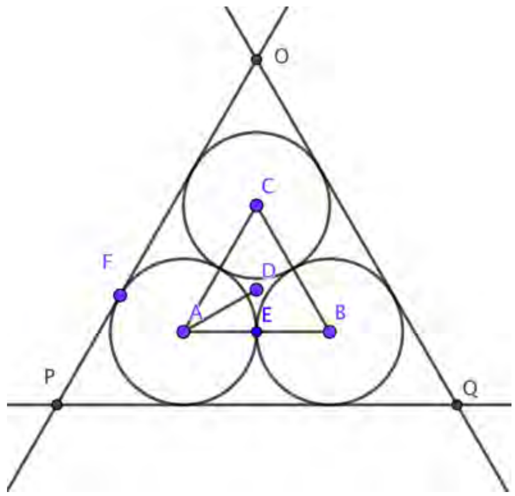
Paper 3 Exploration Questions



Mathematics IB Higher Level
Analysis and Approaches

(For first examination in 2021).

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Author's note

I've made these investigations specifically for the IB HL Analysis Paper 3 (first exam 2021). My aim in each investigation was to have some element of discovery – so new mathematics and new ideas may be introduced (as they will on the real exam). I've also created a full typed mark scheme – which you can download from my site if you get stuck. Daniel Hwang has also created a set of excellent (and challenging) Paper 3 questions – so be sure to also try these. Good luck!

Andrew Chambers

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Page 13 : Circle packing density [Also suitable for Applications]

The mathematics used here is trigonometry and using equations of tangents. Students explore different methods of filling a space with circles to find different circle packing densities.

Note: Fellow IB maths teacher Daniel Hwang has also made a large number of practice paper 3s. He explores a similar topic with focus on 3D packing.

Page 15: A sliding ladder investigation [Most parts suitable for Applications]

The mathematics used here is trigonometry and differentiation. Students find the general equation of the midpoint of a slipping ladder and calculate the length of the astroid formed.

Note: This was also first inspired by the video from a Mind Your Decisions video: Can you solve an Oxford interview question?

Page 18: Exploring the Si(x) function [Not suitable for Applications]

The mathematics used here is Maclaurin series, integration, summation notation, sketching graphs. Students explore methods for approximating non-integrable functions and conclude by approximating $\frac{\pi^2}{12}$.

Page 20: Volume optimization of a cuboid [Most parts suitable for Applications]

The mathematics used here is optimization, graph sketching, extended binomial series, limits to infinity. Students start with a simple volume optimization problem but extend this to a general case.

Note: This task was originally discussed in an Nrich problem number 6399.

Page 22: Exploring Riemann sums [Some parts suitable for Applications]

The mathematics used here is integration, logs, differentiation and functions. Students explore the use of Riemann sums to find upper and lower bounds of functions – finding both an approximation for π and also for ln (1.1).

Note: Even though Riemann sums are no longer on the syllabus this is accessible with scaffolding. This task is based on an old IB Calculus Option question.

Page 24 : Optimisation of area [Most parts suitable for Applications]

The mathematics used here is trigonometry and calculus (differentiation and L'Hopital's rule) to find optimum solutions to an optimization problem.

Page 26: Quadruple Proof [Some parts suitable for Applications]

The mathematics used here is trigonometry (identities) and complex numbers. Students explore 4 different ways of proving the same geometrical relationship.

Note: This proof task was originally discussed in an Nrich problem number 1335.

Page 28: Circumscribed and inscribed polygons [Most parts suitable for Applications]

The mathematics used here is trigonometry and calculus (differentiation and L'Hopital's rule). Students explore different methods for achieving an upper and lower bound for π.

Note: After I had half finished this task I saw the specimen paper 3 provided by the IB and realized it was the same topic. I guess that means I must be thinking along the same lines as the IB examiners! I have included it here because I have added an extra couple of sections on alternative methods for bounding π which I think make it worthwhile trying.

Page 31: Using the binomial expansion for bounds of accuracy [Not suitable for Applications]

The mathematics used here is the extended binomial expansion for fractional and negative powers and integration. Students explore methods of achieving lower and upper bounds for π and non-calculator methods for calculating logs.

Page 33: Radioactive decay [Not suitable for Applications]

The mathematics used here is integration, probability density functions and Euler's method of approximation. Students explore a discrete approximation to radioactive decay, model with Carbon-14 and consider a more advanced model of a decay chain.

Rotating curves [40 marks]

1. *[Maximum marks: 15]*

This question asks you to investigate the rotation of a coordinate point.

(a) *The line $y = 0$ is rotated θ radians anti-clockwise about the origin. Find the equation of the new line in terms of y, x and θ.* *[2]*

(b) *If we have a coordinate point (a, b) rotated θ radians anti-clockwise about the origin we can find the x, y coordinates of the new point by using the following parametric equations:*

$$x = acos\theta - bsin\theta \ (1)$$

$$y = asin\theta + bcos\theta \ (2)$$

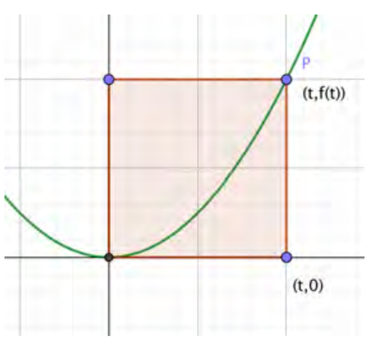
Rotate the point $(1,1)$ anti-clockwise around the origin by $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{2}$ radians. Give your answers as coordinates. *[4]*

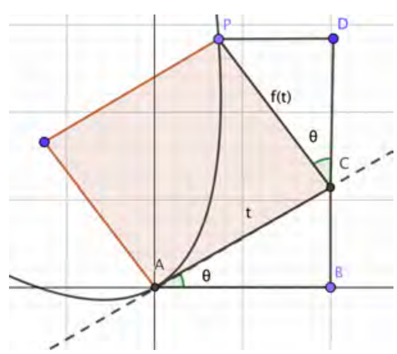
(c) *Draw a sketch of your points. What is the locus of points for this transformation?* *[4]*

(d) *By squaring both equations (1) and (2) obtain an equation in terms of x and y only. What is the geometrical significance of this equation?* *[5]*

2. *[Maximum marks: 25]*

This question expands this method to rotating a curve. We can derive the general equation as follows:
We start with a function $f(t)$, and draw a rectangle centred at the origin through point P: $(t, f(t))$. We then rotate the curve and the rectangle by θ radians anti-clockwise from the horizontal. Angles ABC and CDP are right angles.





