TECHNICAL UNIVERSITY HAMBURG-HARBURG
AB - MATHEMATICS

NUMERICAL MODELLING IN COMPUTATIONAL FLUID DYNAMICS

Mária Lukáčová - Medviďová

CONTENTS

The book of nature is written in the language of mathematics.
Galileo Galilei

INTRODUCTION

Mankind has always been attracted by the idea of being able to fly. Flying seemed to be an unreachable goal, which is clearly exemplified by the myth of Ikarus.

One of the first men who tried to change this situation was famous Leonardo da Vinci. In 1505 he drew the first designs of flying machines with an aviator in horizontal or vertical position. However he realized that human power was not sufficient to move these flying machines.

Many others tried to continue in this work but they did not succeed. Next 400 years it was believed that flying with a motor will not be possible.

The first who actually left the ground were Wright brothers in 1903. The secret of their success was hidden in using of a wind-tunnel in the design of their aeroplane.

From this time the using of wind-tunnels for experimental purposes increases. During the Second World War the first simple mathematical models were computed on large machines to simulate flow problems.

Nowadays Computational Fluid Dynamics (CFD) plays an important role. Due to the development of highly efficient computers we are able to obtain the behaviour of a flow passing any part of machine. This allows us to choose the best numerical design of plane which is then experimentally tested.

But the application of CFD does not lie only in the aeroplane industry. Liquids have also very similar behaviour as gases, and CFD techniques are used for study of motion of water, blood or oil. We can meet its application in the design of turbines, constructions of ships, oil pipelines, channels or in special part of medicine, dealing with flow of blood, hemodynamics.

To the long list of applications of CFD we can add one more that is very important from theoretical point of view. There are still some open theoretical problems concerning the fundamental questions on existence and uniqueness of a solution to equations describing the motion of fluids. Numerical experiments can help us to understand behaviour of various types of flows and to give at least some "verified" hypothesis concerning the answers to these open problems.

This scriptum is devoted to the study of numerical solution of inviscid compressible flow. It is governed by the system of conservation laws consisting of the continuity equation, the Euler equations of motion of inviscid flow, the energy equation and the state equation. If the viscous effects are included we get the so-called

Navier-Stokes equations. Nowadays there are well-developed numerical techniques, such as the *finite volume methods,* and *finite element methods*, which play an important role for mathematical and engineering modelling of technical problems.

The aim of this scriptum is to give an introduction and brief description of several numerical techniques used in the computational fluid dynamics of compressible inviscid flows. These lecture notes were developed from my notes on *Mathematical Methods in Fluid Dynamics*, which I have taught at the Mathematical Institute, University of Technology in Brno for students of mathematical engineering (9th semester). The overall emphasis of this notes is on studying the mathematical tools that are essential in developing, analyzing, and successfully using numerical methods for nonlinear systems of conservation laws, particularly for inviscid problems involving shock waves. An understanding of mathematical structure of the governing equations is first required. Afterwards a reasonable scheme for solving the system of partial differential equations can be suggested and studied. In this notes I have stressed the underlying ideas used in various classes of methods rather than presenting the most sophisticated methods in great details. My aim was to provide a sufficient background that students could then approach the current research literature with the necessary tools for understanding.

Some sections have been reworked several times by now, some are still preliminary. I can only hope that the errors will not cause misunderstanding. I hope of eventually expanding the presented notes into a book, going in deep discussions in some areas. For these reasons I am interested in receiving suggestions, comments and corrections. I can be reached via email at *lukacova@tu-harburg.de*.

This notes are organized as follows. In Chapter I we will introduce some basic notation and physical quantities describing the motion of fluids. We will formulate the system of conservation laws which govern inviscid as well as viscous compressible flow.

In Chapter II we will deal with the Euler equations. We summarize known theoretical results for existence and uniqueness for hyperbolic conservations laws. Particularly, we deal with multidimensional scalar equation and one-dimensional system. Further we study the specific so-called Riemann problem for linear as well as nonlinear systems.

For numerical solution, the finite volume method (FVM), applied on fully unstructured grid consisting of the so-called dual finite volumes, is used. We will discuss typical problems of finding the weak solution satisfying the entropy inequality. This is in a close relation to the second law of thermodynamics. Further, we will present results of numerical experiments where the Vijayasundaram FVM is used to compute the Euler equations in 2D. We will particularly discuss some improvements obtained by the second order TVD-MUSCL Vijayasundaram scheme and by the suitable mesh refinement. There is a wide class of literature concerning hyperbolic conservation laws and/or their numerical solution. See, e.g., [Feistauer], [Godlewski, Raviart(1),(2)], [Hirsch], [Kröner], [LeVeque(1)], [LeVeque(2)], [Morton, Mayers], [Smoller], [Sonar(1),(2)], [Toro], [Warnecke] to mention some of them.

In Chapter III we will give a list of unsolved problems which can be used in seminars and for students' individual work. Some other problems, which corresponds to particular theorems of Chapter I, II were already included therein.

Finally I would like to thank Alexander Ženíšek, Technical University Brno, for his support and encouragement in writing these lecture notes. They were further developed during my visiting Sofia-Kovalevskaja professorship at the Institute of Computational Mathematics, University of Kaiserslautern. I want to express my thank to Helmut Neunzert for initiating a valuable project of Sofia-Kovalevskaja professorship in order to support female mathematicians. I also wish to thank Libor Čermák and Jitka Saibertová, Technical University Brno, for reading of several versions of the manuscripts, fruitful discussions and helpful comments which improved the final version of the notes substantially.

I want to express my thanks to my academic teacher, Miloslav Feistauer, Charles University Prague, for introducing me to a fascinated field of mathematical and numerical modelling in fluid dynamics. Further, I want to express my deep thank to Gerald Warnecke, Otto-von-Guericke-Univerität Magdeburg for the hours he devoted me in various discussions, for his continual support and encouragement. I also wish to thank Bill Morton, Bath/Oxford Universities, for his valuable advices, informations and fruitful discussions.

At the end I would like to thank my husband L'ubo and my parents for their unfailing support, great patience and intuitive understanding of my work.

Hamburg, summer 2003 *Mária Lukáčová*

CHAPTER I

BASIC EQUATIONS DESCRIBING FLOWS

Throughout this scriptum we will deal with the complete system of equations describing the motion of compressible viscous and inviscid fluids. In this chapter we will give physical and mathematical formulation of conservation laws, which accompanied by the constitutive relations and thermodynamical state equations lead to the system of Navier–Stokes or Euler equations describing the motion of viscous or inviscid fluids, respectively. We will formulate the initial-boundary value problems for considered compressible flows.

At the beginning we write a list of some symbols and notations we will use:

$\Omega \subset \mathbb{R}^d$	...	bounded domain occupied by the fluid
$d \in \mathbb{N}$	...	dimension, in practice d = 2,3
$\langle 0, T\rangle$	...	time interval, $T > 0$
$\mathbf{x} = (x_1, \dots, x_d) \in \Omega$	...	any point from Ω
$t \in \langle 0, T\rangle$	...	time instant from time interval $\langle 0, T\rangle$
$Q_T = \Omega \times (0, T)$	...	space-time cylinder
$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (v_1, \dots, v_d)$	...	velocity vector
$\rho = \rho(\mathbf{x}, t)$	...	density
$p = p(\mathbf{x}, t)$	...	pressure
$\theta = \theta(\mathbf{x}, t)$	...	absolute temperature
$\varepsilon = \varepsilon(\mathbf{x}, t)$	...	total specific energy i.e. per unit of mass
$\mathbf{f} = \mathbf{f}(\mathbf{x}, t) = (f_1, \dots, f_d)$	...	vector of external (volume) forces per unit of mass (given)
$q = q(\mathbf{x}, t)$	...	rate of external heat sources (given)
$\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$	...	heat flux.

These are basic physical quantities describing the moving fluid, other quantities will be defined later. Let us note that we will often use the summation convention, i.e. one has to sum up over the index occurring twice in some term. We will recall a well-known definition of the following operators.

