R^2 dm$ $dm = \frac{M}{[\text{length}]} dx$ $dm = \frac{M}{[\text{area}]} dA,$ $dm = \frac{M}{[\text{volume}]} dV$	**(12.33)**
PARALLEL-AXIS THEOREM where M is the total mass and d is the distance from CM axis.	$I = I_{CM} + Md^2$	**(12.38)**
KINETIC ENERGY OF ROTATION	$K = \frac{1}{2} I\omega^2$	**(12.28)**

QUESTIONS FOR DISCUSSION

1. A spinning flywheel in the shape of a disk suddenly shatters into many small fragments. Draw the trajectories of a few of these small fragments; assume that the fragments do not interfere with each other.
2. You may have noticed that in some old movies the wheels of moving carriages or stagecoaches seem to rotate backwards. How does this come about?
3. Relative to an inertial reference frame, what is your angular velocity right now about an axis passing through your center of mass?
4. Consider the wheel of an accelerating automobile. Draw the instantaneous acceleration vectors for a few points on the rim of the wheel.
5. The hands of a watch are small rectangles with a common axis passing through one end. The minute hand is long and thin; the hour hand is short and thicker. Assume both hands have the same mass. Which has the greater moment of inertia? Which has the greater kinetic energy and angular momentum?
6. What configuration and what axis would you choose to give your body the smallest possible moment of inertia? The greatest?
7. About what axis through the center of mass is the moment of inertia of this book largest? Smallest? (Assume the book is closed.)
8. A circular hoop made of thin wire has a radius R and mass M. About what axis perpendicular to the plane of the hoop must you rotate this hoop to obtain the minimum moment of inertia? What is the value of this minimum?
9. Automobile engines and other internal combustion engines have flywheels attached to their crankshafts. What is the purpose of these flywheels? (Hint: Each explosive combustion in one of the cylinders of such an engine gives a sudden push to the crankshaft. How would the crankshaft respond to this push if it had no flywheel?)
10. Suppose you pump a mass M of seawater into a pond on a hill at the equator. How does this change the moment of inertia of the Earth?

PROBLEMS

12.2 Rotation About A Fixed Axis†

1. The minute hand of a wall clock has a length of 20 cm. What is the angular velocity of this hand? What is the speed of the tip of this hand?
2. Quito is on the Earth's equator; New York is at latitude 41° north. What is the angular velocity of each city about the Earth's axis of rotation? What is the linear speed of each?
3. An automobile has wheels with a radius of 30 cm. What are the angular velocity (in radians per second) and the frequency (in revolutions per second) of the wheels when the automobile is traveling at 88 km/h?

† For help, see Online Concept Tutorial 15 at www.wwnorton.com/physics

4. In an experiment at the Oak Ridge Laboratory, a carbon fiber disk of 0.70 m in diameter was set spinning at 37 000 rev/min. What was the speed at the edge of this disk?

5. The rim of a phonograph record is at a distance of 15 cm from the center, and the rim of the paper label on the record is at a distance of 5 cm from the center.
   (a) When this record is rotating at $33\frac{1}{3}$ rev/min, what is the translational speed of a point on the rim of the record? The translational speed of a point on the rim of the paper label?
   (b) What are the centripetal accelerations of these points?

6. An electric drill rotates at 5000 rev/min. What is the frequency of rotation (in rev/s)? What is the time for one revolution? What is the angular velocity (in radians/s)?

7. An audio compact disk (CD) rotates at 210 rev/min when playing an outer track of radius 5.8 cm. What is the angular velocity in radians/s? What is the tangential speed of a point on the outer track? Because the CD has the same linear density of bits on each track, the drive maintains a constant tangential speed. What is the angular velocity (in radians/s) when playing an inner track of radius 2.3 cm? What is the corresponding rotational frequency (in rev/s)?

8. An automobile travels one-fourth of the way around a traffic circle in 4.5 s. The diameter of the traffic circle is 50 m. The automobile travels at constant speed. What is that speed? What is the angular velocity in radians/s?

9. When a pottery wheel motor is switched on, the wheel accelerates from rest to 90 rev/min in 5.0 s. What is its angular velocity at $t = 5.0$ s (in radians/s)? What is the linear speed of a piece of clay 10 cm from the center of the wheel at $t = 5.0$ s? What is its average angular acceleration during the acceleration?

10. A grinding wheel of radius 6.5 cm accelerates from rest to its operating speed of 3450 rev/min in 1.6 s. When up to speed, what is its angular velocity in radians/s? What is the linear speed at the edge of the wheel? What is its average angular acceleration during this 1.6 s? When turned off, it decelerates to a stop in 35 s. What is its average angular acceleration during this time?

11. When drilling holes, manufacturers stay close to a recommended linear cutting speed in order to maintain efficiency while avoiding overheating. The rotational speed of the drill thus depends on the diameter of the hole. For example, recommended linear cutting speeds are typically 20 m/min for steel and 100 m/min for aluminum. What is the corresponding rotational rate (in rev/s) when drilling a 3.0-mm-diameter hole in aluminum? When drilling a 2.5-cm-diameter hole in steel?

12. An electric blender accelerates from rest to 500 radians/s in 0.80 s. What is the average angular acceleration? What is the corresponding average tangential acceleration for a point on the tip of a blender blade a distance 3.0 cm from the axis? If this point has that tangential acceleration when the blender's angular velocity is 50 radians/s, what is the corresponding total acceleration of the point?

13. The angular position of a ceiling fan during the first two seconds after start-up is given by $\phi = C[t^2 - (t^3/4\ \text{s})]$, where $C = 20/\text{s}^2$ and t is in seconds. What are the angular position, angular frequency, and angular acceleration at $t = 0$ s? At $t = 1.0$ s? At $t = 2.0$ s?

*14. An aircraft passes directly over you with a speed of 900 km/h at an altitude of 10 000 m. What is the angular velocity of the aircraft (relative to you) when directly overhead? Three minutes later?

*15. The outer edge of the grooved area of a long-playing record is at a radial distance of 14.6 cm from the center; the inner edge is at a radial distance of 6.35 cm. The record rotates at $33\frac{1}{3}$ rev/min. The needle of the pickup arm takes 25 min to play the record, and in that time interval it moves uniformly and radially from the outer edge to the inner edge. What is the radial speed of the needle? What is the speed of the outer edge relative to the needle? What is the speed of the inner edge relative to the needle?

*16. Consider the phonograph record described in Problem 15. What is the total length of the groove in which the needle travels?

12.3 Motion with Constant Angular Acceleration

17. The blade of a circular saw of diameter 20 cm accelerates uniformly from rest to 7000 rev/min in 1.2 s. What is the angular acceleration? How many revolutions will the blade have made by the time it reaches full speed?

18. A large ceiling fan has blades of radius 60 cm. When you switch this fan on, it takes 20 s to attain its final steady speed of 1.0 rev/s. Assume a constant angular acceleration.
   (a) What is the angular acceleration of the fan?
   (b) How many revolutions does it make in the first 20 s?
   (c) What is the distance covered by the tip of one blade in the first 20 s?

19. When you switch on a PC computer, the disk in the disk drive takes 5.0 s to reach its final steady speed of 7200 rev/min. What is the average angular acceleration?

20. When you turn off the motor, a phonograph turntable initially rotating at $33\frac{1}{3}$ rev/min makes 25 revolutions before it stops. Calculate the angular deceleration of this turntable; assume it is constant.

21. A large merry-go-round rotates at one revolution each 9.0 seconds. When shut off, it decelerates uniformly to a stop in 16 s. What is the angular acceleration? How many revolutions does the merry-go-round make during the deceleration?

22. A cat swipes at a spool of thread, which then rolls across the floor with an initial speed of 1.0 m/s. The spool decelerates uniformly to a stop 3.0 m from its initial position. The spool has a radius of 1.5 cm and rolls without slipping. What is the

initial angular velocity? Through what total angle does the spool rotate while slowing to a stop? What is the angular acceleration during this motion?

23. If you lift the lid of a washing machine during the rapid spin–dry cycle, the cycle stops (for safety), typically after 5.0 revolutions. If the clothes are spinning at 6.0 rev/s initially, what is their constant angular acceleration during the slowing motion? How long do they take to come to a stop?

24. A toy top initially spinning at 30 rev/s slows uniformly to a stop in 25 seconds. What is the angular acceleration during this motion? Through how many revolutions does the top turn while slowing to a stop?

*25. The rotation of the Earth is slowing down. In 1977, the Earth took 1.01 s longer to complete 365 rotations than in 1900. What was the average angular deceleration of the Earth in the time interval from 1900 to 1977?

*26. An automobile engine accelerates at a constant rate from 200 rev/min to 3000 rev/min in 7.0 s and then runs at constant speed.

(a) Find the angular velocity and the angular acceleration at $t = 0$ (just after acceleration begins) and at $t = 7.0$ s (just before acceleration ends).

(b) A flywheel with a radius of 18 cm is attached to the shaft of the engine. Calculate the tangential and the centripetal acceleration of a point on the rim of the flywheel at the times given above.

(c) What angle does the net acceleration vector make with the radius at $t = 0$ and at $t = 7.0$ s? Draw diagrams showing the wheel and the acceleration vector at these times.

12.4 Motion with Time-Dependent Angular Acceleration

27. A disk has an initial angular velocity of $\omega_0 = 8.0$ radians/s. At $t = 0$, it experiences a time-dependent angular acceleration given by $\alpha = Ct^2$, where $C = 0.25$ radian/s^4. What is the instantaneous angular velocity at $t = 3.0$ s? What is the change in angular position between $t = 0$ and $t = 1.0$ s?

28. A rigid body is initially at rest. Beginning at $t = 0$, it begins rotating, with an angular acceleration given by $\alpha = \alpha_0\{1 - [t^2/(4\text{ s}^2)]\}$ for $0 \le t \le 2.0$ s and $\alpha = 0$ thereafter. The initial value is $\alpha_0 = 20$ radians/s^2. What is the body's angular velocity after 1.0 s? After a long time? How many revolutions have occurred after 1.0 s?

*29. A sphere is initially rotating with angular velocity ω_0 in a viscous liquid. Friction causes an angular deceleration that is proportional to the instantaneous angular velocity, $\alpha = -A\omega$, where A is a constant. Show that the angular velocity as a function of time is given by

$$\omega = \omega_0 e^{-At}$$

12.5 Kinetic Energy of Rotation; Moment of Inertia

30. Find the moment of inertia of an orange of mass 300 g and diameter 9.0 cm. Treat the orange as a uniform sphere.

31. The original Ferris wheel built by George Ferris (see Fig. 12.20) had a radius of 38 m and a mass of 1.9×10^6 kg. Assume that all of the mass of the wheel was uniformly distributed along its rim. If the wheel was rotating at 0.050 rev/min, what was its kinetic energy?

FIGURE 12.20 The original Ferris wheel.

32. What is the moment of inertia of a broomstick of mass 0.50 kg, length 1.5 m, and diameter 2.5 cm about its longitudinal axis? About an axis at right angles to the broomstick, passing through its center?

33. According to spectroscopic measurements, the moment of inertia of an oxygen molecule about an axis through the center of mass and perpendicular to the line joining the atoms is 1.95×10^{-46} kg·m^2. The mass of an oxygen atom is 2.66×10^{-26} kg. What is the distance between the atoms? Treat the atoms as pointlike particles.

34. The moment of inertia of the Earth about its polar axis is $0.331 M_E R_E^2$, where M_E is the mass and R_E the equatorial radius. Why is the moment of inertia smaller than that of a sphere of uniform density? What would the radius of a sphere of uniform density have to be if its mass and moment of inertia are to coincide with those of the Earth?

35. Problem 41 in Chapter 10 gives the dimensions of a molecule of nitric acid (HNO_3). What is the moment of inertia of this molecule when rotating about the symmetry axis passing through the H, O, and N atoms? Treat the atoms as pointlike particles.

36. The water molecule has a shape shown in Fig. 12.21. The distance between the oxygen and the hydrogen atoms is d, and the angle between the hydrogen atoms is θ. From spectroscopic investigations it is known that the moment of inertia of the molecule is 1.93×10^{-47} kg·m^2 for rotation about the axis AA' and 1.14×10^{-47} kg·m^2 for rotation about the axis BB'. From this information and the known values of the masses of the atoms, determine the values of d and θ. Treat the atoms as pointlike.

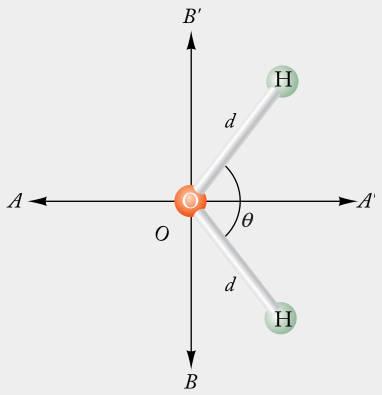


FIGURE 12.21 Atoms in a water molecule.

37. What is the moment of inertia (about the axis of symmetry) of a bicycle wheel of mass 4.0 kg, radius 0.33 m? Neglect the mass of the spokes.

38. An airplane propeller consists of three radial blades, each of length 1.2 m and mass 6.0 kg. What is the kinetic energy of this propeller when rotating at 2500 rev/min? Assume that each blade is (approximately) a uniform rod.

39. Estimate the moment of inertia of a human body spinning rigidly about its longitudinal axis. Treat the body as a uniform cylinder of mass 70 kg, length 1.7 m, and average diameter 23 cm.

40. Use the parallel-axis theorem to determine the moment of inertia of a solid disk or cylinder of mass M and radius R rotating about an axis parallel to its symmetry axis but tangent to its surface.

41. The moment of inertia of the Earth is approximately $0.331 M_E R_E^2$ (see also Problem 34). Calculate the rotational kinetic energy of the Earth.

42. Assume that a potter's kickwheel is a disk of radius 60 cm and mass 120 kg. What is its moment of inertia? What is its rotational kinetic energy when revolving at 2.0 rev/s?

43. A flywheel energy-storage system designed for the International Space Station has a maximum rotational rate of 53 000 rev/min. The cylindrical flywheel has a mass of 75 kg and a radius of 16 cm. For simplicity, assume the cylinder is solid and uniform. What is the moment of inertia of the flywheel? What is the maximum rotational kinetic energy stored in the flywheel?

*44. An empty beer can has a mass of 15 g, a length of 12 cm, and a radius of 3.3 cm. Find the moment of inertia of the can about its axis of symmetry. Assume that the can is a perfect cylinder of sheet metal with no ridges, indentations, or holes.

*45. Suppose that a supertanker transports 4.4×10^8 kg of oil from a storage tank in Venezuela (latitude 10° north) to a storage tank in Holland (latitude 53° north). What is the change of the moment of inertia of the Earth–oil system?

*46. A dumbbell consists of two uniform spheres of mass M and radius R joined by a thin rod of mass m. The distance between the centers of the sphers is l (Fig. 12.22). What is the moment of inertia of this device about an axis through the center of the rod perpendicular to the rod? About an axis along the rod?

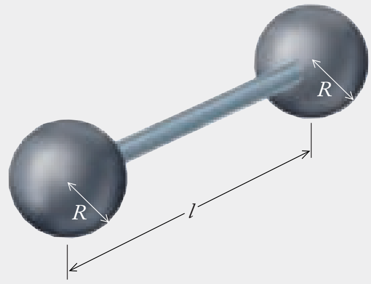


FIGURE 12.22 A dumbbell.

*47. Suppose that the Earth consists of a spherical core of mass $0.22 M_E$ and radius $0.54 R_E$ and a surrounding mantle (a spherical shell) of mass $0.78 M_E$ and outer radius R_E. Suppose that the core is of uniform density and the mantle is also of uniform density. According to this simple model, what is the moment of inertia of the Earth? Express your answer as a multiple of $M_E R_E^2$.

*48. In order to increase her moment of inertia about a vertical axis, a spinning figure skater stretches her arms out horizontally; in order to reduce her moment of inertia, she brings her arms down vertically along her sides. Calculate the change of moment of inertia between these two configurations of the arms. Assume that each arm is a thin, uniform rod of length 0.60 m and mass 2.8 kg hinged at the shoulder a distance of 0.20 m from the axis of rotation.

*49. Find the moment of inertia of a thin rod of mass M and length L about an axis through the center inclined at an angle θ with respect to the rod.

*50. Given that the moment of inertia of a sphere about a diameter is $\frac{2}{5}MR^2$, show that the moment of inertia about an axis tangent to the surface is $\frac{7}{5}MR^2$.

*51. Find a formula for the moment of inertia of a uniform thin square plate (mass m, dimension $l \times l$) rotating about an axis that coincides with one of its edges.

*52. A conical shell has mass M, height h, and base radius R. Assume it is made from a thin sheet of uniform thickness. What is its moment of inertia about its symmetry axis?

*53. Suppose a peach of radius R and mass M consists of a spherical pit of radius $0.50R$ and mass $0.050M$ surrounded by a spherical shell of fruit of mass $0.95M$. What is the moment of inertia of the peach?

*54. Find the moment of inertia of the flywheel shown in Fig. 12.23 rotating about its axis. The flywheel is made of material of uniform thickness; its mass is M.

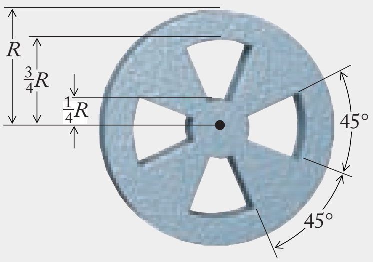


FIGURE 12.23 A flywheel.

*55. A solid cylinder capped with two solid hemispheres rotates about its axis of symmetry (Fig. 12.24). The radius of the cylinder is R, its height is h, and the total mass (hemispheres included) is M. What is the moment of inertia?

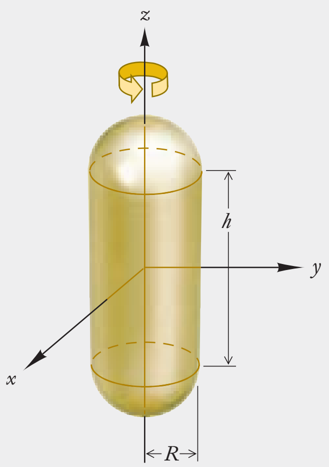


FIGURE 12.24 A solid cylinder capped with two solid hemispheres.

*56. A hole of radius r has been drilled in a circular, flat plate of radius R (Fig. 12.25). The center of the hole is at a distance d from the center of the circle. The mass of this body is M. Find the moment of inertia for rotation about an axis through the center of the circle, perpendicular to the plate.

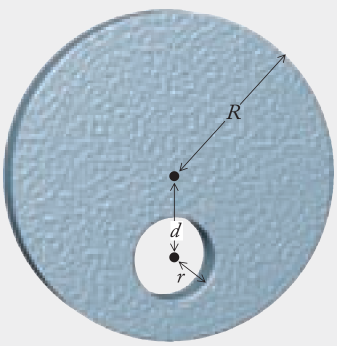


FIGURE 12.25 Circular plate with a hole.

*57. Derive a formula for the moment of inertia of a uniform spherical shell of mass M, inner radius R_1, outer radius R_2, rotating about a diameter.

*58. Find the moment of inertia of a flywheel of mass M made by cutting four large holes of radius r out of a uniform disk of radius R (Fig. 12.26). The holes are centered at a distance $R/2$ from the center of the flywheel.

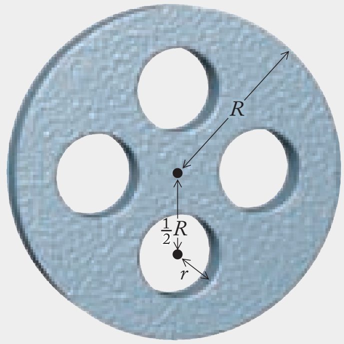


FIGURE 12.26 Disk with four holes.

*59. Show that the moment of inertia of a long, very thin cone (Fig. 12.27) about an axis through the apex and perpendicular to the centerline is $\frac{3}{5}Ml^2$, where M is the mass and l the height of the cone.

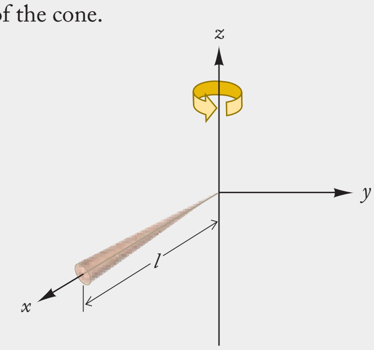


FIGURE 12.27 A long, thin cone rotating about its apex.

*60. The mass distribution within the Earth can be roughly approximated by several concentric spherical shells, each of constant density. The following table gives the outer and the inner radius of each shell and its mass (expressed as a fraction of the Earth's mass):

SHELL	OUTER RADIUS	INNER RADIUS	FRACTION OF MASS
1	6400 km	5400 km	0.28
2	5400	4400	0.25
3	4400	3400	0.16
4	3400	2400	0.20
5	2400	0	0.11

Use these data to calculate the moment of inertia of the Earth about its axis.

61. The drilling pipe of an oil rig is 2.0 km long and 15 cm in diameter, and it has a mass of 20 kg per meter of length. Assume that the wall of the pipe is very thin.
    (a) What is the moment of inertia of this pipe rotating about its longitudinal axis?
    (b) What is the kinetic energy when rotating at 1.0 rev/s?

62. Engineers have proposed that large flywheels be used for the temporary storage of surplus energy generated by electric power plants. A suitable flywheel would have a diameter of 3.6 m and a mass of 300 metric tons and would spin at 3000 rev/min. What is the kinetic energy of rotation of this flywheel? Give the answer in both joules and kilowatt-hours. Assume that the moment of inertia of the flywheel is that of a uniform disk.

63. An automobile of mass 1360 kg has wheels 76.2 cm in diameter of mass 27.2 kg each. Taking into account the rotational kinetic energy of the wheels about their axles, what is the total kinetic energy of the automobile when traveling at 80.0 km/h? What percentage of the kinetic energy belongs to the rotational motion of the wheels about their axles? Pretend that each wheel has a mass distribution equivalent to that of a uniform disk.

*64. The Oerlikon Electrogyro bus uses a flywheel to store energy for propelling the bus. At each bus stop, the bus is briefly connected to an electric power line, so that an electric motor on the bus can spin up the flywheel to 3000 rev/min. If the flywheel is a disk of radius 0.60 m and mass 1500 kg, and if the bus requires an average of 40 hp for propulsion at an average speed of 20 km/h, how far can it move with the energy stored in the rotating flywheel?

*65. Pulsars are rotating stars made almost entirely of neutrons closely packed together. The rate of rotation of most pulsars gradually decreases because rotational kinetic energy is gradually converted into other forms of energy by a variety of complicated "frictional" processes. Suppose that a pulsar of mass 1.5×10^{30} kg and radius 20 km is spinning at the rate of 2.1 rev/s and is slowing down at the rate of 1.0×10^{-15} rev/s^2. What is the rate (in joules per second, or watts) at which the rotational energy is decreasing? If this rate of decrease of the energy remains constant, how long will it take the pulsar to come to a stop? Treat the pulsar as a sphere of uniform density.

66. For the sake of directional stability, the bullet fired from a rifle is given a spin angular velocity about its axis by means of spiral grooves ("rifling") cut into the barrel. The bullet fired by a Lee–Enfield rifle is (approximately) a uniform cylinder of length 3.18 cm, diameter 0.790 cm, and mass 13.9 g. The bullet emerges form the muzzle with a translational velocity of 628 m/s and a spin angular velocity of 2.47×10^3 rev/s. What is the translational kinetic energy of the bullet? What is the rotational kinetic energy? What fraction of the total kinetic energy is rotational?

*67. Find a formula for the moment of inertia of a thin disk of mass M and radius R rotating about a diameter.

*68. Derive the formula for the moment of inertia of a thin hoop of mass M and radius R rotating about a diameter.

*69. Find a formula for the moment of inertia of a uniform thin square plate (mass M, dimension $l \times l$) rotating about an axis through the center and perpendicular to the plate.

*70. Find the moment of inertia of a uniform cube of mass M and edge l. Assume the axis of rotation passes through the center of the cube and is perpendicular to two of the faces.

*71. What is the moment of inertia of a thin, flat plate in the shape of a semicircle rotating about the straight side (Fig. 12.28)? The mass of the plate is M and the radius is R.

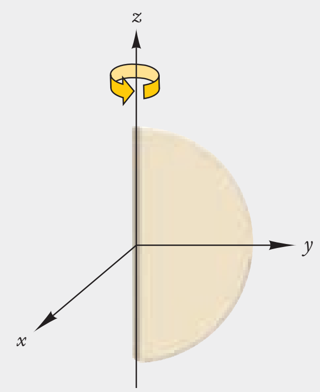


FIGURE 12.28 A semicircle rotating about its straight edge.

**72. Find the moment of inertia of the thin disk with two semicircular cutouts shown in Fig. 12.29 rotating about its axis. The disk is made of material of uniform thickness; its mass is M.

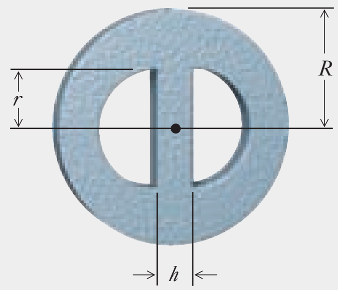


FIGURE 12.29 Disk with two semicircular cutouts.

**73. A cone of mass M has a height h and a base diameter R. Find its moment of inertia about its axis of symmetry.

**74. Derive the formula given in Table 12.3 for the moment of inertia of a sphere.

REVIEW PROBLEMS

75. An automobile has wheels of diameter 0.63 m. If the automobile is traveling at 80 km/h, what is the instantaneous velocity vector (relative to the ground) of a point at the top of the wheel? At the bottom? At the front?

76. The propeller of an airplane is turning at 2500 rev/min while the airplane is cruising at 200 km/h. The blades of the propeller are 1.5 m long. Taking into account both the rotational motion of the propeller and the translational motion of the aircraft, what is the velocity (magnitude and direction) of the tip of the propeller?

77. An automobile accelerates uniformly from 0 to 80 km/h in 6.0 s. The automobile has wheels of radius 30 cm. What is the angular acceleration of the wheels? What is their final angular velocity? How many turns do they make during the 6.0-s interval?

78. The minute hand of a wall clock is a rod of mass 5.0 g and length 15 cm rotating about one end. What is the rotational kinetic energy of the minute hand?

79. What is the kinetic energy of rotation of a phonograph record of mass 170 g and radius 15.2 cm rotating at $33\frac{1}{3}$ revolutions per minute? To give this phonograph record a translational kinetic energy of the same magnitude, how fast would you have to throw it?

80. The wheel of a wagon consists of a rim of mass 20 kg and eight spokes in the shape of rods of length 0.50 m and mass 0.80 kg each.
    (a) What is the moment of inertia of this wheel about its axle?
    (b) What is the kinetic energy of this wheel when rotating at 1.0 rev/s?

*81. A solid body consists of two uniform solid spheres of mass M and radius R welded together where they touch (see Fig. 12.30). What is the moment of inertia of this rigid body about the longitudinal axis through the center of the spheres? About the transverse axis through the point of contact?

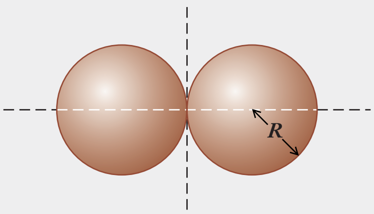


FIGURE 12.30 Two connected solid spheres.

*82. A .22-caliber bullet is a solid cylinder of length 7.0 mm and radius 2.7 mm capped at its front with a hemisphere of the same radius. The mass of the bullet is 15 g.
    (a) What is the moment of inertia of this bullet when rotating about its axis of symmetry?
    (b) What is the rotational kinetic energy of the bullet when rotating at 1.2×10^3 rev/s?

*83. Find the moment of inertia of the wheel shown in Fig. 12.31 rotating about its axis. The wheel is made of material of uniform thickness, its mass is M, and its radius is R. Treat the spokes as thin rods of length $R/2$ and width $R/12$.

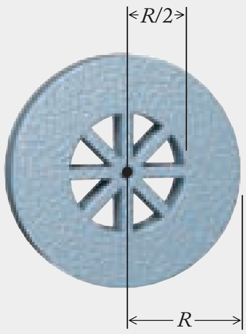


FIGURE 12.31 A wheel.

*84. The total kinetic energy of a rolling body is the sum of its translational kinetic energy $\frac{1}{2}Mv^2$ and its rotational kinetic energy $\frac{1}{2}I\omega^2$. Suppose that a cylinder, a sphere, and a pipe (a cylindrical shell) of equal masses 2.0 kg are rolling with equal speeds of 1.0 m/s. What is the total kinetic energy of each?

*85. A uniform solid cylinder is initially at rest at the top of a ramp of height 1.5 m. If the cylinder rolls down this ramp without slipping, what will be its speed at the bottom? (Hint: Use energy conservation. The kinetic energy of the cylinder at the bottom of the ramp is the sum of its translational kinetic energy $\frac{1}{2}Mv^2$ and its rotational kinetic energy $\frac{1}{2}I\omega^2$.)

**86. An airplane propeller (Fig. 12.32) is rotating at 3000 rev/min when one of the blades breaks off at the hub. Treat the blade as a rod, of length 1.2 m. The blade is horizontal and swinging upward at the instant it breaks.
    (a) What is the velocity (magnitude and direction) of the motion of the center of mass of the blade immediately after this instant?
    (b) What is the angular velocity of the rotational motion of the blade about its own center of mass?
    (c) Suppose that this happens while the aircraft is on the ground, with the hub of the propeller 2.4 m above the ground. How high above the ground does the center of mass of the broken propeller blade rise? Neglect air resistance.

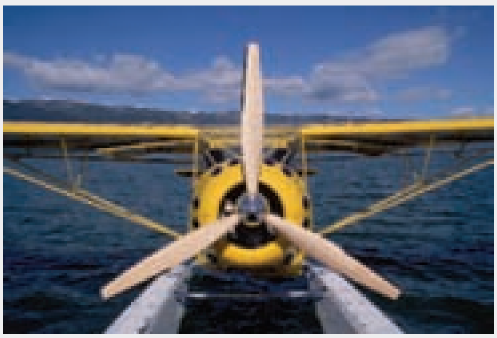


FIGURE 12.32 An airplane propeller.

Answers to Checkups

Checkup 12.1

1. The swinging door executes only rotational motion about its (fixed) hinges. The motions of the wheel of a train and of the propeller of an airplane involve both rotational and translational motion; the wheel and propeller rotate as the vehicle moves through space.
2. Yes, the motion is describable as rotation about an axis and simultaneous translational motion. The rotational motion is rotation about an axis through the end of the hammer; the translational motion, however, is not along a parabolic path, but involves more complicated looping motion (see Fig. 12.1).
3. An automobile exhibits roll motion when driving on a banked surface; the auto is then tilted. Pitch motion can occur during sudden braking, when the front of the auto dives downward. Turning motion occurs whenever the auto is being driven around a curve (compare Fig. 12.2).
4. (D) Swinging door. The axis of rotation is through the hinges, along the edge of the door.

Checkup 12.2

1. The point P has the larger instantaneous speed (it travels through a greater distance per unit time). Both points have the same instantaneous angular velocity ω and the same angular acceleration α (as do all points on the same rigid body). Hence the point P has the larger tangential acceleration ($a_{\text{tangential}} = \alpha R$) and also the larger centripetal acceleration ($a_{\text{centripetal}} = \omega^2 R$).
2. The radius R for circular motion is the perpendicular distance from the axis of rotation, and so is equal to the Earth's radius only at the equator, and is increasingly smaller as one moves toward the poles; at a pole, R is zero. All points have the same angular velocity ω, as for any rigid body. The velocity is not the same for all points; since $v = \omega R$, v is largest at the equator. All points do not have the same centripetal acceleration; since $a_{\text{centripetal}} = \omega^2 R$, the centripetal acceleration is largest at the equator.
3. There is a centripetal acceleration; at the top of the arc, this is directed downward ($a_{\text{centripetal}} = v^2/R$). There is no tangential acceleration at the top (no forces act in this direction). Some distance beyond the highest point, there will be both a centripetal acceleration (since the car still moves along an arc) and a tangential acceleration (since now a component of the gravitational force is tangent to the path).
4. (D) Handle end; handle end. Since the rotation is about an axis through the center of mass (near the hammer head), the end of the handle is furthest from the axis. Thus both the speed $v = \omega R$ and the centripetal acceleration $a_{\text{centripetal}} = \omega^2 R$ are largest at the end of the handle, since R is largest there (and ω is a constant for all points on a rigid body).

Checkup 12.3

1. The centripetal acceleration always points toward the center of curvature of the circular arc of the problem; here, this is vertically down. The tangential acceleration points perpendicular to a radius at any point; since the elevator accelerates upward, the tangential acceleration at the top of the wheel points horizontally toward the left.
2. Yes to both. As long as there is no slipping, we have $\omega = v/R$ and $\alpha = a/R$, so the behavior of an angular quantity is the same as the corresponding translational quantity.
3. (D) 20. For constant acceleration and starting from rest, the angular position is $\phi = \frac{1}{2}\alpha t^2$. Since this is proportional to t^2, the angular position will be four times greater in twice the time. Thus the total number of revolutions in the first two seconds is $4 \times 5 = 20$.

Checkup 12.4

1. Since the angular velocity is proportional to $t^{1/2}$, the angular position, which is the integral of the angular velocity over time [Eq. (12.23)], will be proportional to $t^{1/2+1} = t^{3/2}$. Thus the angular position will be $4^{3/2} = 8$ times as large at $t = 4$ s as it was at $t = 1$ s.
2. (D) t^4. If the angular acceleration α increases in proportion to the time t, then the angular velocity $\omega = \int \alpha\, dt$ increases in proportion to t^2. The centripetal acceleration is given by $a_{\text{centripetal}} = v^2/R = \omega^2 R$, and so increases in proportion to the fourth power of the time.

Checkup 12.5

1. Since all of the mass M is at the same distance from the axis of rotation, the moment of inertia is simply $I = MR^2$.
2. Rotation about an axis perpendicular to the rod through its end gives the largest moment of inertia, since more mass is located at a greater distance from the axis of rotation. Rotation about an axis along the rod must give the smallest moment of inertia, since in this case all of the mass is very close to the axis.
3. About an axis along one edge or through its center parallel to one edge, the distribution of mass (relative to the axis of rotation) in each case is the same as for the corresponding rod (imagine viewing Fig. 12.19b from above, that is, along the axis of rotation). Thus the moment of inertia of the square about an axis along one edge is $I = \frac{1}{3}ML^2$; about an axis through its center parallel to one edge, it is $\frac{1}{12}ML^2$.
4. About an axis through the center, each particle is a distance $l/2$ from the axis, and so the moment of inertia is $I = m(l/2)^2 + m(l/2)^2 = \frac{1}{2}ml^2$. About an axis through one particle, one particle is a distance l from the axis and the other is at zero distance, so $I = ml^2 + 0 = ml^2$. Since we have shifted the axis by $d = l/2$ in the second case, we indeed have $I = I_{\text{CM}} + Md^2 = \frac{1}{2}ml^2 + (2m)(l/2)^2 = ml^2$, so the parallel-axis theorem is satisfied (notice we must use the total mass $M = 2m$).
5. (B) $2MR^2$. Since the axis is shifted by a distance $d = R$, the parallel-axis theorem gives $I = I_{\text{CM}} + Md^2 = MR^2 + MR^2 = 2MR^2$ for rotation about a point on the hoop.

CHAPTER

13

Dynamics of a Rigid Body

CONCEPTS IN CONTEXT

Concepts in Context

The *Gravity Probe B* satellite, containing four high-precision gyroscopes, was recently placed in orbit by a rocket. These gyroscopes are used for a delicate test of Einstein's theory of General Relativity. The rotor of one of these gyroscopes is shown here. It consists of a nearly perfect sphere of quartz, 3.8 cm in diameter, suspended electrically and spinning at 10 000 revolutions per minute.

Some of the questions we can address with the concepts developed in this chapter are:

? When initially placed in orbit, the rotor is at rest. What torque and what force are needed to spin up this gyroscope with a given angular acceleration? (Example 4, page 401)

? A rotating body, such as this rotor, has not only kinetic energy, but also an angular momentum, which is the rotational analog of the linear momentum introduced in Chapter 10. How is the angular momentum of the gyroscope expressed in terms of its angular velocity? (Example 8, page 406)

? The gyroscope is used like a compass, to establish a reference direction in space. How does a gyroscope maintain a fixed reference direction? (Physics in Practice: The Gyrocompass, page 414)

As we saw in Chapter 5, Newton's Second Law is the equation that determines the translational motion of a body. In this chapter, we will derive an equation that determines the rotational motion of a rigid body. Just as Newton's equation of motion gives us the translational acceleration and permits us to calculate the change in velocity and position, the analogous equation for rotational motion gives us the angular acceleration and permits us to calculate the change in angular velocity and angular position. The equation for rotational motion is not a new law of physics, distinct from Newton's three laws. Rather, it is a consequence of these laws.