(a) *Explain why angle CAB and PCD are equal.* *[2]*

(b) *Show that the x and y coordinates of P can be written as:*

$$x = tcos\theta - f(t)sin\theta \quad (3)$$

$$y = tsin\theta + f(t)cos\theta \quad (4)$$

[4]

(c) *By multiplying equation (3) by $cos(\theta)$ and equation (4) by $sin(\theta)$ show that this can be written as:*

$$ycos\theta - xsin\theta = f(ysin\theta + xcos\theta)$$

[6]

(d) *By taking $f(t) = 2t + 3$, show that the equation when $y = 2x + 3$ is rotated $\frac{\pi}{4}$ radians anticlockwise around (0,0) is given by $y = -3x - 3\sqrt{2}$.*

[5]

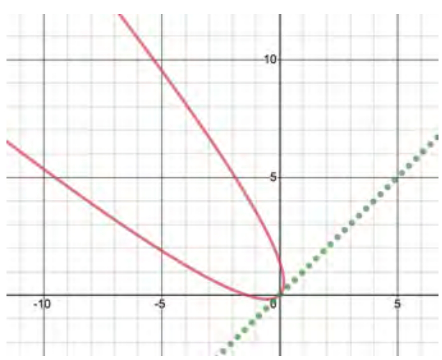
For 2 straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, the acute angle between them is given by:

$$tan\theta = \left|\frac{a_1b_2 - b_1a_2}{a_1a_2 + b_1b_2}\right|$$

(e) *Show that the angle between $y = 2x + 3$ and $y = -3x - 3\sqrt{2}$ is indeed $\frac{\pi}{4}$.*

[3]

(f) *By taking $f(t) = t^2$ find the equation of $y = x^2$ when it is rotated $\frac{\pi}{4}$ radians anticlockwise around (0,0).*



Leave your answer in the form:

$$ax^2 + bx + cy^2 + dy + gxy = 0$$

[5]

Who killed Mr. Potato? [37 marks]

1. *[Maximum marks: 12]*

Cooling rates are an essential tool in forensics in determining time of death. We can use differential equations to model the cooling of a body using Newton's Law of Cooling. The rate of change in the body temperature with respect to time is proportional to the difference in temperature between the body and the ambient room temperature.

In the following investigation we will determine when a potato was removed from an oven. If the ambient room temperature is 75 degrees Fahrenheit and T is the temperature of the potato at time t, *then we have:*

$$\frac{dT}{dt} = -k(T - 75)$$

a) *Show that the solution to this differential equation is:*

$$T - 75 = Ae^{-kt}$$

[4]

We arrive at a room at midday 12:00 to discover a potato on the kitchen counter. The ambient room temperature is 75 degrees Fahrenheit and we know the initial temperature of the potato before being taken out of the oven was 194 degrees Fahrenheit. We take the following measurements:

Time after 12:00 (t mins)	*Temperature of potato (T degrees Fahrenheit)*	*Difference from ambient room temperature* T_d *(T – 75)*
0	*133*	*58*
30	*116*	*41*
60	*104*	*29*
90	*98*	*23*
120	*91*	*16*

(b) *By using the measurements when* $t = 0$ *and* $t = 120$, *find A and k. Use your model to predict when the potato was first removed from the oven.*

[8]

2. *[Maximum marks: 12]*

In this question we use an alternative method to estimate k.
If we take $T_d = T - 75$, then we have:

$$T_d = Be^{-kt}$$

(a) *By finding the regression line of $\ln(T_d)$ on t, find an estimate for k, B. State the correlation coefficient and comment on this value.*

[8]

(b) *Use your model to predict to the nearest minute when the potato was first removed from the oven. Compare your answer with (1c). Which method of finding k is likely to be more accurate?*

[4]

3. *[Maximum marks: 14]*

In this question we use a modified version of Newton's law of cooling introduced by Marshall and Hoare.
The modified equation includes an extra exponential term to account for an initial, more rapid cooling. We can write this as:

$$\frac{dT_d}{dt} + kT_d = 58ke^{-pt}$$

(a) *Show that the solution to this differential equation is:*

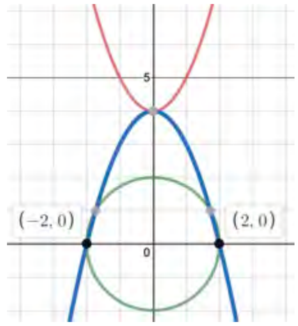
$$T_d = \frac{58k}{k-p}e^{-pt} + Ae^{-kt}$$

[7]

(b) *By considering the measurements when $t = 60, t = 120$, we can find that $k \approx 0.085$, $p \approx 0.012$. By considering the measurements when $t = 0$, find A and use your model to predict the time when the potato was first removed from the oven to the nearest minute. Compare your 3 results.*

[7]

Graphically understanding complex roots: an investigation [35 marks]



1. *[Maximum marks: 8]*

In this question we will explore the complex roots of $f(x) = x^2 + 4x + 5$

(a) *Write* $f(x)$ *in the form* $(x - a)^2 + b$.

[1]

(b) *By solving* $f(x) = 0$*, find the complex roots of* $f(x)$.