For a vector function $\mathbf{u} = (u_1, \dots, u_n) : \mathbb{R}^m \to \mathbb{R}^n$ we put

$$\text{grad } u_i := \Big(\frac{\partial u_i}{\partial x_1}, \dots, \frac{\partial u_i}{\partial x_m}\Big), \qquad i = 1, 2, \dots n;$$

$$\text{grad } \mathbf{u} := (\text{ grad } u_1, \dots, \text{ grad } u_n)^T,$$

(note that grad $\mathbf{u}$ is an $n \times m$ matrix, having in the i-th row a vector grad u_i)

$$\text{if} \quad m = n \quad \text{then} \quad \text{div } \mathbf{u} := \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}.$$

In what follows we will assume that the students are familiar with definitions and properties of L^p -spaces, Sobolev spaces and Bochner spaces. There is a lot of literature concerning these topics (see, e.g., [Feistauer] [Kufner, Fučík, John], [Málek, Nečas, Rokyta, Růžička, [Nečas], [Ženíšek], etc.).

1.1 Conservation Laws

Eulerian and Lagrangian description of fluid motion

Consider the motion of each particular fluid particle. Trajectories of the particles can be described by the equation

$$\mathbf{x} = \varphi(\mathbf{X}, t).$$

Here $\mathbf{X} = (X_1, \ldots, X_d)$ represents the reference of the particle that we consider. Thus, in d-dimensional space we have together with time variable the following so-called *Lagrangian coordinates* $\mathbf{X}, t$. The above equation determinates the position of the particle given by the reference $\mathbf{X}$ at time t.

However, if we investigate fluid we are rarely interested in the motion of each particular fluid particle. Instead we are interested in the state of flow and its change in time. Therefore, we usually work with the so-called *Eulerian coordinates* $\mathbf{x}, t$, which are based on the determination of the velocity $\mathbf{v}(\mathbf{x}, t)$ of the fluid particle passing through the point $\mathbf{x}$ at time t. We can write

$$\mathbf{v}(\mathbf{x}, t) = \frac{\partial \varphi}{\partial t}(\mathbf{X}, t),$$

where $\mathbf{x} = \varphi(\mathbf{X}, t)$.

Now, let us consider some physical quantity F (e.g. temperature, density, etc.) transported by moving particles in fluid. In the Lagrangian concept this quantity is viewed as $F(\mathbf{X}, t)$, which describes the value of the quantity considered, bound to the particular fluid particle given by $\mathbf{X}$ at time t.

On the other hand, in the Eulerian description the quantity is represented by a function $F(\mathbf{x}, t)$, which denotes the value of quantity at the point $\mathbf{x}$ at time t. Describing the path of a fluid particle by the equation $\mathbf{x} = \varphi(\mathbf{X}, t)$, we can express the rate of change of the quantity $F(\mathbf{x}, t)$ as

$$\begin{aligned}
\frac{dF(\mathbf{x}, t)}{dt} &= \frac{dF(\varphi(\mathbf{X}, t), t)}{dt} \\
&= \frac{\partial F(\varphi(\mathbf{X}, t), t)}{\partial t} + \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}(\varphi(\mathbf{X}, t), t) \frac{\partial \varphi_i(\mathbf{X}, t)}{\partial t} \\
&= \frac{\partial F(\mathbf{x}, t)}{\partial t} + \mathbf{v}(\mathbf{x}, t) \cdot \text{grad } F(\mathbf{x}, t).
\end{aligned}$$

We see that the rate of change of the quantity F is equal to the so-called material derivative

$$\frac{d}{dt}F\left(\mathbf{x},t\right):=\frac{\partial F}{\partial t}\left(\mathbf{x},t\right)+\sum_{i=1}^{d}\frac{\partial F}{\partial x_i}v_i\left(\mathbf{x},t\right).$$

The local derivative $\partial F/\partial t$ results from the dependence of F on time, the convective derivative $\mathbf{v}\cdot\text{grad}\ F$ is a consequence of the transport of the quantity F by moving fluid. In this context the material derivative is also sometimes called the *derivative along a trajectory* of a fluid particle.

Transport theorem

Let us consider *transport* of the physical quantity $F(\mathbf{x},t)$ (Eulerian description) in a control volume $\sigma(t)$. The total amount of the quantity given by the function F that is contained in the volume $\sigma(t)$ at time t equals

$$\mathcal{F}(t)=\int_{\sigma(t)}F(\mathbf{x},t)d\mathbf{x}.$$

In what follows we will need to calculate the rate of change of the quantity F bound on the system of particles considered. Thus, we are interested in

$$\frac{d\mathcal{F}(t)}{dt}=\frac{d}{dt}\int_{\sigma(t)}F(\mathbf{x},t)d\mathbf{x}.$$

In textbooks of mathematical analysis one can find theorems for differentiation of integrals with respect to a parameter, but the domain is fixed. In our situation however we have both the integrand F as well as the integration domain $\sigma(t)$ depending on t. Therefore we need to use the **Reynolds transport theorem**.

THEOREM 1.1.1. *Let $F:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function and the mapping $\varphi:\mathbb{R}^d\to\mathbb{R}^d$ be also continuously differentiable. Then for each control volume $\sigma(t)$ the following equality holds*

$$\begin{aligned}\frac{d\mathcal{F}(t)}{dt}&=\frac{d}{dt}\int_{\sigma(t)}F(\mathbf{x},t)dx=\\&=\int_{\sigma(t)}\left[\frac{\partial F}{\partial t}\left(\mathbf{x},t\right)+\mathbf{v}(\mathbf{x},t)\cdot\ \text{grad}\ F(\mathbf{x},t)+F(\mathbf{x},t)\ \text{div}\ \mathbf{v}(\mathbf{x},t)\right]d\mathbf{x}.\end{aligned}\tag{1.1.2}$$

PROOF. (see, e.g., [Feistauer], [Warnecke]).

□

Flux formulation of the transport theorem

It is clear that identity (1.1.2) can be rewritten in the form

$$\frac{d\mathcal{F}(t)}{dt}=\int_{\sigma(t)}\left[\frac{\partial F}{\partial t}\left(\mathbf{x},t\right)+\ \text{div}\ (F\mathbf{v})(\mathbf{x},t)\right]d\mathbf{x}.$$

By Green's theorem we get

$$\frac{d\mathcal{F}(t)}{dt} = \int_{\sigma(t)} \frac{\partial F}{\partial t}(\mathbf{x},t) + \int_{\partial\sigma(t)} F(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})dS,$$

where $\mathbf{n}(\mathbf{x})$ denotes a unit outer normal to $\partial\sigma(t)$ at the point $\mathbf{x}$. The first integral on the right-hand side determines the rate of change of the quantity $\mathcal{F}$ in virtue of the dependence of F on t. The second integral represents the flux of the quantity F through the boundary $\partial\sigma(t)$.

Formulation of basic conservation laws

The first physical law describing the fluid motion is **the law of conservation of mass** and can be formulated in the following way:

The mass of a piece of fluid formed by the same particles at any time instant is constant in time. In other words : mass is neither created nor destroyed.

Let $\rho(\mathbf{x},t)$ be the density of the fluid at a position $\mathbf{x}$ and time t. Then the mass $m(\sigma(t))$ of fluid occupying the control volume $\sigma(t)$ is given as $\int_{\sigma(t)}\rho(\mathbf{x},t)d\mathbf{x}$. The fact that the mass is constant in time naturally means that

$$\frac{d}{dt}\int_{\sigma(t)}\rho(\mathbf{x},t)d\mathbf{x} = 0.$$

Using the Transport theorem 1.1.1 with $F = \rho$ we get under the assumptions on smooth density ρ

$$\int_{\sigma(t)}\left[\frac{\partial\rho}{\partial t} + \text{ div } (\rho\mathbf{v})\right] = 0.$$

But the above integral identity holds for any control volume $\sigma(t)\subset\Omega$, where $\Omega\in\mathbb{R}^d$ is a fixed domain occupied by the fluid, $t\in(0,T)$. Thus the integrand must be equal to zero itself. Now the differential form of the mass conservation, also called the continuity equation, reads

$$\frac{\partial\rho}{\partial t} + \text{ div } (\rho\mathbf{v}) = 0 \qquad \text{in } \Omega\times(0,T)\,. \tag{1.1.3}$$

Incompressible flow

An incompressible flow is a flow in which density of each material particle remains the same during the motion

$$\rho(\varphi(\mathbf{X},t),t) = \rho(\mathbf{X},0), \qquad \mathbf{X} = \varphi(\mathbf{X},0).$$

Hence $\dfrac{d\rho}{dt} = 0$ and because div $(\rho\mathbf{v}) = \rho\,\text{div }\mathbf{v} + \text{grad }\rho\cdot\mathbf{v}$ we get from (1.1.3) the incompressibility condition

$$\text{div } \mathbf{v} = 0. \tag{1.1.3'}$$

Sometimes incompressibility is erroneously taken to be a property of the fluid instead of the flow. But compressibility depends only on the speed of the flow. If

the magnitude of the velocity of the flow is of order of the speed of sound in the fluid (≈ 340 m/s in air) the flow is compressible. If the velocity is much smaller than the speed of sound, incompressibility is a good approximation. It is true than in liquids flow velocities anywhere near the speed of sound cannot normally be reached, due to enormous pressures involved and the phenomenon of cavitation.