13.1 WORK, ENERGY, AND POWER IN ROTATIONAL MOTION; TORQUE

We begin with a calculation of the work done by an external force on a rigid body constrained to rotate about a fixed axis. Figure 13.1 shows the body, with the axis of rotation perpendicular to the page. The force is applied at some point of the body at a distance R from the axis of rotation. For a start, we will assume that the force has no component parallel to the axis; any such component is of no interest in the present context since the body does not move in the direction parallel to the axis, and so a force parallel to the axis can do no work. In Fig. 13.1, the force is shown entirely in the plane of the page. The work done by this force during a small displacement of the point at which the force acts is the product of the force F, the displacement ds, and the cosine of the angle between the force and the displacement [see Eq. (7.5)]. The cosine of this angle is equal to the sine of the angle θ between the force and the radial line (see Fig. 13.1). Hence, we can write the work as

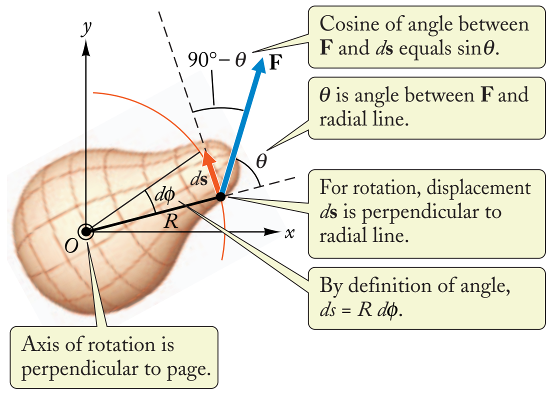


FIGURE 13.1 Force applied to a rigid body rotating about a fixed axis. As in Chapter 12, the axis of rotation is indicated by a circled dot. The force makes an angle θ with the radial line and an angle $90° - \theta$ with the instantaneous displacement d**s**.

$$dW = F\,ds\sin\theta$$

If the body rotates through a small angle $d\phi$, the displacement is $ds = R\,d\phi$, and therefore

$$dW = FR\,d\phi\,\sin\theta \qquad (13.1)$$

The product $FR\sin\theta$ is called the **torque** of the force F, usually designated by the symbol τ (the Greek letter *tau*):

torque

$$\tau = FR\sin\theta \qquad (13.2)$$

With this notation, the work done by the force, or *the work done by the torque,* is simply

work done by torque

$$dW = \tau\,d\phi \qquad (13.3)$$

This is the rotational analog of the familiar equation $dW = F\,dx$ for work done in translational motion. The torque τ is analogous to the force F, and the angular displacement $d\phi$ is analogous to the translational displacement dx. The analogy between torque and force extends beyond the equation for the work. As we will see in the next section, a torque applied to a rigid body causes angular acceleration, just as a force applied to a particle causes translational acceleration.

According to Eq. (13.3), each contribution to the work is the product of the torque τ and the small angular displacement $d\phi$. Thus the total work done in rotating a body from an initial angle ϕ_1 to a final angle ϕ_2 is

$$W = \int dW = \int_{\phi_1}^{\phi_2} \tau \, d\phi \tag{13.4}$$

In the special case of a constant torque, the torque may be brought outside the integral to obtain

$$W = \tau \int_{\phi_1}^{\phi_2} d\phi = \tau(\phi_2 - \phi_1)$$

or simply

work done by constant torque

$$W = \tau \, \Delta\phi \qquad (\text{for } \tau = \text{constant}) \tag{13.5}$$

where $\Delta\phi = \phi_2 - \phi_1$ is the change in angular position during the time that the torque is applied. Equation (13.5) is analogous to the equation for the work done by a constant force on a body in one-dimensional translational motion, $W = F\,\Delta x$.

From Eq. (13.2), we see that the unit of torque is the unit of force multiplied by the unit of distance; *this SI unit of torque is the newton-meter* (N·m).

Note that according to Eq. (13.2), for a force of given magnitude, the torque is largest if the force acts at right angles to the radial line ($\theta = 90°$) and if the force acts at a large distance from the axis of rotation (large R). This dependence of the torque (and of the work) on the distance from the axis of rotation and on the angle of the push agrees with our everyday experience in pushing doors open or shut. A door is a rigid body, which rotates about a vertical axis through the hinges. If you push perpendicularly against the door, near the edge farthest from the hinge (largest R; see Fig. 13.2a), you produce a large torque, which does work on the door, increases its kinetic energy, and swings the door quickly on its hinges. If you push at a point near the hinge (small R; see Fig. 13.2b), the door responds more sluggishly. You produce a smaller torque, and you have to push harder to do the same amount of work and attain the same amount of kinetic energy and the same final angular velocity. Finally, if you push in a direction that is not perpendicular to the door (small θ; see Fig. 13.2c), the door again responds sluggishly, because the torque is small.

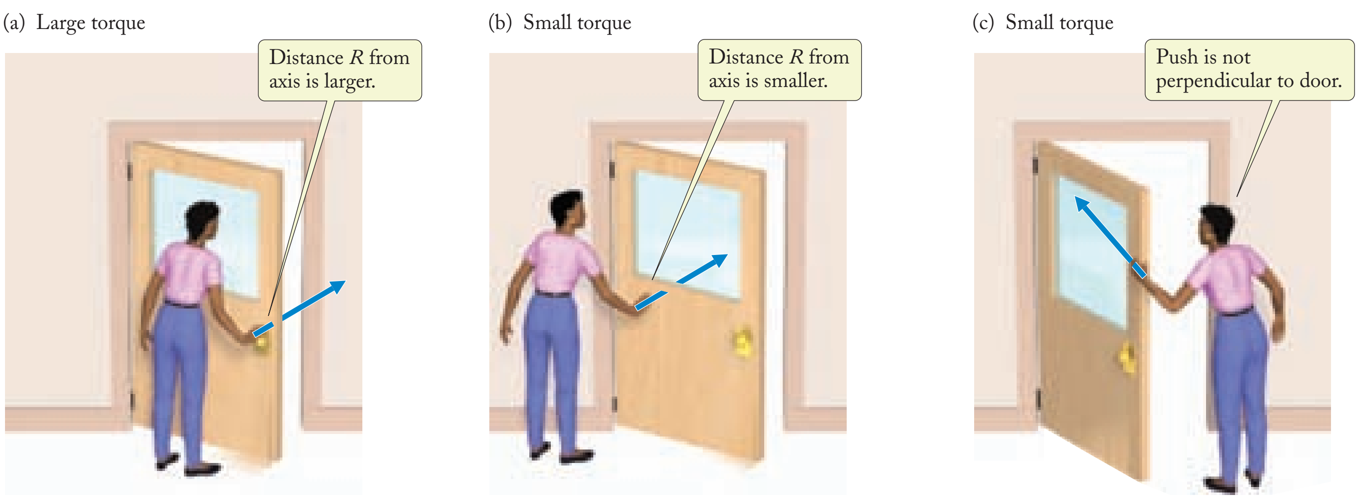


FIGURE 13.2 (a) A push against the door far from the hinge produces a large angular acceleration. (b) The same push near the hinge produces a small angular acceleration. (c) A push against the door at a small angle also produces a small angular acceleration.

EXAMPLE 1 Suppose that while opening a 1.0-m-wide door, you push against the edge farthest from the hinge, applying a force with a steady magnitude of 0.90 N at right angles to the surface of the door. How much work do you do on the door during an angular displacement of 30°?

SOLUTION: For a constant torque, the work is given by Eq. (13.5), $W = \tau\,\Delta\phi$. The definition of torque, Eq. (13.2), with $F = 0.90$ N, $R = 1.0$ m, and $\theta = 90°$, gives

$$\tau = FR \sin 90° = 0.90 \text{ N} \times 1.0 \text{ m} \times 1 = 0.90 \text{ N·m}$$

To evaluate the work, the angular displacement must be expressed in radians; $\Delta\phi = 30° \times (2\pi \text{ radians}/360°) = 0.52$ radian. Then

$$W = \tau\,\Delta\phi = 0.90 \text{ N·m} \times 0.52 \text{ radian}$$
$$= 0.47 \text{ J}$$

The equation for the power in rotational motion and the equations that express the work–energy theorem and the conservation law for energy in rotational motion are analogous to the equations we formulated for translational motion in Chapters 7 and 8. If we divide both sides of Eq. (13.3) by dt, we find *the instantaneous power delivered by the torque:*

$$P = \frac{dW}{dt} = \tau\frac{d\phi}{dt}$$

or

$$P = \tau\omega \tag{13.6}$$

power delivered by torque

where $\omega = d\phi/dt$ is the angular velocity. Obviously, this equation is analogous to the equation $P = Fv$ obtained in Section 8.5 for the power in one-dimensional translational motion.

The work done by the torque changes the rotational kinetic energy of the body. Like the work–energy theorem for translational motion, the work–energy theorem for rotational motion says that the work done on the body by the external torque equals the change in rotational kinetic energy (the internal forces and torques in a rigid body do no net work):

$$W = K_2 - K_1 = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2 \tag{13.7}$$

If the force acting on the body is conservative—such as the force of gravity or the force of a spring—then the work equals the negative of the change in potential energy, and Eq. (13.7) becomes

$$-U_2 + U_1 = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2 \tag{13.8}$$

or

$$\tfrac{1}{2}I\omega_1^2 + U_1 = \tfrac{1}{2}I\omega_2^2 + U_2 \tag{13.9}$$

This expresses the **conservation of energy in rotational motion**: the sum of the kinetic and potential energies is constant, that is,

$$E = \tfrac{1}{2}I\omega^2 + U = [\text{constant}] \tag{13.10}$$

conservation of energy in rotational motion

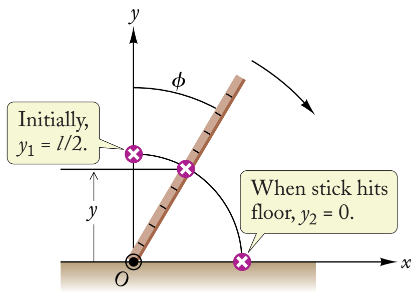


FIGURE 13.3 Meterstick rotating about its lower end.

EXAMPLE 2 A meterstick is initially standing vertically on the floor. If the meterstick falls over, with what angular velocity will it hit the floor? Assume that the end in contact with the floor does not slip.

SOLUTION: The motion of the meterstick is rotation about a fixed axis passing through the point of contact with the floor (see Fig. 13.3). The stick is a uniform rod of mass M and length $l = 1.0$ m. Its moment of inertia about the end is $Ml^2/3$ (see Table 12.3), and its rotational kinetic energy is therefore $\frac{1}{2}I\omega^2 = Ml^2\omega^2/6$. The gravitational potential energy is Mgy, where y is the height of the center of mass above the floor. When the meterstick is standing vertically, the initial angular velocity is $\omega_1 = 0$ and $y_1 = l/2$, so the total energy is

$$E = \tfrac{1}{6}Ml^2\omega_1^2 + Mgy_1 = 0 + Mgl/2 \tag{13.11}$$

Just before the meterstick hits the floor, the angular velocity is ω_2 and $y_2 = 0$. The energy is

$$E = \tfrac{1}{6}Ml^2\omega_2^2 + Mgy_2 = \tfrac{1}{6}Ml^2\omega_2^2 + 0 \tag{13.12}$$

Conservation of energy therefore implies

$$\tfrac{1}{6}Ml^2\omega_2^2 = Mgl/2$$

from which we obtain

$$\omega_2^2 = \frac{3g}{l} \tag{13.13}$$

Taking the square root of both sides, we find

$$\omega_2 = \sqrt{\frac{3g}{l}} = \sqrt{\frac{3 \times 9.81\,\text{m/s}^2}{1.0\text{ m}}} = 5.4\ \text{radians/s}$$

EXAMPLE 3 At what instantaneous rate is gravity delivering energy to the meterstick of Example 2 just before it hits the floor? The mass of the meterstick is 0.15 kg.

SOLUTION: The rate of energy delivery is the power,

$$P = \tau\omega$$

From Example 2, we know $\omega = 5.4$ radians/s just before the stick hits the floor. At that instant, gravity acts perpendicular to the stick at the center of mass (in the next chapter we will see that the weight acts as if concentrated at the center of mass), a distance $R = l/2 = 0.50$ m from the end. So the torque exerted by gravity is

$$\tau = FR\sin\theta = mg\frac{l}{2}\sin 90^\circ = 0.15\text{ kg} \times 9.81\text{ m/s}^2 \times 0.50\text{ m} \times 1$$

$$= 0.74\text{ N}\cdot\text{m}$$

Thus the instantaneous power delivered by the torque due to gravity is

$$P = \tau\omega = 0.74\ \text{N}\cdot\text{m} \times 5.4\ \text{radians/s} = 4.0\text{ W}$$

Checkup 13.1

QUESTION 1: You are trying to tighten a bolt with a wrench. Where along the handle should you place your hand so you can exert maximum torque? In what direction should you push?

QUESTION 2: A force is being exerted against the rim of a freely rotating wheel, but the work done by this force is zero. What can you conclude about the direction of the force? What is the torque of the force?

QUESTION 3: Consider the meterstick falling over, as in Example 2. What is the torque that the weight exerts on the meterstick when it is in the upright, initial position? After the stick begins to fall over, the torque increases. When is the torque maximum?

QUESTION 4: Suppose you first push a door at its outer edge at right angles to the surface of the door with a force of magnitude F. Next you push the door at its center, again at right angles to the surface, with a force of magnitude $F/2$. In both cases you push the door as it moves through 30°. The ratio of the work done by the second push to the work done by the first push is:

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 4

13.2 THE EQUATION OF ROTATIONAL MOTION

Our intuition tells us that a torque acting on a wheel or some other body free to rotate about an axis will produce an angular acceleration. For instance, the push of your hand against a crank on a wheel (see Fig. 13.4) exerts a torque or "twist" that starts the wheel turning. The angular acceleration depends on the magnitude of your push on the crank and also on its direction (as well as on the inertia of the wheel). Your push will be most effective if exerted tangentially, at right angles to the radius (at $\theta = 90°$; see Fig. 13.4a). It will be less effective if exerted at a smaller or larger angle (see Fig. 13.4b). And it will be completely ineffective if exerted parallel to the radius (at $\theta = 0$ or 180°; see Fig. 13.4c)—such a push in the radial direction produces no rotation at all. These qualitative considerations are in agreement with the definition of torque,

$$\tau = FR \sin\theta \qquad (13.14)$$

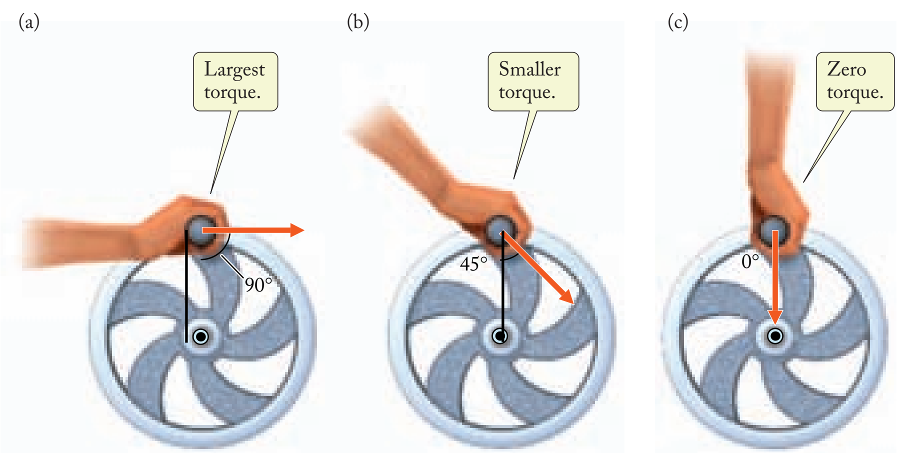


FIGURE 13.4 (a) A push at right angles to the radius is most effective in producing rotation. (b) A push at 45° is less effective. (c) A push parallel to the radius produces no rotation.

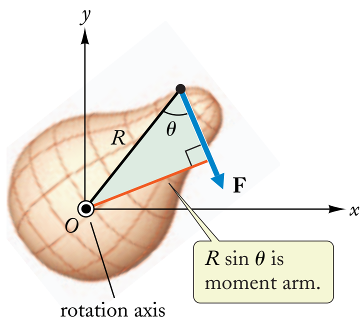


FIGURE 13.5 The distance between the center of rotation and the point of application of the force is R. The perpendicular distance between the center of rotation and the line of action of the force is $R \sin\theta$.

According to this equation, the torque provided by a force of a given magnitude F is maximum if the force is at right angles to the radius ($\theta = 90°$), and it is zero if the force is parallel to the radius ($\theta = 0°$ or $180°$).

The quantity $R \sin\theta$ appearing in Eq. (13.14) has a simple geometric interpretation: it is the perpendicular distance between the line of action of the force and the axis of rotation (see Fig. 13.5); this perpendicular distance is called the **moment arm of the force.** Hence, Eq. (13.14) states that the torque equals the magnitude of the force multiplied by the moment arm.

To find a quantitative relationship between torque and angular acceleration, we recall from Eq. (13.6) that the power delivered by a torque acting on a body is

$$\frac{dW}{dt} = \tau\omega \tag{13.15}$$

The work–energy theorem tells us that the work dW equals the change of kinetic energy in the small time interval dt. The small change in the kinetic energy is $dK = d(\frac{1}{2}I\omega^2) = \frac{1}{2}I \times 2\omega\, d\omega = I\omega\, d\omega$. Thus,

$$dW = I\omega\, d\omega \tag{13.16}$$

Inserting this into the left side of Eq. (13.15), we find

$$\frac{I\omega\, d\omega}{dt} = \tau\omega \tag{13.17}$$

Canceling the factor of ω on both sides of the equation, we obtain

$$I\frac{d\omega}{dt} = \tau \tag{13.18}$$

But $d\omega/dt$ is the angular acceleration α; hence

equation of rotational motion

$$I\alpha = \tau \tag{13.19}$$

This is the equation for rotational motion. As we might have expected, this equation says that *the angular acceleration is directly proportional to the torque.* Equation (13.19) is mathematically analogous to Newton's Second Law, $m\mathbf{a} = \mathbf{F}$, for the translational motion of a particle; the moment of inertia takes the place of the mass, the angular acceleration the place of the acceleration, and the torque the place of the force.

In our derivation of Eq. (13.19) we assumed that only one external force is acting on the rigid body. If several forces act, then each produces its own torque. If an individual torque would produce an angular acceleration in the rotational direction chosen as positive, it is reckoned as positive, and if a torque would produce an angular acceleration in the opposite direction, it is reckoned as negative. The net torque is the sum of these individual torques, and the angular acceleration is proportional to this net torque:

equation of rotational motion for net torque

$$I\alpha = \tau_{\text{net}} \tag{13.20}$$

In the evaluation of the net torque, we need to take into account all the external forces acting on the rigid body, but we can ignore the internal forces that particles in the body exert on other particles also in the body. The torques of such internal forces cancel (this is an instance of the general result mentioned in Section 10.4: for a rigid body, the work of internal forces cancels).

EXAMPLE 4 The rotor of the gyroscope of the *Gravity Probe B* experiment (see the chapter photo and Fig. 13.6) is a quartz sphere of diameter 3.8 cm and mass 7.61×10^{-2} kg. To start this sphere spinning, a stream of helium gas flowing in an equatorial channel in the surface of the housing is blown tangentially against the rotor. What torque must this stream of gas exert on the rotor to accelerate it uniformly from 0 to 10 000 rpm (revolutions per minute) in 30 minutes? What force must it exert on the equator of the sphere?

Concepts in Context

FIGURE 13.6 A gyroscope sphere for *Gravity Probe B*.

SOLUTION: The final angular velocity is $2\pi \times 10000$ radians/60 s $= 1.05 \times 10^3$ radians/s, and therefore the angular acceleration is

$$\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

$$= \frac{1.05 \times 10^3 \text{ radians/s} - 0}{30 \times 60 \text{ s} - 0} = 0.582 \text{ radian/s}^2$$

The moment of inertia of the rotor is that of a sphere (see Table 12.3):

$$I = \tfrac{2}{5}MR^2$$

$$= \tfrac{2}{5} \times 7.61 \times 10^{-2} \text{ kg} \times (0.019 \text{ m})^2 = 1.1 \times 10^{-5} \text{ kg}\cdot\text{m}^2$$

Hence the required torque is, according to Eq. (13.19),

$$\tau = I\alpha = 1.1 \times 10^{-5} \text{ kg}\cdot\text{m}^2 \times 0.582 \text{ radian/s}^2$$

$$= 6.4 \times 10^{-6} \text{ N}\cdot\text{m}$$

The driving force is along the equator of the rotor—that is, it is perpendicular to the radius—so $\sin\theta = 1$ and Eq. (13.2) reduces to $\tau = FR$, which yields

$$F = \frac{\tau}{R} = \frac{6.4 \times 10^{-6} \text{ N}\cdot\text{m}}{0.019 \text{ m}} = 3.4 \times 10^{-4} \text{ N}$$

EXAMPLE 5 Two masses m_1 and m_2 are suspended from a string that runs, without slipping, over a pulley (see Fig. 13.7a). The pulley has a radius R and a moment of inertia I about its axle, and it rotates without friction. Find the accelerations of the masses.

SOLUTION: We have already found the motion of this system in Example 10 of Chapter 5, where the two masses were an elevator and its counterweight, and where we neglected the inertia of the pulley. Now we will take this inertia into account.

Figure 13.7c shows the "free-body" diagrams for the masses m_1 and m_2. In these diagrams, T_1 and T_2 are the tensions in the two parts of the string attached to the two masses. (Note that now T_1 and T_2 are not equal. For a pulley of zero moment of inertia, these tensions would be equal; but for a pulley of nonzero moment of inertia, a difference between T_1 and T_2 is required to produce the angular acceleration of the pulley.) If the acceleration of mass m_1 is a (reckoned as positive if upward), then the acceleration of mass m_2 is $-a$, and the equations of motion of the two masses are

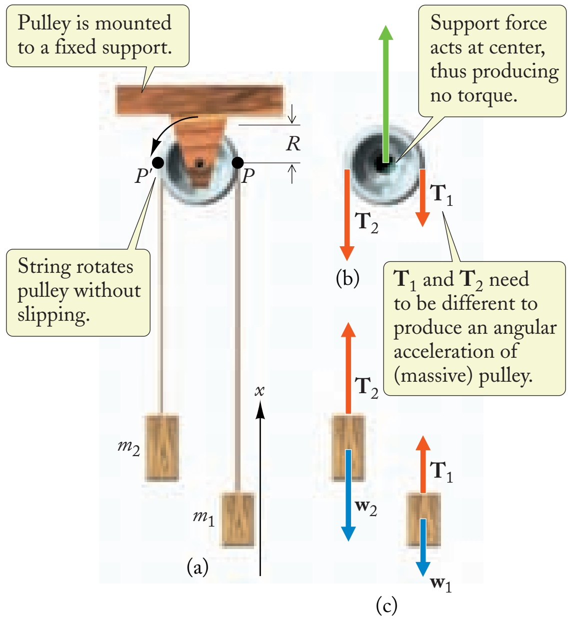


FIGURE 13.7 (a) Two masses m_1 and m_2 suspended from a string that runs over a pulley. (b) "Free-body" diagram for the pulley. (c) "Free-body" diagrams for the masses m_1 and m_2.

$$m_1 a = T_1 - m_1 g \tag{13.21}$$

$$-m_2 a = T_2 - m_2 g \tag{13.22}$$

Figure 13.7b shows the "free-body" diagram for the pulley. The tension forces act at the ends of the horizontal diameter (since the string does not slip, it behaves as though instantaneously attached to the pulley at the point of first contact; see points P and P' in Fig. 13.7a). The upward supporting force of the axle acts at the center of the pulley, and it generates no torque about the center of the pulley. The tensions act perpendicular to the radial direction, so $\sin\theta = 1$ in Eq. (13.2). Taking the positive direction of rotation as counterclockwise (to match the positive direction for the motion of mass m_1), we see that the tension forces T_1 and T_2 generate torques $-RT_1$ and RT_2 about the center. The equation of rotational motion of the pulley is

$$I\alpha = \tau_{\text{net}} = -RT_1 + RT_2 \tag{13.23}$$

The translational acceleration of each hanging portion of the string must match the instantaneous translational acceleration of the point of first contact (for the given condition of no slipping). Hence the translational acceleration a of the masses is related to the angular acceleration α by $a = \alpha R$, or $\alpha = a/R$ [see Eq. (12.13)]. Furthermore, according to Eqs. (13.21) and (13.22), $T_1 = m_1 g + m_1 a$ and $T_2 = m_2 g - m_2 a$. With these substitutions, Eq. (13.23) becomes

$$I(a/R) = -R(m_1 g + m_1 a) + R(m_2 g - m_2 a)$$

Solving this for a, we find

$$a = \frac{m_2 - m_1}{m_1 + m_2 + (I/R^2)}\, g \tag{13.24}$$

COMMENT: If the mass of the pulley is small, then I/R^2 can be neglected; with this approximation, Eq. (13.24) reduces to Eq. (5.44), which was obtained without taking into account the inertia of the pulley.

A device of this kind, called **Atwood's machine,** can be used to determine the value of g. For this purpose, it is best to use masses m_1 and m_2 that are nearly equal. Then a is much smaller than g and easier to measure; the value of g can be calculated from the measured value of a according to Eq. (13.24).

Atwood's machine

In some cases—for instance, the rolling motion of a wheel—the axis of rotation is in motion, perhaps accelerated motion, and is *not a fixed axis*. For such problems, some further arguments can be used to demonstrate that Eq. (13.19) remains valid for rotation about an axis in accelerated translational motion, *provided the axis passes through the center of mass of the rotating body*. When this condition is met, we can use the equation of rotational motion (13.19) as in the following examples.

EXAMPLE 6 An automobile with rear-wheel drive is accelerating at 4.0 m/s^2 along a straight road. Consider one of the front wheels of this automobile (see Fig. 13.8a). The axle pushes the wheel forward, providing an acceleration of 4.0 m/s^2. Simultaneously, the friction force of the road pushes the bottom of the wheel backward, providing a torque that gives the wheel an angular acceleration. The wheel has a radius of 0.38 m and a mass of 25 kg. Assume that the wheel is (approximately) a uniform disk, and assume it rolls without slipping. Find the backward force that the friction force exerts on the wheel, and find the forward force that the axle exerts on the wheel.

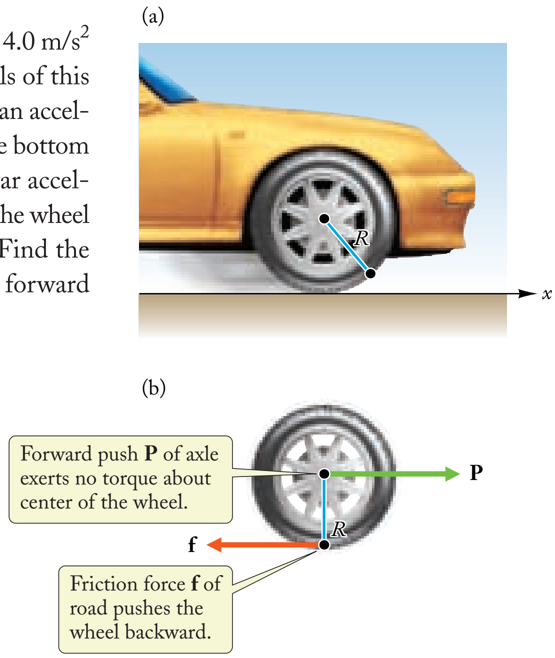


FIGURE 13.8 (a) Front wheel of an automobile. (b) "Free-body" diagram for the wheel. The friction force of the road pushes the wheel backward. The axle pushes the wheel forward.

SOLUTION: Figure 13.8b shows a "free-body" diagram of the wheel, with the horizontal forces acting on it (besides these horizontal forces, there are also a vertical downward push exerted by the axle and a vertical upward normal force exerted by the road; these forces exert no torque and cancel, so they need not concern us here). The forward push of the axle is P, and the rearward push of the ground is f. The force P, acting at the center of the wheel, exerts no torque; the force f, acting at the rim, exerts a torque Rf. Thus, the equation for the rotational motion of the wheel is

$$I\alpha = Rf$$

or, since $I = \frac{1}{2}MR^2$ for a uniform disk (see Table 12.3),

$$\tfrac{1}{2}MR\alpha = f$$

As we have seen in Example 4 of Chapter 12, the angular acceleration of a rolling wheel is related to the translational acceleration by $\alpha = a/R$. Hence

$$\tfrac{1}{2}Ma = f$$

from which

$$f = \tfrac{1}{2}Ma = \tfrac{1}{2} \times 25 \text{ kg} \times 4.0 \text{ m/s}^2$$
$$= 50 \text{ N}$$

To find the force P, we need to examine the equation for the translational motion. The net horizontal force is $F_{net} = P - f$. Hence the equation for the translational motion of the wheel is

$$Ma = P - f$$

from which

$$P = Ma + f = 25 \text{ kg} \times 4.0 \text{ m/s}^2 + 50 \text{ N}$$
$$= 150 \text{ N}$$

Thus, the force required to accelerate a rolling wheel is larger than the force required for a wheel that slips on a frictionless surface without rolling—for such a wheel the force would be only $Ma = 25 \text{ kg} \times 4.0 \text{ m/s}^2 = 100$ N. Here, the additional rotational inertia $\frac{1}{2}MR^2$ adds an additional amount $f = \frac{1}{2}Ma$ to the required force, so the total required force is $\frac{3}{2}$ that for sliding without rolling.

EXAMPLE 7 A solid cylinder of mass M and radius R rolls down a sloping ramp that makes an angle β with the ground (see Fig. 13.9a). What is the acceleration of the cylinder? Assume that the cylinder is uniform and rolls without slipping.

SOLUTION: Figure 13.9b shows the "free-body" diagram for the cylinder. The forces on the cylinder are the normal force **N** exerted by the ramp, the friction force **f** exerted by the ramp, and the weight **w**. The friction force is exerted on the rim of the cylinder, and the weight is effectively exerted at the center of the cylinder (in the next chapter we will see that the weight can always be regarded as concentrated at the center of mass). As axis of rotation, we take the axis that passes through the center of the cylinder. The weight exerts no torque about this axis, and neither does the normal force (zero moment arm). Hence, the only force that exerts a torque is the friction force, and so

$$\tau = Rf$$

The equation of rotational motion is then

$$I\alpha = Rf$$

The moment of inertia of a uniform cylinder is the same as that of a disk, $I = \frac{1}{2}MR^2$. Furthermore, for rolling motion without slipping, $\alpha = a/R$. Hence

$$\tfrac{1}{2}MRa = Rf$$

or

$$a = \frac{2f}{M} \tag{13.25}$$

To evaluate the acceleration, we need to eliminate the friction force f from this equation. We can do this by appealing to the equation for the component of the translational motion along the ramp (the motion along the x direction in Fig. 13.9b).

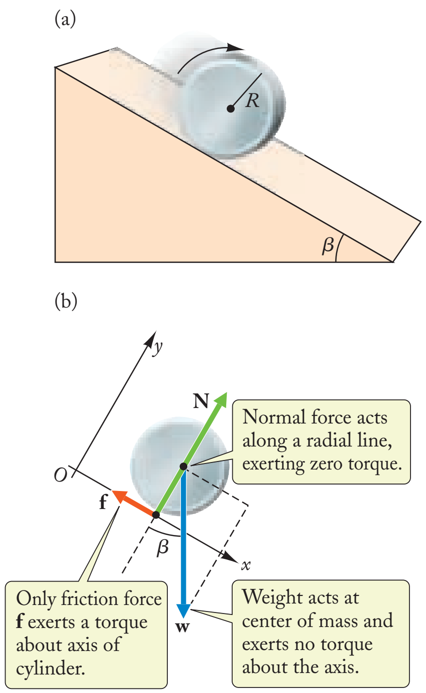


FIGURE 13.9 (a) A cylinder rolling down an inclined plane. (b) "Free-body" diagram for the cylinder.

The components of the forces along the ramp are $-f$ for the friction force and $Mg \sin \beta$ for the weight. Hence

$$Ma = Mg \sin \beta - f$$

or

$$f = Mg \sin \beta - Ma$$

Substituting this into Eq. (13.25), we find

$$a = 2g \sin \beta - 2a$$

which we can immediately solve for a:

$$a = \tfrac{2}{3} g \sin \beta$$

COMMENT: Note that the force $Mg \sin \beta$ along the ramp here produces an acceleration that is two-thirds of the acceleration that the cylinder would have if it were to slip down a frictionless ramp without rolling. This is consistent with the last example, where we saw that a force $\frac{3}{2}$ as large was required to produce a given acceleration. The same factor occurs in both cases, because both the disk and the cylinder have the same moment of inertia, $\frac{1}{2}MR^2$.

PROBLEM-SOLVING TECHNIQUES — TORQUES AND ROTATIONAL MOTION

The general techniques for the solution of problems of rotational motion are similar to the techniques we learned in Chapters 5 and 6 for translational motion.

1. The first step is always a careful enumeration of all the forces. Make a complete list of these forces, and label each with a vector symbol.
2. Identify the body whose motion or whose equilibrium is to be investigated and draw the "free-body" diagram showing the forces acting on this body. If there are several distinct bodies in the problem (as in Example 5), then you need to draw a separate "free-body" diagram for each. When drawing the arrows for the forces acting on a rotating body, be sure to draw the head or the tail of the arrow at the actual point of the body where the force acts, since this will be important for the calculation of the torque. Note that the weight acts at the center of mass (we will establish this in the next chapter).
3. Select which direction of rotation will be regarded as positive (for instance, in Example 5, we selected the counterclockwise direction of rotation as positive). If the problem involves joint rotational and translational motions, select coordinate axes for the translational motion, preferably placing one of the axes along the direction of motion.
4. Select an axis for the rotation of the rigid body, either an axis through the center of mass, or else a fixed axis (such as an axle or a pivot mounted on a support) about which the body is constrained to rotate. Calculate the torque of each force acting on the body about this center. Remember that the sign of the torque is positive or negative depending on whether it produces an angular acceleration in the positive or the negative direction of rotation.
5. Then apply the equation of rotational motion, $I\alpha = \tau$, to each rotating body, where τ is the net torque on a given body.
6. If the rigid body has a translational motion besides the rotational motion, apply Newton's Second Law, $\mathbf{F} = m\mathbf{a}$, for the translational motion (see Examples 5 and 6). For rolling without slipping, the translational and the rotational motions are related by $v = \omega R$ and $a = \alpha R$.
7. If there are several distinct bodies in the problem, you need to apply the equation of rotational motion or Newton's Second Law separately for each (see Example 5).

Checkup 13.2

QUESTION 1: Consider a meterstick falling over, as in Example 2. At what instant is the angular acceleration produced by the weight force maximum?

QUESTION 2: A rolling cylinder has both rotational kinetic energy (reckoned about its center of mass) and translational kinetic energy. Which is larger?

QUESTION 3: Consider the rolling cylinder of Example 7. When this cylinder reaches the bottom of the ramp, is its kinetic energy larger, smaller, or the same as that of a similar cylinder that slips down a frictionless ramp without rolling?

QUESTION 4: A sphere and a cylinder of equal masses roll down an inclined plane without slipping. Will they have equal kinetic energies when they reach the bottom? Which will get to the bottom first?

QUESTION 5: A thin hoop and a solid cylinder roll down an inclined plane without slipping. When they reach the bottom, the translational speed of the hoop is

(A) Less than that of the cylinder
(B) Greater than that of the cylinder
(C) Equal to that of the cylinder

13.3 ANGULAR MOMENTUM AND ITS CONSERVATION

In Chapter 10 we saw how to express the equation for the translational motion in terms of the momentum: the rate of change of the momentum equals the force ($dp_x/dt = F_x$). Likewise, we can express the equation for rotational motion in terms of **angular momentum.** *The angular momentum of a body rotating about a fixed axis is defined as the product of the moment of inertia and the angular velocity,*

angular momentum

$$L = I\omega \tag{13.26}$$

This equation for angular momentum is analogous to the equation $p = mv$ for translational momentum. *The SI unit of angular momentum is* kg·m²/s, which can also be written in the alternative form J·s. Table 13.1 gives some examples of typical values of angular momenta.

Concepts in Context

EXAMPLE 8 According to the data given in Example 4, what is the angular momentum of the rotor of the *Gravity Probe B* gyroscope when spinning at 10 000 revolutions per minute?

SOLUTION: From Example 4, the angular velocity is $\omega = 1.05 \times 10^3$ radians/s, and the moment of inertia is $I = 1.1 \times 10^{-5}$ kg·m². So

$$L = I\omega = 1.1 \times 10^{-5}\ \text{kg}\cdot\text{m}^2 \times 1.05 \times 10^3\ \text{radians/s}$$
$$= 1.2 \times 10^{-2}\ \text{kg}\cdot\text{m}^2/\text{s}$$

To express the equation for rotational motion in terms of angular momentum, we proceed as we did in the translational case. We note that if the change of angular velocity is $d\omega$, then $dL = I\,d\omega$. Dividing both sides of this relation by dt, we see

$$\frac{dL}{dt} = I\frac{d\omega}{dt}$$

If we compare this with Eq. (13.18), we see that the right side can be expressed as the torque, so

$$\frac{dL}{dt} = \tau \qquad (13.27)$$

equation of rotational motion in terms of angular momentum

This says that *the rate of change of angular momentum equals the torque.* Obviously, this equation is analogous to the equation $dp_x/dt = F_x$ for translational motion.

We now see that the analogy between rotational and translational quantities mentioned in Section 12.3 can be extended to angular momentum and momentum. Table 13.2 lists analogous quantities, including the quantities for work, power, and kinetic energy.