[1]

(c) *Reflect* $f(x)$ *in the line* $y = b$. *Call this new graph* $g(x)$.

[1]

(d) *Find the roots* x_1, x_2 *of* $g(x)$.

[1]

(e) *Rotate* x_1, x_2 *90 degrees anticlockwise around the point* $(a, 0)$. *Write down the coordinates of the points. How is this related to part (a)?*

[2]

(f) *For a graph* $g(x)$ *with real roots* $x_1 = a + \sqrt{b}, x_2 = a - \sqrt{b}$*, what are the coordinates that represent the complex roots of* $f(x)$ *in the complex plane?*

[2]

2. *[Maximum marks: 11]*

In this question we will explore the complex roots of $f(x) = x^3 - 9x^2 + \beta x - 17$

(a) $f(x)$ *has only 1 real root,* $x_1 = 1$. *Write down equations for the sum and product of the roots of* $f(x)$*,*

[2]

(b) ***Hence*** *find the 2 complex roots,* x_2, x_3.

[3]

(c) *By using the factor theorem or otherwise, show that* $\beta = 25$.

[2]

(d) *By method of polynomial long division, factorise $f(x)$ into a linear and quadratic polynomial.*

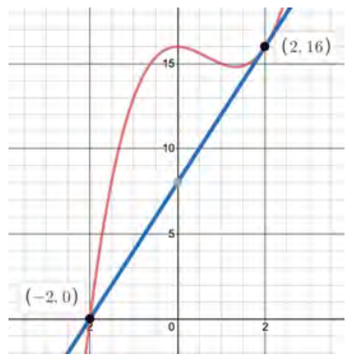
[2]

(e) ***Hence** show an alternative method for finding the 2 complex roots, x_2, x_3.*

[2]

3. *[Maximum marks: 16]*

In this question we will investigate a graphical method of finding complex roots of cubics.



If we have a cubic $f(x)$ with one real root we can graphically find the complex roots by the following method:

Draw the line through the real root which is also tangent to the curve $f(x)$ at another point. If the x-coordinate of the point of intersection between the tangent and the curve is a, and the gradient of the tangent is m, then the complex roots are $a \pm \sqrt{m}i$

(a) *The graph above shows a cubic, with a tangent to the curve drawn so that it also passes through its real root. Write down the 3 roots of this cubic and **hence** find the equation of this cubic.*

[6]

(b) *For the curve $f(x) = x^3 - 4x^2 + 6x - 4$, verify that $(x-2)$ is a factor*

[1]

(ii) Given that the line through the real root is a tangent to the curve at $x = a$, find the equation of the tangent and show that:

$$0 = -2a^3 + 10a^2 - 16a + 8$$

[6]

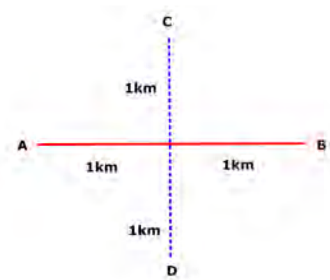
*(iii) **Hence** find the complex roots of $f(x)$*

[3]

Avoiding a magical barrier. [30 marks]

1. *[Maximum marks: 30]*

In this question we explore a scenario where we walk from Town A to Town B, a distance of 2km.
50% of the time there is no obstruction along this route. However, 50% of the time there is a magical barrier perpendicular to the route exactly half way between A and B, extending for 1km in both directions. This barrier is invisible and can only be sensed when you meet it.



Your task is to investigate the optimum strategy for minimizing your average journey.

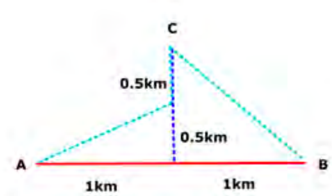
(a) *You set out directly on the route AC. You then walk in the line CB. Find the distance travelled.*

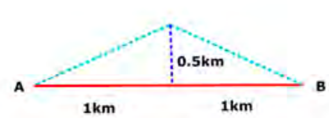
[1]

(b) *You set out directly on the route AB. If there is no barrier you continue on this route to B. If you meet the barrier you walk up to C and then in the line CB. Find the average distance travelled.*

[3]

(c) *This time you set off from A in a straight line to a point 0.5 km along the perpendicular bisector of AB. If you meet the barrier you continue up to C before travelling in the line CB. If you don't meet the barrier you head straight for point B.*

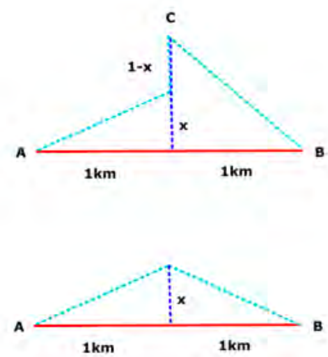




Find the average distance travelled.

[3]

(d) *This time you set off from A in a straight line to a point x km along the perpendicular bisector of AB. If you meet the barrier you continue up to C before travelling in the line CB. If you don't meet the barrier you head straight for point B.*



(i) Find an equation for the average distance travelled in terms of x.

[3]

(ii) Use ***calculus*** *to find the exact value of x which minimizes the average distance travelled.*

[5]

(iii) Sketch a graph to verify your result graphically. What is the minimum average distance?

[2]

(e) *The barrier now appears n % of the time. Use calculus to find the optimum distance x, in terms of n.*

[7]

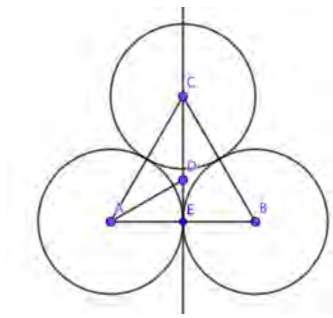
(f) *For a given integer value of n, the optimum strategy is simply to head in a straight line from A to C. Find this value of n.*

[6]

Circle packing density [31 marks]

1. *[Maximum marks: 31]*

In this question we investigate the density of circles in a given space.
Below we have 3 circles tangent to each other, each with radius 1. Point A has coordinates (0,0). Point D is at the intersection of the three angle bisectors of the triangle.