The balance of momentum is just the second Newton's law $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ denotes the vector of forces, m stays for the mass and $\mathbf{a}$ is the acceleration of the fluid. It can be also formulated as follows

The rate of change of total momentum of a piece of fluid formed by the same particles at any time instant is equal to the forces acting on this piece of fluid.

There are two types of forces acting in flows: body forces and surface forces. A body force acts on a material particle, and is proportional to its mass. Let the density of the body force be denoted by $\mathbf{f}(\mathbf{x}, t)$, e.g. the density of gravity force is $(0, 0, -g)$, g is the acceleration of gravity. Then the body force acting on the fluid in the control volume $\sigma(t)$ is given as

$$\int_{\sigma(t)} \rho \mathbf{f}(\mathbf{x}, t) d\mathbf{x}.$$

The action of the surface forces can be expressed by means of the stress tensor $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^{d}$, which results from the inner interactions between fluid volumes through their boundaries. Thus, the surface forces are expressed in the form

$$\int_{\partial\sigma(t)} \boldsymbol{\tau} \cdot \mathbf{n} dS \equiv \int_{\partial\sigma(t)} (\boldsymbol{\tau}_{ij}\mathbf{n}_j)_{i=1}^{d} dS.$$

Note that we have used here the Einstein summation convention in the stress tensor term, i.e. we sum up over the index j occurring twice in the stress tensor term. In order to simplify expressions we will often use this convention in what follows.
Now the law of conservation of momentum can be for an arbitrary control volume $\sigma(t)$ expressed as follows

$$\frac{d}{dt}\int_{\sigma(t)} \rho v_i \, d\mathbf{x} = \int_{\sigma(t)} \rho f_i \, d\mathbf{x} + \int_{\partial\sigma(t)} \tau_{ij} n_j \, dS, \qquad i = 1, \ldots, d.$$

Using the Transport theorem 1.1.1 for $F = \rho v_i$ and the Gauss theorem for the surface integral of the stress tensor we can rewrite the above identity as

$$\int_{\sigma(t)} \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) \, d\mathbf{x} = \int_{\sigma(t)} \rho f_i \, d\mathbf{x} + \int_{\sigma(t)} \frac{\partial \tau_{ij}}{\partial x_j} \, d\mathbf{x}, \qquad i = 1, \ldots, d.$$

Since this holds for every control volume $\sigma(t) \subset \Omega$, $t \in (0, T)$ we get the differential form of the momentum conservation law

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = \rho f_i + \frac{\partial \tau_{ij}}{\partial x_j} \quad \text{in } \Omega \times (0, T), \; i = 1, \ldots, d. \tag{1.1.4}$$

In order to specify further properties of the stress tensor we use the balance of moment of momentum, which says:

the rate of change of the moment of momentum of a piece of fluid occupied by the same particles at any time instant is equal to the sum of the moments of the forces acting on this piece of fluid.

It can be proved (see, e.g., [Gurtin]) that this law is equivalent to the symmetry of the stress tensor $\boldsymbol{\tau}$, i.e.

$$\tau_{ij} = \tau_{ji}, \qquad i, j = 1, 2, \ldots, d.$$

In order to complete the system of the equations (1.1.3) - (1.1.4) it is necessary to relate the stress tensor to other quantities describing the motion of the fluid. Such a relation is called the *constitutive relation* and we will discuss this point in the next section.

The law of conservation of energy, which is the last conservation law used for description of the motion of fluid, follows directly from the first law of thermodynamics:

The rate of change of total energy of a piece of fluid formed by the same particles at any time instant is equal to the sum of powers of the volume and surface forces acting on this piece of fluid and to the amount of heat transmitted to this piece of fluid. In other words one can say: energy is neither created nor destroyed.

For an arbitrary control volume $\sigma(t)$ we obtain the following integral formulation

$$\frac{d}{dt}\int_{\sigma(t)} \rho\varepsilon d\mathbf{x} = W(\sigma(t)) + Q(\sigma(t)), \tag{1.1.5}$$

where W is the power of the body and surface forces acting on the fluid volume $\sigma(t)$ and Q denotes the rate of heat addition. The total specific energy ε consists of the specific internal energy, denoted by $u = u(\mathbf{x}, t)$, and the specific kinetic energy $\frac{1}{2}|\mathbf{v}(\mathbf{x}, t)|^2$, where $|\mathbf{v}| = \left(\sum_{i=1}^{d} v_i^2\right)^{1/2}$ denotes a usual Euclidean norm of a vector. The power of the body and surface forces can be expressed in the form

$$W(\sigma(t)) = \int_{\sigma(t)} \rho f_i v_i d\mathbf{x} + \int_{\partial\sigma(t)} \tau_{ji} n_j v_i dS = \int_{\sigma(t)} \rho f_i v_i + \frac{\partial}{\partial x_j}\tau_{ji} v_i d\mathbf{x}.$$

Assuming that heat is added to each material particle at a rate q per unit of mass, and that there is a heat flux $\mathbf{q}$ per unit of area through $\partial\sigma(t)$, we find

$$Q(\sigma(t)) = \int_{\sigma(t)} \rho q d\mathbf{x} - \int_{\partial\sigma(t)} \mathbf{q}\cdot\mathbf{n} dS = \int_{\sigma(t)} \rho q - \frac{\partial}{\partial x_i} q_i d\mathbf{x}.$$

Substituting the above expressions into (1.1.5) and applying the Transport theorem 1.1.1 on the left-hand side of (1.1.5) we get

$$\int_{\sigma(t)} \frac{\partial}{\partial t}(\rho\varepsilon) + \frac{\partial}{\partial x_i}(\rho\varepsilon v_i)\, d\mathbf{x} = \int_{\sigma(t)} \rho f_i v_i + \frac{\partial}{\partial x_j}(\tau_{ij} v_i) + \rho q - \frac{\partial}{\partial x_i} q_i d\mathbf{x}.$$

Since this hold for every control volume $\sigma(t)$, we have the differential formulation of the law of conservation of energy

$$\frac{\partial}{\partial t}(\rho\varepsilon) + \frac{\partial}{\partial x_i}(\rho\varepsilon v_i) = \rho f_i v_i + \frac{\partial}{\partial x_j}(\tau_{ij} v_i) + \rho q - \frac{\partial}{\partial x_i} q_i \qquad \text{in } \Omega\times(0,T). \tag{1.1.6}$$

1.2 Constitutive Relations

Putting (1.1.3), (1.1.4) and (1.1.6) together we see that the number of unknown quantities is larger than the number of equations. To complete the whole system of conservation laws we have to add some constitutive or closing relations that will specify our fluids.

The first and natural question is the following one: What is the relation between the stress tensor and other quantities describing the fluid motion? Such a relation is called rheological. The simplest situation is obtained in the case of inviscid fluids, where

$$\tau_{ij} = -p\delta_{ij}, \qquad i,j,= 1,2,\dots,d, \tag{1.2.1}$$

here δ_{ij} is a Kronecker delta.