If there is no torque acting on the rotating body, $\tau = 0$ and therefore $dL/dt = 0$, which means that the angular momentum does not change:

$$L = [\text{constant}] \qquad (\text{when } \tau = 0) \qquad (13.28)$$

conservation of angular momentum

This is the **Law of Conservation of Angular Momentum.** Since $L = I\omega$, we can also write this law as

$$I\omega = [\text{constant}] \qquad (13.29)$$

TABLE 13.1 SOME ANGULAR MOMENTA

Orbital motion of Earth	2.7×10^{40} J·s
Rotation of Earth	5.8×10^{33} J·s
Helicopter rotor (320 rev/min)	5×10^{4} J·s
Automobile wheel (90 km/h)	1×10^{2} J·s
Electric fan	1 J·s
Frisbee	1×10^{-1} J·s
Toy gyroscope	1×10^{-2} J·s
Phonograph record (33.3 rev/min)	6×10^{-3} J·s
Compact disc (plating outer track)	2×10^{-3} J·s
Bullet fired from rifle	2×10^{-3} J·s
Orbital motion of electron in atom	1.05×10^{-34} J·s
Spin of electron	0.53×10^{-34} J·s

TABLE 13.2 FURTHER ANALOGIES BETWEEN 1D TRANSLATIONAL AND ROTATIONAL QUANTITIES

$dW = F\,dx$	→	$dW = \tau\,d\phi$
$P = Fv$	→	$P = \tau\omega$
$K = \frac{1}{2}mv^2$	→	$K = \frac{1}{2}I\omega^2$
$ma = F$	→	$I\alpha = \tau$
$p = mv$	→	$L = I\omega$
$\frac{dp}{dt} = F$	→	$\frac{dL}{dt} = \tau$

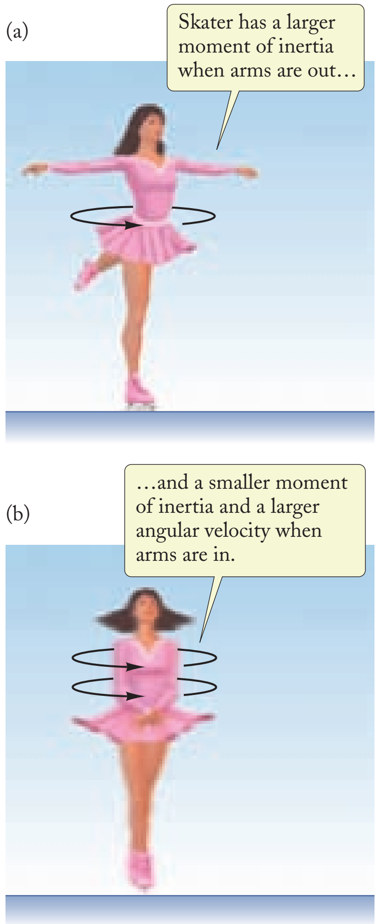


FIGURE 13.10 Figure skater performing a pirouette. (a) Arms extended. (b) Arms folded against body.

FIGURE 13.11 A figure skater whirling at high speed.

A pirouette performed by a figure skater on ice provides a nice illustration of the conservation of angular momentum. The skater begins the pirouette by spinning about her vertical axis with her arms extended horizontally (see Fig. 13.10a); in this configuration, the arms have a large moment of inertia. She then brings her arms close to her body (see Fig. 13.10b), suddenly decreasing her moment of inertia. Since the ice is nearly frictionless, the external torque on the skater is nearly zero, and therefore the angular momentum is conserved. According to Eq. (13.26), a decrease of I requires an increase of ω to keep the angular momentum constant. Thus, the change of configuration of her arms causes the skater to whirl around her vertical axis with a dramatic increase of angular velocity (see Fig. 13.11).

Like the law of conservation of translational momentum, the Law of Conservation of Angular Momentum is often useful in the solutions of problems in which the forces are not known in detail.

EXAMPLE 9 Suppose that a pottery wheel is spinning (with the motor disengaged) at 80 rev/min when a 6.0-kg ball of clay is suddenly dropped down on the center of the wheel (see Fig. 13.12). What is the angular velocity after the drop? Treat the ball of clay as a uniform sphere of radius 8.0 cm. The pottery wheel has a moment of inertia $I = 7.5 \times 10^{-2}$ kg·m^2. Ignore the (small) friction force in the axle of the turntable.

SOLUTION: Since there is no external torque on the system of pottery wheel and clay, the angular momentum of this system is conserved. The angular momentum before the drop is

$$L = I\omega \tag{13.30}$$

where ω is the initial angular velocity and I the moment of inertia of the pottery wheel. The angular momentum after the drop is

$$L' = I'\omega' \tag{13.31}$$

where ω' is the final angular velocity and I' the moment of inertia of pottery wheel and clay combined. Hence

$$I\omega = I'\omega' \tag{13.32}$$

from which we find

$$\omega' = \frac{I}{I'}\omega \tag{13.33}$$

The wheel is initially rotating with angular velocity

$$\omega = 2\pi \times f = 2\pi \times \frac{80\ \text{rev}}{60\ \text{s}} = 8.4\ \text{radians/s}$$

The moment of inertia of the pottery wheel is given,

$$I = 7.5 \times 10^{-2}\ \text{kg·m}^2$$

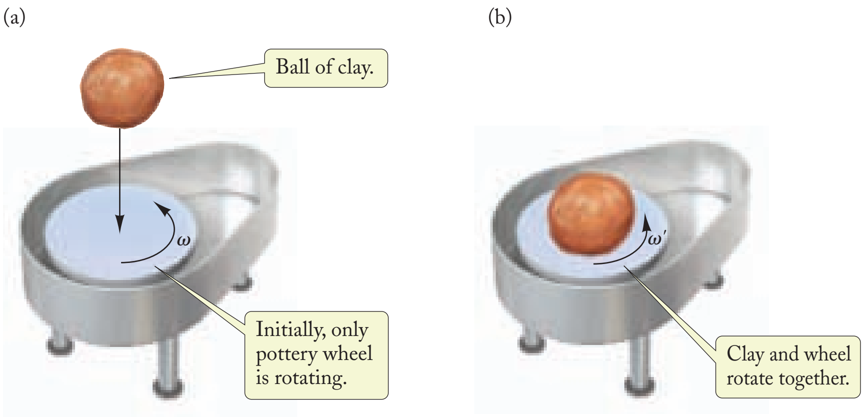


FIGURE 13.12 (a) A pottery wheel rotates with angular velocity ω; (b) when a ball of clay is dropped on the wheel, the angular velocity slows to ω'.

and the moment of inertia of the clay is that of a uniform sphere (see Table 12.3):

$$I_{clay} = \tfrac{2}{5}MR^2 = \tfrac{2}{5} \times 6.0\ \text{kg} \times (0.080\ \text{m})^2$$

$$= 1.5 \times 10^{-2}\ \text{kg}\cdot\text{m}^2$$

Accordingly,

$$\omega' = \frac{I}{I'}\omega = \frac{I}{I + I_{clay}}\omega$$

$$= \frac{7.5 \times 10^{-2}\ \text{kg}\cdot\text{m}^2}{7.5 \times 10^{-2}\ \text{kg}\cdot\text{m}^2 + 1.5 \times 10^{-2}\ \text{kg}\cdot\text{m}^2} \times 8.4\ \text{radians/s}$$

$$= 7.0\ \text{radians/s}$$

As already mentioned in Chapter 9, the Law of Conservation of Angular Momentum also applies to a single particle moving in an orbit under the influence of a central force. Such a force is always directed along the radial line, and it therefore exerts no torque. If the particle is moving along a circle of radius r with velocity v (see Fig. 13.13), its moment of inertia is mr^2 and its angular velocity is $\omega = v/r$. Hence $I\omega = mr^2 \times v/r = mvr$, and the angular momentum of the particle is

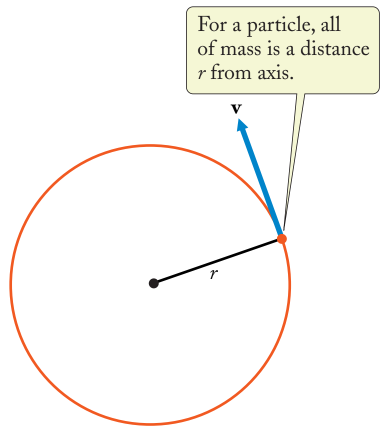


FIGURE 13.13 A particle moving with speed v along a circle of radius r. The moment of inertia of this particle with respect to the center of the circle is $I = mr^2$.

$$L = mvr \qquad \text{(circular orbit)} \qquad (13.34)$$

angular momentum for circular orbit

This formula is valid not only for a circular orbit, but also for the perihelion and aphelion points of an elliptical orbit, where the instantaneous velocity is perpendicular to the radius. In Chapter 9 we took advantage of the conservation of the angular momentum $L = mvr$ to compare the speeds of a planet at perihelion and at aphelion.

The angular momentum defined by Eq. (13.34) is called the **orbital angular momentum** to distinguish it from **spin angular momentum** of a body rotating about its own axis. For instance, the Earth has both an orbital angular momentum (due to its motion around the Sun) and a spin angular momentum (due to its rotation about its own axis). Table 13.1 includes examples of both kinds of angular momentum.

orbital angular momentum and spin angular momentum

PROBLEM-SOLVING TECHNIQUES — CONSERVATION OF ANGULAR MOMENTUM

The use of conservation of angular momentum in a problem involving rotational motion involves the familiar three steps we used with conservation of momentum or of energy in translational motion:

1. First write an expression for the angular momentum at one instant of the motion [Eq. (13.30)].
2. Then write an expression for the angular momentum at another instant [Eq. (13.31)].
3. And then rely on conservation of angular momentum to equate the two expressions [Eq. (13.32)]. This yields one equation, which can be solved for an unknown quantity, such as the final angular speed.

Checkup 13.3

QUESTION 1: A hoop and a uniform disk have equal radii and equal masses. Both are spinning with equal angular speeds. Which has the larger angular momentum? By what factor?

QUESTION 2: Two automobiles of equal masses are traveling around a traffic circle side by side, with equal angular velocities. Which has the larger angular momentum?

QUESTION 3: You sit on a spinning stool with your legs tucked under the seat. You then stretch your legs outward. How does your angular velocity change?

QUESTION 4: Consider the spinning skater described in Fig. 13.10. While she brings her arms close to her body, does the rotational kinetic energy remain constant?

QUESTION 5: Three children sit on a tire swing (see Fig. 13.14), leaning backward as the wheel rotates about a vertical axis. What happens to the rotational frequency if the children sit up straight?

(A) Frequency increases (B) Frequency decreases
(C) Frequency remains constant

FIGURE 13.14 Children on a spinning tire swing.

13.4 TORQUE AND ANGULAR MOMENTUM AS VECTORS

The rotational motion of a rigid body about a fixed axis is analogous to one-dimensional translational motion. More generally, if the axis of rotation is not fixed but changes in direction, the motion becomes three-dimensional. A wobbling, spinning top provides an example of such a three-dimensional rotational motion. In this case, the torque and the angular momentum must be treated as vectors, analogous to the force vector and the momentum vector. The definitions of the torque vector and the angular-momentum vector involve the vector cross product that we introduced in Section 3.4. When a force $\mathbf{F}$ acts at some point with position vector $\mathbf{r}$, *the resulting* **torque vector** *is the cross product of the position vector and the force vector:*

torque vector

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \tag{13.35}$$

According to the definition of the cross product, the magnitude of $\boldsymbol{\tau}$ is

$$\tau = rF\sin\theta \qquad (13.36)$$

and the direction of $\boldsymbol{\tau}$ is perpendicular to the force vector and the position vector, as specified by the right-hand rule (see Fig. 13.15). Note that since the position vector depends on the choice of origin, *the torque* also *depends on the choice of origin.* We will usually place the origin on some axis or some pivot, and the torque (13.35) is then reckoned in relation to this pivot. For instance, for rotation about a fixed axis, we place the origin on that axis, so **r** is in the plane of the circular motion of the point at which the force acts; then $r = R$, and Eq. (13.36) agrees with Eq. (13.2).

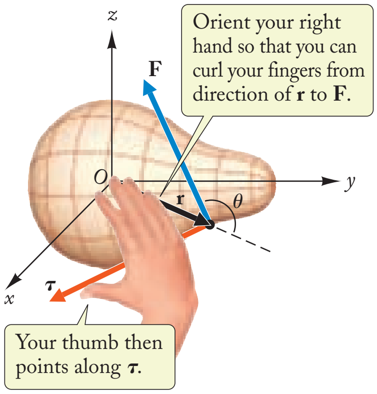


FIGURE 13.15 The torque vector $\boldsymbol{\tau}$ is perpendicular to the force **F** and the position vector **r**, in the direction specified by the right-hand rule: place the fingers of your right hand along the direction of **r** and curl toward **F** along the smaller angle between these vectors; your thumb will then point in the direction of $\mathbf{r} \times \mathbf{F}$.

The definition of the angular-momentum vector of a rigid body is based on the definition of the angular-momentum vector for a single particle. If a particle has translational momentum **p** at position **r**, then *its* **angular-momentum vector** *is defined as the cross product of the position vector and the momentum vector:*

angular-momentum vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad (13.37)$$

As in the case of the torque, *the angular momentum vector depends on the choice of origin.* For instance, if the particle is moving along a circle, we place the origin at the center of the circle, so **r** and **p** are in the plane of the circular motion. Since the vectors **r** and **p** are perpendicular, the magnitude of their cross product is then $L = rp \sin 90° = rp = rmv$. By the right-hand rule, the direction of $\mathbf{r} \times \mathbf{p}$ is perpendicular to the plane of the circular motion, parallel to the axis of rotation. (see Fig. 13.16).

For a rigid body rotating about some (instantaneous) axis, the angular-momentum vector is defined as the sum of the angular-momentum vectors of all the particles in the body,

$$\mathbf{L} = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 + \cdots \qquad (13.38)$$

As in the case of a single particle, the value of the angular momentum obtained from this formula depends on the choice of the origin of coordinates. For the calculation of the angular momentum of a rigid body rotating about a fixed axis, it is usually convenient to choose an origin on the axis of rotation.

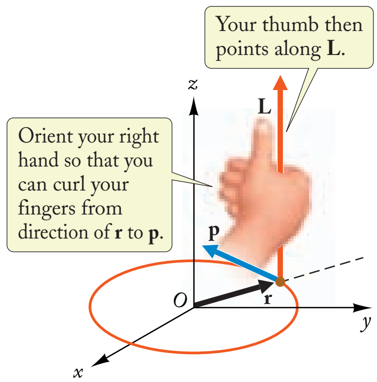


FIGURE 13.16 Angular-momentum vector for a particle.

EXAMPLE 10 Figure 13.17 shows a dumbbell, a rigid body consisting of two particles of mass m attached to the ends of a massless rigid rod of length $2r$. The body rotates with angular velocity ω about a perpendicular axis through the center of the rod. Find the angular momentum about this center.

SOLUTION: Each particle executes circular motion with speed $v = r\omega$. Hence the angular momentum of each has a magnitude $L = rmv = mr^2\omega$ (compare the case of a single particle, illustrated in Fig. 13.13). The direction of each angular-momentum vector is parallel to the axis of rotation (see Fig. 13.16). Thus the direction of the vector sum of the two angular-momentum vectors is also parallel to the axis of rotation, and its magnitude is

$$L = mr^2\omega + mr^2\omega = 2mr^2\omega$$

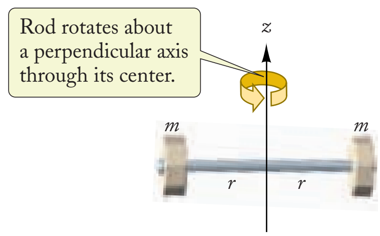


FIGURE 13.17 A rotating dumbbell.

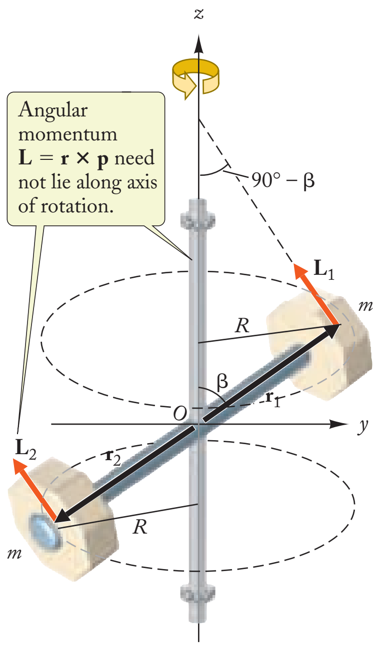


FIGURE 13.18 A rotating dumbbell oriented at an angle β with the axis of rotation.

EXAMPLE 11 Suppose that the rod of the dumbbell described in the preceding example is welded to an axle inclined at an angle β with respect to the rod. The dumbbell rotates with angular velocity ω about this axis, which is supported by fixed bearings (see Fig. 13.18). Find the angular momentum about an origin on the axis, at the center of mass.

SOLUTION: Each particle executes circular motion, but since the origin is not at the center of the circle, the angular momentum is not the same as in Example 10. The distance between each particle and the axis of rotation is

$$R = r \sin \beta$$

and the magnitude of the velocity of each particle is

$$v = \omega R = \omega r \sin \beta$$

The direction of the velocity is perpendicular to the position vector. Hence the angular-momentum vector of each mass has a magnitude

$$|\mathbf{L}_1| = |\mathbf{L}_2| = m|\mathbf{r} \times \mathbf{v}| = mrv = m\omega r^2 \sin \beta \tag{13.39}$$

The direction of the angular-momentum vector of each mass is perpendicular to both the velocity and the position vectors, as specified by the right-hand rule. The angular-momentum vector of each mass is shown in Fig. 13.18; these vectors are parallel to each other, they are in the plane of the axis and the rod, and they make an angle of $90° - \beta$ with the axis. The total angular momentum is the vector sum of these individual angular momenta. This vector is in the same direction as the individual angular-momentum vectors, and it has a magnitude twice as large as either of those in Eq. (13.39):

$$L = 2m\omega r^2 \sin \beta \tag{13.40}$$

As the body rotates, so does the angular-momentum vector, remaining in the plane of the axis and the rod. If at one instant the angular momentum lies in the z–y plane, a quarter of a cycle later it will lie in the z–x plane, etc.

COMMENT: Note that the z component of the angular momentum is

$$L_z = L\cos(90° - \beta) = 2m\omega r^2 \sin \beta \cos(90° - \beta) = 2m\omega r^2 \sin^2\beta$$

This can also be written as

$$L_z = 2m\omega R^2 \tag{13.41}$$

where $R = r \sin \beta$ is the perpendicular distance between each mass and the axis of rotation. Since $2mR^2$ is simply the moment of inertia of the two particles about the z axis, Eq. (13.41) is the same as

$$L_z = I\omega \tag{13.42}$$

As we will see below, this formula is of general validity for rotation around a fixed axis.

The preceding example shows that *the angular-momentum vector of a rotating body need not always lie along the axis of rotation.* However, if the body is symmetric about the axis of rotation, then the angular-momentum vector will lie along this axis. In such a symmetric body, each particle on one side of the axis has a counterpart on the other side of the axis, and when we add the angular-momentum vectors contributed by these two particles (or any other pair of particles), the resultant lies along the axis of rotation (see Fig. 13.19).

Since Newton's Second Law for translational motion states that the rate of change of the momentum equals the force, the analogy between the equations for translational and rotational motion suggests that the rate of change of the angular momentum should equal the torque. It is easy to verify this for the case of a single particle. With the usual rule for the differentiation of a product,

$$\begin{aligned}\frac{d}{dt}\mathbf{L} &= \frac{d}{dt}(\mathbf{r}\times\mathbf{p}) \\ &= \frac{d\mathbf{r}}{dt}\times\mathbf{p}+\mathbf{r}\times\frac{d\mathbf{p}}{dt}\end{aligned} \tag{13.43}$$

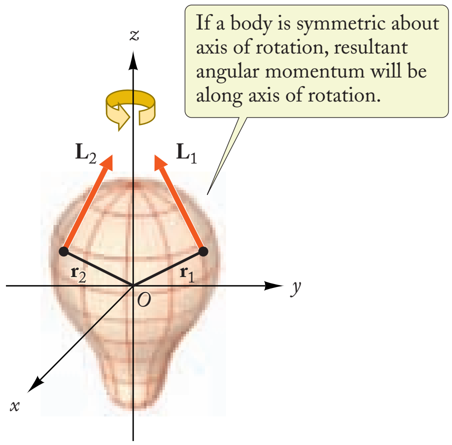


FIGURE 13.19 For a rotating symmetric body, the angular momentum is always along the axis of rotation.

The first term on the right side is

$$\frac{d\mathbf{r}}{dt}\times\mathbf{p} = \mathbf{v}\times(m\mathbf{v}) = m(\mathbf{v}\times\mathbf{v}) = 0 \tag{13.44}$$

This is zero because the cross product of a vector with itself is always zero. According to Newton's Second Law, the second term on the right side of Eq. (13.43) is

$$\mathbf{r}\times\frac{d\mathbf{p}}{dt} = \mathbf{r}\times\mathbf{F} \tag{13.45}$$

where **F** is the force acting on the particle. Therefore, Eq. (13.43) becomes

$$\frac{d\mathbf{L}}{dt} = \mathbf{r}\times\mathbf{F} = \boldsymbol{\tau} \tag{13.46}$$

In the case of a rigid body, the angular momentum is the sum of all the angular momenta of the particles in the body, and the rate of change of this total angular momentum can be shown to equal the net external torque:

equation of rotational motion for vector angular momentum

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} \tag{13.47}$$

This equation for the rate of change of the angular momentum of a rigid body is analogous to the equation $d\mathbf{p}/dt = \mathbf{F}$ for the rate of change of the translational momentum of a particle.

To compare the vector equation (13.47) with our earlier equation $I\alpha = \tau$, we must focus our attention on the component of the angular momentum along the axis of rotation, that is, the z axis. Figure 13.20 shows an arbitrary rigid body rotating about a fixed axis, which coincides with the z axis. As in Example 11, the angular-momentum vector of this body makes an angle with the axis of rotation. However, as we discussed in Example 11, the z component of the angular momentum of each particle in the rotating body is simply equal to its moment of inertia about the z axis multiplied by the angular velocity [see Eq. (13.42)]. Hence, when we sum the contributions of all the particles in the rotating body, we find that the z component of the net angular momentum of the entire rotating body equals the net moment of inertia of the entire body multiplied by the angular velocity. This establishes that the equation

$$L_z = I\omega \tag{13.48}$$

is of general validity.

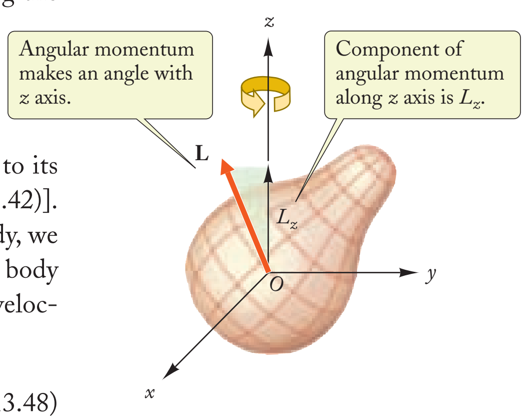


FIGURE 13.20 A body rotating about the z axis.

PHYSICS IN PRACTICE THE GYROCOMPASS

Concepts in Context

A gyroscope is a flywheel suspended in gimbals (pivoted rings; see Fig. 1). The angular-momentum vector of the flywheel lies along its axis of rotation. Since there are no torques on this flywheel, except for the very small and negligible frictional torques in the pivots of the gimbals, the angular-momentum vector remains constant in both magnitude and direction. Hence the direction of the axis of spin remains fixed in space—the gyroscope can be carried about, its base can be twisted and turned in any way, and yet the axis always continues to point in its original direction. Thus, the gyroscope serves as a compass. High-precision gyroscopes are used in the inertial-guidance systems for ships, aircraft, rockets, and spacecraft (see Fig. 2). They provide an absolute reference direction relative to which the orientation of the vehicle can be established. In such applications, three gyroscopes aimed along mutually perpendicular axes define the absolute orientation of an x, y, z coordinate grid.

The best available high-precision gyroscopes, such as those used in the inertial-guidance system of the Hubble Space Telescope, are capable of maintaining a fixed reference direction with a deviation, or drift, of no more than 10 arc-seconds per hour. The special gyroscopes developed for the *Gravity Probe B* experiment are even better than that; their drift is less than 1 milliarcsecond per year!

FIGURE 1 Gyroscope mounted in gimbals.

FIGURE 2 Internal-guidance system for an Atlas rocket. This system contains gyroscopes to sense the orientation of the rocket and accelerometers to measure the instantaneous acceleration. From these measurements, computers calculate the position of the rocket and guide it along the intended flight path.

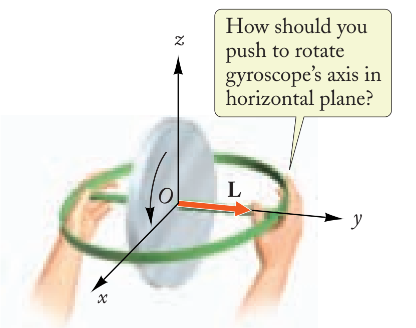


FIGURE 13.21 A gyroscope held in both hands. The axis of the gyroscope is horizontal, and the hands twist this axis sideways through an angle in the x–y plane.

EXAMPLE 12 You grasp the gimbals of a spinning gyroscope with both hands and you forcibly twist the axis of the gyroscope through an angle in the horizontal plane (see Fig. 13.21). If the angular momentum of the gyroscope spinning about its axis is 3.0×10^{-2} J·s, what are the magnitude and the direction of the torque you need to exert to twist the axis of the gyroscope at a constant rate through 90° in the horizontal plane in 1.0s?

SOLUTION: Figure 13.22a shows the angular-momentum vector $\mathbf{L}$ of the spinning gyroscope at an initial time and the new angular-momentum vector $\mathbf{L} + d\mathbf{L}$ after you have turned the gyroscope through a small angle $d\beta$. From the figure, we see that $d\mathbf{L}$ is approximately perpendicular to $\mathbf{L}$, and that the magnitude of $d\mathbf{L}$ is

$$dL = L\,d\beta$$

Hence

$$\frac{dL}{dt} = L\frac{d\beta}{dt} \tag{13.49}$$

According to Eq. (13.49), the magnitude of the torque is

$$\tau = \frac{dL}{dt} = L\frac{d\beta}{dt}$$

With $L = 3.0 \times 10^{-2}$ J·s and $d\beta/dt = (90°)/(1.0\ \text{s}) = \pi/2$ radians/s,

$$\tau = 3.0 \times 10^{-2}\ \text{J·s} \times \frac{\pi}{2}\ \text{radians/s} = 4.7 \times 10^{-2}\ \text{N·m}$$

Since $\boldsymbol{\tau} = d\mathbf{L}/dt$, the direction of the torque vector $\boldsymbol{\tau}$ must be the direction of $d\mathbf{L}$; that is, the torque vector must be perpendicular to **L**, or initially into the plane of the page (see Fig. 13.22b). To produce such a torque, your left hand must push up, and your right hand must pull down. This is contrary to intuition, which would suggest that to twist the axis in the horizontal plane, you should push forward with your right hand and pull back with your left! This surprising behavior also explains why a downward gravitational force causes the slow precession of a spinning top, as considered in the next example.

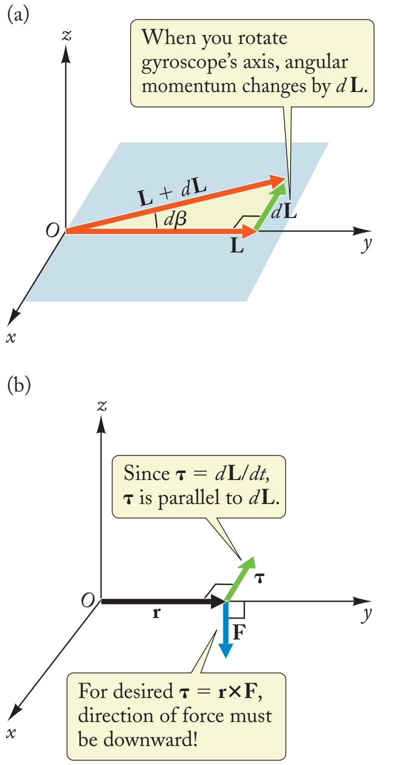


FIGURE 13.22 (a) $d\mathbf{L}$ is approximately perpendicular to **L**, in the x–y plane. (b) The torque $\boldsymbol{\tau}$ is parallel to $d\mathbf{L}$, also in the x–y plane.

EXAMPLE 13 A toy top spins with angular momentum of magnitude L; the axis of rotation is inclined at an angle θ with respect to the vertical (see Fig. 13.23). The spinning top has mass M; its point of contact with the ground remains fixed, and its center of mass is a distance r from the point of contact. The top *precesses;* that is, its angular-momentum vector rotates about the vertical. Find the angular velocity Ω_p of this precessional motion. If a top has $r = 4.0$ cm and moment of inertia $I = MR^2/4$, where $R = 3.0$ cm, find the period of the precessional motion when the top is spinning at 250 radians/s.

SOLUTION: From Fig. 13.24a, we see that the weight, Mg, acting at the center of mass, produces a torque τ of magnitude

$$\tau = rMg\sin\theta \tag{13.50}$$

As in Example 12, the change in angular momentum $d\mathbf{L}$ will be parallel to the torque, since $\boldsymbol{\tau} = d\mathbf{L}/dt$. In a time dt, the top will precess though an angle $d\beta$ given by (see Fig. 13.24b)

$$d\beta = \frac{dL}{L\sin\theta}$$

Using $dL = \tau\, dt = rMg \sin\theta\, dt$, we thus have

$$d\beta = \frac{rMg\sin\theta\, dt}{L\sin\theta} = \frac{rMg}{L}dt$$

The precessional angular velocity is the rate of change of this angle:

$$\Omega_p = \frac{d\beta}{dt} = \frac{rMg}{L} \tag{13.51}$$

Thus the angular velocity of precession is independent of the tilt angle θ.

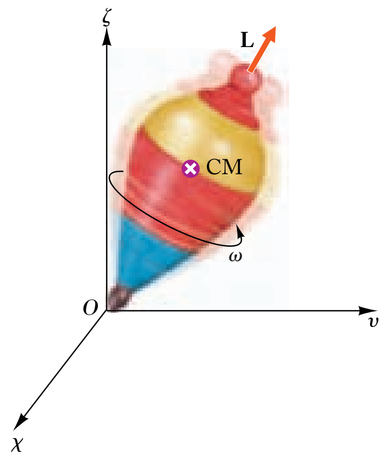


FIGURE 13.23 A tilted top spinning with angular velocity ω.

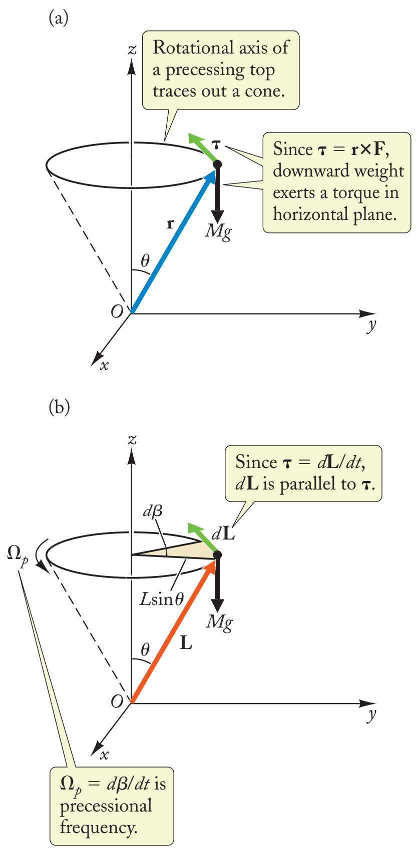


FIGURE 13.24 (a) The weight of the top, acting at the center of mass (a distance r from the point of contact), produces a torque perpendicular to $\mathbf{r}$ and to the weight. (b) The torque is parallel to $d\mathbf{L}$, which results in a slow precession around a vertical axis at an angular velocity Ω_p.

The period of the precession is related to the precessional angular velocity by

$$T = \frac{2\pi}{\Omega_p} = \frac{2\pi L}{rMg}$$

For the particular top described, we insert the angular momentum

$$L = I\omega = \frac{MR^2}{4}\omega$$

and obtain

$$T = \frac{2\pi MR^2\omega}{4rMg} = \frac{\pi R^2\omega}{2rg}$$

$$= \frac{\pi \times (0.030\,\text{m})^2 \times 250\,\text{radians/s}}{2 \times 0.040\,\text{m} \times 9.81\,\text{m/s}^2}$$

$$= 0.90\,\text{s}$$

Since this precessional period is proportional to ω, we see that as the spinning of the top slows down, the top will precess with a shorter period, that is, more quickly.

✔ Checkup 13.4

QUESTION 1: A particle has a nonzero position vector $\mathbf{r}$ and a nonzero momentum $\mathbf{p}$. Can the angular momentum of this particle be zero?

QUESTION 2: What is the angle between the momentum vector $\mathbf{p}$ and the angular-momentum vector $\mathbf{L}$ of a particle?

QUESTION 3: Suppose that instead of calculating the angular momentum of the dumbbell shown in Fig. 13.17 about the center, we calculate it about an origin on the z axis at some distance below the center. What are the directions of the individual angular-momentum vectors of the two masses m in this case? What is the direction of the total angular momentum?

QUESTION 4: Is a torque required to keep the dumbbell in Fig. 13.18 rotating around the z axis at constant angular velocity?

QUESTION 5: What is the direction of the angular-momentum vector of the rotating minute hand on your watch (calculated with respect to an origin at the center of the watch face)?

(A) In the direction that the minute hand points
(B) Antiparallel to the direction that the minute hand points
(C) In the plane of the watch face, but perpendicular to the minute hand
(D) Perpendicularly out of the face of the watch
(E) Perpendicularly into the face of the watch

SUMMARY

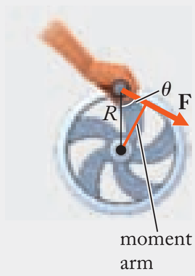
PROBLEM-SOLVING TECHNIQUES Torques and Rotational Motion (page 405)

PROBLEM-SOLVING TECHNIQUES Conservation of Angular Momentum (page 410)

PHYSICS IN PRACTICE The Gyrocompass (page 414)

TORQUE
where θ is the angle between the force **F** and the radial line of length R.

$$\tau = FR \sin\theta \qquad (13.3)$$



WORK DONE BY TORQUE

$$W = \int \tau \, d\phi \qquad (13.4)$$

WORK DONE BY A CONSTANT TORQUE

$$W = \tau \Delta\phi \qquad (13.5)$$

POWER DELIVERED BY TORQUE where ω is the angular velocity.

$$P = \tau\omega \qquad (13.6)$$

CONSERVATION OF ENERGY IN ROTATIONAL MOTION

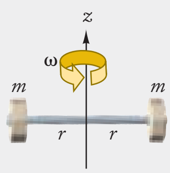
$$E = \tfrac{1}{2}I\omega^2 + U = [\text{constant}] \qquad (13.10)$$

EQUATION OF ROTATIONAL MOTION (Fixed axis)
where I is the moment of inertia and α is the angular acceleration.

$$I\alpha = \tau \qquad (13.19)$$

ANGULAR MOMENTUM OF ROTATION

$$L = I\omega \qquad (13.26)$$



CONSERVATION OF ANGULAR MOMENTUM

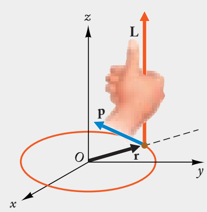
$$I\omega = [\text{constant}] \qquad (13.29)$$

ANGULAR MOMENTUM OF PARTICLE (In circular orbit)

$$L = mvr \qquad (13.34)$$

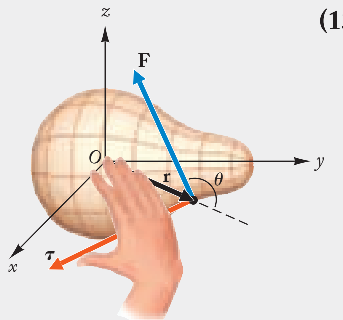
ANGULAR MOMENTUM VECTOR

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad (13.37)$$



TORQUE VECTOR

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \qquad (13.35)$$



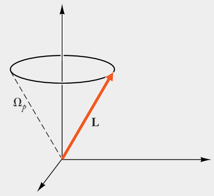
EQUATION OF ROTATIONAL MOTION FOR VECTOR ANGULAR MOMENTUM

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} \qquad (13.47)$$

GYROSCOPIC PRECESSION ANGULAR VELOCITY

where r is the distance from the point of contact to the center of mass.

$$\Omega_p = \frac{rMg}{L} \qquad (13.51)$$



QUESTIONS FOR DISCUSSION

1. Suppose you push down on the rim of a stationary phonograph turntable. What is the direction of the torque you exert about the center of the turntable?
2. Many farmers have been injured when their tractors suddenly flipped over backward while pulling a heavy piece of farm equipment. Can you explain how this happens?
3. Rifle bullets are given a spin about their axis by spiral grooves ("rifling") in the barrel of the gun. What is the advantage of this?
4. You are standing on a frictionless turntable (like a phonograph turntable, but sturdier). How can you turn 180° without leaving the turntable or pushing against any exterior body?
5. If you give a hard-boiled egg resting on a table a twist with your fingers, it will continue to spin. If you try doing the same with a raw egg, it will not. Why?
6. A tightrope walker uses a balancing pole to keep steady (Fig. 13.25). How does this help?
7. Why do helicopters need a small vertical propeller on their tail?
8. The rate of rotation of the Earth is subject to small seasonal variations. Does this mean that angular momentum is not conserved?
9. Why does the front end of an automobile dip down when the automobile is braking sharply?
10. The friction of the tides against the ocean coasts and the ocean shallows is gradually slowing down the rotation of the Earth. What happens to the lost angular momentum?
11. An automobile is traveling on a straight road at 90 km/h. What is the speed, relative to the ground, of the lowermost point on one of its wheels? The topmost point? The midpoint?
12. A sphere and a hoop of equal masses roll down an inclined plane without slipping. Which will get to the bottom first? Will they have equal kinetic energy when they reach the bottom?
13. A yo-yo rests on a table (Fig. 13.26). If you pull the string horizontally, which way will it move? If you pull vertically?