(a) *Find the coordinates of Point B, Point C and Point D* [5]

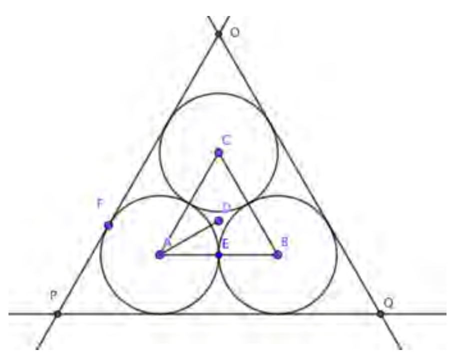
(b) *Find the percentage of the triangle ABC that is not filled by the circles.* [4]

(c) *The general equation of a circle with radius r and centered at (a,b) is given by:*

$$(x-a)^2 + (y-b)^2 = r^2$$

Write down the equation of the three circles. [2]

(d) *The tangents to the three circles make an equilateral triangle OPQ.*



Show that the coordinate point F has coordinates $(\frac{-\sqrt{3}}{2}, \frac{1}{2})$. [7]

(e) *By first finding the equation of the tangent PQ, find the coordinates of P and O,*
[7]

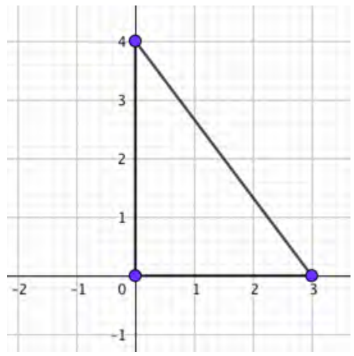
(f) *Hence find the area of the triangle OPQ.*
[3]

(g) *Find the percentage of the triangle OPQ that is not filled by the circles. Comment on which triangle has a higher circle packing density.*
[3]

A sliding ladder investigation [40 marks]

1. *[Maximum marks: 16]*

In this question you investigate the motion of the midpoint of a sliding ladder.
You have a 5 metre long ladder which ends rest against a perpendicular wall at (0,4) and (3,0).



(a) *Explain why the midpoint of the ladder is given by the coordinate (1.5, 2).* *[1]*

(b) *The ladder begins to slip down the wall such that the new base coordinate is (3+t, 0). Find the new height in terms of t.* *[1]*

(c) *Find the new midpoint (x,y) in terms of t. Write down an equation for the x coordinate in terms of t and an equation for the y coordinate in terms of t.* *[2]*

(d) *Eliminate t and show that an equation for the midpoint can be written as*

$$y^2 + x^2 = \frac{25}{4} \qquad x \geq 0, y \geq 0.$$

[3]

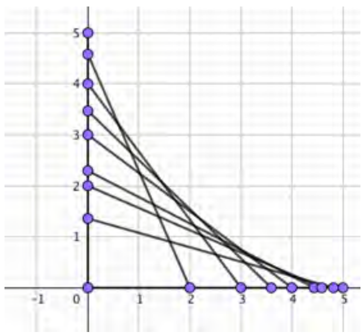
(e) *Sketch this equation and comment on its geometrical significance.* [3]

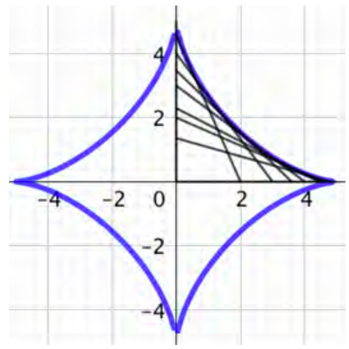
(f) *Find the general equation of the midpoint of a slipping ladder when the ladder rests against a perpendicular wall at (0,a) and (b,0).*

[6]

2. *[Maximum marks: 14]*

This time we take a ladder with length 5 and draw the family (envelope) of lines we get as it slips down the wall. The boundary of these lines creates the first quadrant of an astroid.





The equation of this particular astroid is given by:

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5^{\frac{2}{3}}$$

(a) *Find an equation for* $\frac{dy}{dx}$ *in terms of x only.*

[5]

(b) *Sketch* $\frac{dy}{dx}$ $x > 0$.

[3]

(c) *Find the gradient of the astroid when* $y = x,\ x > 0, y > 0$.

[4]

(d) *What is the significance of the line y = x in relation to the astroid in the first quadrant?*

[2]

This question continues on the next page.

3. *[Maximum marks: 10]*

We start with the astroid formed by a ladder of length m

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = m^{\frac{2}{3}}$$

We can define the curve parametrically as:

$$x = mcos^3\theta$$

$$y = msin^3\theta$$

with the length L of the astroid given by:

$$\frac{L}{4} = \int_0^{\frac{\pi}{2}} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\ d\theta$$

(a) *Find* $\frac{dx}{d\theta}$ *and* $\frac{dy}{d\theta}$

[2]

(b) *Hence show that:*

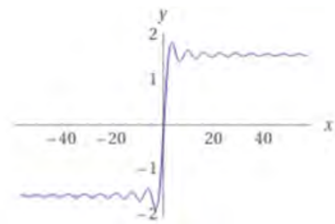
$$\frac{L}{4} = 3m\int_0^{\frac{\pi}{2}} sin\theta cos\theta\ d\theta$$

[4]

(c) *Find an equation for L in terms of m. Find the first quadrant length of the astroid made by a ladder of length 5.*

[4]

Exploring the Si(x) function [36 marks]



(Graph drawn from Wolfram Alpha)

In this question we investigate key features about the Si(x) function.
***1.** [Maximum marks: 9]*