Substituting (1.2.1) into the equations of conservation of momentum (1.1.4) we derive the Euler equations of motion of ideal, i.e. inviscid, fluids

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = \rho f_i - \frac{\partial p}{\partial x_i}, \qquad i = 1,\dots,d. \tag{1.2.2}$$

These equations are written in the conservative form; using the continuity equation (1.1.3) we easily derive the convective form of the Euler equations

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i}, \qquad i = 1,\dots,d. \tag{1.2.2'}$$

Real fluids are more or less viscous. Under the Newton's hypothesis of linear dependence of the stress tensor on the deformation velocity tensor $\mathbf{e}$, where

$$\mathbf{e} = (e_{ij})_{i,j=1}^{d}, \qquad e_{ij} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big),$$

one can derive the following form of $\boldsymbol{\tau}$:

$$\begin{aligned} &\tau_{ij} = -p\delta_{ij} + \tau_{ij}^V \\ &\tau_{ij}^V = \lambda \text{ div } \mathbf{v}\, \delta_{ij} + 2\mu e_{ij}, \qquad i,j = 1,2,\dots,d. \end{aligned} \tag{1.2.3}$$

Here $\boldsymbol{\tau}^V = \big(\tau_{ij}^V\big)_{i,j=1}^{d}$ is said to be a viscous part of stress tensor. The viscosity is expressed by μ, λ, the first and second viscosity coefficient, respectively. μ is also called dynamical viscosity coefficient. There is often used the following relation

$$3\lambda + 2\mu = 0, \qquad \mu \geq 0, \tag{1.2.4}$$

which is derived from the kinetic theory for one-atomic gas. To simplify the situation we will suppose that viscosity coefficients μ and λ are constant. Note that in many fluids they can depend on density ρ or temperature θ, but not on pressure p.

Fluids satisfying (1.2.3) are called *Newtonian fluids*. Examples are gasses and liquid such as water or mercury. But there is a large amount of fluids, for which

(1.2.3) do not hold. Such fluids are called *non-Newtonian* fluids, which are nowadays studied extensively from the experimental as well as theoretical point of view. Examples of those are polymers or blood.

Substitution of (1.2.3) into the equations of conservation of momentum (1.1.4) gives for $i = 1, \dots, d$

$$(1.2.5)\quad \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}(\lambda \text{div } \mathbf{v}) + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right).$$

These are the Navier-Stokes equations, which describe the motion of viscous fluids. Similarly as for the Euler equations (1.2.2) we can write the convective form of the Navier-Stokes equations using the continuity equation (1.1.3)

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}(\lambda \text{div } \mathbf{v}) + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right), \quad i = 1, \dots, d.$$

If the flows are incompressible div $\mathbf{v} = 0$ and the above identity yields the Navier-Stokes equations for incompressible flows:

$$(1.2.5')\qquad \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta v_i, \qquad i = 1, \dots, d,$$

where $\nu = \mu/\rho$ is the so-called kinematic viscosity, assumed to be constant. These equations were firstly derived by Navier in 1823 and later by Stokes in 1845. Note that until now the question of global existence and uniqueness of the solution stays for $d = 3$ a great open problem in the theory of partial differential equations. In 2000, the year of mathematics, the Clay Mathematical Institute declared the problem of existence and regularity of the solution to the Navier-Stokes equations to be one of the seven greatest open problems in mathematics. For each of them the Millennium Prize of one-million dollars was offered.

The next constitutive relation is obtained from Fourier's law

$$(1.2.6)\qquad \mathbf{q} = -k \ \text{grad}\ \theta,$$

where $k \geq 0$ is called the heat conductivity and is supposed to be constant.

The Reynolds number and similarity of flows

Let L^*, U^* and t^* be typical length, velocity and time scales for a given flow problem, respectively. We introduce the following dimensionless variables

$$\mathbf{v}' = \frac{\mathbf{v}}{U^*}, \quad x_i' = \frac{x_i}{L^*}, \quad t' = \frac{t}{t^*}$$

and rewrite the continuity equation (1.1.3') and the Navier-Stokes equations (1.2.5') (of incompressible flow) in the form

$$\frac{\partial v_i'}{\partial x_i'} = 0$$

$$\left(\frac{L^*}{t^* U^*}\right)\frac{\partial v_i'}{\partial t'} + v_j'\frac{\partial v_i'}{\partial x_j'} = \frac{L^*}{U^{*2}} f_i - \frac{1}{U^{*2}}\frac{1}{\rho}\frac{\partial p}{\partial x_i'} + \frac{\nu}{L^* U^*}\Delta v_i', \quad i = 1, \dots, d.$$

Thus, if we set $t^* = L^*/U^*$, $p' = p/(\rho {U^*}^2)$, $f_i' = f_i L^*/{U^*}^2$, $\rho = 1$ and $\nu' = \nu/(L^*U^*)$, then we the continuity equation and the Navier-Stokes equations have the same form in the variables with primes as in the original variables without primes.

We define the dimensionless Reynolds number

$$Re = \frac{1}{\nu'} = \frac{L^*U^*}{\nu} \tag{1.2.7}$$

which play an important role in the theory of similarity flows.

The above transformation shows that Re is a measure of the ratio of inertial (convective) and viscous forces in the flow. This can be seen immediately from the equation (1.2.5') with $1/Re$ instead of ν. For $Re \gg 1$ inertia or convective term dominates, for $Re \ll 1$ friction or viscosity term dominates. Both are balanced by the pressure. This is an example of the convection-diffusion equation. The left-hand side represents the transport by convection term $v_j\, \partial v_i/\partial x_j$, the term $1/Re\, \Delta v_i$ represents transport by diffusion. Many aspects of numerical approximation in computational fluid dynamics already show up in the numerical analysis of relatively simple convection-diffusion equation.

Let us consider two flows in geometrically similar domains Ω_1 and Ω_2 such that Ω_1 is L-times larger than Ω_2 ("$\Omega_1 = L\Omega_2$"). We call this flow dynamically similar, if they have the same Reynolds numbers. Then their dimensionless Navier-Stokes equations are identical. Similarity of flows allow us to carry out computational experiments on small models and transfer the results to the original real flows. We see that the solution to the incompressible Navier-Stokes equation depends actually on the single parameter Re only.

What values does Re have in practice?

In the International Civil Aviation Organization Standard Atmosphere, $\nu = \mu/\rho = 4.9 \cdot 10^{-5} m^2/s$ at 12.5 km. This gives for the flow over an aircraft wing in cruise condition at 12.5 km with wing length $L = 3$ m and velocity $U = 900$km/h: $Re = 1.5 \cdot 10^7$. In a windtunnel experiment at sea level with $L = 0.5$m and $U = 25$m/s we obtain $Re = 8.3 \cdot 10^5$. For landing aircraft at sea level with $L = 3$m and $U = 220$km/h we obtain $Re = 1.2 \cdot 10^7$. For a house in a light wind with $L = 10$m and $U = 0.5$m/s we have $Re = 3.3 \cdot 10^5$. Air circulation in a room with $L = 4$m and $U = 0.1$m/s gives $Re = 2.7 \cdot 10^4$.

Large ship has $Re \approx 10^8$, small yacht $Re \approx 10^7$ and a small fish only $Re \approx 10^4$. All these examples have in common that $Re \gg 1$, which is almost the rule in all environmental and industrial flows. One might think that flow around a given shape will be quite similar for different values of Re, as long as $Re \gg 1$. But nothing is farther from the truth! Therefore computational fluid dynamics plays an important role in the prediction of flow behaviour for full scale of Re numbers. The rich variety of solutions that evolves as $Re \to \infty$ is one of the most surprising and interesting features of fluid dynamics, with important consequences for technological applications.

A 'route to chaos' develops as $Re \to \infty$, resulting in turbulence. Turbulent flow is a complicated example of chaotic dynamical system. The difficulty is that turbulence is both nonlinear and stochastic. Since turbulence is governed by the

Navier-Stokes equations, turbulent flow can be computed from the basic conservation principles. But nowadays this is still not feasible for general engineering computations. We will explain this more closely in what follows.