FIGURE 13.25 A tightrope walker.

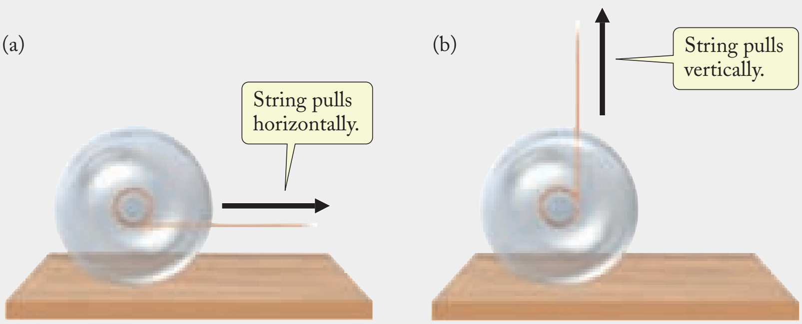


FIGURE 13.26 Yo-yo resting on a table. (a) String pulls horizontally. (b) String pulls vertically.

14. Stand a pencil vertically on its point on a table and let go. The pencil will topple over.
    (a) If the table is very smooth, the point of the pencil will slip in the direction opposite to that of the toppling. Why?
    (b) If the table is somewhat rough, or covered with a piece of paper, the point of the pencil will jump in the direction of the toppling. Why? (Hint: During the early stages of the toppling, friction holds the point of the pencil fixed; thus the pencil acquires horizontal momentum.)
15. An automobile travels at constant speed along a road consisting of two straight segments connected by a curve in the form of an arc of a circle. Taking the center of the circle as origin, what is the direction of the angular momentum of the automobile? Is the angular momentum constant as the automobile travels along this road?
16. Is the angular momentum of the orbital motion of a planet constant if we choose an origin of coordinates on the Sun?
17. A pendulum is swinging back and forth. Is the angular momentum of the pendulum bob constant?
18. What is the direction of the angular-momentum vector of the rotation of the Earth?
19. A bicycle is traveling east along a level road. What are the directions of the angular-momentum vectors of its wheels?

PROBLEMS

13.1 Work, Energy, and Power in Rotational Motion; Torque

1. The operating instructions for a small crane specify that when the boom is at an angle of 20° above the horizontal (Fig. 13.27), the maximum safe load for the crane is 500 kg. Assuming that this maximum load is determined by the maximum torque that the pivot can withstand, what is the maximum torque for 20° in terms of length R of the boom? What is the maximum safe load for 40°? For 60°?

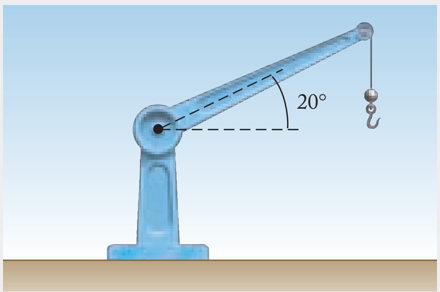


FIGURE 13.27 Small crane.

2. A simple manual winch consists of a drum of radius 4.0 cm to which is attached a handle of radius 25 cm (Fig. 13.28). When you turn the handle, the rope winds up on the drum and pulls the load. Suppose that the load carried by the rope is 2500 N. What force must you exert on the handle to hold this load?

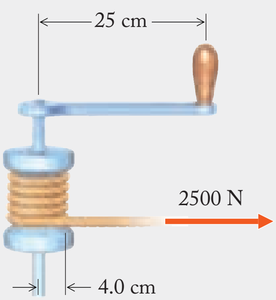


FIGURE 13.28 Manual winch.

3. The repair handbook for an automobile specifies that the cylinder-head bolts are to be tightened to a torque of 62 N·m. If a mechanic uses a wrench of length 20 cm on such a bolt, what perpendicular force must he exert on the end of this wrench to achieve the correct torque?
4. A 2.0-kg trout hangs from one end of a 2.0-m-long stiff fishing pole that the fisherman holds with one hand by the other end. If the pole is horizontal, what is the torque that the weight of the trout exerts about the end the fisherman holds? If the pole is tilted upward at an angle of 60°?
5. You hold a 10-kg book in your hand with your arm extended horizontally in front of you. What is the torque that the weight of this book exerts about your shoulder joint, at a distance of 0.60 m from the book?
6. If you bend over, so your trunk is horizontal, the weight of your trunk exerts a rather strong torque about the sacrum, where your backbone is pivoted on your pelvis. Assume that the mass of your trunk (including arms and head) is 48 kg, and that the weight effectively acts at a distance of 0.40 m from the sacrum. What is the torque that this weight exerts?
7. The engine of an automobile delivers a maximum torque of 203 N·m when running at 4600 rev/min, and it delivers a maximum power of 142 hp when running at 5750 rev/min. What power does the engine deliver when running at maximum torque? What torque does it deliver when running at maximum power?
8. The flywheel of a motor is connected to the flywheel of a pump by a drive belt (Fig. 13.29). The first flywheel has a radius R_1, and the second a radius R_2. While the motor wheel is rotating at a constant angular velocity ω_1, the tensions in the upper and the lower portions of the drive belt are T and T', respectively. Assume that the drive belt is massless.
    (a) What is the angular velocity of the pump wheel?
    (b) What is the torque of the drive belt on each wheel?

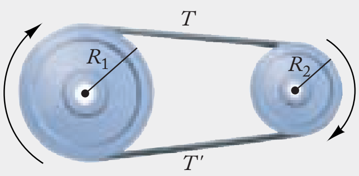


FIGURE 13.29 Motor and pump wheels connected by a drive belt.

(c) By taking the product of torque and angular velocity, calculate the power delivered by the motor to the drive belt, and the power removed by the pump from the drive belt. Are these powers equal?

9. The Wright Cyclone engine on a DC-3 airplane delivers a power of 850 hp with the propeller revolving steadily at 2100 rev/min. What is the torque exerted by air resistance on the propeller?

10. A woman on an exercise bicycle has to exert an (average) tangential push of 35 N on each pedal to keep the wheel turning at constant speed. Each pedal has a radial length of 0.18 m. If she pedals at the rate of 60 rev/min, what is the power she expends against the exercise bicycle? Express your answer in watts and in kilocalories per minute.

11. With what translational speed does the upper end of the meterstick in Example 2 hit the floor? If, instead of a 1.0-m stick, we use a 2.0-m stick, with what translational speed does it hit?

12. A ceiling fan uses 0.050 hp to maintain a rotational frequency of 150 rev/min. What torque does the motor exert?

13. The motor of a grinding wheel exerts a torque of 0.65 N·m to maintain an operating speed of 3450 rev/min. What power does the motor deliver?

14. From the human-body data of Fig. 10.17, calculate (a) the torque about the shoulder for an arm held horizontally and (b) the torque about the hip for a leg held horizontally.

15. A large grinding table is used to thin large batches of silicon wafers in the final stage of semiconductor manufacturing, a process called *backlap*. If the driving motor exerts a torque of 250 N·m while rotating the table 1200 times for one batch of wafers, how much work does the motor do?

16. Recently, a microfabricated torque sensor measured a torque as small as 7.5×10^{-24} N·m. If the torque is produced by a force applied perpendicular to the sensor at a distance of 25 μm from the axis of rotation, what is the smallest force that the sensor can detect?

*17. The angular position of a ceiling fan during the first two seconds after start-up is given by $\phi = Ct^2$, where $C =$ 7.5 radians/s^2 and t is in seconds. If the fan motor exerts a torque of 2.5 N·m, how much work has the motor done after $t = 1.0$ s? After $t = 2.0$ s?

*18. While braking, a 1500-kg automobile decelerates at the rate of 8.0 m/s^2. What is the magnitude of the braking force that the road exerts on the automobile? What torque does this force generate about the center of mass of the automobile? Will this torque tend to lift the front end of the automobile or tend to depress it? Assume that the center of mass of the automobile is 60 cm above the surface of the road.

*19. A tractor of mass 4500 kg has rear wheels of radius 0.80 m. What torque and what power must the engine supply to the rear axle to move the tractor up a road of slope 1:3 at a constant speed of 4.0 m/s?

*20. A bicycle and its rider have a mass of 90 kg. While accelerating from rest to 12 km/h, the rider turns the pedals through three full revolutions. What torque must the rider exert on the pedals? Assume that the torque is constant during the acceleration and ignore friction within the bicycle mechanism.

*21. A meterstick is held to a wall by a nail passing through the 60-cm mark (Fig. 13.30). The meterstick is free to swing about this nail, without friction. If the meterstick is released from an initial horizontal position, what angular velocity will it attain when it swings through the vertical position?

FIGURE 13.30 A meterstick.

*22. A uniform solid sphere of mass M and radius R hangs from a string of length $R/2$. Suppose the sphere is released from an initial position making an angle of 45° with the vertical (Fig. 13.31).

(a) Calculate the angular velocity of the sphere when it swings through the vertical position.

(b) Calculate the tension in the string at this instant.

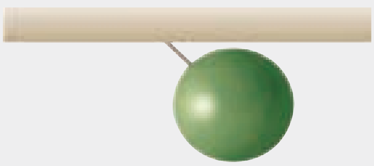


FIGURE 13.31 A hanging sphere.

*23. The maximum (positive) acceleration an automobile can achieve on a level road depends on the maximum torque the engine can deliver to the wheels.

(a) The engine of a Maserati sports car delivers a maximum torque of 441 N·m to the gearbox. The gearbox steps down the rate of revolution by a factor of 2.58; that is, whenever the engine makes 2.58 revolutions, the wheels make 1 revolution. What is the torque delivered to the wheels? Ignore frictional losses in the gearbox.

(b) The mass of the car (including fuel, driver, etc.) is 1770 kg, and the radius of its wheels is 0.30 m. What is the maximum acceleration? Ignore the moment of inertia of the wheels and frictional losses.

*24. An automobile of mass 1200 kg has four brake drums of diameter 25 cm. The brake drums are rigidly attached to the wheels of diameter 60 cm. The braking mechanism presses brake pads against the rim of each drum, and the friction between the pad and the rim generates a torque that slows the rotation of the wheel. Assume that all four wheels contribute equally to the braking. What torque must the brake pads exert on each drum in order to decelerate the automobile at 7.8 m/s^2? If the coefficient of friction between the pad and the drum is $\mu_k = 0.60$, what normal force must the brake pad exert on the rim of the drum? Ignore the masses of the wheels.

*25. In one of the cylinders of an automobile engine, the gas released by internal combustion pushes on the piston, which, in turn, pushes on the crankshaft by means of a piston rod (Fig. 13.32). If the crankshaft experiences a torque of 31 N·m and if the dimensions of the crankshaft and piston rod are as in Fig. 13.32, what must be the force of the gas on the piston when the crankshaft is in the horizontal position as in Fig. 13.32? Ignore friction, and ignore the masses of the piston and rod.

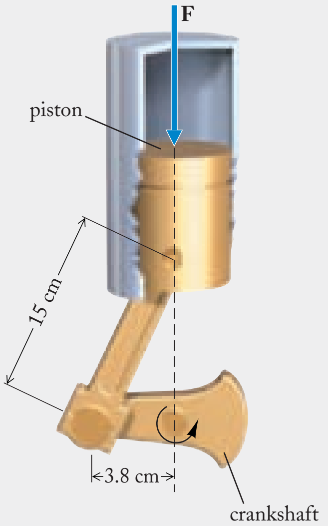


FIGURE 13.32 Automobile piston and crankshaft.

13.2 The Equation of Rotational Motion

26. While starting up a roulette wheel, the croupier exerts a torque of 100 N·m with his hand on the spokes of the wheel. What angular acceleration does this produce? Treat the wheel as a disk of mass 30 kg and radius 0.25 m.

27. The center span of a revolving drawbridge consists of a uniform steel girder of mass 300 metric tons and length 25 m. This girder can be regarded as a uniform thin rod. The bridge opens by rotating about a vertical axis through its center. What torque is required to open this bridge in 60 s? Assume that the bridge first accelerates uniformly through an angular interval of 45° and then the torque is reversed, so the bridge decelerates uniformly through an angular interval of 45° and comes to rest after rotating by 90°.

28. The original Ferris wheel, built by George Ferris, had a radius of 38 m and a mass of 1.9×10^6 kg. Assume that all of its mass was uniformly distributed along the rim of the wheel. If the wheel was initially rotating at 0.050 rev/min, what constant torque had to be applied to bring it to a full stop in 30 s? What force exerted on the rim of the wheel would have given such a torque?

29. The pulley of an Atwood machine for the measurement of g is a brass disk of mass 120 g. When using masses $m_1 = 0.4500$ kg and $m_2 = 0.4550$ kg, an experimenter finds that the larger mass descends 1.6 m in 8.0 s, starting from rest. What is the value of g?

30. A hula hoop rolls down a slope of 1:10 without slipping. What is the (linear) acceleration of the hoop?

31. A uniform cylinder rolls down a plane inclined at an angle θ with the horizontal. Show that if the cylinder rolls without slipping, the acceleration is $a = \frac{2}{3}g \sin\theta$.

32. The spare wheel of a truck, accidentally released on a straight road leading down a steep hill, rolls down the hill without slipping. The mass of the wheel is 60 kg, and its radius is 0.40 m; the mass distribution of the wheel is approximately that of a uniform disk. At the bottom of the hill, at a vertical distance of 120 m below the point of release, the wheel slams into a telephone booth. What is the total kinetic energy of the wheel just before impact? How much of this kinetic energy is translational energy of the center of mass of the wheel? How much is rotational kinetic energy about the center of mass? What is the speed of the wheel?

33. Galileo measured the acceleration of a sphere rolling down an inclined plane. Suppose that, starting from rest, the sphere takes 1.6 s to roll a distance of 3.00 m down a 20° inclined plane. What value of g can you deduce from this?

34. A yo-yo consists of a uniform disk with a string wound around the rim. The upper end of the string is held fixed. The yo-yo unwinds as it drops. What is its downward acceleration?

35. A man is trying to roll a barrel along a level street by pushing forward along its top rim. At the same time another man is pushing backward at the middle, with a force of equal magnitude F (see Fig. 13.33). The barrel rolls without slipping. Which way will the barrel roll? Find the magnitude and direction of the friction force at the point of contact with the street. The barrel is a uniform cylinder of mass M and radius R.

FIGURE 13.33 One man pushes horizontally at a cylinder's top; another pushes with equal force in the opposite direction at its middle. Which way does it roll?

36. An electric blender uniformly accelerates from rest beginning at $t = 0$; at $t = 0.50$ s, the blender has reached 250 radians/s and continues accelerating. If the rotating components have a moment of inertia of 2.0×10^{-4} kg·m^2, at what instantaneous rate is the motor delivering energy at $t = 0.50$ s?

37. A basketball is released from rest on a 15° incline. How many revolutions will the basketball undergo in 4.0 s? Assume the basketball is a thin spherical shell with a diameter of 23 cm, and that it rolls without slipping.

38. A 25-cm length of thin string is wound on the axle of a toy gyroscope that rotates in fixed bearings; the radius of the winding is 2.0 mm. If the string is pulled with a steady force of 5.0 N until completely unwound, how long does it take to complete the pull? What is the final angular velocity? The moment of inertia of the gyroscope (including axle) is 5.0×10^{-5} kg·m^2.

39. A phonograph turntable driven by an electric motor accelerates at a constant rate from 0 to 33.3 revolutions per minute in a time of 2.0 s. The turntable is a uniform disk of metal, of mass 1.2 kg and radius 15 cm. What torque is required to produce this acceleration? If the driving wheel makes contact with the turntable at its outer rim, what force must it exert?

*40. A bowling ball sits on the smooth floor of a subway car. If the car has a horizontal acceleration a, what is the acceleration of the ball? Assume that the ball rolls without slipping.

*41. A hoop rolls down an inclined ramp. The coefficient of static friction between the hoop and the ramp is μ_s. If the ramp is very steep, the hoop will slip while rolling. Show that the critical angle of inclination at which the hoop begins to slip is given by $\tan\theta = 2\mu_s$.

*42. A solid cylinder rolls down an inclined plane. The angle of inclination θ of the plane is large so that the cylinder slips while rolling. The coefficient of kinetic friction between the cylinder and the plane is μ_k. Find the rotational and translational accelerations of the cylinder. Show that the translational acceleration is the same as that of a block sliding down the plane.

**43. Suppose that a tow truck applies a horizontal force of 4000 N to the front end of an automobile similar to that described in Problem 63 of Chapter 12. Taking into account the rotational inertia of the wheels and ignoring frictional losses, what is the acceleration of the automobile? What is the percentage difference between this value of the acceleration and the value calculated by neglecting the rotational inertia of the wheels?

**44. A cart consists of a body and four wheels on frictionless axles. The body has a mass m. The wheels are uniform disks of mass M and radius R. Taking into account the moment of inertia of the wheels, find the acceleration of this cart if it rolls without slipping down an inclined plane making an angle θ with the horizontal.

**45. When the wheels of a landing airliner touch the runway, they are not rotating initially. The wheels first slide on the runway (and produce clouds of smoke and burn marks on the runway, which you may have noticed; see Fig. 13.34), until the sliding friction force has accelerated the wheels to the rotational speed required for rolling without slipping. From the following data, calculate how far the wheel of an airliner slips before it begins to roll without slipping: the wheel has a radius of 0.60 m and a mass of 160 kg, the normal force acting on the wheel is 2.0×10^5 N, the speed of the airliner is 200 km/h, and the coefficient of sliding friction for the wheel on the runway is 0.80. Treat the wheel as a uniform disk.

FIGURE 13.34 A landing airliner.

13.3 Angular Momentum and its Conservation

46. You spin a hard-boiled egg on a table, at 5.0 rev/s. What is the angular momentum of the egg? Treat the egg as a sphere of mass 70 g and mean diameter 5.0 cm.

47. The Moon moves around the Earth in an (approximately) circular orbit of radius 3.8×10^8 m in a time of 27.3 days. Calculate the magnitude of the orbital angular momentum of the Moon. Assume that the origin of coordinates is centered on the Earth.

48. At the Fermilab accelerator, protons of momentum 5.2×10^{-16} kg·m/s travel around a circular path of diameter 2.0 km. What is the orbital angular momentum of one of these protons? Assume that the origin is at the center of the circle.

49. Prior to launching a stone from a sling, a Bolivian native whirls the stone at 3.0 rev/s around a circle of radius 0.75 m. The mass of the stone is 0.15 kg. What is the angular momentum of the stone relative to the center of the circle?

50. A communications satellite of mass 100 kg is in a circular orbit of radius 4.22×10^7 m around the Earth. The orbit is in the equatorial plane of the Earth, and the satellite moves along it from west to east with a speed of 4.90×10^2 m/s. What is the magnitude of the angular momentum of this satellite?

51. According to Bohr's (oversimplified) theory, the electron in the hydrogen atom moves in one or another of several possible circular orbits around the nucleus. The radii and the orbital

velocities of the three smallest orbits are, respectively, 0.529×10^{-10} m, 2.18×10^6 m/s; 2.12×10^{-10} m, 1.09×10^6 m/s; and 4.76×10^{-10} m, 7.27×10^5 m/s. For each of these orbits calculate the orbital angular momentum of the electron, with the origin at the center. How do these angular momenta compare?

52. A high-speed meteoroid moves past the Earth along an (almost) straight line. The mass of the meteoroid is 150 kg, its speed relative to the Earth is 60 km/s, and its distance of closest approach to the center of the Earth is 1.2×10^4 km.
    (a) What is the angular momentum of the meteoroid in the reference frame of the Earth (origin at the center of the Earth)?
    (b) What is the angular momentum of the Earth in the reference frame of the meteoroid (origin at the center of the meteoroid)?

53. A train of mass 1500 metric tons runs along a straight track at 85 km/h. What is the angular momentum of the train about a point 50 m to the side of the track, left of the train? About a point on the track?

54. The electron in a hydrogen atom moves around the nucleus under the influence of the electric force of attraction, a central force pulling the electron toward the nucleus. According to the Bohr theory, one of the possible orbits of the electron is an ellipse of angular momentum $2h$ with a distance of closest approach $(1 - 2\sqrt{2}/3)a_0$ and a distance of farthest recession $(1 + 2\sqrt{2}/3)a_0$, where h and a_0 are two atomic constants with the numerical values 1.05×10^{-34} kg·m^2/s ("Planck's constant") and 5.3×10^{-11} m ("Bohr radius"), respectively. In terms of h and a_0, find the speed of the electron at the points of closest approach and farthest recession; then evaluate numerically.

55. According to a simple (but erroneous) model, the proton is a uniform rigid sphere of mass 1.67×10^{-27} kg and radius 1.0×10^{-15} m. The spin angular momentum of the proton is 5.3×10^{-35} J·s. According to this model, what is the angular velocity of rotation of the proton? What is the linear velocity of a point on its equator? What is the rotational kinetic energy? How does this rotational energy compare with the rest-mass energy mc^2?

56. What is the angular momentum of a Frisbee spinning at 20 rev/s about its axis of symmetry? Treat the Frisbee as a uniform disk of mass 200 g and radius 15 cm.

57. A phonograph turntable is a uniform disk of radius 15 cm and mass 1.4 kg. If this turntable accelerates from 0 rev/min to 78 rev/min in 2.5 s, what is the average rate of change of the angular momentum in this time interval?

58. The propeller shaft of a cargo ship has a diameter of 8.8 cm, a length of 27 m, and a mass of 1200 kg. What is the rotational kinetic energy of this propeller shaft when it is rotating at 200 rev/min? What is the angular momentum?

59. The Sun rotates about its axis with a period of about 25 days. Its moment of inertia is $0.20M_S R_S^2$, where M_S is its mass and R_S its radius. Calculate the angular momentum of rotation of the Sun. Calculate the total orbital angular momentum of all the planets; make the assumption that each planet moves in a circular orbit of radius equal to its mean distance from the Sun listed in Table 9.1. What percentage of the angular momentum of the Solar System is in the rotational motion of the Sun?

60. Suppose we measure the speed v_1 and the radial distance r_1 of a comet when it reaches perihelion. Use conservation of angular momentum and conservation of energy to determine the speed and the radius at aphelion.

61. A playground merry-go-round is rotating at 2.0 radians/s. Consider the merry-go-round to be a uniform disk of mass 20 kg and radius 1.5 m. A 25-kg child, moving along a radial line, jumps onto the edge of the merry-go-round. What is its new angular velocity? The child then kicks the ground until the merry-go-round (with the child) again rotates at 2.0 radians/s. If the child then walks radially inward, what will the angular velocity be when the child is 0.50 m from the center?

62. The moment of inertia of the Earth is approximately $0.331 M_E R_E^2$. If an asteroid of mass 5.0×10^{18} kg moving at 150 km/s struck (and stuck in) the Earth's surface, by how long would the length of the day change? Assume the asteroid was traveling westward in the equatorial plane and struck the Earth's surface at 45°.

63. In a popular demonstration, a professor rotates on a stool at 0.50 rev/s, holding two 10-kg masses, each 1.0 m from the axis of rotation. If she pulls the weights inward until they are 10 cm from the axis, what is the new rotational frequency? Without the weights, the professor and stool have a moment of inertia of 6.0 kg·m^2 with arms extended and 4.0 kg·m^2 with arms pulled in.

64. In a demonstration, a bicycle wheel with moment of inertia 0.48 kg·m^2 is spun up to 18 radians/s, rotating about a vertical axis. A student holds the wheel while sitting on a rotatable stool. The student and stool are initially stationary and have a moment of inertia of 3.0 kg·m^2. If the student turns the bicycle wheel over so its axis points in the opposite direction, with what angular velocity will the student and stool rotate? For simplicity, assume the wheel is held overhead, so that the student, wheel, and stool all have the same axis of rotation.

65. A very heavy freight train made up of 250 cars has a total mass of 7700 metric tons. Suppose that such a train accelerates from 0 to 65 km/h on a track running exactly east from Quito, Ecuador (on the equator). The force that the engine exerts on the Earth will slow down the rotational motion of the Earth. By how much will the angular velocity of the Earth have decreased when the train reaches its final speed? Express your answer in revolutions per day. The moment of inertia of the Earth is $0.33M_E R_E^2$.

66. There are 1.1×10^8 automobiles in the United States, each of an average mass of 2000 kg. Suppose that one morning all these automobiles simultaneously start to move in an eastward direction and accelerate to a speed of 80 km/h.

(a) What total angular momentum about the axis of the Earth do all these automobiles contribute together? Assume that the automobiles travel at an average latitude of 40°.

(b) How much will the rate of rotation of the Earth change because of the action of these automobiles? Assume that the axis of the Earth remains fixed. The moment of inertia of the Earth is 8.1×10^{37} kg·m^2.

*67. Two artificial satellites of equal masses are in circular orbits of radii r_1 and r_2 around the Earth. The second has an orbit of larger radius than the first ($r_2 > r_1$). What is the speed of each? What is the angular momentum of each? Which has the larger speed? Which has the larger angular momentum?

*68. Consider the motion of the Earth around the Sun. Take as origin the point at which the Earth is today and treat the Earth as a particle.

(a) What is the angular momentum of the Earth about this origin today?

(b) What will be the angular momentum of the Earth about the same origin three months from now? Six months from now? Nine months from now? Is the angular momentum conserved?

*69. The friction of the tides on the coastal shallows and the ocean floors gradually slows down the rotation of the Earth. The period of rotation (length of a sidereal day) is gradually increasing by 0.0016 s per century. What is the angular deceleration (in radians/s^2) of the Earth? What is the rate of decrease of the rotational angular momentum? What is the rate of decrease of the rotational kinetic energy? The moment of inertia of the Earth about its axis is $0.331 M_E R_E^2$, where M_E is the mass of the Earth and R_E its equatorial radius.

*70. Phobos is a small moon of Mars. For the purposes of the following problem, assume that Phobos has a mass of 5.8×10^{15} kg and that it has a shape of a uniform sphere of radius 7.5×10^3 m. Suppose that a meteoroid strikes Phobos 5.0×10^3 m off center (Fig. 13.35) and remains stuck. If the momentum of the meteoroid was 3×10^{13} kg·m/s before impact and the mass of the meteoroid is negligible compared with the mass of Phobos, what is the change in the rotational angular velocity of Phobos?

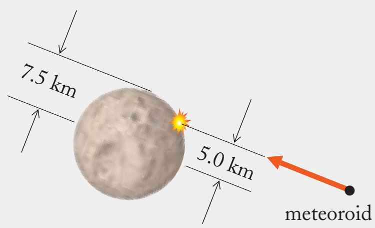


FIGURE 13.35 A meteoroid strikes Phobos.

*71. A woman stands in the middle of a small rowboat. The rowboat is floating freely and experiences no friction against the water. The woman is initially facing east. If she turns around 180° so that she faces west, through what angle will the rowboat turn? Assume that the woman performs her turning movement at constant angular velocity and that her moment of inertia remains constant during this movement. The moment of inertia of the rowboat about the vertical axis is 20 kg·m^2 and that of the woman is 0.80 kg·m^2.

*72. Two automobiles both of 1200 kg and both traveling at 30 km/h collide on a frictionless icy road. They were initially moving on parallel paths in opposite directions, with a center-to-center distance of 1.0 m (Fig. 13.36). In the collision, the automobiles lock together, forming a single body of wreckage; the moment of inertia of this body about its center of mass is 2.5×10^3 kg·m^2.

(a) Calculate the angular velocity of the wreck.

(b) Calculate the kinetic energy before the collision and after the collision. What is the change of kinetic energy?

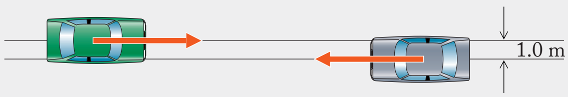


FIGURE 13.36 Two automobiles collide.

*73. In one experiment performed under weightless conditions in Skylab, the three astronauts ran around a path on the inside wall of the spacecraft so as to generate artificial gravity for their bodies (Fig. 13.37). Assume that the center of mass of each astronaut moves around a circle of radius 2.5 m; treat the astronauts as particles.

(a) With what speed must each astronaut run if the average normal force on his feet is to equal his normal weight (mg)?

(b) Suppose that before the astronauts begin to run, Skylab is floating in its orbit without rotating. When the astronauts begin to run clockwise, Skylab will begin to rotate counterclockwise. What will be the angular velocity of Skylab when the astronauts are running steadily with the speed calculated above? Assume that the mass of each astronaut is 70 kg and that the moment of inertia of Skylab about its longitudinal axis is 3×10^5 kg·m^2.

(c) How often must the astronauts run around the inside if they want Skylab to rotate through an angle of 30°?

FIGURE 13.37 Three astronauts about to start running around inside Skylab.

*74. A flywheel rotating freely on a shaft is suddenly coupled by means of a drive belt to a second flywheel sitting on a parallel shaft (Fig. 13.38). The initial angular velocity of the first flywheel is ω; that of the second is zero. The flywheels are uniform disks of masses M_1, M_2 and of radii R_1, R_2, respectively. The drive belt is massless and the shafts are frictionless.

(a) Calculate the final angular velocity of each flywheel.

(b) Calculate the kinetic energy lost during the coupling process. What happens to this energy?

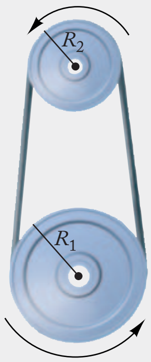


FIGURE 13.38 Two flywheels coupled by a drive belt.

*75. A thin rod of mass M and length l hangs from a pivot at its upper end. A ball of clay of mass m and of horizontal velocity v strikes the lower end at right angles and remains stuck (a totally inelastic collision). How high will the rod swing after this collision?

**76. If the melting of the polar ice caps were to raise the water level on the Earth by 10 m, by how much would the day be lengthened? Assume that the moment of inertia of the ice in the polar ice caps is negligible (they are very near the axis), and assume that the extra water spreads out uniformly over the entire surface of the Earth (that is, neglect the area of the continents compared with the area of the oceans). The moment of inertia of the Earth (now) is 8.1×10^{37} kg·m^2.

77. Consider a projectile of mass m launched with a speed v_0 at an elevation angle of 45°. If the launch point is the origin of coordinates, what is the angular momentum of the projectile at the instant of launch? At the instant it reaches maximum height? At the instant it strikes the ground? Is the angular momentum conserved in this motion with this choice of origin?

13.4 Torque and Angular Momentum as Vectors

78. Show that for a flat plate rotating about an axis perpendicular to the plate, the angular-momentum vector lies along the axis of rotation, even if the body is not symmetric.

79. A child's toy top consists of a uniform thin disk of radius 5.0 cm and mass 0.15 kg with a thin spike passing through its center. The lower part of the spike protrudes 6.0 cm from the disk. If you stand this top on its spike and start it spinning at 200 rev/s, what will be its precession frequency?

80. Suppose that the flywheel of a gyroscope is a uniform disk of mass 250 g and radius 3.0 cm. The distance of the center of this flywheel from the point of support is 4.0 cm. What is the precession frequency if the flywheel is spinning at 120 rev/s?

81. If a bicycle in forward motion begins to tilt to one side, the torque exerted by gravity will tend to turn the bicycle. Draw a diagram showing the angular momentum of a (slightly tilted) front wheel, the weight of the wheel, and the resulting torque. In which direction is the instantaneous change in angular momentum? Will this change make the tilt worse or better?

82. Slow precession can be used to determine a much more rapid rotational frequency. Consider a top made by inserting a small pin radially into a ball (a uniform sphere) of radius $R = 6.0$ cm. The pin extends 1.0 cm from the surface of the ball and supports the top. When set spinning, the top is observed to precess with a period of 0.75 s. What is the rotational frequency of the top?

*83. The wheel of an automobile has a mass of 25 kg and a diameter of 70 cm. Assume that the wheel can be regarded as a uniform disk.

(a) What is the angular momentum of the wheel when the automobile is traveling at 25 m/s (90 km/h) on a straight road?

(b) What is the rate of change of the angular momentum of the wheel when the automobile is traveling at the same speed along a curve of radius 80 m?

(c) For this rate of change of the angular momentum, what must be the torque on the wheel? Draw a diagram showing the path of the automobile, the angular-momentum vector of the wheel, and the torque vector.

*84. Consider the airplane propeller described in Problem 38 in Chapter 12. If the airplane is flying around a curve of radius 500 m at a speed of 360 km/h, what is the rate of change of the angular momentum of the propeller? What torque is required to change the angular momentum at this rate? Draw a diagram showing $\mathbf{L}$, $d\mathbf{L}/dt$, and $\boldsymbol{\tau}$.

*85. A large flywheel designed for energy storage at a power plant has a moment of inertia of 5×10^5 kg·m^2 and spins at 3000 rev/min. Suppose that this flywheel is mounted on a horizontal axle oriented in the east–west direction. What are the magnitude and direction of its angular momentum? What is the rate of change of this angular momentum due to the rotational motion of the Earth and the consequent motion of the axle of the flywheel? What is the torque that the axle of the flywheel exerts against the bearings supporting it? If the bearings are at a distance of 0.60 m from the center of the flywheel on each side, what are the forces associated with this torque?

REVIEW PROBLEMS

86. A door is 0.80 m wide. What is the torque you exert about the axis passing through the hinges if you push against this door with a perpendicular force of 200 N at its middle? What is it if you push at the edge? A wind is blowing against the other side of the door and trying to push it open. Where should you push to keep the door closed?

87. An elevator of mass 900 kg is being lifted at constant speed by a cable wrapped around a wheel (see Fig. 13.39). The radius of the wheel is 0.35 m. What torque does the cable exert on the wheel?

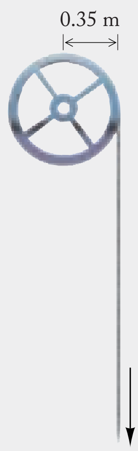


FIGURE 13.39 Elevator cable attached to a wheel.

88. Each of the two fuel turbopumps in the Space Shuttle delivers a power of 700 hp. The rotor of this pump rotates at 37 000 rev/min. What is the torque that the rotor exerts while pushing against the fuel?

89. A manual winch has a crank of length (radius) 0.25 m. If a laborer pushes against its handle tangentially with a force of 200 N, how much work does the laborer do while turning the crank through 10 revolutions?

90. A meterstick is initially standing vertically on the floor. If the meterstick falls over, with what angular velocity will it hit the floor? Assume that the end in contact with the floor experiences no friction and slips freely.

*91. A heavy hatch on a ship is made of a uniform plate of steel that measures 1.2 m × 1.2 m and has a mass of 400 kg. The hatch is hinged along one side; it is horizontal when closed, and it opens upward. A torsional spring assists in the opening of the hatch. The spring exerts a torque of 2.00×10^3 N·m when the hatch is horizontal and a torque of 0.30×10^3 N·m when the hatch is vertical; in the range of angles between horizontal and vertical, the torque decreases linearly (e.g., the torque is 1.15×10^3 N·m when the hatch is at 45°).
    (a) At what angle will the hatch be in equilibrium so the spring exactly compensates the torque due to the weight?
    (b) What minimum push must a sailor exert on the hatch to open it from the closed position? To close it from the open position? Assume that the sailor pushes perpendicularly on the hatch at the edge that is farthest from the hinge.

92. With your bicycle upside down on the ground, and the wheel free to rotate, you grasp the front wheel at the top and give it a horizontal push of 20 N. What is the instantaneous angular acceleration of the wheel? The wheel is a hoop of mass 4.0 kg and radius 0.33 m; ignore the mass of the spokes.

93. A toy top consists of a disk of radius 4.0 cm with a reinforced rim (a ring). The mass of the disk is 20 g, and the mass of the rim is 15 g. The mass of the pivot of this top is negligible.
    (a) What is the moment of inertia of this top?
    (b) When you give this top a twist and start it rotating at 100 rev/min on the floor, friction slows the top to a stop in 1.5 min. Assuming that the angular deceleration is uniform, what is the angular deceleration?
    (c) What is the frictional torque on the top?
    (d) What is the work done by the frictional torque?

94. The turntable of a record player is a uniform disk of radius 0.15 m and mass 1.2 kg. When in operation, it spins at $33\frac{1}{3}$ rev/min. If you switch the record player off, you find that the turntable coasts to a stop in 45 s.
    (a) Calculate the frictional torque that acts on the turntable. Assume the torque is constant, that is, independent of the angular speed.
    (b) Calculate the power that the motor of the record player must supply to keep the turntable in operation at $33\frac{1}{3}$ rev/min.

*95. A barrel of mass 200 kg and radius 0.50 m rolls down a 40° ramp without slipping. What is the value of the friction force acting at the point of contact between the barrel and the ramp? Treat the barrel as a cylinder of uniform density.

*96. A disk of mass M is free to rotate about a fixed horizontal axis. A string is wrapped around the rim of the disk, and a mass m is attached to this string (see Fig. 13.40). What is the downward acceleration of the mass?