The $Si(x)$, which is drawn above can be defined as:

$$\int \frac{\sin(x)}{x} dx = Si(x) + c$$

(a) *Write down $\frac{d}{dx}(Si(x))$*

[1]

(b) *Find the x coordinates of the local maximums of $Si(x)$ for $x > 0$*

[3]

(c) *By finding $\frac{d^2}{dx^2}(Si(x))$ find the x coordinate of the first point of inflection of $Si(x)$, for $x > 0$.*

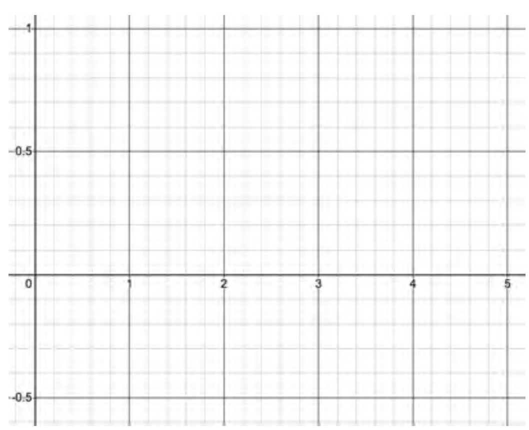
[5]

***2.** [Maximum marks: 9]*

(a) *By considering the Maclaurin expansion of $\sin(x)$ give a polynomial approximation for $\frac{\sin(x)}{x}$ up to the x^4 term. Explain why this approximation is not valid for all x.*

[4]

(b) *Draw a sketch of $y = \frac{\sin(x)}{x}$ and your polynomial approximation on the graph paper below for $0 < x \leq 4$.* *[4]*



(c) *For what domain does your polynomial closely approximate* $\frac{\sin(x)}{x}$ *?*

[1]

3. *[Maximum marks: 9]*

(a) *Find a polynomial approximation for* $Si(x)$ *up to the* x^5 *term.*

[3]

(b) *If we have our integral start from* $x = 0$ *then we can write:*

$$\int_0^a \frac{\sin(x)}{x}\,dx = Si(a)$$

Using your polynomial find an approximation for $Si\left(\frac{\pi}{2}\right)$ *to 6 significant figures.*

[3]

(c) *We can use a computer to calculate* $Si\left(\frac{\pi}{2}\right) \approx 1.37076$. *What is the percentage error in using your approximation?*

[3]

4. *[Maximum marks: 9]*

(a) *Show that*

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$$

[5]

(b) *By considering the first 5 terms of this series find an approximation to* $\int_0^1 \frac{\ln(1+x)}{x}\,dx$ *to 6 significant figures.*

[2]

(c) *Given that:*

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2} = \frac{\pi^2}{12}$$

Find the percentage error when using your approximation to find $\frac{\pi^2}{12}$.

[2]

Volume optimization of a cuboid [36 marks]

1. *[Maximum marks: 11]*

A piece of paper has 4 square edges of size x cut out. It is then folded up to make a cuboid with an open top.

(a) Find an expression for the volume of the cuboid formed when the original paper was a square of length 20cm.

[1]

(b) Use calculus to find the value of x that gives the maximum volume.

[3]

(c) By sketching a graph, find the value of x that gives the maximum volume when the original paper is a square with sides:

(i) 10cm.

(ii) 30cm.

[5]

(d) Hence find the value of x which gives a maximum volume of an m *by* m square.

[2]

2. *[Maximum marks: 25]*

Next we will explore a 10 *by* n rectangle.

(a) Show that the value of x which gives the maximum volume is given by:

$$x = \frac{40 + 4n - 4\sqrt{(n-5)^2 + 75}}{24}$$

[7]

(b) Use this equation to find the value of x which gives the maximum volume for a 10 *by* 30 rectangle.

[2]

(c) We will now investigate what happens to this value of x as n tends to infinity, ie:

$$\lim_{n\to\infty} \frac{40+4n-4\sqrt{(n-5)^2+75}}{24}$$

By making the substitution $n = \frac{\sqrt{75}}{u} + 5$ show that this can be written as:

$$\lim_{u\to 0} \frac{15+\frac{\sqrt{75}}{u}-\sqrt{75}\frac{1}{u}\sqrt{1+u^2}}{6}$$

[4]

(d) Find the binomial expansion of $\sqrt{1+u^2}$ until the u^4 term

[3]

(e) Hence find the value that x approaches as n tends to infinity for an n *by* n rectangle

[5]

(f) By a similar process we can arrive at the following equation for an m *by* n rectangle

$$\lim_{u\to 0} \frac{\left(4m+4\left(\frac{\sqrt{\frac{3m^2}{4}}}{u}+\frac{m}{2}\right)\right)-4\sqrt{\frac{3m^2}{4}}\frac{1}{u}\sqrt{1+u^2}}{24}$$

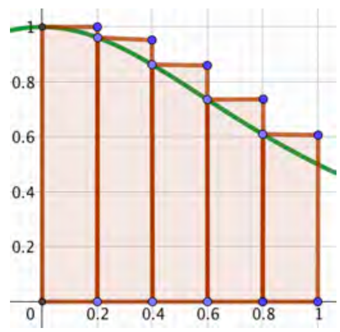
Find the value that x approaches as n tends to infinity for an m *by* n rectangle

[4]

Exploring Riemann Sums [31 marks]

1. *[Maximum marks: 19]*

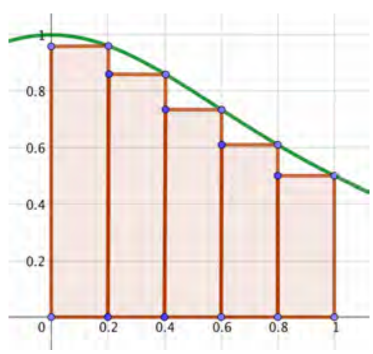
We have the function $f(x) = \frac{1}{1+x^2}$ *and draw 5 rectangles as shown below. The width of each rectangle is* 0.2*, and the height of the rectangles are given by* $f(0)$*,* $f(0.2)$*,* $f(0.4)$ *etc.*