Let $\mathcal{U}$ and $\mathcal{L}$ be velocity and length scale of large eddies in a turbulent flow, respectively, and let as define the macroscale Reynolds number by

$$\mathcal{R}e = \mathcal{U}\mathcal{L}/\nu.$$

The length scale of smallest eddies is called the Kolmogorov scale, and is denoted by η. It has been shown by Kolmogorov that $\eta/\mathcal{L} = \mathcal{O}(\mathcal{R}e^{-3/4})$. Let us consider flow in a pipe with diameter L. After some computation one can show that away from the pipe wall

$$\eta/L \approx \mathcal{O}(Re^{-21/32}), \quad \mathcal{R}e = \frac{1}{50}Re^{7/8}.$$

To resolve the Kolmogorov scale, the mesh size h of the computational domain must satisfy $h < \eta$. With $h = \eta/2$, we obtain for the number of grid cells in one direction $N = L/h \approx Re^{21/32}$, so that the total number of cells for three-dimensional computations is

$$N^3 \approx Re^{62/32} \approx Re^2.$$

Let us take, for example, $Re = 10^5$; this give us $N^3 \approx 10^{10}$ number of mesh cells, which is clearly not feasible on the computer infrastructures available at present and in the foreseeable future. For complicated geometries and higher Reynolds numbers the required number of grid cells is even larger. But at moderate Reynolds numbers and in simple geometries direct numerical simulation is a valuable tool for studying the fundamental properties of turbulence. However, in order to be able to predict, at least approximately, behaviour of flows at high Reynolds numbers several turbulence models have been developed for industrial computations. In engineering practice in order to achieve computer time and memory requirements that are feasible, turbulent flows are generally modelled by the Reynolds-averaged Navier-Stokes equations with, e.g., the so-called $k - \varepsilon$ model. See, e.g., [Wesseling] or [Zienkiewicz, Taylor] for more details.

1.3 Thermodynamic State Equations

The relations (1.2.1) , (1.2.3) and (1.2.6) still do not give enough information to obtain the closed system of conservation laws. Choosing as thermodynamic unknowns the density ρ and the temperature θ, to close the system we must add the following state equations.

(1.3.1) $$p = p\,(\rho, \theta)\,,$$

$$u = u(\rho, \theta). \tag{1.3.2}$$

Remind that generally we can also assume λ, μ, k to be functions of ρ and θ, i.e.

$$\lambda = \lambda(\rho, \theta), \qquad \mu = \mu(\rho, \theta), \qquad k = k(\rho, \theta). \tag{1.3.3}$$

As we have already said we will restrict to the case of constant functions.

In what follows we will always consider perfect gas, which means that (1.3.1) can be explicitly written as

$$p = R\,\rho\,\theta. \tag{1.3.4}$$

This equation is called the state equation of perfect gas. $R > 0$ is the specific gas constant and it can be expressed in the next form

$$R = c_p - c_v, \tag{1.3.5}$$

where c_p and c_v are the specific heat at constant pressure and volume, respectively. We assume that c_p and c_v are constant and it follows from experiments that $c_p > c_v$. The quantity

$$\kappa = \frac{c_p}{c_v} > 1 \tag{1.3.6}$$

is called the Poisson adiabatic constant, $\kappa = 1.4$ for dry air.

Now we will specify the relation (1.3.2). A perfect gas is said to be polytropic, if

$$u = c_v \theta. \tag{1.3.7}$$

We will work with a perfect polytropic gas, i.e. thermodynamic state equations (1.3.4), (1.3.7) hold. Let us note that we have just closed the whole system of conservation laws describing motion of compressible viscous fluids. We have the same number of unknown functions as the number of equations, i.e. $d + 2$. Let $\rho, \mathbf{v}, \theta$ be unknown. Then the system of conservation laws (1.1.3), (1.1.4), and (1.1.6) in which the constitutive relations (1.2.3), (1.2.6) and thermodynamic state equations (1.3.4), (1.3.7) are included, can be rewritten in the following form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0; \tag{1.3.8}$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}(\lambda \mathrm{div}\ \mathbf{v}) + \frac{\partial}{\partial x_j}(2\mu e_{ij}), \qquad i = 1, 2, \ldots, d;$$

$$c_v \left(\rho \frac{\partial \theta}{\partial t} + \rho \mathbf{v} \mathrm{grad}\ \theta \right) = \mathrm{div}\ (k \mathrm{grad}\ \theta) - p \mathrm{div}\ \mathbf{v} + \lambda (\mathrm{div}\ \mathbf{v})^2 +$$
$$\frac{1}{2}\mu \sum_{i,j=1}^{d} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 + \rho q; \qquad \text{in } Q_T \equiv \Omega \times (0, T).$$

Since p is a function of ρ, θ, in virtue of the state equation (1.3.4), the system (1.3.8) is closed. We have derived the system of Navier–Stokes equations describing the motion of viscous compressible fluids. The motion of inviscid compressible fluids is governed by the Euler equations, which are obtained from (1.3.8), if we put $\lambda = 0 = \mu$. Note that inviscid fluids are non-heat conductive, thus $k = 0$.

For incompressible flows system (1.3.8) splits into two separate subsystems. First we have the continuity equation (1.1.3') with the momentum equations (1.2.2') or (1.2.5') for inviscid or viscous fluids, respectively. These are $d + 1$ equations for unknowns: $\mathbf{v} = (v_1, \ldots, v_d)$ and p. Without lost of generality we can put $\rho = 1$. Next we have the energy equation for unknown temperature θ.

Speaking about thermodynamic relations one should not omit a very important thermodynamic quantity, namely the entropy, denoted by η (specific entropy). It is defined by the relation

$$\theta \, d\eta = du + p \, dV, \tag{1.3.9}$$

where $V = 1/\rho$ is the so-called specific volume. This formula expresses the fact that the internal energy gained by the medium during a change from one state to another is equal to the heat contributed to the medium plus the work done on the medium by compressive action of pressure forces. (See, e.g., [Courant, Friedrichs]).

For a perfect polytropic gas the entropy η can be expressed in the form (see [Feistauer])

$$\eta = c_v \ln \frac{p/p_0}{(\rho/\rho_0)^{\kappa}} + \text{ const.}, \tag{1.3.10}$$

where $p_0, \rho_0 > 0$ are some fixed values of pressure and density, respectively.

There is an important law, the second law of thermodynamics (or the so-called Clausius–Duhem inequality), which is used for a selection of admissible processes. Differential form of this law is

$$\rho \frac{d\eta}{dt} \geq -\text{div } \left(\frac{\mathbf{q}}{\theta}\right) + \frac{\rho q}{\theta}. \tag{1.3.11}$$

Every material satisfying (1.3.11) is said to be thermodynamic compatible. The perfect polytropic gas is such a material.

1.4 Boundary and Initial Conditions

In this section we will formulate the initial-boundary value problems for considered inviscid-viscous flows. It means that the system (1.3.8) will be completed by some boundary and initial conditions. Firstly, we will rewrite (1.3.8) in a very useful so-called conservative form.

In what follows we omit for simplicity the heat sources (i.e. $q = 0$) and the volume forces (i.e. $\mathbf{f} = 0$). Let us denote by e the total energy of a moving fluid, i.e.

$e = \rho\varepsilon$. We can denote the unknown functions by the vector $\mathbf{w} = (\rho, \rho v_1, \ldots, \rho v_d, e)$ and rewrite the system (1.3.8) in the following way:

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \mathbf{f}_i(\mathbf{w}) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \mathbf{R}_i(\mathbf{w}, \operatorname{grad} \mathbf{w}) \quad \text{in } Q_T. \tag{1.4.1}$$

Here $\mathbf{f}_i(\mathbf{w}) := \begin{pmatrix} \rho v_i \\ \rho v_i v_1 + p\delta_{i1} \\ \vdots \\ \rho v_i v_d + p\delta_{id} \\ (e+p) v_i \end{pmatrix}$, $\mathbf{R}_i(\mathbf{w}, \operatorname{grad} \mathbf{w}) := \begin{pmatrix} 0 \\ \tau_{1i}^V \\ \vdots \\ \tau_{di}^V \\ \tau_{ij}^V v_j + k \dfrac{\partial \theta}{\partial x_i} \end{pmatrix}$,

$i = 1, 2, \ldots, d$.