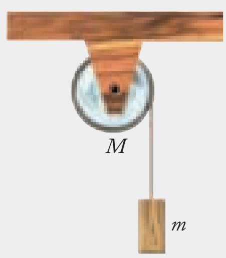


FIGURE 13.40 A mass m hanging from a disk.

*97. A hoop of mass M and radius R rolls down a sloping ramp that makes an angle of 30° with the ground. What is the acceleration of the hoop if it rolls without slipping?

*98. An automobile has the arrangement of the wheels shown in Fig. 13.41. The mass of this automobile is 1800 kg, the center of mass is at the midpoint of the rectangle formed by the wheels, and the moment of inertia about a vertical axis through the center of mass is 2200 kg·m^2. Suppose that during braking in an emergency, the left front and rear wheels lock and begin to skid while the right wheels continue to rotate just short of skidding. The coefficient of static friction between the wheels and the road is $\mu_s = 0.90$, and the coefficient of kinetic friction is $\mu_k = 0.50$. Calculate the instantaneous angular acceleration of the automobile about the vertical axis through the center of mass.

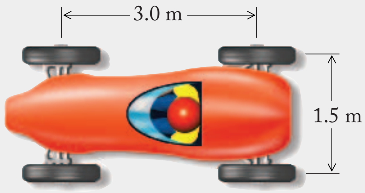


FIGURE 13.41 An automobile.

*99. Neutron stars, or pulsars, spin very quickly about their axes. Their high rate of spin is the result of the conservation of angular momentum during the formation of the neutron star by the gradual contraction (shrinking) of an initially normal star.

(a) Suppose that the initial star is similar to the Sun, with a radius of 7.0×10^8 m and a rate of rotation of 1.0 revolution per month. If this star contracts to a radius of 1.0×10^4 m, by what factor does the moment of inertia increase? Assume that the relative distribution of mass in the initial and the final stars is roughly the same.

(b) By what factor does the angular velocity increase? What is the final angular velocity?

*100. A rod of mass M and length l is lying on a flat, frictionless surface. A ball of putty of mass m and initial velocity v at right angles to the rod strikes the rod at a distance $l/4$ from the center (Fig. 13.42). The collision is inelastic, and the putty adheres to the rod.

(a) Where is the center of mass of the rod with adhering putty?

(b) What is the velocity of this center of mass after the collision?

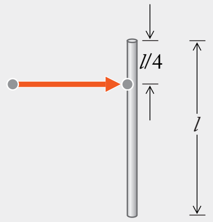


FIGURE 13.42 A ball of putty strikes a rod.

(c) What is the angular momentum about this center of mass? What is the moment of inertia, and what is the angular velocity?

*101. A communications satellite of mass 1000 kg is in a circular orbit of radius 4.22×10^7 m around the Earth. The orbit is in the equatorial plane of the Earth, and the satellite moves along it from west to east. What are the magnitude and the direction of the angular-momentum vector of this satellite?

*102. The spin angular momentum of the Earth has a magnitude of 5.9×10^{33} kg·m^2/s. Because of forces exerted by the Sun and the Moon, the spin angular momentum gradually changes direction, describing a cone of half-angle 23.5° (Fig. 13.43). The angular-momentum vector takes 26 000 years to swing once around this cone. What is the magnitude of the rate of change of the angular-momentum vector; that is, what is the value of $|d\mathbf{L}/dt|$?

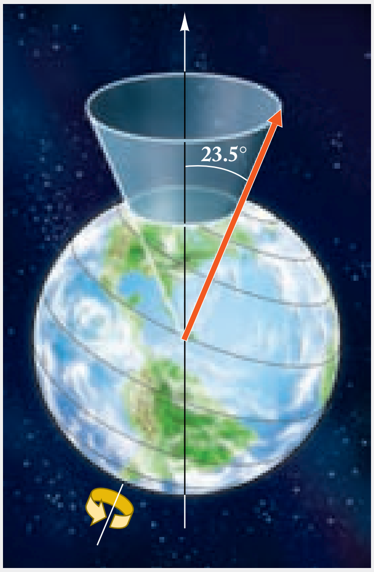


FIGURE 13.43 The precessing Earth.

Answers to Checkups

Checkup 13.1

1. You should place your hand at the end of the handle farthest from the bolt; this will provide the largest R in Eq. (13.2) and maximize the torque. Similarly, your push should be perpendicular to the wrench handle, in order to maximize $\sin\theta$ to the value $\sin 90° = 1$ in Eq. (13.2).
2. The direction must be toward the axis (along a radius), so that $\sin\theta = \sin 0° = 0$; thus both the torque and the work done will be zero.
3. Initially, when the stick is upright, the weight acts downward, along the radial direction, and so the torque is zero. As the stick falls, the weight (mg) and the point at which it acts ($R = y_{CM} = l/2$) remain constant. Only the angle between the force and the radial line changes; the sine of this angle is maximum just as the meterstick hits the floor (when $\sin\theta = \sin 90° = 1$), so the torque is maximum then.
4. (A) $\frac{1}{4}$. The work done is $W = \tau\,\Delta\phi = FR\sin\theta\,\Delta\phi$. In both cases, pushing at right angles implies $\sin\theta = 1$, and both angular displacements $\Delta\phi$ are the same. But with half the force applied at half the radius for the second push, the work will be one-fourth of that for the first push.

Checkup 13.2

1. The angular acceleration results from the torque exerted by gravity at the center of mass; this is maximum when the weight is perpendicular to the radial direction (when $\sin\theta = 1$). That occurs when the meterstick is horizontal, just before it hits the floor.
2. The translational kinetic energy is twice as large for a (uniform) rotating cylinder, because the rotational kinetic energy is $\frac{1}{2}I\omega^2 = \frac{1}{2} \times \frac{1}{2}MR^2 \times (v/R)^2 = \frac{1}{2} \times \frac{1}{2}Mv^2$.
3. The rolling cylinder's total kinetic energy is the same as for a slipping cylinder; in each case, it is equal to the change in potential energy Mgh. For the rolling cylinder, one-third of the total kinetic energy is rotational kinetic energy, and two-thirds is translational kinetic energy; thus, the rolling cylinder's translational speed is smaller when it reaches the bottom than that of a slipping cylinder (by a factor of $\sqrt{2/3}$).
4. The sphere and cylinder must have equal kinetic energies when they reach the bottom; each kinetic energy is equal to the change in potential energy Mgh. The sphere's moment of inertia is only $\frac{2}{5}MR^2$, compared with $\frac{1}{2}MR^2$ for the cylinder, so the sphere will achieve a higher speed and get to the bottom first.
5. (A) Less than that of the cylinder. For the thin hoop ($I = MR^2$), only one-half of its kinetic energy is translational; for the cylinder ($I = \frac{1}{2}Mr^2$), two-thirds of its kinetic energy will be translational. Since the total kinetic energy in each case will equal the change in potential energy (Mgh), the speed of the hoop will be smaller.

Checkup 13.3

1. Since the angular momentum is $L = I\omega$ and the angular speeds (ω) are equal, the hoop (with moment of inertia $I = MR^2$; see Table 12.3) has a larger angular momentum by a factor of 2 compared with the uniform disk (which has $I = \frac{1}{2}MR^2$).
2. Since the angular velocities are equal and the angular momentum is $L = I\omega$, the car with the larger moment of inertia $I = MR^2$ has the greater angular momentum. Since the masses are equal, this is the car on the outside, with the greater value of R.
3. Since there are no external torques on you, angular momentum $L = I\omega$ is conserved. Since you increase your moment of inertia I by stretching your legs outward (increasing R^2), your angular velocity ω must decrease.
4. No. Since angular momentum $L = I\omega$ is conserved and she decreases her moment of inertia I, her angular velocity ω increases. But her rotational kinetic energy is $K = \frac{1}{2}I\omega^2 = \frac{1}{2}I\omega \times \omega$. Since $I\omega$ is constant and ω increases, the kinetic energy increases. Thus the skater must do work to bring her arms close to her body.
5. (A) Frequency increases. The moment of inertia decreases when the children sit up, since more of their mass is closer to the axis. Since the angular momentum $L = I\omega$ is conserved, a smaller moment of inertia requires a larger angular frequency.

Checkup 13.4

1. Yes; since the angular-momentum vector is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, it will be zero when $\mathbf{r}$ and $\mathbf{p}$ are parallel (or antiparallel).
2. Since the angular-momentum vector is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, $\mathbf{L}$ is always perpendicular to $\mathbf{p}$; the angle between them is 90°.
3. The individual angular-momentum vectors will be inclined at an angle with respect to the z axis; each, however, will point toward the z axis, like the angular-momentum vectors $\mathbf{L}_1$ and $\mathbf{L}_2$ in Fig. 13.19. In this case, the horizontal components of the two angular-momentum vectors will cancel, and the total angular-momentum vector will point along the z axis.
4. Yes; the total angular momentum is changing as the dumbbell rotates about the z axis (because the direction of $\mathbf{L}$ is changing), so a torque is required to produce that change in angular momentum.
5. (E) Perpendicularly into the face of the watch. By the right-hand rule, with $\mathbf{r}$ pointing along the minute hand and $\mathbf{p}$ in the direction of motion, the clockwise rotation implies that the angular-momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is perpendicularly *into* the face of the watch.

CHAPTER

14

Statics and Elasticity

Concepts *in* Context

CONCEPTS IN CONTEXT

Tower cranes are widely used at construction sites. The K-10000 tower crane shown here is the largest commercially available tower crane. Its central tower is 110 m high, and its long horizontal arm reaches out to 84 m. It can lift 120 tons at the end of the long arm, and more than twice as much at the middle of the long arm. The short arm holds a fixed counterweight of 100 tons (at the end, above the arm) and two additional mobile counterweights (below the arm). For the lift of a small load, the mobile counterweights are parked in the inboard position, near the central tower. For the lift of a large load, the mobile counterweights are moved outward to keep the crane in balance.

The concepts discussed in this chapter permit us to examine many aspects of the operation of such a crane:

? Where must the mobile counterweights be placed to keep the crane in balance for a given load? (Example 2, page 435)

? What is the tension in the tie-rod (stretched diagonally from the top of the tower to the end of the arm) that holds the short arm in place? (Example 3, page 435)

? What is the elongation of the lifting cable when subjected to a given load? (Example 8, page 448)

Engineers and architects concerned with the design of bridges, buildings, and other structures need to know under what conditions a body will remain at rest, even when forces act on it. For instance, the designer of a railroad bridge must make sure that the bridge will not tip over or break when a heavy train passes over it. *A body that remains at rest, even though several forces act on it, is said to be in equilibrium.* The branch of physics that studies the conditions for the equilibrium of a body is called **statics.** Statics is the oldest branch of physics. The ancient Egyptians, Greeks, and Romans had a good grasp of the basic principles of statics, as is evident from their construction of elegant arches for doorways and bridges. The oldest surviving physics textbook is a treatise on the statics of ships by Archimedes.

In the first three sections of this chapter, we will rely on the assumption that the "rigid" structural members—such as beams and columns—indeed remain rigid; that is, they do not deform. In essence, this means that we assume that the forces are not so large as to produce a significant bending or compression of the beams or columns. However, in the last section, we will take a brief look at the phenomenon of the elastic deformation of solid bodies when subjected to the action of large forces.

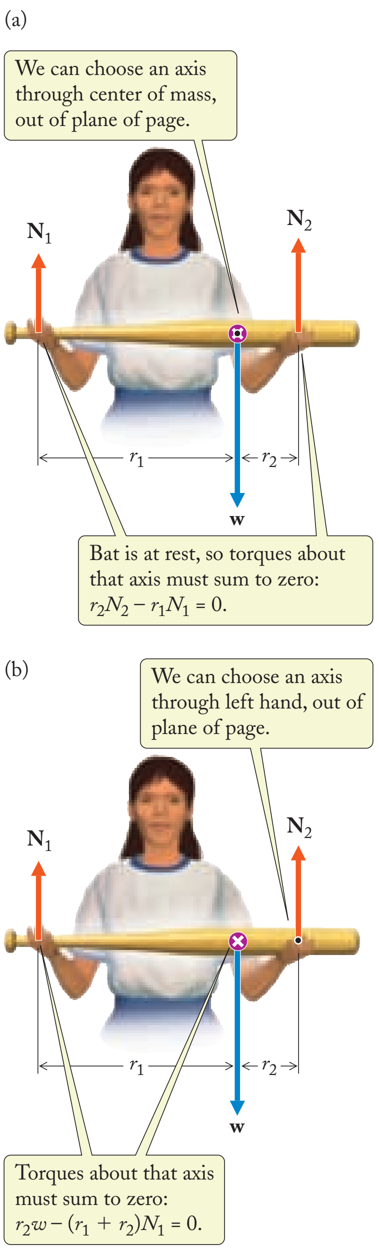


FIGURE 14.1 A baseball bat at rest in your hands. The external forces are the downward weight **w** and the upward pushes $\mathbf{N}_1$ and $\mathbf{N}_2$ of the right and left hands, respectively. These external forces add to zero. The external torques about any axis also add to zero. (a) Axis is through center of mass. (b) Axis is through left hand.

14.1 STATICS OF RIGID BODIES

If a rigid body is to remain at rest, its translational and rotational accelerations must be zero. Hence, the condition for the **static equilibrium** of a rigid body is that *the sum of external forces and the sum of external torques on the body must be zero*. This means that the forces and the torques are in balance; each force is compensated by some other force or forces, and each torque is compensated by some other torque or torques. For example, when a baseball bat rests in your hands (Fig. 14.1), the external forces on the bat are its (downward) weight **w** and the (upward) pushes $\mathbf{N}_1$ and $\mathbf{N}_2$ of your hands. If the bat is to remain at rest, the sum of these external forces must be zero—that is, $\mathbf{w} + \mathbf{N}_1 + \mathbf{N}_2 = 0$, or, in terms of magnitudes, $-w + N_1 + N_2 = 0$. Likewise, the sum of the torques of the external forces must be zero. Since the angular acceleration of the bat is zero about any axis of rotation whatsoever that we might choose in Fig. 14.1, the sum of torques must be zero about any such axis. For example, we might choose a horizontal axis of rotation through the center of mass of the bat, out of the plane of the page, as in Fig. 14.1a. With this choice of axis, the force $\mathbf{N}_2$ produces a counterclockwise torque r_2N_2 and the force $\mathbf{N}_1$ produces a clockwise torque r_1N_1, whereas the weight **w** (acting at the axis) produces no torque. The equilibrium condition for the torque is then $r_2N_2 - r_1N_1 = 0$. Alternatively, we might choose a horizontal axis of rotation through, say, the left hand, out of the plane of the page, as in Fig. 14.1b. With this choice, the force $\mathbf{N}_1$ produces a clockwise torque $(r_1 + r_2)N_1$, the weight produces a counterclockwise torque r_2w, and the force $\mathbf{N}_2$ produces no torque. The equilibrium condition for the torques is then $-(r_1 + r_2)N_1 + r_2w = 0$. With other choices of axis of rotation, we can generate many more equations than there are unknown forces or torques in a static equilibrium problem. However, the equations obtained with different choices of axis of rotation are related, and they can always be shown to be consistent.

From this discussion, we conclude that for the purposes of static equilibrium, *any line through the body or any line passing at some distance from the body can be thought of as a conceivable axis of rotation, and the torque about every such axis must be zero*. This means we have complete freedom in the choice of the axis of rotation, and we can make whatever choice seems convenient. With some practice, one learns to recognize which choice of axis will be most useful for the solution of a problem in statics.

The force of gravity plays an important role in many problems of statics. The force of gravity on a body is distributed over all parts of the body, each part being subjected to a force proportional to its mass. However, for the calculation of the torque exerted by gravity on a rigid body, *the entire gravitational force may be regarded as acting on the center of mass*. We relied on this rule in Fig. 14.1, where we assumed that the weight acts at the center of mass of the bat. The proof of this rule is easy: Suppose that we release some arbitrary rigid body and permit it to fall freely from an initial condition of rest. Since all the particles in the body fall at the same rate, the body will not change its orientation as it falls. If we consider an axis through the center of mass, the absence of angular acceleration implies that gravity does not generate any torque about the center of mass. Hence, if we want to simulate gravity by a single force acting at one point of the rigid body, that point will have to be the center of mass.

Given that in a rigid body the force of gravity effectively acts on the center of mass, we see that a rigid body supported by a single force acting at its center of mass or acting on the vertical line through its center of mass is in equilibrium, since the support force is then collinear with the effective force of gravity, and such collinear forces of equal magnitudes and opposite directions exert no net torque. This provides us with a simple method for the experimental determination of the center of mass of a body of complicated shape: Suspend the body from a string attached to a point on its surface (Fig. 14.2); the body will then settle into an equilibrium position such that the center of mass is on the vertical downward prolongation of the string (this vertical prolongation is marked dashed in Fig. 14.2). Next, suspend the body from a string attached at another point of its surface, and mark a new vertical downward prolongation of the string. The center of mass is then at the intersection of the new and the old prolongations of the string.

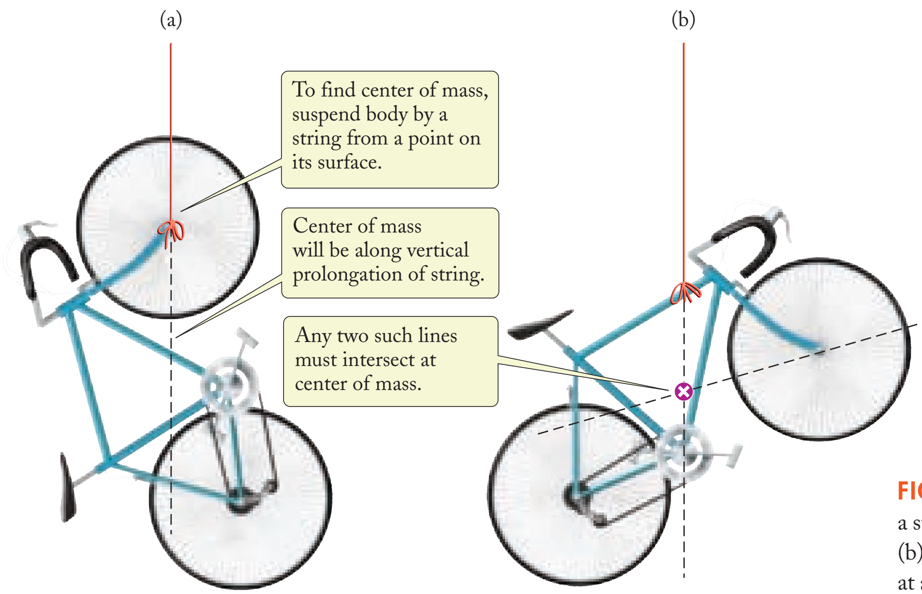


FIGURE 14.2 (a) Bicycle suspended by a string attached at a point on its "surface." (b) Bicycle suspended by a string attached at a different point.

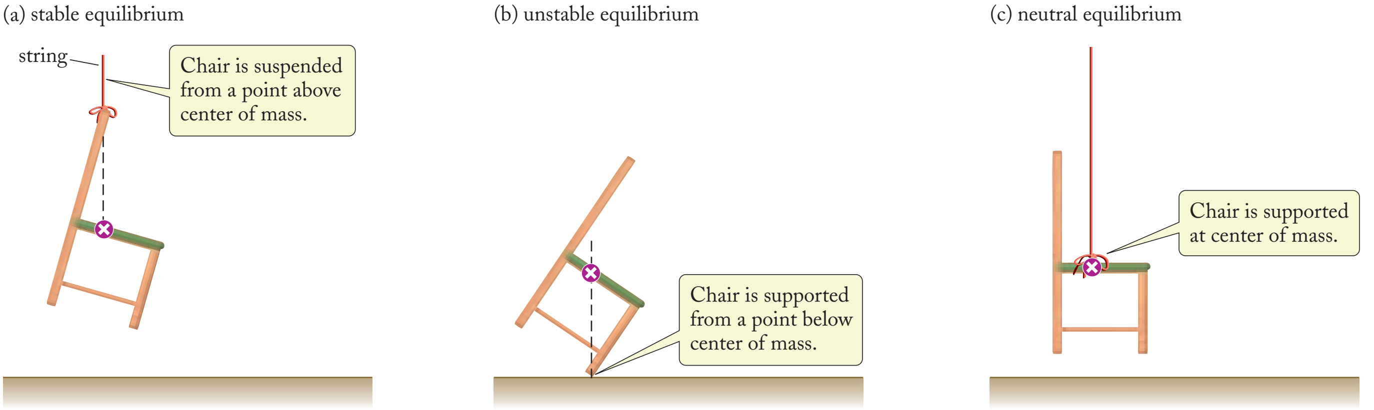


FIGURE 14.3 A body (a) in stable equilibrium; (b) in unstable equilibrium; (c) in neutral equilibrium.

A body suspended from a point above its center of mass, as in Fig. 14.3a, is in **stable equilibrium** (see also Section 8.2). If we turn this body through some angle, so the center of mass is no longer vertically below the point of support, the force of gravity and the supporting force will produce a torque that tends to return the body to the equilibrium position. In contrast, *if this body is supported by a single force applied at a point below the center of mass, as in Fig. 14.3b, the body is in* **unstable equilibrium**. If we turn the body ever so slightly, the force of gravity and the supporting force will produce a torque that tends to turn the body farther away from the equilibrium position—the body tends to topple over. Finally, *a body supported by a single force at its center of mass, as in Fig. 14.3c, is in* **neutral equilibrium.** If we turn such a body, it remains in equilibrium in its new position, and exhibits no tendency to return to its original position or to turn farther away.

Similar stability criteria apply to the translational motion of a body moving on a surface. A body is in stable equilibrium if it resists small disturbances and tends to return to its original position when the disturbance ceases. A car resting at the bottom of a dip in the road is an example of this kind of equilibrium; if we displace the car forward and then let go, the car rolls back to its original position. A body is in unstable equilibrium if it tends to move away from its original position when disturbed. A car resting on the top of a hill is an example of this second kind of equilibrium. If we displace the car forward, it continues to roll down the hill. A car resting on a flat street is in neutral equilibrium with respect to translational displacements. If we displace the car along the street, it merely remains at the new position, without any tendency to return to its original position or to move away from it (see Fig. 14.4).

The first four examples of the next section involve stable or neutral equilibrium; the next two examples involve unstable equilibrium. Engineers take great care to avoid unstable equilibrium in the design of structures and machinery, since an unstable configuration will collapse or come apart at the slightest provocation.

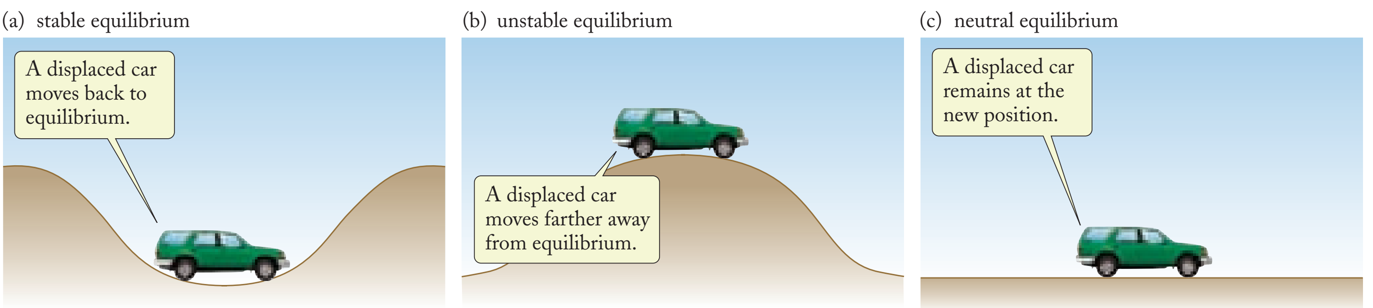


FIGURE 14.4 Stationary automobile in (a) stable, (b) unstable, and (c) neutral equilibrium.

Checkup 14.1

QUESTION 1: Is a cyclist balanced on an upright bicycle in stable or unstable equilibrium? Assume the cyclist sits rigidly, and makes no effort to avoid whatever might befall (see Fig. 14.5).

QUESTION 2: You sit in a swing, with your knees bent. If you now extend your legs fully, how will this change the equilibrium position of the swing and your body?

QUESTION 3: (a) You hold a fishing pole with both hands and point it straight up. Is the support force aligned with the weight? (b) You point the fishing pole horizontally. Is the support force aligned with the weight? Is there a single support force?

QUESTION 4: Consider a cone on a table (a) lying flat on its curved side, (b) standing on its base, (c) standing on its apex. Respectively, the equilibrium of each position is

(A) Stable, unstable, neutral
(B) Stable, neutral, unstable
(C) Unstable, stable, neutral
(D) Neutral, stable, unstable
(E) Neutral, unstable, stable

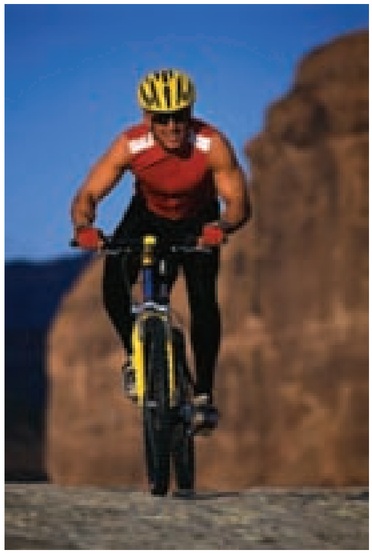


FIGURE 14.5 Is an upright bicycle in unstable equilibrium?

14.2 EXAMPLES OF STATIC EQUILIBRIUM

The following are some examples of solutions of problems in statics. In these examples, the conditions of a zero sum of external forces,

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = 0 \tag{14.1}$$

and a zero sum of external torques,

$$\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3 + \cdots = 0 \tag{14.2}$$

are used either to find the magnitudes of the forces that hold the body in equilibrium, or to find whether the body can achieve equilibrium at all.

EXAMPLE 1 A locomotive of mass 90 000 kg is one-third of the way across a bridge 90 m long. The bridge consists of a uniform iron girder of mass 900 000 kg, which is supported by two piers (see Fig. 14.6a). What is the load on each pier?

SOLUTION: The body whose equilibrium we want to investigate is the bridge. Figure 14.6b is a "free-body" diagram for the bridge, showing all the forces acting on it: the weight of the bridge, the downward push exerted by the locomotive, and the upward thrust exerted by each pier. The weight of the bridge can be regarded as acting at its center of mass. The bridge is static, and hence the net torque on the bridge reckoned about any point must be zero.

Let us first consider the torques about the point P_2, at the right pier. These torques are generated by the weight of the bridge acting at a distance of 45 m, the downward push of the locomotive acting at a distance of 30 m, and the upward thrust $\mathbf{F}_1$ of the pier at P_1 acting at a distance of 90 m (the upward thrust $\mathbf{F}_2$ has zero moment arm and generates no torque about P_2). The weight of the bridge is $m_{\text{bridge}}\,g = 9.0 \times 10^5 \text{ kg} \times g$, and the downward push exerted by the locomotive equals its weight, $m_{\text{loc}}\,g = 9.0 \times 10^4 \text{ kg} \times g$. Since each of the forces

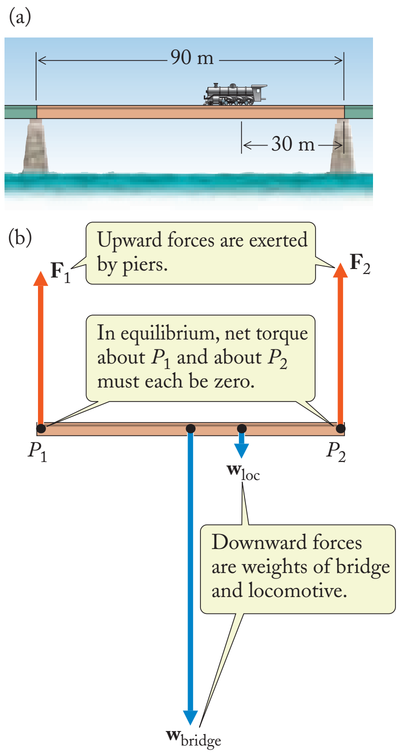


FIGURE 14.6 (a) Bridge with a locomotive on it. (b) "Free-body" diagram for the bridge.

acts at right angles to the (horizontal) line from P_2 to the point of application of the force, the magnitude of the torque $\tau = rF \sin 90°$ for each force is simply the product of the distance and the force, $\tau = rF$. According to the equilibrium condition, we must set the sum of the three torques equal to zero:

$$\boldsymbol{\tau}_{\text{bridge}} + \boldsymbol{\tau}_{\text{loc}} + \boldsymbol{\tau}_{\text{pier}} = 0 \tag{14.3}$$

$$45 \text{ m} \times 9.0 \times 10^5 \text{ kg} \times g + 30 \text{ m} \times 9.0 \times 10^4 \text{ kg} \times g - 90 \text{ m} \times F_1 = 0 \tag{14.4}$$

Here, we have chosen to reckon the first two torques as positive, since they tend to produce counterclockwise rotation about P_2, and the last torque must then be reckoned as negative, since it tends to produce clockwise rotation. Equation (14.4) contains only the single unknown force F_1. Note that we were able to isolate this unknown force by evaluating the torques about P_2: the other unknown force F_2 is absent because it produces no torque about P_2. Solving this equation for the unknown F_1, we find

$$\begin{aligned} F_1 &= \frac{(45\text{ m} \times 9.0 \times 10^5\text{ kg} + 30 \text{ m} \times 9.0 \times 10^4\text{ kg}) \times g}{90 \text{ m}} \\ &= 4.8 \times 10^5\text{ kg} \times g \\ &= 4.8 \times 10^5\text{ kg} \times 9.81 \text{ m/s}^2 = 4.7 \times 10^6\text{ N} \end{aligned}$$

Next, consider the torques about the point P_1. These torques are generated by the weight of the bridge, the weight of the locomotive, and the upward thrust $\mathbf{F}_2$ at point P_2 (the upward thrust of $\mathbf{F}_1$ has zero moment arm and generates no torque about P_1). Setting the sum of these three torques about the point P_1 equal to zero, we obtain

$$-45 \text{ m} \times 9.0 \times 10^5 \text{ kg} \times g - 60 \text{ m} \times 9.0 \times 10^4 \text{ kg} \times g + 90 \text{ m} \times F_2 = 0$$

This equation contains only the single unknown force F_2 (the force F_1 is absent because it produces no torque about P_1). Solving for the unknown F_2, we find

$$F_2 = 5.0 \times 10^6\text{ N}$$

The loads on the piers (the downward pushes of the bridge on the piers) are opposite to the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ (these downward pushes of the bridge on the piers are the reaction forces corresponding to the upward thrusts of the piers on the bridge). Thus, the magnitudes of the loads are 4.7×10^6 N and 5.0×10^6 N, respectively.

COMMENT: Note that the net vertical upward force exerted by the piers is $F_1 + F_2 = 9.7 \times 10^6$ N. It is easy to check that this matches the sum of the weights of the bridge and the locomotive; thus, the condition for zero net vertical force, as required for translational static equilibrium, is automatically satisfied. This automatic result for the equilibrium of vertical forces came about because we used the condition for rotational equilibrium twice. Instead, we could have used the condition for rotational equilibrium once [Eq. (14.4)] and then evaluated F_2 by means of the condition for translational equilibrium [Eq. (14.1)]. The result for zero net torque about the point P_1 would then have emerged automatically.

Also note that instead of taking the bridge as the body whose equilibrium is to be investigated, we could have taken the bridge plus locomotive as a combined body. The downward push of the locomotive on the bridge would then not be an external force, and would not be included in the "free-body" diagram. Instead, the weight of the locomotive would be one of the external forces acting on the combined body and would have to be included in the "free-body" diagram. The vectors in Fig. 14.6b would therefore remain unchanged.

EXAMPLE 2 A large tower crane has a fixed counterweight of 100 tons at the end of its short arm, and it also has a mobile counterweight of 120 tons. The length of the short arm is 56 m, and the length of the long arm is 84 m; the total mass of both arms is 100 tons, and this mass is uniformly distributed along their combined length. The crane is lifting a load of 80 tons hanging at the end of the long arm. Where should the crane operator position the mobile counterweight to achieve a perfect balance of the crane, that is, a condition of zero (external) torque?

Concepts in Context

SOLUTION: To find the position of the counterweight, we consider the equilibrium condition for the entire crane (alternatively, we could consider the upper part of the crane, that is, the arms and the tie-rods that hold them rigid). Figure 14.7 is a "free-body" diagram for the crane. The external forces are the support force of the base and the weights of the load, the tower, the horizontal arms, the fixed counterweight, and the mobile counterweight. The weight of the arms acts at the center of mass of the combined arms. The total length of these arms is 84 m + 56 m = 140 m, and the center of mass is at the midpoint, 70 m from each end, that is, 14 m from the centerline of the tower.

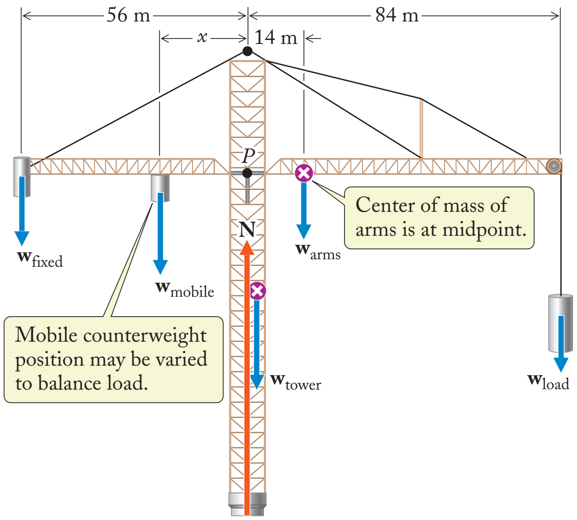


FIGURE 14.7 "Free-body" diagram of a tower crane. The crane is balanced, so that no torque is exerted by the base.

To examine the balance of torques, it is convenient to select the point P at the intersection of the arms and the midline of the tower. All the forces then act at right angles to the line from P to the point of application of the force, and the torque for each is simply the product of the distance and the force. The weight of the tower and the support force of the base do not generate any torques, since they act at zero distance. The equilibrium condition for the sum of the torques generated by the weights of the load, the arms, the fixed counterweight, and the mobile counterweight is

$$\boldsymbol{\tau}_{\text{load}} + \boldsymbol{\tau}_{\text{arms}} + \boldsymbol{\tau}_{\text{fixed}} + \boldsymbol{\tau}_{\text{mobile}} = 0 \qquad (14.5)$$

Inserting the values of the weights and moment arms, we have

$$-84\text{ m} \times 80\text{ t} \times g - 14\text{ m} \times 100\text{ t} \times g + 56\text{ m} \times 100\text{ t} \times g$$
$$+ x \times 120\text{ t} \times g = 0$$

where we have again chosen to reckon counterclockwise torques as positive and clockwise torques as negative. When we solve this equation for x, we obtain

$$x = \frac{84\text{ m} \times 80\text{ t} + 14\text{ m} \times 100\text{ t} - 56\text{ m} \times 100\text{ t}}{120\text{ t}}$$
$$= 21\text{ m}$$

EXAMPLE 3 The short arm of the tower crane is held in place by a steel tie-rod stretched diagonally from the top of the tower to the end of the arm, as shown in Fig. 14.8a. The top part of the tower is 30 m high, and the short arm has a length of 56 m and a mass of 40 metric tons. The joint of the arm and the tower is somewhat flexible, so the joint acts as a pivot. Suppose

Concepts in Context

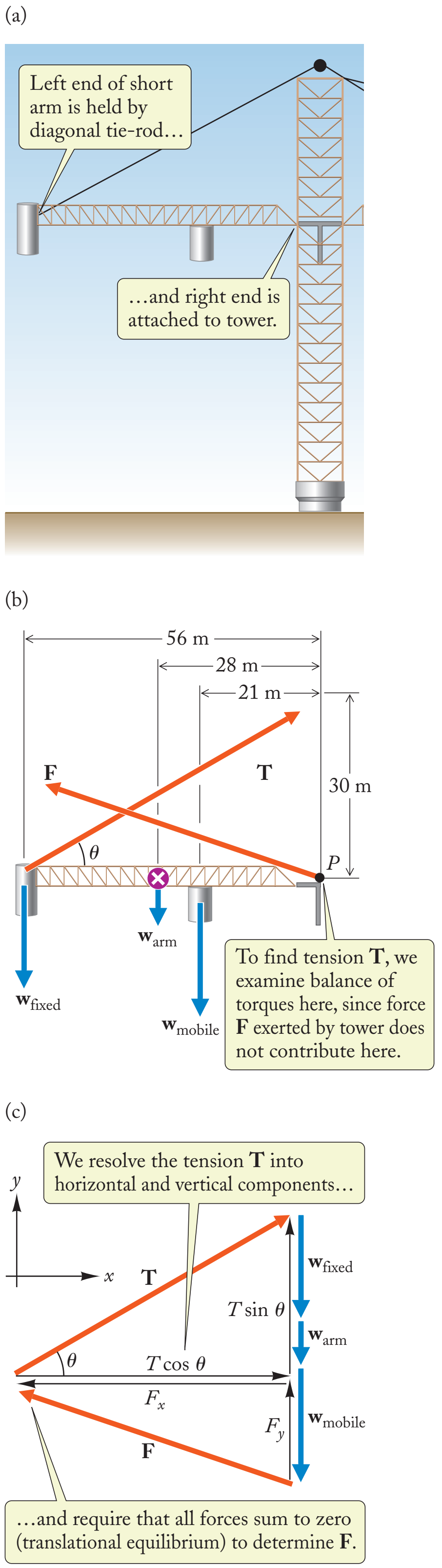


FIGURE 14.8 (a) Steel tie-rod supporting the short tower crane arm. (b) "Free-body" diagram for the short tower crane arm. (c) The *x* and *y* components of the forces.

that the counterweights are placed on the short arm as in the preceding example: the fixed counterweight of 100 metric tons is at the end of the arm, and the mobile counterweight of 120 metric tons is at a distance of 21 m from the centerline. (a) What is the tension in the tie-rod? (b) What is the force that the short arm exerts against the tower at the joint?