(a) *Use the rectangles above to find an upper bound estimate for the area under the curve* $f(x)$*. Give your answer to 5 significant figures.*

[3]

(b) *It is also possible to draw rectangles from the right hand side. The width of each rectangle is* 0.2*, and the height of the rectangles are given by* $f(0.2)$*,* $f(0.4)$*,* $f(0.6)$ *etc.*



Use the rectangles above to find a lower bound estimate for the area under the curve $f(x)$*. Give your answer to 5 significant figures.*

[2]

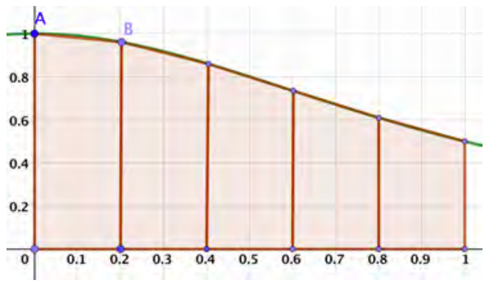
(c) *We can refine our approximation for the are under the curve by using the trapezium rule. The trapezium rule provides an underestimation of the area when the curve is concave down and an overestimation when the curve is concave up. Use calculus to find the x coordinate when* $f(x)$ *changes concavity.*

[5]

(d) *In this case the trapezium rule provides an underestimation of the area under $f(x)$. By referring to your previous answer give a justification why this might be.*

[2]

(e) *For the trapezium rule here we split our area into 5 trapezia each with width 0.2 and with top vertices on the function $f(x)$.*



Use the 5 trapezia above to find a better approximation for the area under $f(x)$. Give your answer to 5 significant figures.

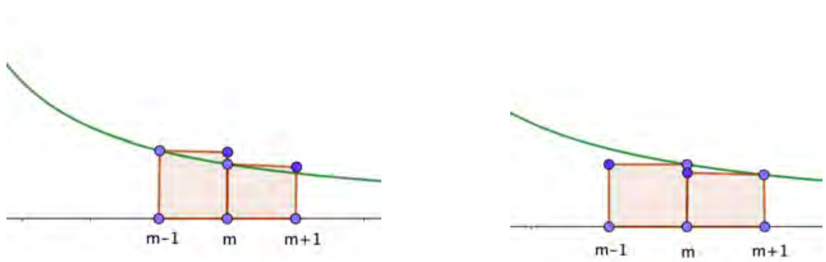
[3]

(f) *By considering $\int \frac{1}{1+x^2} dx$, use your previous results to find a lower and upper bound for π. Give your bounds to 3 significant figures.*

[4]

2. *[Maximum marks: 12]*

Below we have the function $g(x) = \frac{1}{x}, x > 0$ and two possible Riemann approximations.



(a) *Use the diagram above to show that:*

$$\frac{2m+1}{m^2+m} < ln\left(\frac{m+1}{m-1}\right) < \frac{2m-1}{m^2-m}$$

[6]

(b) *Hence find a lower and upper bound approximation to 5 significant figures for $ln(1.1)$. Find the maximum percentage error when using the upper bound. Explain why $ln(1.1)$ would have a more accurate bound than $ln(3)$.*

[6]

Optimization of area: an investigation [42 marks]

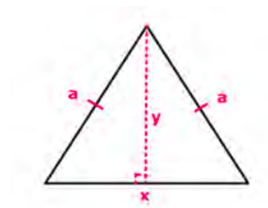
A farmer has 40m of fencing in every case. In this task you will explore the maximum area he can enclose.

1. *[Maximum marks: 23]*

(a) *What is the maximum area when the fence is in the shape of a circle? Leave your answer in terms of π.*
[3]

(b) *What is the maximum area when the fence is in the shape of a rectangle?*
[4]

The farmer now makes a fence in the shape of an isosceles triangle.



(c) *Find an equation for a in terms of x and y.*
[3]

(d) *Find an equation for y in terms of x only.*
[3]

(e) *Show that the area of the triangle can be written as:*

$$A = \sqrt{5}\sqrt{20x^2 - x^3}$$

[5]

(f) *Hence use **calculus** to find the value of x that maximizes the area and the area of the triangle. Comment on the value of x that you obtain.*
[5]

2. *[Maximum marks: 19]*

(a) *The farmer makes a fence in the shape of a regular pentagon.*

Find an expression for x, the distance from the centre to a vertex and hence find the area of this pentagon.
[4]

(b) *Find an equation in terms of n for the area of an n sided regular polygon, n≥3.*
[6]

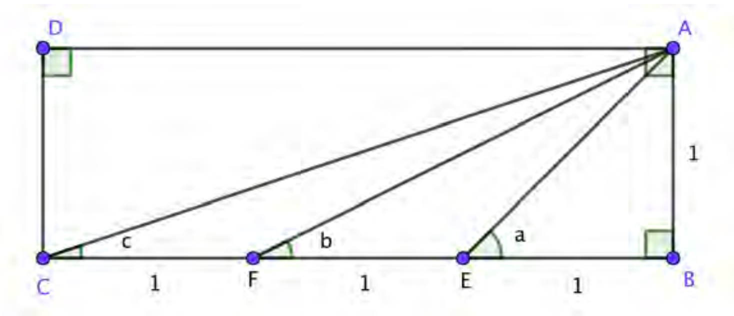
(c) *Sketch a graph of the area of an n sided regular polygon (y-axis) versus n (x axis). Hence find the limit of the maximum area as n approaches infinity. Can you explain the significance of this result geometrically?*
[3]

(d) *By making use of the substitution* $u = \frac{1}{n}$ *or otherwise, can you use L'Hopital's rule to find the limit as n approaches infinity?*
[6]

Quadruple Proof. [34 marks]

*[This entire investigation is intended to be completed **without** a calculator and hence non-calculator method should be shown throughout]*

***1.** [Maximum marks: 8]*



The diagram above shows a rectangle ABCD. CF=FE=EB = 1. $a \leq \frac{\pi}{4}$ *and* $a > b > c$.