The functions $\mathbf{f}_i(\mathbf{w})$ are called inviscid Euler fluxes and $\mathbf{R}_i(\mathbf{w}, \operatorname{grad} \mathbf{w})$ are viscous fluxes.

The system (1.4.1) gives the conservative form of the complete Navier–Stokes equations for viscous fluids. In the case of inviscid fluids, i.e. $\lambda = \mu = k = 0$, we get $\mathbf{R}_i = \mathbf{0}$ and thus, the conservative form of the Euler equations can be written in the following way

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{i=1}^{d} \frac{\partial \mathbf{f}_i(\mathbf{w})}{\partial x_i} = \mathbf{0} \quad \text{in } Q_T. \tag{1.4.2}$$

Moreover, the state equation should be added to close the systems (1.4.1) or (1.4.2). Using (1.3.4) and (1.3.7) one gets

$$p = (\kappa - 1)\left(e - \frac{1}{2}\rho|\mathbf{v}|^2\right). \tag{1.4.3}$$

1.4.4 Boundary conditions. Several boundary conditions can be considered with respect to different physical situations. In the sequel we will consider the most frequently used, describing the motion of a fluid which flows through a rigid domain Ω. This means that we obtain the so-called inflow-outflow problem. The problem of a fluid moving through a channel or the problem of a flow past an air profile belongs to this class.

We distinguish the inlet part of the boundary $\partial\Omega$

$$\Gamma_I = \{\mathbf{x} \in \partial\Omega; \quad \mathbf{v}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0, \, \mathbf{n} \text{ is the unit outer normal to } \partial\Omega\},$$

through which the fluid enters to the domain Ω , the outlet part

$$\Gamma_O = \{\mathbf{x} \in \partial\Omega; \quad \mathbf{v}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) > 0\},$$

through which the fluid exits the domain Ω and the rest of the boundary $\partial\Omega$, called the solid, impermeable wall, denoted by Γ_W.

The different structure of equations leads to the necessity of distinguishing between viscous and inviscid fluids.

(i) Case $\lambda > 0, \mu > 0, k > 0$ (viscous, conductive fluids).

In this case, the system (1.4.1) is of hyperbolic-parabolic type. The continuity equation is hyperbolic for the unknown ρ, and the equations of motion and energy equation are of parabolic type for the unknowns $\mathbf{v}$ and θ, respectively.

Due to the viscosity of fluid the particles adhere to the walls Γ_W and one has to prescribe no-slip boundary conditions, i.e.

$$\mathbf{v} = \mathbf{0} \qquad \text{on } \Gamma_W \times (0,T)\,.$$

Since the continuity equation is hyperbolic, the density ρ should be prescribed only on inlet part Γ_I, while other quantities $\mathbf{v}, \theta$ are prescribed on the whole boundary $\partial\Omega$ due to the parabolicity of equations. It is relatively easy to get information on prescribed physical quantities on the inflow part of the domain. However it happens frequently that physically no outflow boundary condition is known, but this is required mathematically. In practice "artificial" boundary conditions of Neumann type are preferred above one of Dirichlet type on outflow boundary. For example we can assume the following boundary conditions for system (1.4.1)

(1.4.5)

$$\begin{aligned}
&\text{on } \Gamma_I \qquad \rho = \rho^*, \qquad \mathbf{v} = \mathbf{v}^*, \qquad \theta = \theta^*, \qquad \rho^*, \mathbf{v}^*, \theta^* \text{ are prescribed,}\\
&\text{on } \Gamma_W\text{:} \qquad \mathbf{v} = 0, \qquad \frac{\partial \theta}{\partial n} = 0,\\
&\text{on } \Gamma_O\text{:} \qquad \sum_{i=1}^{d} \tau_{ij} n_i = 0, \qquad j = 1,2,\ldots,d, \qquad \frac{\partial \theta}{\partial n} = 0, \qquad \text{for any } t \in (0,T)\,.
\end{aligned}$$

These special conditions will be used in our numerical experiments, which we present in Chapter III. For the temperature θ we prescribed on Γ_W, Γ_O the so-called adiabatic condition; mathematically it is the homogeneous Neumann boundary condition. Note that we can prescribe also the Dirichlet boundary condition if, for example, the solid wall is heated and we know temperature of a heat source.

(ii) Case $\lambda = \mu = k = 0$ (inviscid, non-conductive fluids).

The system of Euler equations (1.4.2) is hyperbolic (cf. Theorem 2.1.7). In this case the number of boundary conditions on inlet Γ_I and outlet Γ_O is different if the flow is subsonic or supersonic. The local sound speed in fluid is given by

(1.4.6)
$$a = \sqrt{\frac{\kappa p}{\rho}}.$$

Defining the Mach number as

$$M = \frac{|\mathbf{v}|}{a}$$

we speak about hypersonic flow if $M \gg 1$, supersonic flow if $M > 1$, sonic flow if $M = 1$ and subsonic if $M < 1$. Flow is transonic if there are regions with

$M > 1$ as well as $M < 1$. In practice for $M \approx< 0.3$ the incompressibility is a good approximation and it is satisfactory to solve the Euler equations of incompressible flows instead of the complete system (1.4.2).

In order to prescribe boundary conditions for the Euler equations (1.4.2) we need to take into account a local character of this hyperbolic system. The analysis depends on the sign of eigenvalues of the associated characteristic matrix to (1.4.2). General rule, which follows from the theory of characteristics, says:

The number of boundary conditions in a point on the boundary must be equal to the number of incoming characteristics at that point.

This yields that the number of boundary conditions on inflow boundary must be $d+2$ or $d+1$ ($d =$ dimension), depending on supersonic or subsonic character of the flow. The number of boundary conditions on outflow boundary is either zero or one, again depending if the flow is supersonic or subsonic. In fact, we prescribe so many boundary conditions as the number of negative local eigenvalues is. Let us postpone more detailed explanation until Section 2.3, where we will deal with the question of boundary conditions for hyperbolic conservation laws more deeply.

On the other hand the situation on impermeable parts of boundary Γ_W is easier. Since the fluid is inviscid there is no reason for adhesion of the fluid to the walls. We do not put any condition on the tangential component of velocity, but due to impermeability of the wall, the normal component of the velocity must be zero, i.e.

$$\mathbf{v} \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma_W \times (0,T)\,. \tag{1.4.7}$$

No boundary conditions has to be imposed on θ on the solid wall since in this case the temperature is not subjected to transport phenomena through Γ_W.

1.4.8 Initial conditions. If we are concerned with non-stationary problems, suitable initial conditions have to be added. Looking at the preceding evolution equations (1.4.1) or (1.4.2), we see that it is necessary to assign

$$\rho(\cdot,0) = \rho_0,\ \mathbf{v}(\cdot,0) = \mathbf{v}_0,\ \theta(\cdot,0) = \theta_0 \qquad \text{in } \Omega. \tag{1.4.9}$$

Let us note that the initial conditions (1.4.9) as well as the boundary conditions (1.4.5) clearly give the initial and boundary conditions for the vector $\mathbf{w} = (\rho, \rho\mathbf{v}, e)$ for the system of the Navier–Stokes equations (1.4.1). It means we get from (1.4.5), (1.4.9) :

$$\begin{aligned} &\text{a)} \quad B(\mathbf{w}) = \mathbf{b} \qquad \text{on } \partial\Omega \times (0,T)\,, \\ &\text{b)} \quad \mathbf{w}(\cdot,0) = \mathbf{w}_0 \qquad \text{in } \Omega, \end{aligned} \tag{1.4.10}$$

where B is a boundary operator and $\mathbf{b}$ is a R.H.S. function. Further, $\mathbf{w}_0 = (\rho_0, \rho_0\mathbf{v}_0, e_0)$; $e_0 = \rho_0\left(c_v\theta_0 + \frac{1}{2}|\mathbf{v}_0|^2\right)$.

It is easy to see that the analogous situation is obtained for the case of the Euler equations (1.4.2).