SOLUTION: Figure 14.8b is a "free-body" diagram of the short arm, displaying all the external forces acting on it. These forces are the weight $\mathbf{w}_{arm}$ of the arm, the weights of the counterweights $\mathbf{w}_{fixed}$ and $\mathbf{w}_{mobile}$, the tension **T** of the tie-rod, and the force **F** exerted by the tower at the joint. The force **F** is equal and opposite to the force that the short arm exerts against the tower. The weight of the arm acts at its center of mass, at a distance of 28 m from the centerline; the mobile counterweight acts at a distance of 21 m; and the fixed counterweight and the tension act at the end of the short arm, at a distance of 56 m.

(a) To find the tension **T**, it is convenient to examine the balance of torques about a point *P* that coincides with the joint. The force **F** does not generate any torque about this point, and hence the condition for the balance of the torques will contain **T** as the sole unknown. The weight of the short arm and the counterweights act at right angles to the line from *P* to the point of application of the force, so the torque for each is the product of the distance and the force. From Fig. 14.8b, we see that the tension acts at an angle θ, given by

$$\tan\theta = \frac{30\text{ m}}{56\text{ m}} = 0.54$$

which corresponds to $\theta = 28°$. With the same sign convention for the direction of the torques as in the preceding example, the equilibrium condition for the torques exerted by the weight of the arm, the counterweights, and the tension is then

$$28\text{ m} \times 40\text{ t} \times g + 21\text{ m} \times 120\text{ t} \times g + 56\text{ m} \times 100\text{ t} \times g - 56\text{ m} \times T \times \sin 28° = 0$$

We can solve this equation for T, with the result

$$T = \frac{28\text{ m} \times 40\text{ t} \times g + 21\text{ m} \times 120\text{ t} \times g + 56\text{ m} \times 100\text{ t} \times g}{56\text{ m} \times \sin 28°}$$

$$= 351\text{ t} \times g$$

$$= 351 \times 1000\text{ kg} \times 9.8\text{ m/s}^2 = 3.4 \times 10^6\text{ N}$$

(b) To find the components of the force **F** (Fig. 14.8c), we simply use the conditions for translational equilibrium: the sum of the horizontal components of all the forces and the sum of the vertical components of all the forces must each be zero. The weights of the short arm and the counterweights have vertical components, but no horizontal components. The tension force has a horizontal component $T\cos\theta$ and a vertical component $T\sin\theta$. Hence

$$3.4 \times 10^6\text{ N} \times \cos 28° + F_x = 0$$

and

$$3.4 \times 10^6\text{ N} \times \sin 28° - 40\text{ t} \times g - 120\text{ t} \times g - 100\text{ t} \times g + F_y = 0$$

When we solve these equations for F_x and F_y, we find

$$F_x = -3.0 \times 10^6\text{ N}$$

PROBLEM-SOLVING TECHNIQUES — STATIC EQUILIBRIUM

From the preceding examples we see that the steps in the solution of a problem of statics resemble the steps we employed in Chapter 5.

1. The first step is the selection of the body that is to obey the equilibrium conditions. The body may consist of a genuine rigid body (for instance, the bridge in Example 1), or it may consist of several pieces that act as a single rigid body for the purposes of the problem (for instance, the bridge plus the locomotive in Example 1). It is often helpful to mark the boundary of the selected rigid body with a distinctive color or with a heavy line; this makes it easier to recognize which forces are external and which internal.
2. Next, list all the external forces that act on this body, and display these forces on a "free-body" diagram.
3. If the forces have different directions, it is usually best to draw coordinate axes on the diagram and to resolve the forces into x and y components.
4. For each component, apply the static equilibrium condition for forces: the sum of forces is zero.
5. Make a choice of axis of rotation, calculate the torque of each force about this axis ($\tau = RF \sin\theta$), and apply the static equilibrium condition for torques: the sum of torques is zero. Establish and maintain a sign convention for torques; for example, for an axis pointing into the plane of the paper, counterclockwise torques to be positive and clockwise torques to be negative.
6. As mentioned in Section 14.1, any line can be thought of as an axis of rotation; and the torque about every such axis must be zero. You can make an unknown force disappear from the equation if you place the axis of rotation at the point of action or on the line of action of this force, so that this force has zero moment arm. Furthermore, as illustrated in Example 1, sometimes it is convenient to consider two different axes of rotation, and to examine the separate equilibrium conditions of the torques for each of these axes.
7. As recommended in Chapter 2, it is usually best to solve the equations algebraically for the unknown quantities, and to substitute numbers for the known quantities as a last step. But if the equations are messy, with a clutter of algebraic symbols, it may be convenient to substitute some of the numbers before proceeding with the solution of the equations.

and

$$F_y = 9.5 \times 10^5 \text{ N}$$

The x and y components of the force exerted by the short arm on the tower are therefore $+3.0 \times 10^6$ N and -9.5×10^5 N, respectively.

EXAMPLE 4 The bottom of a ladder rests on the floor, and the top rests against a wall (see Fig. 14.9a). If the coefficient of static friction between the ladder and the floor is $\mu_s = 0.40$ and the wall is frictionless, what is the maximum angle that the ladder can make with the wall without slipping?

SOLUTION: Figure 14.9b shows the "free-body" diagram for the ladder, with all the forces. The weight of the ladder acts downward at the center of mass. If the ladder is about to slip, the friction force at the floor has the maximum magnitude for a static friction force, that is,

$$f = \mu_s N_1 \tag{14.6}$$

If we reckon the torques about the point of contact with the floor, the normal force $\mathbf{N}_1$ and the friction force $\mathbf{f}$ exert no torques about this point, since their moment

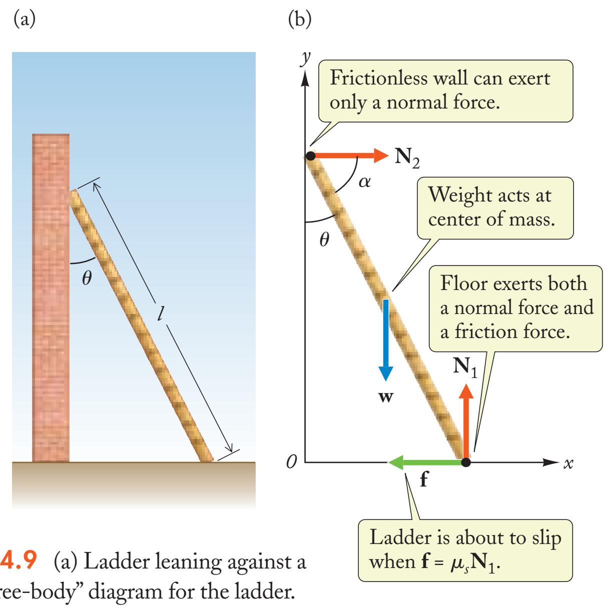


FIGURE 14.9 (a) Ladder leaning against a wall. (b) "Free-body" diagram for the ladder.

arms are zero. The weight $w = mg$ acting at the center of mass exerts a counterclockwise torque of magnitude $(l/2) \times mg \times \sin\theta$, and the normal force $\mathbf{N}_2$ of the wall exerts a clockwise torque of magnitude $l \times N_2 \times \sin\alpha$, where α is the angle between the ladder and the normal force (see Fig. 14.9b); since $\alpha = 90° - \theta$, the sine of α equals the cosine of θ, and the torque equals $l \times N_2 \times \cos\theta$. For equilibrium, the sum of these torques must be zero,

$$+\frac{l}{2} mg \sin\theta - lN_2 \cos\theta = 0 \tag{14.7}$$

or, equivalently,

$$\frac{1}{2} mg \sin\theta = N_2 \cos\theta \tag{14.8}$$

We collect the factors that depend on θ by dividing both sides of this equation by $\frac{1}{2}\, mg \cos\theta$, so

$$\frac{\sin\theta}{\cos\theta} = \frac{2N_2}{mg}$$

or, since $\sin\theta/\cos\theta = \tan\theta$,

$$\tan\theta = \frac{2N_2}{mg} \tag{14.9}$$

To evaluate the angle θ we still need to determine the unknown N_2. For this, we use the condition for translational equilibrium: the net vertical and the net horizontal forces must be zero, or

$$N_1 - mg = 0 \tag{14.10}$$

$$N_2 - \mu_s N_1 = 0 \tag{14.11}$$

From the first of these equations, $N_1 = mg$; therefore, from the second equation, $N_2 = \mu_s mg$. Inserting this into our expression (14.9) for the tangent of the angle θ, we obtain the final result

$$\tan\theta = \frac{2\,\mu_s mg}{mg} = 2\mu_s \qquad (14.12)$$

With $\mu_s = 0.40$, this yields $\tan\theta = 0.80$. With a calculator, we find that the angle with this tangent is

$$\theta = 39°$$

For any angle larger than this, equilibrium is impossible, because the maximum frictional force is not large enough to prevent slipping of the ladder.

EXAMPLE 5 A uniform rectangular box 2.0 m high, 1.0 m wide, and 1.0 m deep stands on a flat floor. You push the upper end of the box to one side and then release it (see Fig. 14.10a). At what angle of release will the box topple over on its side?

SOLUTION: The forces on the box when it has been released are as shown in the "free-body" diagram in Fig. 14.10b. Both the normal force **N** and the friction force **f** act at the bottom corner, which is the only point of contact of the box with the floor. The weight acts at the center of mass, which is at the center of the box.

Since the box rotates about the bottom corner, let us consider the torque about this point. The only force that produces a torque about the bottom corner is the weight. The weight acts at the center of mass; for a uniform box, this is at the center of the box. The torque exerted by the weight can be expressed as $d \times Mg$, where d is the perpendicular distance from the bottom corner to the vertical line through the center of mass (see Fig. 14.10b). This torque produces counterclockwise rotation if the center of mass is to the left of the bottom corner, and it produces clockwise rotation if the center of mass is to the right of the bottom corner. This means that in the former case, the box returns to its initial position, and in the latter case it topples over on its side. Thus, the critical angle beyond which the box will tip over corresponds to vertical alignment of the bottom corner and the center of the box (see Fig. 14.10c). This critical angle equals the angle between the side of the box and the diagonal. The tangent of this angle is the ratio of the width and the height of the box,

$$\tan\theta = \frac{0.50\ \text{m}}{1.0\ \text{m}} = 0.50$$

With our calculator we find that the critical angle is then

$$\theta = 27°$$

COMMENT: In this example we found that the box begins to topple over if its inclination is such that the center of mass is vertically aligned with the bottom corner. This is a special instance of the general rule that a rigid body resting on a surface (flat or otherwise) becomes unstable when its center of mass is vertically above the outermost point of support.

EXAMPLE 6 A uniform rectangular box 2.0 m high, 1.0 m wide, and 1.0 m deep stands on the platform of a truck (Fig. 14.11a). What is the maximum forward acceleration of the truck that the box can withstand without toppling over? Assume that the coefficient of static friction is large enough that the box will topple over before it starts sliding.

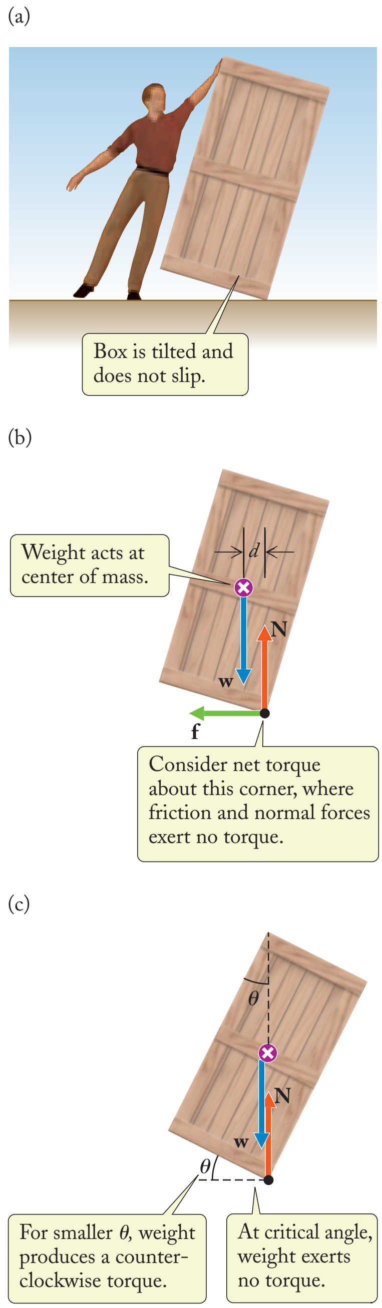


FIGURE 14.10 (a) Box standing on edge. (b) "Free-body" diagram for the box. (c) "Free-body" diagram if the box is tilted at the critical angle. The center of mass is directly above the edge.

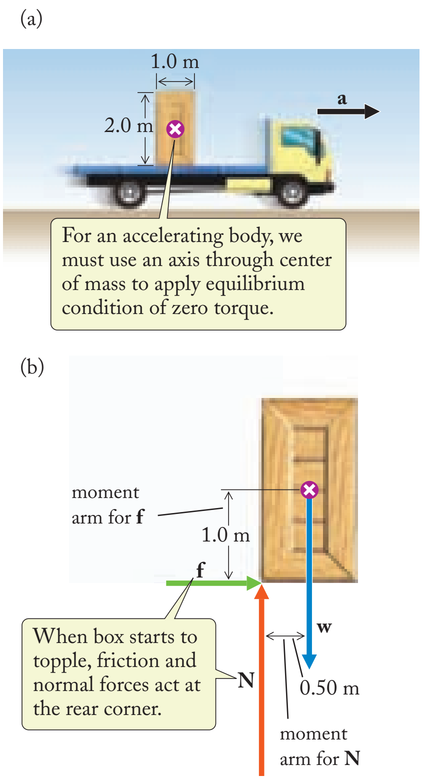


FIGURE 14.11 (a) Box on an accelerating truck. (b) "Free-body" diagram for the box.

SOLUTION: Strictly, this is not a problem of statics, since the translational motion is accelerated; however, the rotational motion involves a question of equilibrium and can be treated by the methods of this section. Under the conditions of the problem, the forces on the box are as shown in Fig. 14.11b. Both the normal force **N** and the friction force **f** act at the rear corner (when the box is about to topple, it makes contact with the platform only along the rear bottom edge). The weight acts at the center of mass; for a uniform box, this is at the center of the box, 1.0 m above and 0.50 m in front of the corner. Since the box is in accelerated motion, we have to be careful about the choice of axis for the calculation of the torque. As mentioned before Example 6 in Chapter 13, for an accelerated body, the equation of rotational motion (and the equilibrium condition of zero torque) is valid only for an axis through the center of mass. The forces that produce a torque about the center of mass are **N** and **f**, and each torque $\tau = RF \sin\theta$ may be expressed as the product of the force and the corresponding moment arm, $R \sin\theta$; the moment arms are the perpendicular distances shown in Fig. 14.11b. For an axis pointing into the page, the normal force tends to produce clockwise rotation and the frictional force counterclockwise; thus the condition of zero torque is

$$-0.50 \text{ m} \times N + 1.0 \text{ m} \times f = 0 \tag{14.13}$$

We can obtain expressions for f and N from the equations for the horizontal and vertical translational motions. The horizontal acceleration is a and the vertical acceleration is zero; accordingly, the horizontal and vertical components of Newton's Second Law are

$$f = ma$$
$$N - mg = 0$$

Inserting these expressions for f and for N into Eq. (14.13), we obtain

$$0.50 \text{ m} \times mg - 1.0 \text{ m} \times ma = 0$$

from which

$$a = 0.50g = 4.9 \text{ m/s}^2$$

If the acceleration exceeds this value, rotational equilibrium fails, and the box topples.

Checkup 14.2

QUESTION 1: Why is it dangerous to climb a ladder that is leaning against a building at a large angle with the vertical? Why is it dangerous to climb a ladder that is leaning against a building at a small angle with the vertical?

QUESTION 2: Suppose that in Example 5 all the mass of the box is concentrated at the midpoint of the bottom surface, so the center of mass is at this midpoint. What is the critical angle at which such a box topples over on its side?

QUESTION 3: Two heavy pieces of lumber lean against each other, forming an A-frame (see Fig. 14.12). Qualitatively, how does the force that one piece of lumber exerts on the other at the tip of the A vary with the angle?

QUESTION 4: You hold a fishing pole steady, with one hand forward, pushing upward to support the pole, and the other hand further back, pushing downward to maintain

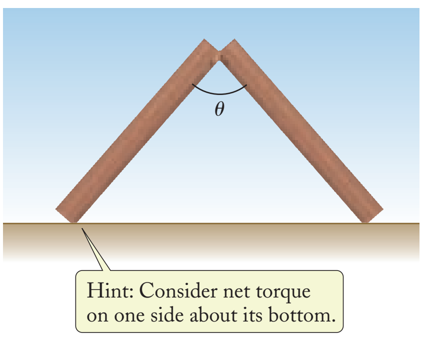


FIGURE 14.12 Two pieces of lumber forming an A-frame.

zero net torque. If a fish starts to pull downward on the far end of the pole, then to maintain equilibrium you must

(A) Increase the upward push and decrease the downward push
(B) Increase the upward push and increase the downward push
(C) Increase the upward push and keep the downward push the same

14.3 LEVERS AND PULLEYS

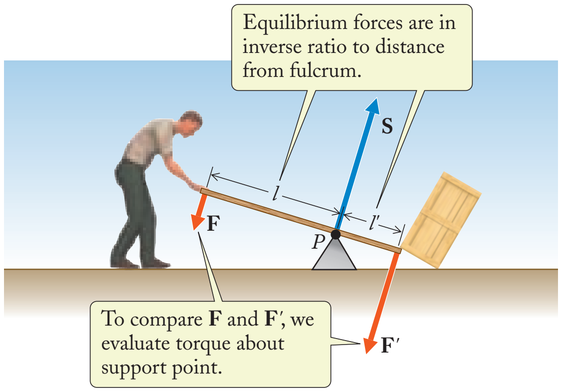


FIGURE 14.13 A lever. The vectors show the forces acting on the lever; **F** is our push, **F**′ is the push of the load, and **S** is the supporting force of the pivot. The force that the lever exerts on the load is of the same magnitude as **F**′, but of opposite direction.

A lever consists of a rigid bar swinging on a pivot (see Fig. 14.13). If we apply a force at the long end, the short end of the bar pushes against a load with a larger force. Thus, the lever permits us to lift a larger load than we could with our bare hands. The relationship between the magnitudes of the forces at the ends follows from the condition for static equilibrium for the lever. Figure 14.13 shows the forces acting on the lever: the force **F** that we exert at one end, the force **F**′ exerted by the load at the other end, and the support force **S** exerted by the pivot point *P*. The net torque about the pivot point *P* must be zero. Since, for the arrangement shown in Fig. 14.13, the forces at the ends are at right angles to the distances l and l', the condition on the net torque is

$$Fl - F'l' = 0 \tag{14.14}$$

from which we find

$$\frac{F'}{F} = \frac{l}{l'} \tag{14.15}$$

mechanical advantage of lever

By Newton's Third Law, the force that the load exerts on the lever is equal in magnitude to the force that the lever exerts on the load (and of opposite direction). Hence Eq. (14.15) tells us the ratio of the magnitudes of the forces we exert and the lever exerts. *These forces are in the inverse ratio of the distances from the pivot point.* For a powerful lever, we must make the lever arm l as long as possible and the lever arm l' as short as possible. The ratio F'/F of the magnitudes of the force delivered by the lever and the force we must supply is called the **mechanical advantage.**

Apart from its application in the lifting of heavy loads, the principle of the lever finds application in many hand tools, such as pliers and bolt cutters. The handles of these tools are long, and the working ends are short, yielding an enhancement of the force exerted by the hand (see Fig. 14.14). A simple manual winch also relies on the principle of the lever. The handle of the winch is long, and the drum of the winch, which acts as the short lever arm, is small (see Fig. 14.15). The force the winch delivers to the rope attached to the drum is then larger than the force exerted by the hand pushing on the handle. Compound winches, used for trimming sails on sailboats, have internal sets of gears that provide a larger mechanical advantage; in essence, such compound winches stagger one winch within another, so the force ratio generated by one winch is further multiplied by the force ratio of the other.

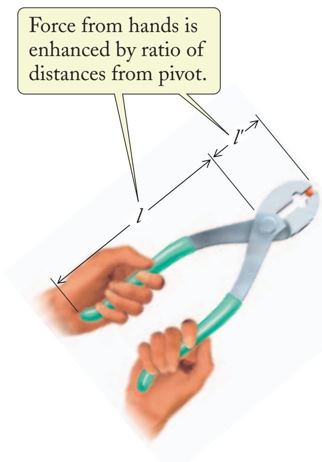


FIGURE 14.14 A pair of pliers serves as levers.

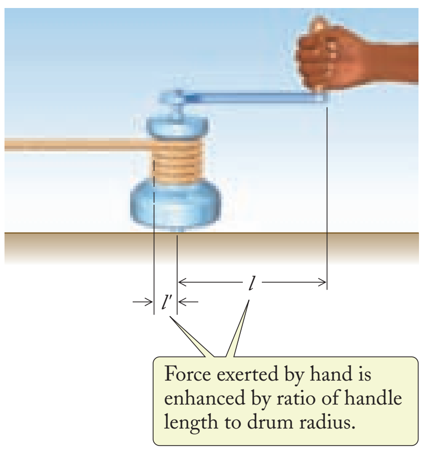


FIGURE 14.15 A manual winch.

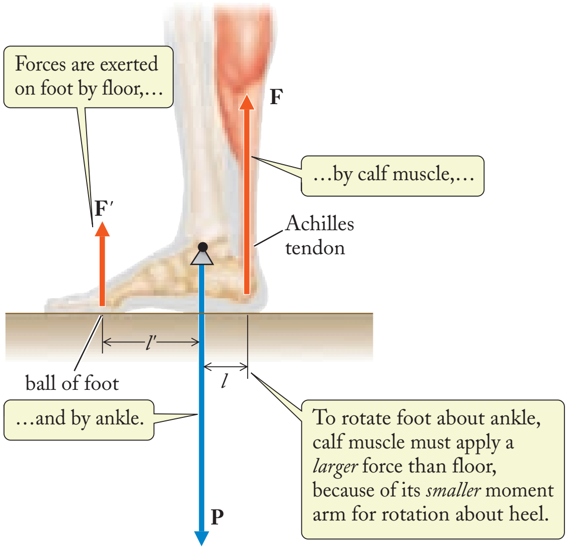


FIGURE 14.16 Bones of the foot acting as a lever.

In the human body, many bones play the role of levers that permit muscles or groups of muscles to support or to move the body. For example, Fig. 14.16 shows the bones of the foot; these act as a lever, hinged at the ankle. The rear end of this lever, at the heel, is tied to the muscles of the calf by the Achilles tendon, and the front end of the lever is in contact with the ground, at the ball of the foot. When the muscle contracts, it rotates the heel about the ankle and presses the ball of the foot against the ground, thereby lifting the entire body on tiptoe. Note that the muscle is attached to the short end of this lever—the muscle must provide a larger force than the force generated at the ball of the foot. At first sight, it would seem advantageous to install a longer projecting spur at the heel of the foot and attach the Achilles tendon to the end of this spur; but this would require that the contracting muscle move through a longer distance. Muscle is good at producing large forces, but not so good at contracting over long distances, and the attachment of the Achilles tendon represents the best compromise. In most of the levers found in the human skeleton, the muscle is attached to the short end of the lever.

Equation (14.15) is valid only if the forces are applied at right angles to the lever. A similar equation is valid if the forces are applied at some other angle, but instead of the lengths l and l' of the lever, we must substitute the lengths of the moment arms of the forces, that is, the perpendicular distances between the pivot point and the lines of action of the forces. These moment arms play the role of effective lengths of the lever.

EXAMPLE 7 When you bend over to pick up something from the floor, your backbone acts as a lever pivoted at the sacrum (see Fig. 14.17). The weight of the trunk pulls downward on this lever, and the muscles attached along the upper part of the backbone pull upward. The actual arrangement of the muscles is rather complicated, but for a simple mechanical model we can pretend that the muscles are equivalent to a string attached to the backbone at an angle of about 12° at a point beyond the center of mass (the other end of the "string" is attached to the pelvis). Assume that the mass of the trunk, including head and arms, is

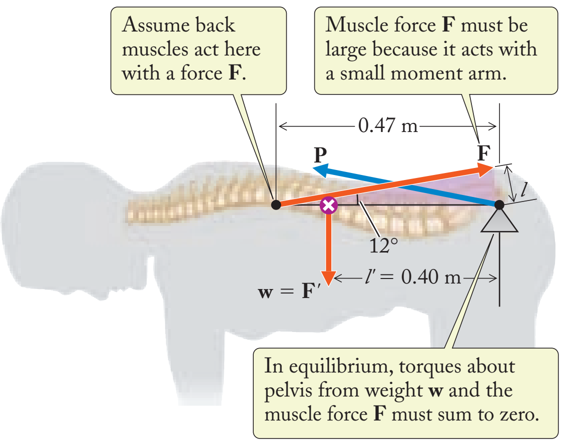


FIGURE 14.17 "Free-body" diagram for the backbone acting as lever. The forces on the backbone are the weight **w** of the trunk (including the weight of the backbone), the pull **F** of the muscles, and the thrust **P** of the pelvis acting as pivot.

48 kg, and that the dimensions are as shown in the diagram. What force must the muscles exert to balance the weight of the trunk when bent over horizontally?

SOLUTION: Figure 14.17 shows a "free-body" diagram for the backbone, with all the forces acting on it. Since the weight **w** of the trunk acts at right angles to the backbone, the lever arm for this weight is equal to the distance $l' = 0.40$ m between the pivot and the center of mass of the trunk. The lever arm for the muscle is the (small) distance l, which equals $l = 0.47 \text{ m} \times \sin 12° = 0.10$ m. According to Eq. (14.15), the force **F** exerted by the muscles then has magnitude

$$F = \frac{l'}{l}F' = \frac{l'}{l}w = \frac{l'}{l}Mg = \frac{0.40 \text{ m}}{0.10 \text{ m}} \times Mg = 4.0 \times Mg$$
$$= 4.0 \times 48 \text{ kg} \times 9.81 \text{ m/s}^2 = 1.9 \times 10^3 \text{ N}$$

This is a quite large force, 4.0 times larger than the weight of the trunk.

COMMENT: Bending over horizontally puts a severe stress on the muscles of the back. Furthermore, it puts an almost equally large compressional stress on the backbone, pulling it hard against the sacrum. The stresses are even larger if you try to lift a load from the floor while your body is bent over in this position. To avoid damage to the muscles and to the lumbosacral disk, it is best to lift by bending the knees, keeping the backbone vertical.

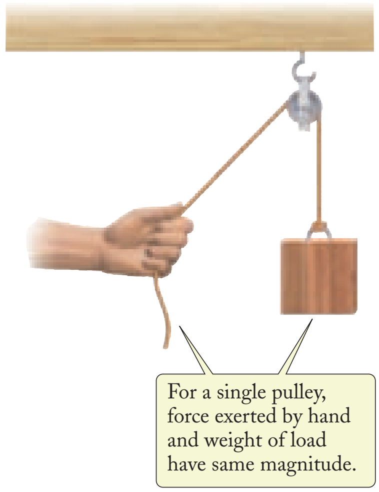


FIGURE 14.18 A single pulley.

Often, a force is applied to a load by means of a flexible rope, or a string. A pulley is then sometimes used to change the direction of the string or rope and the direction of the force exerted on the body. If the pulley is frictionless, the tension at each point of a flexible rope passing over the pulley is the same. For instance, if we want to lift a load with a rope passing over a single pulley attached to the ceiling (see Fig. 14.18), the force we must exert on the rope has the same magnitude as the weight of the load. Thus, there is no gain of mechanical advantage in such an arrangement of a single pulley; the only benefit is that it permits us to pull more comfortably than if we attempted to lift the load directly.

However, an arrangement of several pulleys linked together, called **block and tackle,** can provide a large gain of mechanical advantage. For example, consider the arrangement of three pulleys shown in Fig. 14.19a; the axles of the two upper pulleys are bolted together, and they are linked to each other and to the third pulley by a single rope. If the rope segments linking the pulleys are parallel and there is no friction, then the mechanical advantage of this arrangement is 3; that is, the magnitudes of the forces F and F' are in the ratio of 1 to 3. This can be most easily understood by drawing the "free-body" diagram for the lower portion of the pulley system, including the load (Fig. 14.19b). In this diagram, the three ropes leading upward have been cut off and replaced by the forces exerted on them by the external (upper) portions of the ropes. Since the tension is the same everywhere along the rope, the forces pulling upward on each of the three rope ends shown in the "free-body" diagram all have the same magnitude F, and thus the net upward force is $3F$.

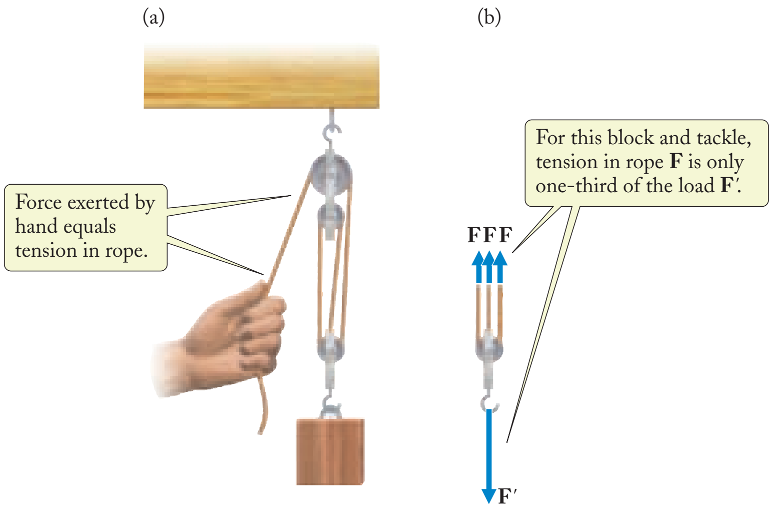


FIGURE 14.19 (a) Block and tackle. (b) "Free-body" diagram for the lower portion of the pulley system.

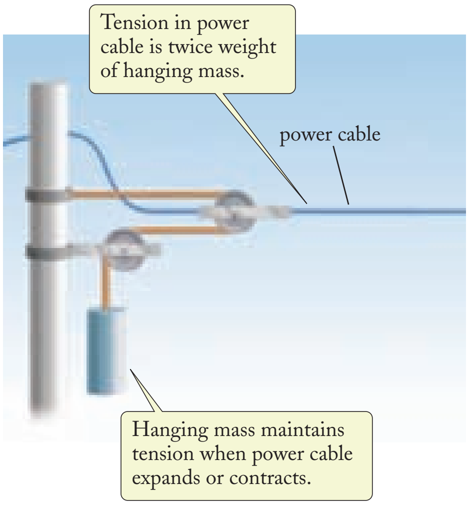


FIGURE 14.20 Block and tackle used for tensioning power line.

Block-and-tackle arrangements have many practical applications. For instance, they are used to provide the proper tension in overhead power cables for electric trains and trams (see Fig. 14.20); without such an arrangement, the cables would sag on warm days when thermal expansion increases their length, and they would stretch excessively tight and perhaps snap on cold days, when they contract. One common cause of power failures on cold winter nights is the snapping of power lines lacking such compensating pulleys.

Another practical application of block and tackle is found in the traction devices used in hospitals to immobilize and align fractured bones, especially leg bones. A typical arrangement is shown in Fig. 14.21; here the pull applied to the leg is twice as large as the magnitude of the weight attached on the lower end to the rope. Also, as in the case of the power line, the tension remains constant even if the leg moves.

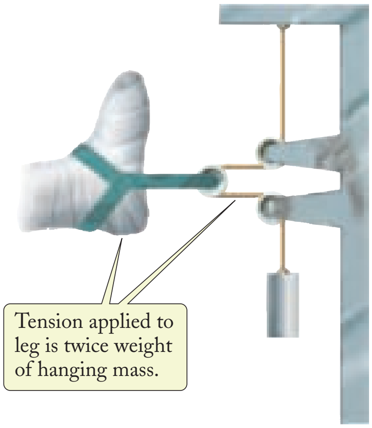


FIGURE 14.21 Block and tackle in traction apparatus for fractured leg.

The mechanical advantage provided by levers, arrangements of pulleys, or other devices can be calculated in a general and elegant way by appealing to the Law of Conservation of Energy. A lever merely transmits the work we supply at one end to the load at the other end. We can express this equality of work input and work output by

$$F'\,\Delta x' = F\,\Delta x \tag{14.16}$$

where Δx is the displacement of our hand and $\Delta x'$ the displacement of the load. According to this equation, the forces F' and F are in the inverse ratio of the displacements,

$$\frac{F'}{F} = \frac{\Delta x}{\Delta x'} \tag{14.17}$$

Consider, now, the rotation of the lever by a small angle (see Fig. 14.22). Since the two triangles included between the initial and final positions of the lever are similar, the distances Δx and $\Delta x'$ are in the same ratio as the lever arms l and l'; thus, we immediately recognize from Eq. (14.17) that the mechanical advantage of the lever is l/l'.

Likewise, we immediately recognize from Eq. (14.17) that the mechanical advantage of the arrangement of pulleys shown in Fig. 14.19 is 3, since whenever our hand pulls a length Δx of rope out of the upper pulley, the load moves upward by a distance of only $\Delta x/3$.

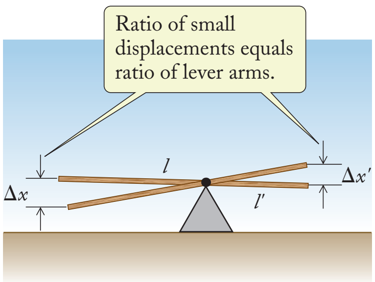


FIGURE 14.22 Rotation of lever by a small angle produces displacements Δx and $\Delta x'$ of the ends.

✔ Checkup 14.3

QUESTION 1: Figure 14.23 shows two ways of using a lever. Which has the larger mechanical advantage?

QUESTION 2: Is Eq. (14.15) for the ratio of the forces F and F' on a lever valid if one or both of these forces are not perpendicular to the lever?

QUESTION 3: Suppose that the pulleys in a block and tackle are of different sizes. Does this affect the mechanical advantage?

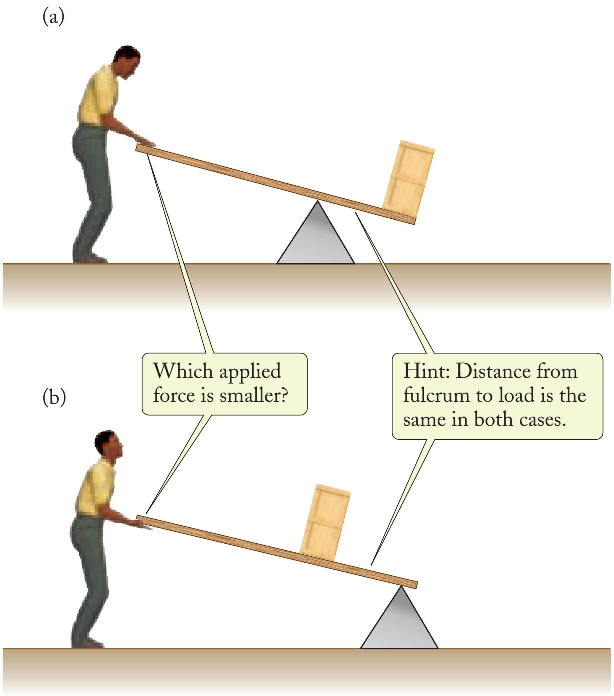


FIGURE 14.23
Two ways of using a lever.

QUESTION 4: A lever is used to lift a 100-kg rock. The distance from the rock to the fulcrum is roughly one-tenth of the distance from the fulcrum to the handle. If the rock has a mass of 100 kg, the downward force at the handle necessary to lift the rock is approximately:

(A) 1 N (B) 10 N (C) 100 N (D) 1000 N

14.4 ELASTICITY OF MATERIALS

In our examples of bridges, tower cranes, etc., we assumed that the bodies on which the forces act are rigid; that is, they do not deform. Although solid bodies, such as bars or blocks of steel, are nearly rigid, they are not exactly rigid, and they will deform by a noticeable amount if a large enough force is applied to them. A solid bar may be thought of as a very stiff spring. If the force is fairly small, this "spring" will suffer only an insignificant deformation, but if the force is large, it will suffer a noticeable deformation. Provided that the force and the deformation remain *within some limits, the deformation of a solid body is* **elastic,** *which means that the body returns to its original shape once the force ceases to act.* Such elastic deformations of a solid body usually obey Hooke's Law: the deformation is proportional to the force. But the constant of proportionality is small, giving a small deformation unless the force is large. The corresponding spring constant is thus very large, meaning that an appreciable deformation requires a large force.