(a) *Find* $\sin(a), \sin(b), \sin(c)$.

[2]

(b) *Hence, by considering* $\sin(b+c)$ *prove that* $a+b+c = \frac{\pi}{2}$.

[6]

***2.** [Maximum marks: 6]*

(a) *Find* $\tan(a), \tan(b), \tan(c)$.

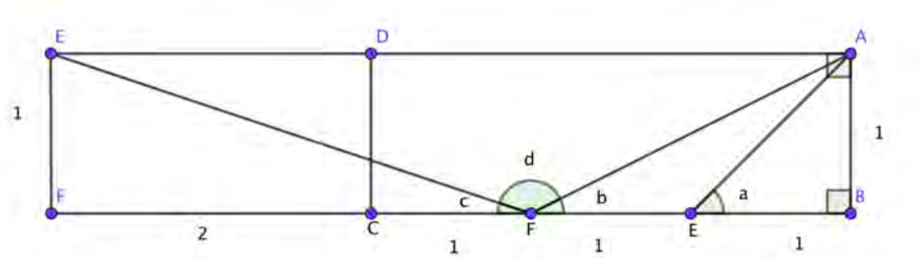
[1]

(b) *By letting* $b = \arctan(X), c = \arctan(Y)$, *find an expression for* $\tan(b+c)$ *and hence prove that* $a+b+c = \frac{\pi}{2}$.

[5]

***3.** [Maximum marks: 5]*

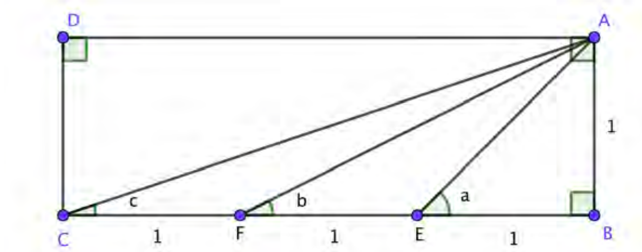
The diagram can be extended as follows:



Angle d is the angle AFE. By considering the triangle AFE, use the cosine rule to find angle d. Hence prove that $a+b+c = \frac{\pi}{2}$.

[5]

***4.** [Maximum marks: 15]*
*This proof should be attempted **without** any trigonometry using a,b,c only.*



(a) *By taking A: (0,0), define E, F and C in terms of complex numbers in Cartesian form.*

[2]

(b) *Find the product ABC of the three complex numbers.*

[3]

(c) *Explain why the complex number E can also be written as* $\sqrt{2}e^{(-\pi+a)i}$

[2]

(d) *Find equivalent expressions for F and C.*

[2]

(e) *Hence find an alternative expression for the product ABC, in terms of a,b,c.*

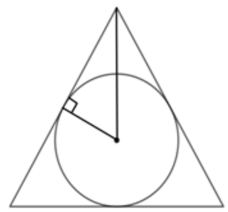
[3]

(f) *Hence prove that* $a+b+c=\frac{\pi}{2}$.

[3]

Approximating pi using circumscribed and inscribed circles. [39 marks]

The perimeter of a circumscribed polygon was used to find an approximation for pi by the ancient Greeks. They used the relationship $\pi \approx \frac{P}{2}$, where P is the perimeter of the regular polygon.

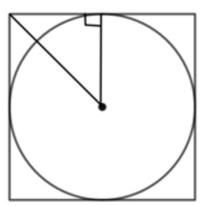


***1.** [Maximum marks: 12]*

(a) By considering the right-angled triangle drawn above, find the perimeter of equilateral triangle, whose sides are 3 tangents to the unit circle (radius 1). Hence use $\pi \approx \frac{P}{2}$ to find an approximation for π.

[3]

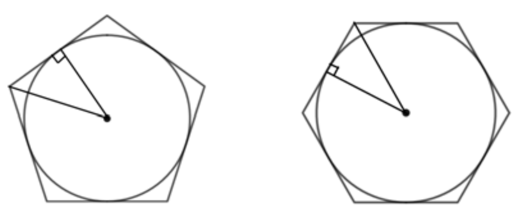
(b) We can draw a square such that the perimeter of the square is made up of sides which are 4 tangents to the unit circle.



By finding the perimeter of this square find a better approximation for π.

[2]

(c) Use a circumscribed regular pentagon and regular hexagon to refine your approximations.



[4]

(d) *Hence find the perimeter of an n-sided regular polygon circumscribed in a unit circle and the approximation for π for this polygon*

[3]

(e) *How many sides would a circumscribed polygon need to have to approximate π to 2 decimal places?*

[3]

2. *[Maximum marks: 11]*

We can use a similar method to approximate π when we have an inscribed regular polygon in a unit circle.

(a) *By considering an appropriate right-angled triangle, show that the perimeter of an n-sided regular polygon inscribed in a unit circle can be given by:*

$$P = 2n\sin\left(\frac{\pi}{n}\right)$$

[2]

(b) *How many sides would an inscribed polygon need to have to approximate π to 2 decimal places? Compare this with your answer to (v).*

[4]

(c) *Use L'Hopital's rule to find the limit as n approaches infinity for this approximation to π.*

[5]

The question continues on the next page

3. *[Maximum marks: 13]*

Archimedes was able to find upper and lower bounds for π by considering inscribed and circumscribed polygons without the need for trigonometry.