CHAPTER II

EULER EQUATIONS AND THEIR NUMERICAL SOLUTION

This chapter builds the core of the notes. It is devoted to the study of hyperbolic conservation laws and the Euler equations as their particular example. Firstly, we will present some known theoretical results for hyperbolic conservation systems. In the second section the finite volume numerical method on unstructured dual grids in 2D will be described. We present theoretical results for the FVM and discuss the discrete entropy condition and its numerical aspects. Since practical problems are formulated on bounded domains, in the third section we will define suitable boundary conditions for the Euler equations. The second order method based on the TVD (total variation diminishing) approach are described in Section 2.4. Finally, we present results on numerical experiments.

There is a wide class of literature devoted to the Euler equations and hyperbolic conservation laws. We will give only a short list of some of them, we often use in our study : [Feistauer], [Godlewski-Raviart (1),(2)], [Hirsch], [Kröner], [LeVeque(1)], [LeVeque(2)], [Morton, Mayers], [Sonar(1),(2)], [Toro], [Warnecke], etc.

2.1 Theoretical Results

We will shortly discuss the problem of existence and uniqueness of a solution to hyperbolic conservation system of PDE's and the Euler equations. The concept of weak solution, the Rankine–Hugoniot jump condition, Riemann problem and entropy weak solution will be defined. We also compare the mathematical and physical concept of entropy.

2.1.1 Euler equations as a hyperbolic system of PDE's. The system of Euler equations can be written in the vector form (cf. (1.4.2)).

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{j=1}^{d} \frac{\partial \mathbf{f}_j(\mathbf{w})}{\partial x_j} = \mathbf{0} \qquad \text{in } Q_T, \tag{2.1.2}$$

which is completed by the initial conditions

$$\mathbf{w}(\cdot, 0) = \mathbf{w}_0 \qquad \text{in } \Omega, \tag{2.1.3}$$

and the boundary conditions

$$B(\mathbf{w}) = \mathbf{b} \qquad \text{on } \partial\Omega \times (0,T). \tag{2.1.4}$$

Here $\mathbf{w}_0$ is a given vector function, B represents a boundary operator (which will be defined in Section 2.3) and $\mathbf{b}$ is a R.H.S. function. For the sake of completeness let us repeat that

$$\mathbf{w} = \begin{pmatrix} \rho \\ \rho v_1 \\ \vdots \\ \rho v_d \\ e \end{pmatrix}, \qquad \mathbf{f}_j(\mathbf{w}) = \begin{pmatrix} \rho v_j \\ \rho v_1 v_j + \delta_{1j}\, p \\ \vdots \\ \rho v_d v_j + \delta_{dj}\, p \\ (e+p)\, v_j \end{pmatrix}, \quad j = 1, 2, \ldots, d.$$

The vector valued function $\mathbf{w}$ is a mapping $\mathbf{w} : Q_T \to D$, where

$$D = \left\{ \mathbf{w} \in \mathbb{R}^{d+2};\ w_1 > 0,\ w_{d+2} > \frac{w_2^2 + \cdots + w_{d+1}^2}{2w_1} \right\},$$

because we consider physically relevant situations, i.e. $\rho > 0$ and $u > 0$. It is easy to realize that $\mathbf{f}_j \in C^1\left(D; \mathbb{R}^{d+2}\right)$ for all $j = 1, 2, \ldots, d$. Thus, we can apply the chain rule to the function $\mathbf{f}_j(\mathbf{w})$ and obtain a first order quasilinear system of PDE's

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{j=1}^{d} \mathbb{A}_j(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_j} = \mathbf{0}, \tag{2.1.5}$$

where $\mathbb{A}_j(\mathbf{w}) = \dfrac{D\mathbf{f}_j(\mathbf{w})}{D\mathbf{w}}$, $j = 1, 2, \ldots, d$, are $(d+2) \times (d+2)$ Jacobi matrices of $\mathbf{f}_j(\mathbf{w}), j = 1, 2, \ldots, d$.

Definition 2.1.6. *System* (2.1.5) *is said to be hyperbolic, if for arbitrary vectors* $\mathbf{w} \in D$ *and* $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{R}^d$ *the matrix*

$$\mathbb{P}(\mathbf{w}, \boldsymbol{\nu}) = \sum_{j=1}^{d} \nu_j\, \mathbb{A}_j(\mathbf{w})$$

has $d+2$ *real eigenvalues* $\lambda_i = \lambda_i(\mathbf{w}, \boldsymbol{\nu})$, $i = 1, 2, \ldots, d+2$, *and is diagonalizable, i.e. there exists a nonsingular matrix* $\mathbb{T} = \mathbb{T}(\mathbf{w}, \boldsymbol{\nu})$, *s.t.*

$$\mathbb{T}^{-1} \cdot \mathbb{P} \cdot \mathbb{T} = \mathbb{D}(\mathbf{w}, \boldsymbol{\nu}) = \begin{pmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_{d+2} \end{pmatrix}.$$

THEOREM 2.1.7. *The system of Euler equations* (2.1.5) *is hyperbolic.*

Note that without loss of generality we can take $\nu \in \mathbb{R}^d$ s.t. $|\nu| = 1$.

PROOF. (see [Feistauer], [Wada, Kubota Ishiguro, Ogawa]).

□

If $d = 2$ then the eigenvalues of the matrix $\mathbb{P}(\mathbf{w}, \boldsymbol{\nu})$ are

$$\lambda_1 = \lambda_2 = \nu_1 v_1 + \nu_2 v_2,$$
$$\lambda_3 = \lambda_1 + a|\boldsymbol{\nu}|,\ \lambda_4 = \lambda_1 - a|\boldsymbol{\nu}|.$$

Here a is a local speed of sound, i.e. $a = \sqrt{\dfrac{\kappa p}{\rho}}$ (cf. (1.4.6)).

The question of the existence and uniqueness of a solution to the initial-boundary value problem (2.1.2) – (2.1.4) still remains open. There are some particular results, e.g., the local existence and uniqueness in time of strong (classical) solutions. See, e.g., [Beiraõ da Veiga], [Valli], [Schochet]. However, in the poblems, we meet in practice, discontinuities, the so-called *shocks and contact discontinuities* may often develop in solution, even for smooth data. This is a fundamental feature of hyperbolic equations. Theoretical results for the Euler equations are the consequences of theoretical results for general hyperbolic systems.

2.1.8 Further examples of hyperbolic conservation laws. Many practical problems in science and engineering involve conserved quantities and lead to PDEs of this class. Although few exact solutions are known a great deal is known for the structure of the solution, which is used to construct adequate numerical schemes.

Wave equation or acoustic equation system. Let us consider the second order wave equation $\varphi_{tt} - a^2(\varphi_{xx} + \varphi_{yy}) = 0$. Here we have used a subscript notation for partial derivatives to simplify the notation. Using a substitution $p = \varphi_t$, $v_1 = -a\varphi_x$, $v_2 = -a\varphi_y$ we obtain the first order linear hyperbolic system

$$\mathbf{w}_t + \mathbb{A}_1 \mathbf{w}_x + \mathbb{A}_2 \mathbf{w}_y = \mathbf{0}, \quad \mathbf{x} = (x, y)^T \in \mathbb{R}^2,$$

where the, noncommuting, coefficient matrices $\mathbb{A}_1, \mathbb{A}_2 \in \mathbb{R}^{3\times 3}$ are defined by

$$\mathbb{A}_1 := \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{A}_2 := \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}.$$

Here $a \in \mathbb{R}$ denotes the speed of sound and $\mathbf{w} = (p, v_1, v_2)^T \in \mathbb{R}^3$ is the vector of dependent variables. We have three eigenvalues $\lambda_1 = -a, \lambda_2 = 0, \lambda_3 = a$ and corresponding linear independent right eigenvectors