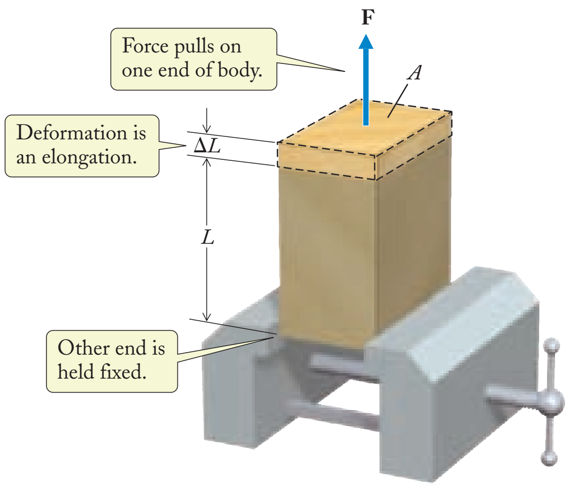


FIGURE 14.24 Tension applied to the end of a block of material causes elongation.

A solid block of material can suffer several kinds of deformation, depending on how the force is applied. If one end of the body is held fixed and the force pulls on the other end, the deformation is a simple **elongation** of the body (see Fig. 14.24). If one side of the body is held fixed and the force pushes tangentially along the other side, then the deformation is a **shear,** which changes the shape of the body from a rectangular parallelepiped to a rhomboidal parallelepiped (see Fig. 14.25a). During this deformation, the parallel layers of the body slide with respect to one another just as the pages of a book slide with respect to one another when we push along its cover (see Fig. 14.25b). If the force is applied from all sides simultaneously, by subjecting the body to the pressure of a fluid in which the body is immersed, then the deformation is a **compression** of the volume of the body, without any change of the geometrical shape (see Fig. 14.26).

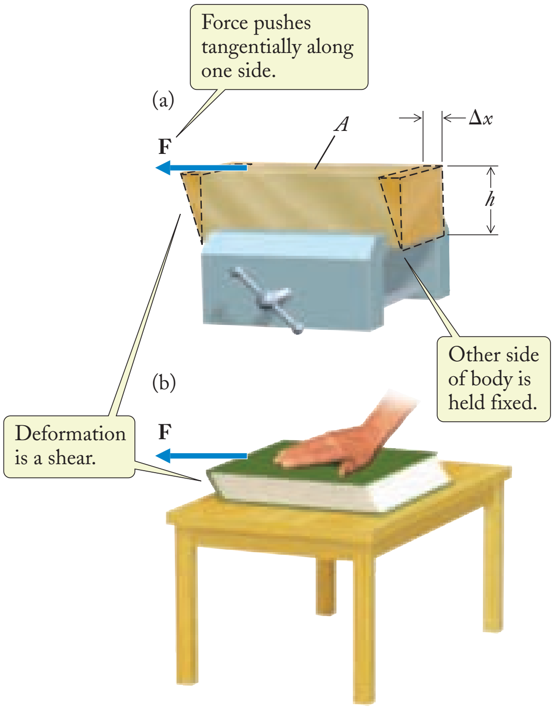


FIGURE 14.25 (a) Tangential force applied to the side of a block of material causes shear. (b) When such a tangential force is applied to the cover of a book, the pages slide past one another.

In all of these cases, *the fractional deformation, or the percent deformation, is directly proportional to the applied force and inversely proportional to the area over which the force is distributed.* For instance, if a given force produces an elongation of 1% when pulling on the end of a block, then the same force pulling on the end of a block of, say, twice the cross-sectional area will produce an elongation of $\frac{1}{2}$%. This can be readily understood if we think of the block as consisting of parallel rows of atoms linked by springs, which represent the interatomic forces that hold the atoms in their places (see Fig. 14.27). When we pull on the end of the block with a given force, we stretch the interatomic springs by some amount; and when we pull on a block of twice the cross-sectional area, we have to stretch twice as many springs, and therefore the force acting on each spring is only half as large and produces only half the elongation in each spring. Furthermore, since the force applied to the end of a row of atoms is communicated to all the interatomic springs in that row, a given force produces a given elongation in each spring in a row. The net elongation of the block is the sum of the elongations of all the interatomic springs in the row, and hence the fractional elongation of the block is the same as the fractional elongation of each spring, regardless of the overall length of the block. For instance, if a block elongates by 0.1 mm when subjected to a given force, then a block of, say, twice the length will elongate by 0.2 mm when subjected to the same force.

To express the relationships among elongation, force, and area mathematically, consider a block of initial length L and cross-sectional area A. If a force F pulls on the end of this block, the elongation is ΔL, and the fractional elongation is $\Delta L/L$. This fractional elongation is directly proportional to the force and inversely proportional to the area A:

elongation and Young's modulus

$$\frac{\Delta L}{L} = \frac{1}{Y}\frac{F}{A} \qquad (14.18)$$

Here the quantity Y is the constant of proportionality. In Eq. (14.18) this constant written as $1/Y$, so it divides the right side, instead of multiplying it (this is analogous to writing Hooke's Law for a spring as $\Delta x = (1/k)\,F$, where Δx is the elongation

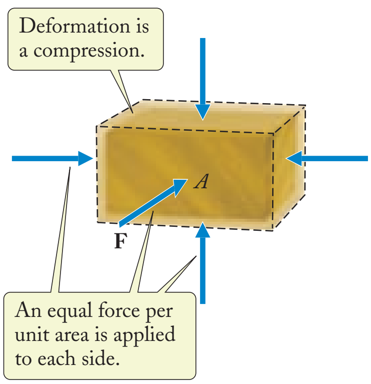


FIGURE 14.26 Pressure applied to all sides of a block of material causes compression.

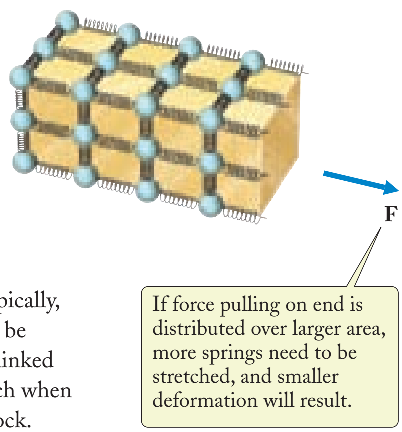


FIGURE 14.27 Microscopically, a block of solid material may be thought of as rows of atoms linked by springs. The springs stretch when a tension is applied to the block.

TABLE 14.1 ELASTIC MODULI OF SOME MATERIALS

MATERIAL	YOUNG'S MODULUS	SHEAR MODULUS	BULK MODULUS
Steel	22×10^{10} N/m^2	8.3×10^{10} N/m^2	16×10^{10} N/m^2
Cast iron	15	6.0	11
Brass	9.0	3.5	6.0
Aluminum	7.0	2.5	7.8
Bone (long)	3.2	1.2	3.1
Concrete	2	—	—
Lead	1.6	0.6	4.1
Nylon	0.36	0.12	0.59
Glycol	—	—	0.27
Water	—	—	0.22
Quartz	9.7(max)	3.1	3.6

produced by an applied force F). Thus, a stiff material, such as steel, that elongates by only a small amount has a large value of Y. The constant Y is called **Young's modulus.** Table 14.1 lists values of Young's moduli for a few solid materials. Note that if, instead of exerting a pull on the end of the block, we exert a push, then F in Eq. (14.18) must be reckoned as negative, and the change ΔL of length will then likewise be negative—the block becomes shorter.

In engineering language, *the fractional deformation is usually called the* **strain,** *and the force per unit area is called the* **stress.** In this terminology, Eq. (14.18) simply states that the strain is proportional to the stress.

This proportionality of strain and stress is also valid for shearing deformations and compressional deformations, provided we adopt a suitable definition of strain, or fractional deformation, for these cases. For shear, the fractional deformation is defined as the ratio of the sideways displacement Δx of the edge of the block to the height h of the block (see Fig. 14.25a). This fractional deformation is directly proportional to the force F and inversely proportional to the area A (note that the relevant area A is now the top area of the block, where the force is applied):

$$\frac{\Delta x}{h} = \frac{1}{S}\frac{F}{A} \tag{14.19}$$

shear and shear modulus

Here, the constant of proportionality S is called the **shear modulus.** Table 14.1 includes values of shear moduli of solids.

For compression, the fractional deformation is defined as the ratio of the change ΔV of the volume to the initial volume, and this fractional deformation is, again, proportional to the force F pressing on each face of the block and inversely proportional to the area A of that face:

$$\frac{\Delta V}{V} = -\frac{1}{B}\frac{F}{A} \tag{14.20}$$

compression and bulk modulus

In this equation, the minus sign indicates that ΔV is negative; that is, the volume decreases. The constant of proportionality B in the equation is called the **bulk modulus.**

Table 14.1 includes values of bulk moduli for solids. This table also includes values of bulk moduli for some liquids. The force per unit area, F/A, is also known as the **pressure:**

pressure

$$[\text{pressure}] = \frac{F}{A} \tag{14.21}$$

The formula (14.20) is equally valid for solids and for liquids—when we squeeze a liquid from all sides, it will suffer a compression. Note that Table 14.1 does not include values of Young's moduli and of shear moduli for liquids. *Elongation and shear stress are not supported by a liquid*—we can elongate or shear a "block" of liquid as much as we please without having to exert any significant force.

Concepts in Context

EXAMPLE 8 The lifting cable of a tower crane is made of steel, with a diameter of 5.0 cm. The length of this cable, from the ground to the horizontal arm, across the horizontal arm, and down to the load, is 160 m (Fig. 14.28). By how much does this cable stretch in excess of its initial length when carrying a load of 60 tons?

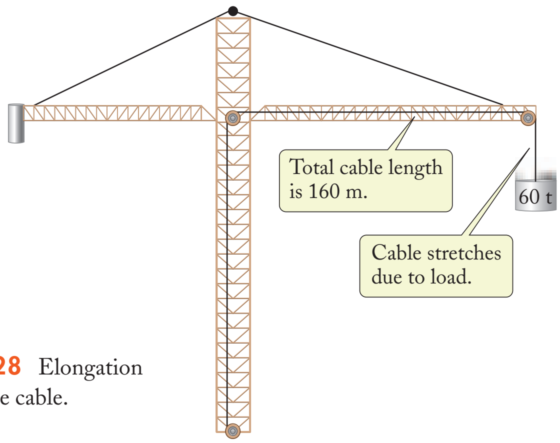


FIGURE 14.28 Elongation of a tower crane cable.

SOLUTION: The cross-sectional area of the cable is

$$A = \pi r^2 = \pi \times (0.025\text{ m})^2 = 2.0 \times 10^{-3}\text{ m}^2$$

and the force per unit area is

$$\frac{F}{A} = \frac{(60\,000\text{ kg} \times 9.81\text{ m/s}^2)}{2.0 \times 10^{-3}\text{ m}^2} = 2.9 \times 10^8\text{ N/m}^2$$

Since we are dealing with an elongation, the relevant elastic modulus is the Young's modulus. According to Table 14.1, the Young's modulus of steel is 22×10^{10} N/m^2. Hence Eq. (14.18) yields

$$\frac{\Delta L}{L} = \frac{1}{Y}\frac{F}{A} = \frac{1}{22 \times 10^{10}\text{ N/m}^2} \times 2.9 \times 10^8\text{ N/m}^2$$

$$= 1.3 \times 10^{-3}$$

The change of length is therefore

$$\Delta L = 1.3 \times 10^{-3} \times L = 1.3 \times 10^{-3} \times 160\text{ m}$$

$$= 0.21\text{ m}$$

EXAMPLE 9 What pressure must you exert on a sample of water if you want to compress its volume by 0.10%?

SOLUTION: For volume compression, the relevant elastic modulus is the bulk modulus B. By Eq. (14.20), the pressure, or the force per unit area, is

$$\frac{F}{A} = -B\frac{\Delta V}{V}$$

For 0.10% compression, we want to achieve a fractional change of volume of $\Delta V/V = -0.0010$. Since the bulk modulus of water is $0.22 \times 10^{10}\ \text{N/m}^2$, the required pressure is

$$\frac{F}{A} = 0.22 \times 10^{10}\ \text{N/m}^2 \times 0.0010 = 2.2 \times 10^6\ \text{N/m}^2$$

The simple uniform deformations of elongation, shear, and compression described above require a rather special arrangement of forces. In general, the forces applied to a solid body will produce nonuniform elongation, shear, and compression. For instance, a beam supported at its ends and sagging in the middle because of its own weight or the weight of a load placed on it will elongate along its lower edge, and compress along its upper edge.

Finally, note that the formulas (14.18)–(14.20) are valid only as long as the deformation is reasonably small—a fraction of a percent or so. If the deformation is excessive, the material will be deformed beyond its elastic limit; that is, the material will suffer a permanent deformation and will *not* return to its original size and shape when the force ceases. If the deformation is even larger, the material will break apart or crumble. For instance, steel will break apart (see Fig. 14.29) if the tensile stress exceeds $5 \times 10^8\ \text{N/m}^2$, or if the shearing stress exceeds $2.5 \times 10^8\ \text{N/m}^2$, and it will crumble if the compressive stress exceeds $5 \times 10^8\ \text{N/m}^2$.

FIGURE 14.29 These rods of steel broke apart when a large tension was applied.

Checkup 14.4

QUESTION 1: When a tension of 70 N is applied to a piano wire of length 1.8 m, it stretches by 2.0 mm. If the same tension is applied to a similar piano wire of length 3.6 m, by how much will it stretch?

QUESTION 2: Is it conceivable that a long cable hanging vertically might snap under its own weight? If so, does the critical length of the cable depend on its diameter?

QUESTION 3: The bulk modulus of copper is about twice that of aluminum. Suppose that a copper and an aluminum sphere have exactly equal volumes at normal atmospheric pressure. Suppose that when subjected to a high pressure, the volume of the aluminum sphere shrinks by 0.01%. By what percentage will the copper sphere shrink at the same pressure?

QUESTION 4: While lifting a load, the steel cable of a crane stretches by 1 cm. If you want the cable to stretch by only 0.5 cm, by what factor must you increase its diameter?

(A) $\sqrt{2}$ (B) 2 (C) $2\sqrt{2}$ (D) 4

SUMMARY

PROBLEM-SOLVING TECHNIQUES Static Equilibrium **(page 437)**

STATIC EQUILIBRIUM The sums of the external forces and of the external torques on a rigid body are zero.

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = 0 \quad (14.1)$$

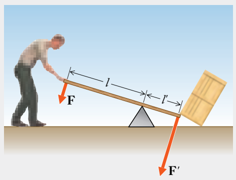
$$\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3 + \cdots = 0 \quad (14.2)$$

STATICS CALCULATION TECHNIQUE To eliminate an unknown force, evaluate torques about the point where that force acts (or about another point where the force has zero moment arm).

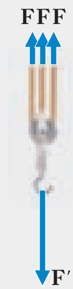
TORQUE DUE TO GRAVITY Gravity effectively acts at the center of mass.

MECHANICAL ADVANTAGE OF LEVER

$$\frac{F'}{F} = \frac{l}{l'} \quad (14.15)$$



BLOCK AND TACKLE An arrangement of several pulleys that provides a mechanical advantage (equal to the ratio of the distance moved where the force is applied to the distance moved by the load).



PRESSURE

$$[\text{pressure}] = \frac{F}{A} \quad (14.21)$$

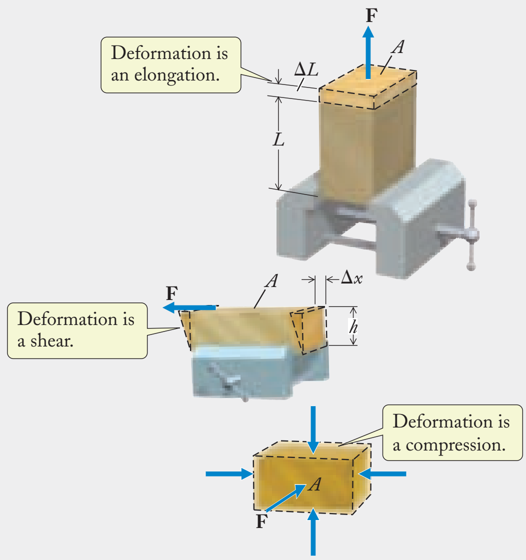
DEFORMATIONS OF ELASTIC MATERIAL

A is cross-sectional area.

Elongation:

Shear:

Compression:



$$\frac{\Delta L}{L} = \frac{1}{Y}\frac{F}{A} \qquad (Y = \text{Young's modulus}) \qquad \textbf{(14.18)}$$

$$\frac{\Delta x}{h} = \frac{1}{S}\frac{F}{A} \qquad (S = \text{shear modulus}) \qquad \textbf{(14.19)}$$

$$\frac{\Delta V}{V} = -\frac{1}{B}\frac{F}{A} \qquad (B = \text{bulk modulus}) \qquad \textbf{(14.20)}$$

QUESTIONS FOR DISCUSSION

1. If the legs of a table are exactly the same length and if the floor is exactly flat, then the weight of the table will be equally distributed over all four legs. But if there are small deviations from exactness, then the weight will not be equally distributed. Is it possible for all of the weight to rest on three legs? On two?
2. List as many examples as you can of joints in the human skeleton that act as pivots for levers. Do any of these levers in the human skeleton have a mechanical advantage larger than 1?
3. Design a block and tackle with a mechanical advantage of 4, and another with a mechanical advantage of 5. If you connect these two arrangements in tandem, what mechanical advantage do you get?
4. Figure 14.30 shows a differential windlass consisting of two rigidly joined drums around which a rope is wound. A pulley holding a load hangs from this rope. Explain why this device gives a very large mechanical advantage if the radii of the two drums are nearly equal.
5. The collapse of several skywalks at the Hyatt Regency hotel in Kansas City on July 17, 1982, with the loss of 114 lives, was due to a defective design of the suspension system. Instead of suspending the beams of the skywalks directly from single, long steel rods anchored at the top of the building, some incompetent engineers decided to use several short steel rods joining the beams of each skywalk to those of the skywalk above (Fig. 14.31). Criticize this design, keeping in mind that the beams are made of a much weaker material than the rods.

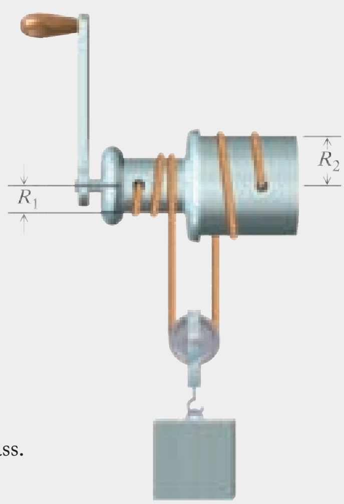


FIGURE 14.30 Differential windlass.

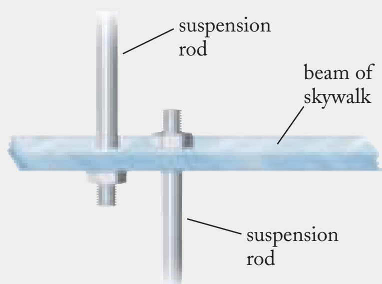


FIGURE 14.31 Beam of skywalk and suspension rods.

6. A steel rod is much less flexible than a woven steel rope of the same strength. Explain this.

7. A carpenter wants to support the (flat) roof of a building with horizontal beams of wood of rectangular cross section. To achieve maximum strength of the roof (least sag), should he install the beams with their narrow side up or with their wide side up?

8. The long bones in the limbs of vertebrates have the shape of hollow pipes. If the same amount of bone tissue had been assembled in a solid rod (of correspondingly smaller cross section), would the limb have been more rigid or less rigid?

PROBLEMS

4.2 Examples of Static Equilibrium

1. At a construction site, a laborer pushes horizontally against a large bucket full of concrete of total mass 600 kg suspended from a crane by a 20-m cable (see Fig. 14.32). What is the force the laborer has to exert to hold the bucket at a distance of 2.0 m from the vertical?

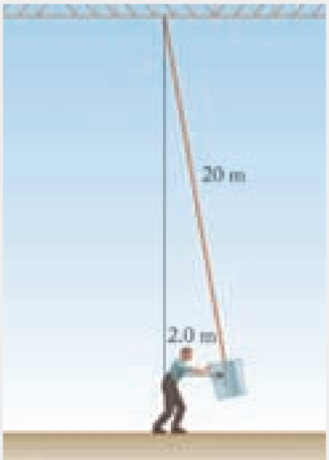


FIGURE 14.32 Bucket hanging from a cable.

2. You are holding a meterstick of 0.20 kg horizontally in one hand. Assume that your hand is wrapped around the last 10 cm of the stick (see Fig. 14.33), so the front edge of your hand exerts an upward force and the rear edge of your hand exerts a downward force. Calculate these forces.

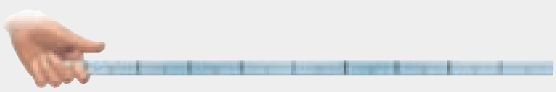


FIGURE 14.33 A meterstick held in a hand.

3. Consider the bridge with the locomotive described in Example 1 and suppose that, besides the first locomotive at 30 m from the right end, there is a second locomotive, also of 90 000 kg, at 80 m from the right end. What is the load on each pier in this case?

4. Repeat the calculations of Example 1 assuming that the bridge has a slope of 1:7, with the left end higher than the right.

5. In order to pull an automobile out of the mud in which it is stuck, the driver stretches a rope taut from the front end of the automobile to a stout tree. He then pushes sideways against the rope at the midpoint (see Fig. 14.34). When he pushes with a force of 900 N, the angle between the two halves of the rope on his right and left is 170°. What is the tension in the rope under these conditions?

6. A mountaineer is trying to cross a crevasse by means of a rope stretched from one side to the other (see Fig. 14.35). The mass of the mountaineer is 90 kg. If the two parts of the rope make angles of 40° and 20° with the horizontal, what are the tensions in the two parts?

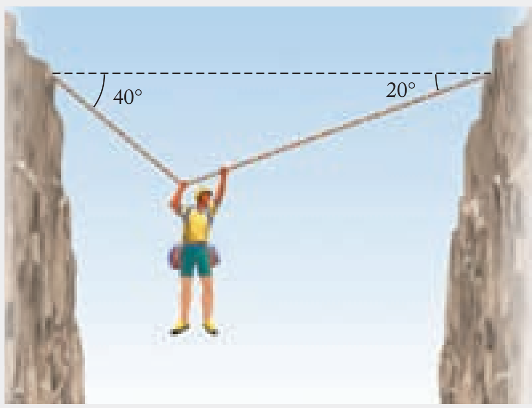
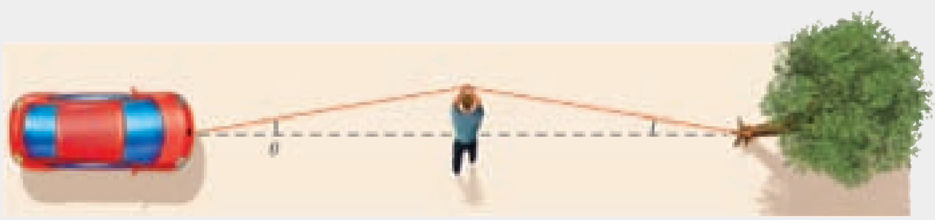


FIGURE 14.35 Mountaineer suspended from a rope.

FIGURE 14.34 The rope is stretched between the automobile and a tree. The driver is pushing at the midpoint.



7. The plant of the foot of an average male is 26 cm, and the height of his center of mass above the floor is 1.03 m. When he is standing upright, the center of mass is vertically aligned with the ankle, 18 cm from the tip of the foot (see Fig. 14.36). Without losing his equilibrium, how far can the man lean forward or backward while keeping his body straight and his feet stiff and immobile?

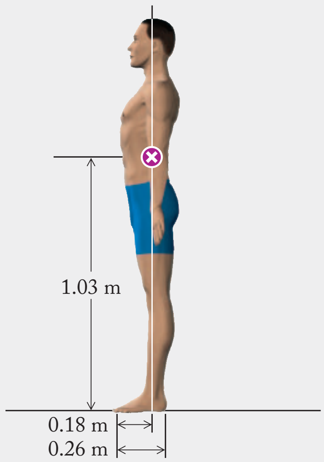


FIGURE 14.36 Man standing on stiff feet.

8. A 50-kg log of uniform thickness lies horizontally on the ground.
   (a) What vertical force must you exert on one end of the log to barely lift this end off the ground?
   (b) If you continue to exert a purely vertical force on the end of the log, what is the magnitude of the force required to hold the log at an angle of 30° to the ground? At an angle of 60°? At an angle of 85°?
   (c) If instead you exert a force at right angles to the length of the log, what is the magnitude of the force required to hold the log at an angle of 30° to the ground? At an angle of 60°? At an angle of 85°?
9. In an unequal-arm balance, the beam is pivoted at a point near one end. With such a balance, large loads can be balanced with small standard weights. Figure 14.37 shows such a balance with an arm of 50 cm swinging on a pivot 1.0 cm from one end. When a package of sugar is deposited in the balance pan, equilibrium is attained with a standard mass of 0.12 kg in the other pan. What is the mass of the sugar? Neglect the masses of the pans.

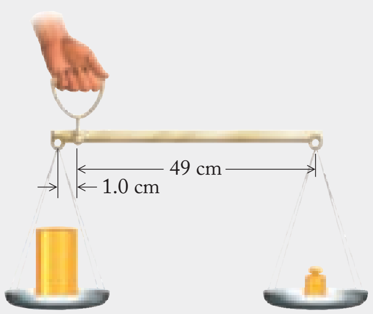


FIGURE 14.37 Unequal-arm balance.

10. One end of a uniform beam of mass 50 kg and length 3.0 m rests on the ground; the other end is held above the ground by a pivot placed 1.0 m from that end (see Fig. 14.38). An 80-kg man walks along the beam, from the low end toward the high end. How far beyond the pivot can the man walk before the high end of the beam swings down?

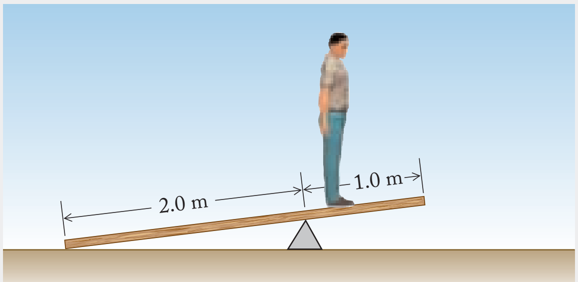


FIGURE 14.38 Man standing on a beam.

11. The mast of a sailboat is held by two steel cables attached as shown in Fig. 14.39. The front cable has a tension of 5.0×10^3 N. The mast is 10 m high. What is the tension in the rear cable? What force does the foot of the mast exert on the sailboat? Assume that the weight of the mast can be neglected and that the foot of the mast is hinged (and therefore exerts no torque).

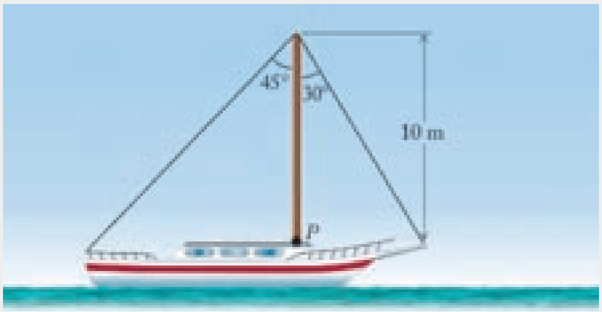


FIGURE 14.39 Steel cables staying a mast.

12. The center of mass of a 45-kg sofa is 0.30 m above its bottom, at its lateral midpoint. You lift one end of the 2.0-m-long sofa to a height of 1.0 m by applying a vertical force at the bottom of one end; the other end stays on the floor without slipping. What force do you apply? Compare this with the force you apply when a friend lifts the other end, also to 1.0 m, so that the weight is shared equally. Based on this, who has the easier task when a short and a tall person share a bulky load?
13. Suppose that you lift the lid of a chest. The lid is a uniform sheet of mass 12 kg, hinged at the rear. What is the smallest force you can apply at the front of the lid to hold it at an angle of 30° with the horizontal? At 60°?
14. A pole-vaulter holds a 4.5-m pole horizontally with her right hand at one end and her left hand 1.5 m from the same end.

The left hand applies an upward force and the right hand a downward force. If the mass of the pole is 3.0 kg, find those two forces.

15. A 50-kg diving board is 3.0 m long; it is a uniform beam, bolted down at one end and supported from below a distance 1.0 m from the same end. A 60-kg diver stands at the other end. Calculate the downward force at the bolted end and the upward support force.

16. A window washer's scaffolding is 12 m long; it is suspended by a cable at each end. Assume that the scaffolding is a horizontal uniform rod of mass 110 kg. The window washer (with gear) has a mass of 90 kg and stands 2.0 m from one end of the scaffolding. Find the tension in each cable.

17. A pencil is placed on an incline, and the angle of the incline is slowly increased. At what angle will the pencil start to roll? Assume the pencil has an exactly hexagonal cross section and does not slip.

18. A 10-kg ladder is 5.0 m long and rests against a frictionless wall, making an angle of 30° with the vertical. The coefficient of friction between the ladder and the ground is 0.35. A 60-kg painter begins to climb the ladder, standing vertically on each rung. How far up the ladder has the painter climbed when the ladder begins to slip?

19. Figure 14.40 shows the arrangement of wheels on a passenger engine of the Caledonian Railway. The numbers give the distances between the wheels in feet and the downward forces that each wheel exerts on the track in short tons (1 short ton = 2000 lbf; the numbers for the forces include both the right and left wheels). From the information given, find how far the center of mass of the engine is behind the front wheel.

20. A door made of a uniform piece of wood measures 1.0 m by 2.0 m and has a mass of 18 kg. The door is entirely supported by two hinges, one at the bottom corner and one at the top corner. Find the force (magnitude and direction) that the door exerts on each hinge. Assume that the *vertical* force on each hinge is the same.

21. You want to pick up a nearly massless rectangular cardboard box by grabbing its top and side between your forefinger and thumb (see Fig. 14.41). Show that this is impossible unless the coefficient of friction between your fingers and the box is at least 1.

FIGURE 14.41 Box held in a hand.

22. A meterstick of wood of 0.40 kg is nailed to the wall at the 75-cm mark. If the stick is free to rotate about the nail, what horizontal force must you exert at the upper (short) end to deflect the stick 30° to one side?

*23. A wheel of mass M and radius R is to be pulled over a step of height h, where $R > h$. Assume that the pulling force is applied at the axis of the wheel. If the pull is horizontal, what force must be applied to barely begin moving? If the pull at the axis is instead in the direction that requires the least force to begin moving, what force must be applied? What is the new direction? (Hint: Consider the torques about the point of contact with the step.)

*24. Consider a heavy cable of diameter d and density ρ from which hangs a load of mass M. What is the tension in the cable as a function of the distance from the lower end?

*25. Figure 14.42 shows two methods for supporting the mast of a sailboat against the lateral force exerted by the pull of the sail. In Fig. 14.42a, the shrouds (wire ropes) are led directly to the top of the mast; in Fig. 14.42b, the shrouds are led around a rigid pair of spreaders. Suppose that the dimensions of the mast

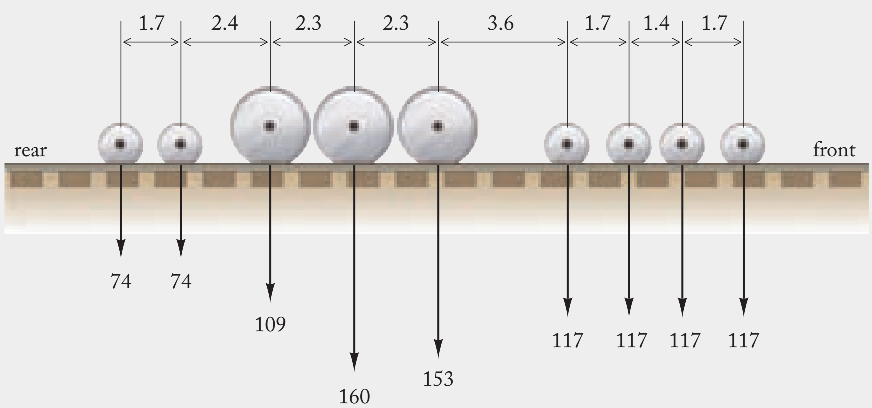


FIGURE 14.40 Wheels of a locomotive.

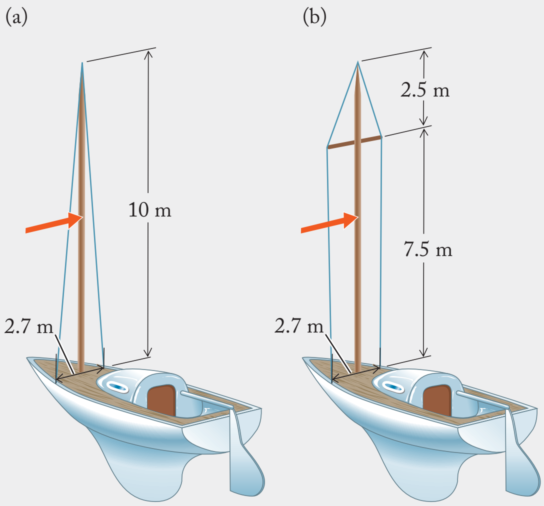


FIGURE 14.42 Two methods for supporting the mast of a boat.

and the boat are as indicated in this figure, and that the pull of the sail is equivalent to a horizontal force of 2400 N acting from the left at half the height of the mast. The foot of the mast permits the mast to tilt, so the only lateral support of the mast is that provided by the shrouds. What is the excess tension in the left shroud supporting the mast in case (a)? In case (b)? Which arrangement is preferable?

*26. A bowling ball of mass 10 kg rests in a groove with smooth, perpendicular walls, inclined at angles of 30° and 60° with the vertical, as shown in Fig. 14.43. Calculate the magnitudes of the normal forces at the points of contact.

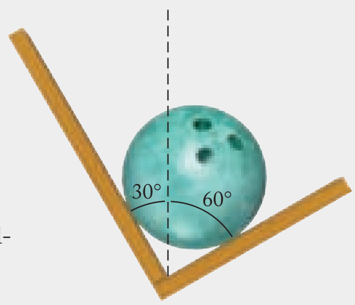


FIGURE 14.43 A bowling ball in a groove.

*27. A tetrahedral tripod consists of three massless legs (see Fig. 14.44). A mass M hangs from the apex of the (regular) tetrahedron. What are the compressional forces in the three legs?

FIGURE 14.44 A tripod.

*28. A sailor is being transferred from one ship to another by means of a bosun's chair (see Fig. 14.45). The chair hangs from a roller riding on a rope strung between the two ships. The distance between the ships is d, and the rope has a length $1.2d$. The mass of the sailor plus the chair is m. If the sailor is at a (horizontal) distance of $0.25d$ from one ship, find the force that must be exerted on the pull rope to keep the sailor in equilibrium. Also find the tension in the long rope. Ignore the masses of the ropes.

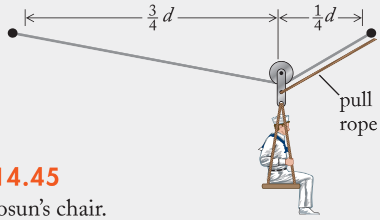


FIGURE 14.45 Sailor in bosun's chair.

*29. A uniform solid disk of mass M and radius R hangs from a string of length l attached to a smooth vertical wall (see Fig. 14.46). Calculate the tension in the string and the normal force acting at the point of contact of disk and wall.

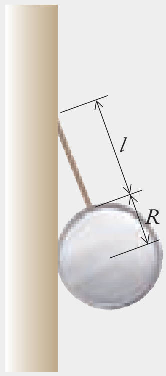


FIGURE 14.46 Disk hanging from string.

*30. Three traffic lamps of equal masses of 20 kg hang from a wire stretched between two telephone poles, 15 m apart (Fig. 14.47). The horizontal spacing of the traffic lamps is uniform. At each pole, the wire makes a downward angle of 10° with the horizontal line. Find the tensions in all the segments of wire, and find the distance of each lamp below the horizontal line.

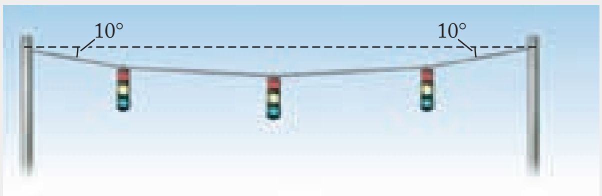


FIGURE 14.47 Three traffic lamps.

*31. Consider the ladder leaning against a wall described in Example 4. If the ladder makes an angle of 30° with the wall, how hard can you push down vertically on the top of the ladder with your hand before slipping begins?

*32. An automobile with a wheelbase (distance from the front wheels to the rear wheels) of 3.0 m has its center of mass at a point midway between the wheels at a height of 0.65 m above the road. When the automobile is on a level road, the force with which each wheel presses on the road is 3100 N. What is the normal force with which each wheel presses on the road when the automobile is standing on a steep road of slope 3:10 with all the wheels locked?

*33. A wooden box is filled with material of uniform density. The box (with its contents) has a mass of 80 kg; it is 0.60 m wide, 0.60 m deep, and 1.2 m high. The box stands on a level floor. By pushing against the box, you can tilt it over (Fig. 14.48). Assume that when you do this, one edge of the box remains in contact with the floor without sliding.