If we define i_n as the perimeter of an n-sided regular polygon inscribed in a circle and c_n as the perimeter of an n-sided regular polygon circumscribed in a circle then we have the following recursive relationship:

$$c_{2n} = \frac{2c_n i_n}{i_n + c_n}$$

$$i_{2n} = \sqrt{i_n c_{2n}}$$

(a) *By using your exact values for c_3 and i_3, and leaving your answers in an exact form, show a non-calculator method for finding*

(i) c_6 and i_6 *[3]*

(ii) c_{12} and i_{12} *[5]*

(b) *The Indian astronomer Madhava discovered the following infinite series representation of π in the 1300s. This series is a convergent, alternating series.*

$$\pi = \sqrt{12}\sum_{k=0}^{\infty}\frac{(-3)^{-k}}{2k+1}$$

Any two consecutive terms of a convergent alternating series will give a lower and upper bound for the series.

Use this series to give both a lower and upper bound for π which are both correct to 2 decimal places. Compare your answer with (1v) and (2ii). *[5]*

Using the Binomial Expansion for bounds of accuracy [29 marks]

1. *[Maximum marks: 8]*

(a) *Find the binomial expansion for* $\frac{1}{1+x}$ *for* $|x| < 1$. *Give your answer in terms of an infinite summation.*

[3]

(b) *Hence show that:*

$$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \ |x| < 1$$

[2]

(c) *By considering the approximation for ln(1+x), how many terms of the expansion are needed to find ln(1.1) correct to 4 decimal places?*

[3]

2. *[Maximum marks: 9]*

(a) *Find the full binomial expansion for* $\frac{1}{1+x^2}$ *for* $|x| < 1$ *and hence show that:*

$$arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \ |x| < 1$$

[4]

(b) *By considering x =*$\frac{\sqrt{3}}{3}$*, and using the approximation for arctan(x) up to* x^7 *term, find a lower bound for π to 3 decimal places.*

[3]

(c) *Find an upper bound for π to 3 decimal places that would be accurate to 2 decimal places.*

[2]

3. *[Maximum marks: 12]*

(a) *By considering:*

$$\int \frac{1}{\sqrt{1-x^2}} \, dx \ \ |x| < 1$$

Find a polynomial approximation for arcsin(x) up to and including the x^7 *term. State clearly the constant of integration.*

[5]

(b) *Find a polynomial approximation for arcos(x) up to and including the* x^7 *term.*

[3]

(c) *Use your answers to part (b) to find* $2(arccos(x) + arcsin(x))$

[2]

(d) *Use your answer to part (c) to explain why the following inequalities hold.*

$$-arccos(x) - arcsin(x) < arctan\,(x) < arccos(x) + arcsin(x)$$

[2]

Radioactive decay [39 marks]

1. [Maximum marks: 31]

We can obtain a discrete model of radioactive decay by collectively rolling a number of dice and then after each roll removing all dice showing a six, before repeating the process with the dice left.

(a) We define the number of dice left after n rolls by $N(n)$. If we start with N_0 dice, find an equation for $N(n)$.

[2]

(b) By considering the relationship $a = e^{\ln(a)}$, find an equation for $N(n)$ in the form $N_0 e^{-\lambda n}$ where λ is a constant you should find.

[3]

The continuous radioactive decay of atoms can be modeled with the following equation:

$$N(t) = N_0 e^{-\lambda t}$$

$N(t)$: The quantity of the element remaining after time t (years). N_0: The initial quantity of the element. λ: The radioactive decay constant.

(c) Carbon-14 has a half-life of 5730 years. This means that after 5730 years exactly half of the atoms of the original quantity will have decayed. Use this information to find the λ, the radioactive decay constant of Carbon-14.

[3]

(d) You find an old manuscript and after testing the levels of Carbon-14 you find that it contains only 30% of Carbon-14 of a new piece of paper. How old is this paper?

[2]

(e) If we define $\int f(t)dt = F(t)$, we can evaluate improper integrals as follows:

$$\int_a^{\infty} f(t)dt = \lim_{b\to\infty} \int_a^b f(t)dt = \lim_{b\to\infty} [F(t)]_a^b$$

Show that $\int_0^{\infty} \frac{1}{e^x} dx = 1$

[4]

(e) The probability density function for the probability of radioactive decay of Carbon-14 can be given by:

$$f(t) = ae^{-\lambda t}, \ \ t \geq 0.$$

By considering $\int_0^\infty f(t)dt$, show that $a = \lambda$.

[6]

(f) Hence show that the median for the probability density function does give 5730 years to 3 significant figures.

[3]

(g) **Use calculus** to find the mean length of time a Carbon-14 atom will exist before decaying.

[7]

(ii) Comment on your result.

[1]

2. [Maximum marks: 8]

In this question we explore radioactive decay chains. In a decay chain, atom A will decay to atom B, which then decays to atom C etc. In our case we will say that Ramanujan-1729 decays into Ramanujan-1728, which then decays into Ramanujan-1727.

(a) We start with 100 atoms of Ramanujan-1729 with a decay constant $\lambda_a = \frac{1}{1729}$. Ramanujan-1728 has decay constant $\lambda_b = \frac{1}{4104}$. Therefore we have the following differential equation for the rate of change of Ramanujan-1728, N_B:

$$\frac{dN_B}{dt} = -\frac{1}{4104}N_B + \frac{1}{1729}(100)e^{-\frac{1}{1729}\mathrm{t}}$$

Given that when $t = 0$, $N_B = 0$ Use Euler's with step size 0.1 to find an approximation for N_B when $t = 0.5$.

[6]

(b) The solution to the differential equation above is given by:

$$N_B(t) = \frac{\frac{100}{1729}}{\frac{1}{4104} - \frac{1}{1729}}\left(e^{-\frac{1}{1729}t} - e^{-\frac{1}{4104}t}\right)$$

Use the equation above to find $N_B(0.5)$ and comment on the accuracy of your approximation.

[3]