(a) Plot the gravitational potential energy of the box as a function of the angle θ between the bottom of the box and the floor.

(b) What is the critical angle beyond which the box will topple over when released?

(c) How much work must you do to push the box to this critical angle?

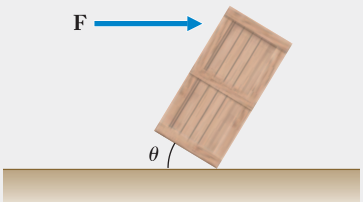


FIGURE 14.48 Tilted box.

*34. A meterstick of mass M hangs from a 1.5-m string tied to the meterstick at the 80-cm mark. If you push the bottom end of the meterstick to one side with a horizontal push of magnitude $Mg/2$, what will be the equilibrium angles of the meterstick and the string?

*35. Five identical books are to be stacked one on top of the other. Each book is to be shifted sideways by some variable amount, so as to form a curved leaning tower with maximum protrusion (see Fig. 14.49). How much must each book be shifted? What is the maximum protrusion? If you had an infinite number of books, what would be the limiting maximum protrusion? (Hint: Try this experimentally; start with the top book, and insert the others underneath, one by one.)

FIGURE 14.49 A stack of books.

**36. A wooden box, filled with a material of uniform density, stands on a concrete floor. The box has a mass of 75 kg and is 0.50 m wide, 0.50 m long, and 1.5 m high. The coefficient of friction between the box and the floor is $\mu_s = 0.80$. If you exert a (sufficiently strong) horizontal push against the side of the box, it will either topple over or start sliding without toppling over, depending on how high above the level of the floor you push. What is the maximum height at which you can push if you want the box to slide? What is the magnitude of the force you must exert to start the sliding?

*37. The left and right wheels of an automobile are separated by a transverse distance of $l = 1.5$ m. The center of mass of this automobile is $h = 0.60$ m above the ground. If the automobile is driven around a flat (no banking) curve of radius $R = 25$ m with an excessive speed, it will topple over sideways. What is the speed at which it will begin to topple? Express your answer in terms of l, h, and R; then evaluate numerically. Assume that the wheels do not skid.

*38. An automobile has a wheelbase (distance from front wheels to rear wheels) of 3.0 m. The center of mass of this automobile is at a height of 0.60 m above the ground. Suppose that this automobile has rear-wheel drive and that it is accelerating along a level road at 6.0 m/s^2. When the automobile is parked, 50% of its weight rests on the front wheels and 50% on the rear wheels. What is the weight distribution when it is accelerating? Pretend that the body of the automobile remains parallel to the road at all times.

*39. Consider a bicycle with only a front-wheel brake. During braking, what is the maximum deceleration that this bicycle can withstand without flipping over its front wheel? The center of mass of the bicycle with rider is 95 cm above the road and 70 cm behind the point of contact of the front wheel with the ground.

*40. A bicycle and its rider are traveling around a curve of radius 6.0 m at a constant speed of 20 km/h. What is the angle at which the rider must lean the bicycle toward the center of the curve (see Fig. 14.50)?

FIGURE 14.50 Bicycle traveling around curve.

**41. An automobile is braking on a flat, dry road with a coefficient of static friction of 0.90 between its wheels and the road. The wheelbase (the distance between the front and the rear wheels) is 3.0 m, and the center of mass is midway between the wheels, at a height of 0.60 m above the road.

(a) What is the deceleration if all four wheels are braked with the maximum force that avoids skidding?

(b) What is the deceleration if the rear-wheel brakes are disabled? Take into account that during braking, the normal force on the front wheels is larger than that on the rear wheels.

(c) What is the deceleration if the front-wheel brakes are disabled?

*42. A square framework of steel hangs from a crane by means of cables attached to the upper corners making an angle of 60° with each other (see Fig. 14.51). The framework is made of beams of uniform thickness joined (loosely) by pins at the corners, and its total mass is M. Find the tensions in the cables and the tensional and compressional forces in each beam at each of its two ends.

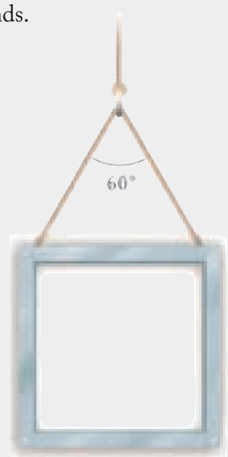


FIGURE 14.51 Hanging framework of beams.

**43. Two smooth balls of steel of mass m and radius R are sitting inside a tube of radius $1.5R$. The balls are in contact with the bottom of the tube and with the wall (at two points; see Fig. 14.52). Find the contact force at the bottom and at the two points on the wall.

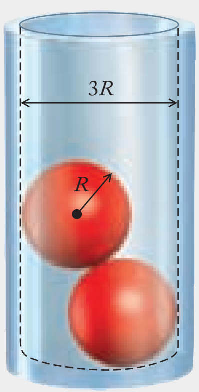


FIGURE 14.52 Two balls in a tube.

**44. One end of a uniform beam of length L rests against a smooth, frictionless vertical wall, and the other end is held by a string of length $l = \frac{3}{2}L$ attached to the wall (see Fig. 14.53). What must be the angle of the beam with the wall if it is to remain at rest without slipping?

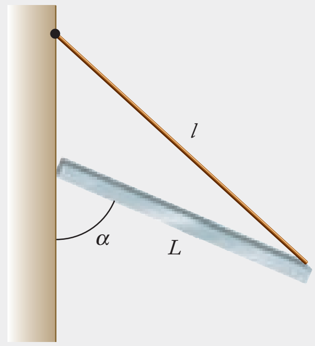


FIGURE 14.53 Beam, string, and wall.

**45. Two playing cards stand on a table leaning against each other so as to form an A-frame "roof." The frictional coefficient between the bottoms of the cards and the table is μ_s. What is the maximum angle that the cards can make with the vertical without slipping?

**46. A rope is draped over the round branch of a tree, and unequal masses m_1 and m_2 are attached to its ends. The coefficient of sliding friction for the rope on the branch is μ_k. What is the acceleration of the masses? Assume that the rope is massless. (Hint: For each small segment of the rope in contact with the branch, the small change in tension across the segment is equal to the friction force.)

**47. The flywheel of a motor is connected to the flywheel of an electric generator by a drive belt (Fig. 14.54). The flywheels are of equal size, each of radius R. While the flywheels are rotating, the tensions in the upper and the lower portions of the drive belt are T_1 and T_2, respectively, so the drive belt exerts a torque $\tau = (T_2 - T_1)R$ on the generator. The coefficient of static friction between each flywheel and the drive belt is μ_s. Assume that the tension in the drive belt is as low as possible with no slipping, and that the drive belt is massless. Show that under these conditions

$$T_1 = \frac{\tau}{R}\frac{1}{e^{\mu_s \pi} - 1}$$

$$T_2 = \frac{\tau}{R}\frac{1}{1 - e^{-\mu_s \pi}}$$

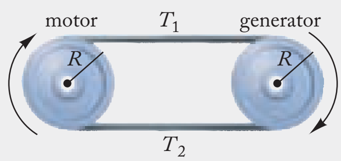


FIGURE 14.54 A drive belt connecting flywheels of a motor and a generator.

**48. A power brake invented by Lord Kelvin consists of a strong flexible belt wrapped once around a spinning flywheel (Fig. 14.55). One end of the belt is fixed to an overhead support; the other end carries a weight w. The coefficient of kinetic friction between the belt and the wheel is μ_k. The radius of the wheel is R, and its angular velocity is ω.

(a) Show that the tension in the belt is

$$T = we^{-\mu_k \theta}$$

as a function of the angle of contact (Fig. 14.55).

(b) Show that the net frictional torque the belt exerts on the flywheel is

$$\tau = wR\,(1 - e^{-2\pi\mu_k})$$

(c) Show that the power dissipated by friction is

$$P = wR\omega\,(1 - e^{-2\pi\mu_k})$$

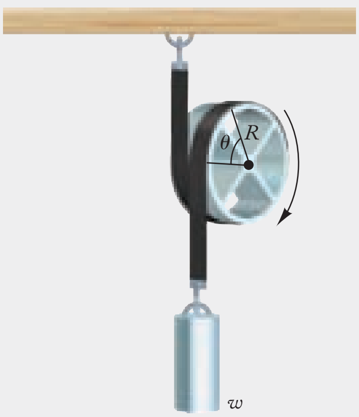


FIGURE 14.55 Belt and flywheel.

14.3 Levers and Pulleys

49. The human forearm (including the hand) can be regarded as a lever pivoted at the joint of the elbow and pulled upward by the tendon of the biceps (Fig. 14.56a). The dimensions of this lever are given in Fig. 14.56b. Suppose that a load of 25 kg rests in the hand. What upward force must the biceps exert to keep the forearm horizontal? What is the downward force at the elbow joint? Neglect the weight of the forearm.

50. Repeat the preceding problem if, instead of being vertical, the upper arm is tilted, so as to make an angle of 135° with the (horizontal) forearm.

51. A simple manual winch consists of a drum of radius 4.0 cm to which is attached a handle of radius 25 cm (Fig. 14.57). When you turn the handle, the rope winds up on the drum and pulls the load. Suppose that the load carried by the rope is 2500 N. What force must you exert on the handle to hold this load?

52. The handle of a crowbar is 60 cm long; the short end is 4.0 cm from a bend, which acts as the fulcrum. If a 75-kg man leans on the handle with all his weight, how much mass can he lift at the short end?

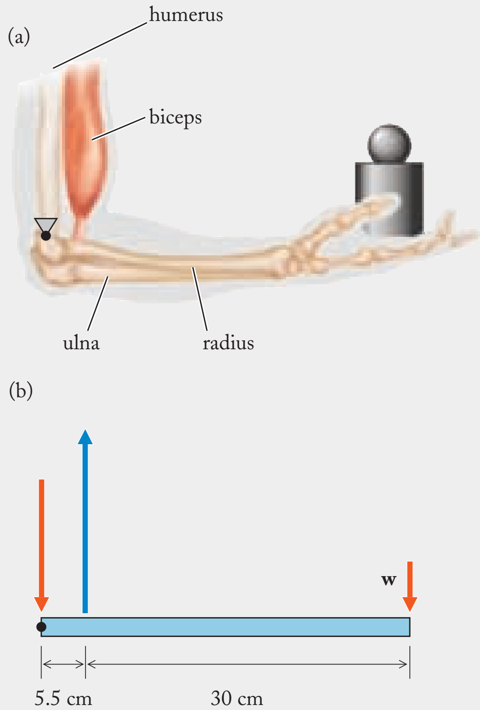


FIGURE 14.56 Forearm as lever.

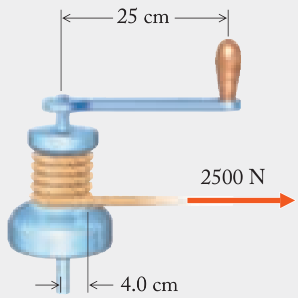


FIGURE 14.57 Manual winch.

53. A 60-kg woman sits 80 cm from the fulcrum of a 4.0-m-long seesaw. The woman's daughter pulls down on the other end of the seesaw. What minimum force must the child apply to hold her mother's end of the seesaw off the ground?

54. The fingers apply a force of 30 N at the handle of a pair of scissors, 4.0 cm from the hinge point. What force is available for cutting when the object to be cut is placed at the far end of the scissors, 12 cm from the hinge point? When the object is placed as close to the hinge point as possible, at a distance of 1.0 cm?

55. A laboratory microbalance has two weighing pans, one hanging 10 times farther away from the fulcrum than the other.

When an unknown mass is placed on the inner pan, the microbalance can measure changes in mass as small as 100 nanograms (1.0×10^{-7} g) and can measure masses up to 2.0 milligrams. What would you expect that the resolution and maximum load for the outer pan might be?

56. A man of 73 kg stands on one foot, resting all of his weight on the ball of the foot. As described in Section 14.3, the bones of the foot play the role of a lever. The short end of the lever (to the heel) measures 5.0 cm and the long end (to the ball of the foot) 14 cm. Calculate the force exerted by the Achilles tendon and the force at the ankle.

57. A rope hoist consists of four pulleys assembled in two pairs with rigid straps, with a rope wrapped around as shown in Fig. 14.58. A load of 300 kg hangs from the lower pair of pulleys. What tension must you apply to the rope to hold the load steady? Treat the pulleys and the rope as massless, and ignore any friction in the pulleys.

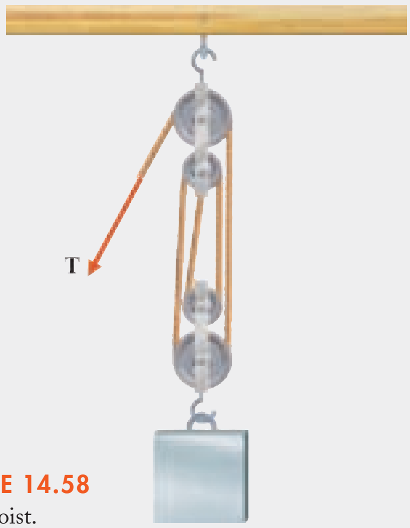


FIGURE 14.58 Rope hoist.

58. A parbuckle is a simple device used by laborers for raising or lowering a barrel or some other cylindrical object along a ramp. It consists of a loop of rope wrapped around the barrel (see Fig. 14.59). One end of the rope is tied to the top of the ramp, and the laborer pulls on the other end. Suppose that the laborer exerts a pull of 500 N on the rope, parallel to the ramp. What is the force that the rope exerts on the barrel? What is the mechanical advantage of the parbuckle?

FIGURE 14.59 Parbuckle used to move a barrel up a ramp.

59. Consider the differential windlass illustrated in Fig. 14.30. Calculate what clockwise torque must be applied to the handle to lift a load of mass m. What tangential force must be exerted on the handle? What is the mechanical advantage of this windlass?

60. Design a block and tackle with a mechanical advantage of 4, and another with a mechanical advantage of 5. If you connect these two arrangements in tandem, what mechanical advantage do you get?

*61. Figure 14.60 shows a compound bolt cutter. If the dimensions are as indicated in this figure, what is the mechanical advantage?

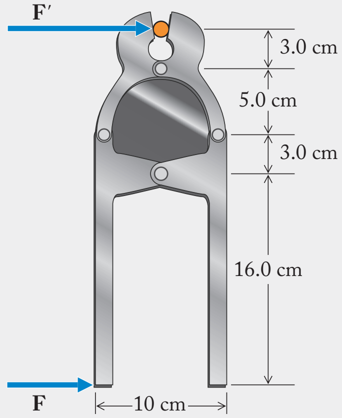


FIGURE 14.60 Compound bolt cutter.

*62. The drum of a winch is rigidly attached to a concentric large gear, which is driven by a small gear attached to a crank. The dimensions of the drum, the gears, and the crank are given in Fig. 14.61. What is the mechanical advantage of this geared winch?

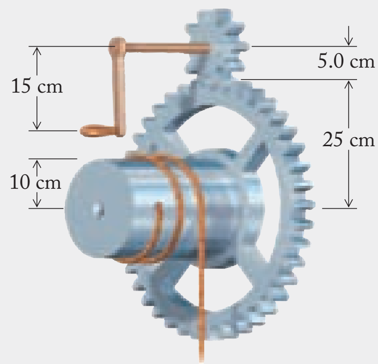


FIGURE 14.61 Geared winch.

*63. The screw of a vise has a *pitch* of 4.0 mm; that is, it advances 4.0 mm when given one full turn. The handle of the vise is 25 cm long, measured from the screw to the end of the handle. What is the mechanical advantage when you push perpendicularly on the end of the handle?

*64. A scissors jack has the dimensions shown in Fig. 14.62. The screw of the jack has a pitch of 5.0 mm (as stated in the previous problem, this is the distance the screw advances when given one full turn). Suppose the scissors jack is partially extended, with an angle of 55° between its upper sides. What is the mechanical advantage provided by the jack?

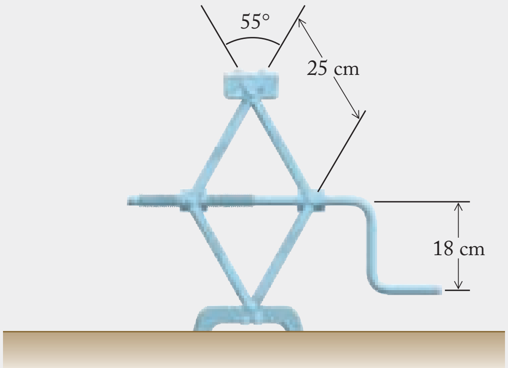


FIGURE 14.62 Scissors jack.

**65. Figure 14.63 shows a tensioning device used to tighten the rear stay of the mast of a sailboat. The block and tackle pulls down a rigid bar with two rollers that squeeze together the two branches of the split rear stay. If the angles are as given in the figure, what is the mechanical advantage?

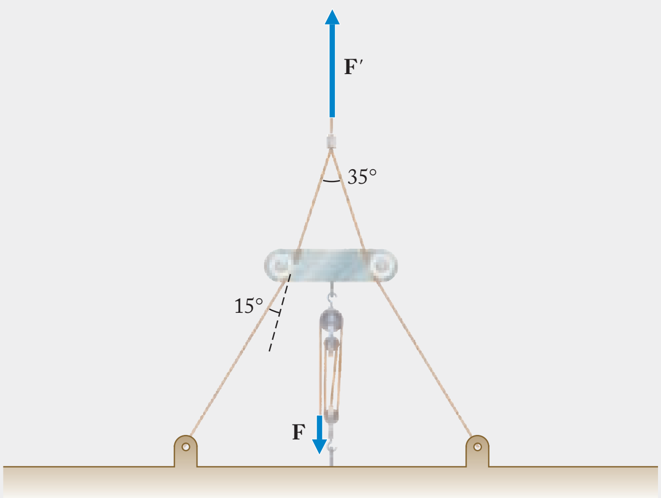


FIGURE 14.63 Tensioning device.

14.4 Elasticity of Materials

66. The anchor rope of a sailboat is a nylon rope of length 60 m and diameter 1.3 cm. While anchored during a storm, the sailboat momentarily pulls on this rope with a force of 1.8×10^4 N. How much does the rope stretch?

67. A piano wire of steel of length 1.8 m and radius 0.30 mm is subjected to a tension of 70 N by a weight attached to its lower end. By how much does this wire stretch in excess of its initial length?

68. An elastic cord is 5.0 m long and 1.0 cm in diameter and acts as a spring with spring constant 70 N/m. What is the Young's modulus for this material?

69. The piano wire described in Problem 67 can be regarded as a spring. What is the effective spring constant of this spring?

70. A simple hand-operated hydraulic press can generate a pressure of 6.0×10^9 N/m^2. If the system is used to compress a small volume of steel, what fraction of the original volume does the final volume of steel occupy?

71. A 10-m length of 1.0-mm-radius copper wire is stretched by holding one end fixed and pulling on the other end with a force of 150 N. What is the change of length? By briefly increasing the force to exceed the limit of elastic behavior (a fractional elongation of approximately 1.0%), the wire may be permanently deformed; this is often done in order to straighten out bends or kinks in a wire. Approximately what force is necessary?

72. A 0.50-mm-radius fishing line made of nylon is 100 m long when no forces are applied. A fish is hooked and pulls with a tension force of 250 N. What is the elongation?

73. In a skyscraper, an elevator is suspended from three equal, parallel 300-m-long steel cables, each of diameter 1.0 cm. How much do these cables stretch if the mass of the elevator is 1000 kg?

74. The length of the femur (thighbone) of a woman is 38 cm, and the average cross section is 10 cm^2. How much will the femur be compressed in length if the woman lifts another woman of 68 kg and carries her piggyback? Assume that, momentarily, all of the weight rests on one leg.

75. If the volume of a sphere subjected to an external pressure shrinks by 0.10%, what is the percent shrinkage of the radius? In general, show that the percent shrinkage of the volume equals three times the percent shrinkage of the radius, provided the shrinkage is small.

76. At the bottom of the Marianas Trench in the Pacific Ocean, at a depth of 10 900 m, the pressure is 1.24×10^8 N/m^2. What is the percent increase of the density of water at this depth as compared with the density at the surface?

77. A slab of stone of mass 1200 kg is attached to the wall of a building by two bolts of iron of diameter 1.5 cm (see Fig. 14.64). The distance between the wall and the slab of stone is 1.0 cm. Calculate by how much the bolts will sag downward because of the shear stress they are subjected to.

78. According to (somewhat oversimplified) theoretical considerations, the Young's modulus, the shear modulus, and the bulk modulus are related by

$$Y = \frac{9BS}{3B + S}$$

Check this for the first four materials listed in Table 14.1.

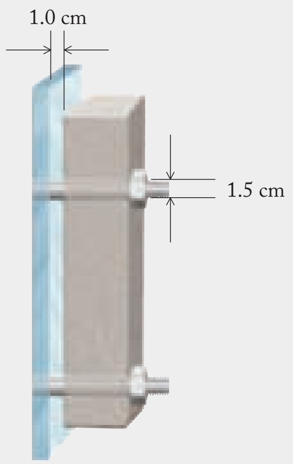


FIGURE 14.64 A slab of stone held by bolts.

79. A nylon rope of diameter 1.3 cm is to be spliced to a steel rope. If the steel rope is to have the same ultimate breaking strength as the nylon, what diameter should it have? The ultimate tensile strength is 2.0×10^9 N/m^2 for the steel and 3.2×10^8 N/m^2 for nylon.

*80. A rod of aluminum has a diameter of 1.000 002 cm. A ring of cast steel has an inner diameter of 1.000 000 cm. If the rod and the ring are placed in a liquid under high pressure, at what value of the pressure will the aluminum rod fit inside the steel ring?

*81. A heavy uniform beam of mass 8000 kg and length 2.0 m is suspended at one end by a nylon rope of diameter 2.5 cm and at the other end by a steel rope of diameter 0.64 cm. The ropes are tied together above the beam (see Fig. 14.65). The unstretched lengths of the ropes are 3.0 m each. What angle will the beam make with the horizontal?

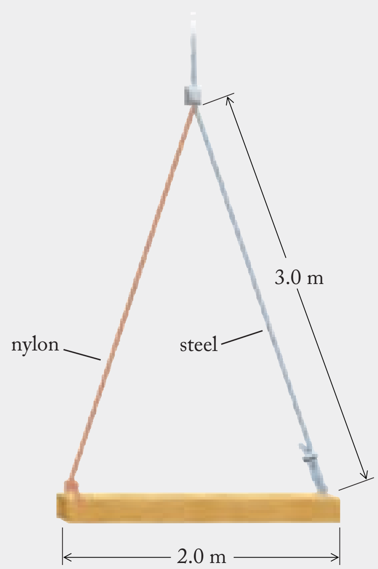


FIGURE 14.65 Beam hanging from two types of rope.

*82. A rod of cast iron is soldered to the upper edges of a plate of copper whose lower edge is held in a vise (see Fig. 14.66). The rod has a diameter of 4.0 cm and a length of 2.0 m. The copper plate measures 6.0 cm $\times$ 6.0 cm $\times$ 1.0 cm. If we pull the free end of the iron rod forward by 3.0 mm, what is the shear strain ($\Delta x / h$) of the copper plate?

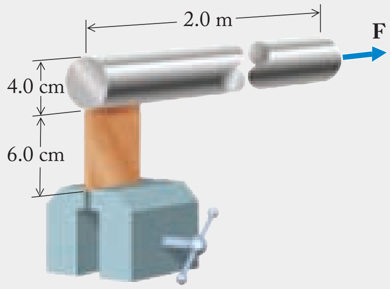


FIGURE 14.66 Iron rod and copper plate.

*83. When a bar of steel is heated, it expands in length by 0.0012% for each degree Celsius of temperature increase. If the length of the heated bar is to be reduced to its original value, a compressive stress must be applied to it. The compressive stress required to cancel the thermal expansion is called **thermal stress.** What is its value for a cylindrical bar of cast steel of cross section 4.0 cm^2 heated by 150°C?

*84. A power cable of copper is stretched straight between two fixed towers. If the temperature decreases, the cable tends to contract (compare Problem 83). The amount of contraction for a free copper cable or rod is 0.0017% per degree Celsius. Estimate what temperature decrease will cause the cable to snap. Pretend that the cable obeys Eq. (14.18) until it reaches its breaking point, which for copper occurs at a tensile stress of 2.4×10^8 N/m^2. Ignore the weight of the cable and the sag and stress produced by the weight.

**85. A meterstick of steel, of density 7.8×10^3 kg/m^3, is made to rotate about a perpendicular axis passing through its middle. What is the maximum angular velocity with which the stick can rotate if its center is to hold? Mild steel will break when the tensile stress exceeds 3.8×10^8 N/m^2.

**86. The wall of a pipe of diameter 60 cm is constructed of a sheet of steel of thickness 0.30 cm. The pipe is filled with water under high pressure. What is the maximum pressure, that is, force per unit area, that the pipe can withstand? See Problem 85 for data on mild steel.

**87. A hoop of aluminum of radius 40 cm is made to spin about its axis of symmetry at high speed. The density of aluminum is 2.7×10^3 kg/m^3, and the ultimate tensile breaking strength is 7.8×10^7 N/m^2. At what angular velocity will the hoop begin to break apart?

**88. A pipe of steel with a wall 0.40 cm thick and a diameter of 50 cm contains a liquid at a pressure of 2.0×10^4 N/m^2. How much will the diameter of the pipe expand due to this pressure?

REVIEW PROBLEMS

89. A traffic lamp of mass 25 kg hangs from a wire stretched between two posts. The traffic lamp hangs at the middle of the wire, and the two halves of the wire sag downward at an angle of 20° (see Fig. 14.67). What is the tension in the wire? Assume the wire is massless.

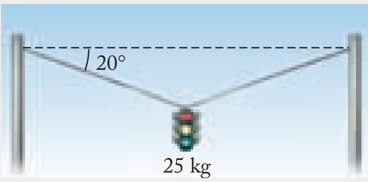


FIGURE 14.67 A traffic lamp.

90. A heavy shop sign hangs from a boom sticking out horizontally from a building (see Fig. 14.68). The boom is hinged at the building and is supported by a diagonal wire, making an angle of 45° with the boom. The mass of the sign is 50 kg, and the boom and the wire are massless. What is the tension in the wire? What is the force with which the end of the boom pushes against the building?

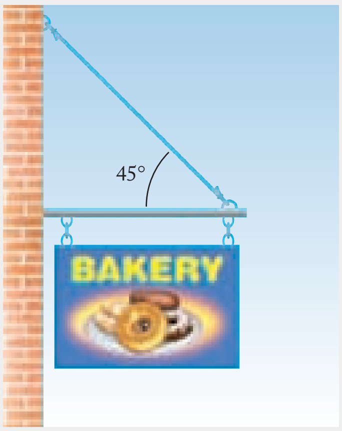


FIGURE 14.68 Sign hanging from a boom.

91. Figure 14.69 shows cargo hanging from the loading boom of a ship. If the boom is inclined at an angle of 30° and the cargo has a mass of 2500 kg, what is the tension in the upper cable? What is the compressional force in the boom? Neglect the mass of the boom.

92. Repeat the calculation of Problem 91, but assume that the mass of the boom is 800 kg, and that this mass is uniformly distributed along the length of the boom.

93. A tractor pulls a trailer along a street (see Fig. 14.70). The rear wheels, which are connected to the engine by means of the axle, have a radius of 0.60 m. Draw a "free-body" diagram for one of the rear wheels; be sure to include the forces and the torque exerted by the axle on the wheel, but neglect the weight of the wheel. If the tractor is to provide a pull of 8000 N (a pull of 4000 N from each rear wheel), what torque must the axle exert on each rear wheel?

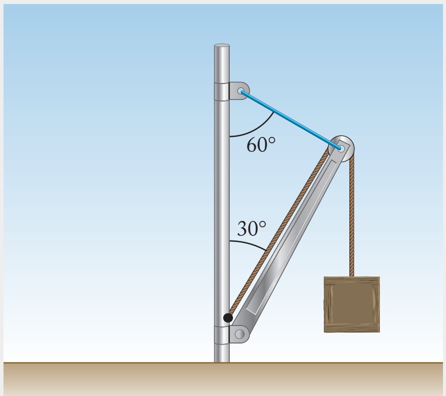


FIGURE 14.69 Cargo hanging from a boom.

FIGURE 14.70 Tractor pulling trailer.

94. One end of a string is tied to a meterstick at the 80-cm mark, and the other end is tied to a hook in the ceiling. You push against the bottom edge of the meterstick at the 30-cm mark, so the stick is held horizontally (see Fig. 14.71). The mass of the meterstick is 0.24 kg. What is the magnitude of the force you must exert? What is the tension in the string?

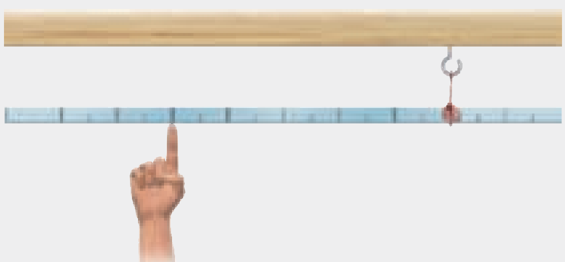


FIGURE 14.71 Meterstick tied to a hook.

95. A beam of steel hangs from a crane by means of cables attached to the upper corners of the beam making an angle of 60° with each other. The mass of the beam is M. Find the tensions in the cables and the compressional force in the beam.

96. Sheerlegs are sometimes used to suspend loads. They consist of two rigid beams leaning against each other, like the legs of the letter A (see Fig. 14.72). The load is suspended by a cable from the apex of the A. Suppose that a pair of sheerlegs, each at an angle of 30° with the vertical, are used to suspend an automobile engine of mass 400 kg. What is the compressional force in each leg? What are the horizontal and vertical forces that each leg exerts on the ground? Neglect the mass of the legs.

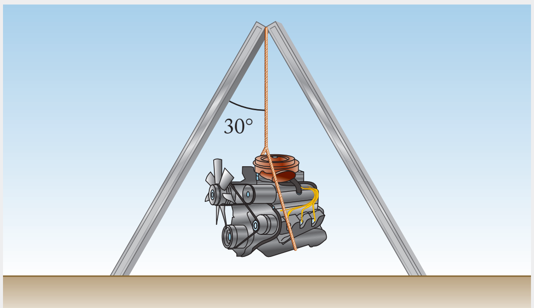


FIGURE 14.72 Sheerlegs supporting a load.

97. A 100-kg barrel is placed on a 30° ramp (see Fig. 14.73). What push, parallel to the ramp, must you exert against the middle of the barrel to keep it from rolling down? Assume that the friction between the barrel and the ramp prevents slipping of the barrel; that is, the barrel would roll without slipping if released.

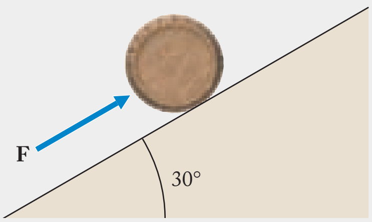


FIGURE 14.73 Barrel on a ramp.

98. Figure. 14.74 shows a pair of pliers and their dimensions. If you push against the handles of the pliers with a force of 200 N from each side, what is the force that the jaws of the pliers exert against each other?

99. To help his horses drag a heavy wagon up a hill, a teamster pushes forward at the top of one of the wheels (see Fig. 14.75). If he pushes with a force of 600 N, what forward force does he generate on the axle of the wagon? (Hint: The diameter of the wheel can be regarded as a lever pivoted at the ground.)

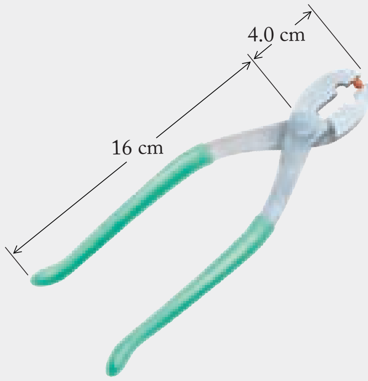


FIGURE 14.74 Pliers.

FIGURE 14.75 Teamster pushing on a wheel.

100. A flagpole points horizontally from a vertical wall. The pole is a uniform rod of mass M and length L. In addition to the pole mount at the wall end (which is hinged and exerts no torque), the pole is supported at its far end by a straight cable; the cable is attached to the wall a distance $L/2$ above the pole mount. What is the tension in the cable? What are the magnitude and direction of the force provided by the pole wall mount?

101. A wire stretches when subjected to a tension. This means that the wire can be regarded as a spring.
    (a) Express the effective spring constant in terms of the length of the wire, its radius, and its Young's modulus.
    (b) If a steel wire of length 2.0 m and radius 0.50 mm is to have the same spring constant as a steel wire of length 4.0 m, what must be the radius of the second wire?

102. If a steel rope and a nylon rope of equal lengths are to stretch by equal amounts when subjected to equal tensions, what must be the ratio of their diameters?

*103. A long rod of steel hangs straight down into a very deep mine shaft. For what length will the rod break off at the top because of its own weight? The density of mild steel is 7.8×10^3 kg/m^3, and its tensile stress for breaking is 3.8×10^8 N/m^2.

104. A rope of length 12 m consists of an upper half of nylon of diameter 1.9 cm spliced to a lower half of steel of diameter 0.95 cm. How much will this rope stretch if a mass of 4000 kg is suspended from it? The Young's modulus for steel rope is 19×10^{10} N/m^2.

105. Suppose you drop an aluminum sphere of radius 10 cm into the ocean and it sinks to a depth of 5000 m, where the pressure is 5.7×10^7 N/m^2. Calculate by how much the diameter of this sphere will shrink.

Answers to Checkups

Checkup 14.1

1. If the bicyclist sits rigidly, the equilibrium is unstable: if tipped slightly, gravity will pull the bicycle and cyclist further over.
2. When you extend your legs while sitting on a swing, you are shifting your center of mass forward. To remain in equilibrium, the swing and your body will shift backward, and tilt, so as to keep your center of mass aligned below the point of support.
3. (a) Yes, the (vertical) support force is along the same line as the weight when holding the pole straight up (more precisely, it is slightly distributed around the edge of the pole). (b) No, the support force is provided by the more forward hand (which pushes up); an additional force from the rear hand pushes down, to balance the torques from the force of gravity and the support force.
4. (D) Neutral, stable, unstable. As our intuition might suggest, a cone on its side is in neutral equilibrium (after a small displacement, it remains on its side). A cone on its base is in stable equilibrium (after being tipped slightly, it will settle back on its base). Finally, a cone balanced on its apex is in unstable equilibrium (after being tipped slightly, the cone will fall over).

Checkup 14.2

1. When a ladder makes a large angle with the vertical, the weight of the ladder and the person climbing it exerts a large torque about the bottom, which can more easily overcome friction and make the ladder slip. When a ladder makes a small angle with the vertical, a person on the ladder can shift the center of mass to a point behind the bottom, causing the ladder to topple backward.
2. With the center of mass on the bottom, the box would have to be rotated 90° before toppling over. In that case, however, the box would then be on its side when it reaches the critical angle, where the center of mass is just above the support point.
3. For each side of the A, the force that one piece of lumber exerts on the other must exert a torque about the other's bottom that balances the torque due to the other's weight. Such a torque increases from zero when the pieces of lumber are vertical (when the tip of the A makes zero angle) to a maximum when the tip approaches 180°. Since the force exerted by one piece on the other acts with a smaller and smaller moment arm as the tip angle approaches 180°, the force must be very large as the tip angle approaches 180°.
4. (B) Increase the upward push and increase the downward push. If we consider the torques about an axis through the forward hand, then the downward pull from the fish must be balanced by increasing the downward push from the rear hand. The upward push of the forward hand must increase to balance those two increased downward forces.

Checkup 14.3

1. The arrangement shown in Fig. 14.23b has the larger ratio l/l', and thus has a greater mechanical advantage.
2. No. If, for example, the force F is not perpendicular to the lever, we must replace l by $l \sin\theta$, where θ is the angle between the force and the lever.
3. No. The pulley transmits tension to a different direction, independent of its size.
4. (C) 100 N. The weight of the rock is $w = mg = 100\text{ kg} \times 9.8\text{ m/s}^2 = 980\text{ N} \approx 1000\text{ N}$. The lever has a mechanical advantage of $l'/l \approx \frac{1}{10}$. So the force required to lift the rock is $F = (l'/l)F' = \frac{1}{10} \times 1000\text{ N} = 100\text{ N}$.

Checkup 14.4

1. The tension determines the *fractional* elongation [see Eq. (14.18)]; thus, for a piano wire of twice the length, the elongation will be twice as long, or 4.0 mm.
2. Yes, a cable can snap under its own weight (the downward weight below any point must be balanced by the upward tension at that point). Since the critical length for breaking is a condition of maximum tensile stress (a force per unit area), this depends on only the material and its mass density, not its area.

3. A material with a larger bulk modulus is stiffer, that is, its volume shrinks less in response to an applied pressure. We can rewrite Eq. (14.20) as $F/A = -B(\Delta V/V)$; thus, at constant pressure, a B that is larger by a given factor results in a fractional volume change that is smaller by the same factor. The volume of the copper sphere then shrinks by 0.005%.

4. (A) $\sqrt{2}$. Since the elongation is inversely proportional to the area of the elastic body [see Eq. (14.18)], if you want to decrease the elongation of a cable by a factor of 2, you must increase the cross-sectional area by a factor of 2; thus, you must increase the diameter by a factor of $\sqrt{2}$.