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ \cos\theta \\ \sin\theta \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 0 \\ \sin\theta \\ -\cos\theta, \end{pmatrix}, \quad \mathbf{r}_3 = \begin{pmatrix} 1 \\ \cos\theta \\ \sin\theta \end{pmatrix}$$

of the matrix pencil $\mathbb{A}(\boldsymbol{\nu}) := \mathbb{A}_1 \cos\theta + \mathbb{A}_2 \sin\theta$ for any unit vector $\boldsymbol{\nu} = (\nu_1, \nu_2)^T = (\cos\theta, \sin\theta)^T \in \mathbb{R}^2$. Wave equation system is sometimes called the acoustic equation system since it describes propagation of acoustic waves in air. Let us consider linearized Euler equations written in primitive variables (ρ, v_1, v_2, p). Here we denote by u, v the x, y components of the velocity vector, respectively.

$$\mathbf{u}_t + \mathbb{B}_1 \mathbf{u}_x + \mathbb{B}_2 \mathbf{u}_y = \mathbf{0}, \quad \mathbf{x} = (x, y)^T \in \mathbb{R}^2,$$

where the vector of unknowns is

$$\mathbf{u} := \begin{pmatrix} \rho \\ v_1 \\ v_2 \\ p \end{pmatrix}$$

and the Jacobians frozen at some state (ρ', v_1', v_2', p') are

$$\mathbb{B}_1 := \begin{pmatrix} v_1' & \rho' & 0 & 0 \\ 0 & v_1' & 0 & \dfrac{1}{\rho'} \\ 0 & 0 & v_1' & 0 \\ 0 & \kappa p' & 0 & v_1' \end{pmatrix} \quad \mathbb{B}_2 := \begin{pmatrix} v_2' & 0 & \rho' & 0 \\ 0 & v_2' & 0 & 0 \\ 0 & 0 & v_2' & \dfrac{1}{\rho'} \\ 0 & 0 & \kappa p' & v_2' \end{pmatrix}.$$

Both systems of the Euler equations in primitive and conservative variables are equivalent if $\rho, v_1, v_2, p \in C^1$. Note that it is the wave equation system, which creates the key part of the Euler equations. Set $\rho' = 1/a'$ and remove the first row corresponding to density as well as first column from the Jacobian matrices $\mathbb{B}_1$, $\mathbb{B}_2$. Then moving the third equation for pressure in the first row leads to the so-called wave equation system with advection. Further, if the advection velocities are $v_1' = v_2' = 0$ and $a' = \text{const.}$ we recover the above linear wave equation system. Thus, by linearizing the Euler equations about some state (ρ', v_1', v_2', p') we can model propagation of sound waves, small disturbances, which propagates with speeds $v_1' \cos\theta + v_2' \sin\theta \pm a'$, i.e. with the velocities $\pm a'$ relative to the background velocity (v_1', v_2'). In fact, our ears are sensitive to the small disturbances in pressure in these waves.

Further, the wave equation system describes, for example, propagation of waves in elastic solids. In exploration seismology one studies the propagation of small amplitude, man-made waves in the earth, and their reflection off geological structures. The hope is to determine the geological structure from measurements at the surface (for example to order oil reservoirs). Reflection of waves at interfaces can lead to *discontinuities* even for linear equations. Earthquakes can cause larger amplitude disturbances and lead to nonlinear effects.

A similar principle as in seismological exploration of the earth is used for *ultrasound exploration* of human tissue. But there are also many other biological processes involving advection and transport, for example *transport of blood cells in vessels*. As we have said hyperbolic conservation laws can typically produce shock waves even if the coefficients and data are smooth. Of course, physiologically, a true shock in arterial circulation is not possible since blood viscosity and elasticity of the vessel wall preclude shock formation. However, it might be possible to generate very steep pressure gradients in the aorta, which are believed to correspond to

the pistol-shut phenomenon, a loud cracking sound heard through a stethoscope by patients with aortic insufficiency. Recently there exists an extensive mathematical research in **hemodynamics** with the aim to help in prediction of optimal strategy in medical treatment.

Traffic flow. Consider a simple problem of traffic flow on a single-line highway. Let $\rho(x,t)$ denotes the density of cars, $0 \leq \rho \leq \rho_{\max}$, where $\rho_{\max}$ is the maximum density of cars staying bumper-to-bumper. Cars are moved with the velocity $u = U(\rho)$. There are different models to determine velocity function with respect to density of cars. The simplest linear model is

$$U(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right).$$

The conservation of cars can be expressed by the following scalar conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0.$$

Thus

$$\frac{\partial \rho}{\partial t} + \frac{\partial f(\rho)}{\partial x} = 0,$$

where $f(\rho) = \rho u_{\max}(1 - \rho/\rho_{\max})$. Now if one car suddenly stops the cars behind start to slam into one another and a shock wave will propagate back through the line of cars. Both density as well as velocity of cars jumps across this shock wave. The propagating shock wave is similar to what seen in a gas tube with two different gases modelled by the Euler equations.

Magnetohydrodynamics. Astrophysical modelling leads to systems of hyperbolic conservation laws. A spiral galaxy can consist of alternating arms of high density and low density, separated by discontinuities, propagating shock waves. In this context the shock width may be of order 10^5 light years. Modelling of dynamics of a single star, or plasma in a fussion reactor, is governed by the conservation laws. However now also electromagnetic effects together with fluid dynamics have to be considered. The magnetohydrodynamic (MHD) equations consist of the Maxwell equation coupled with the Euler equations. In modelling a supernova explosion one must also include gravitational forces so that density should initially be decreasing with radius rather than constant and such effects as radiative transfer are important. Nonetheless, the same basic structure as in simple Riemann problem (shock tube problem) can be found. Note that when gravitational forces directed towards origin are included this dense shell will be separated from lighter gas below by a contact surface. In three-dimensional model this surface would initially be spherical but would be quickly broken up by the so-called Rayleigh-Taylor instabilities.

Shallow water equations can be found in many practical situations. For example, turn on the kitchen faucet full blast and hold a plate or other flat surface underneath. You will see the water rush radially away from the stream in the thin layer along the plate at fairly high speed. At some distance away from the stream, a circular pattern will generally form, outside of which the layer of water is suddenly thicker and relatively slowly mowing. This discontinuity in depth and velocity is called a hydraulic jump, and it is a nice example of a steady shock wave.

But there are many other types of flows not necessarily involving water, which can be characterized as shallow water flows. They describe flows of fluids with a free surface under the influence of gravity, where the vertical dimension is much smaller than any typical horizontal scale. Examples of shallow water are rivers with their flood plains, flows in lakes generated by wind blows, propagation of tsunamis, oceanographic, meteorological and geophysical flows.

Sometimes stronger discontinuities can be observed, e.g. tidal bores in some rivers or the wave resulting from the bursting of a dam, and a moving step front develops, which is comparable to a shock wave in aerodynamics. In this lecture notes we will concentrate on problems involving *bores or hydraulic jumps* and therefore the aim is to use such numerical schemes which take into account the hyperbolic character of the equations and allow modelling of discontinuous flows.

For example, typical lengths of the river Rhine are: length 1000 km, width 100m, depth 5m. As we see there is just one dominant scale and thus one-dimensional equations will be a good model for the river flow. In what follows we derive the one-dimensional shallow water equations.

Let us consider a fluid which is incompressible, non-viscous, non-heat conducting, and neglect the vertical velocity as well as dependency on vertical direction due to the shallow effects. Since the fluid is incompressible the density $\bar{\rho}$ is constant. But the height $h(x,t)$ of the shallow water may vary. Let $\eta(x,t)$ describes the free water surface and $b(x)$ relief of bottom solid surface. Thus $h(x,t) = \eta(x,t) - b(x)$, and the total mass in some volume $\sigma(t) = [x_1(t), x_2(t)]\times$ cross-sectional area at time t is

$$m(\sigma(t)) = \int_{x_1(t)}^{x_2(t)} \bar{\rho} h(x,t) dx.$$

The conservation of mass postulates that $\dfrac{\partial m(\sigma(t))}{\partial t} = 0$. Using the Transport theorem 1.1.1 and the fact that $\bar{\rho} = \text{const.}$ we get

(i) $$h_t + (hu)_x = 0,$$

where $u(x,t)$ denotes the velocity of the shallow water. Further, the one-dimensional momentum equation (1.2.2') with zero outer forces yields

(ii) $$u_t + uu_x = -\frac{1}{\bar{\rho}} p_x.$$

Now the pressure p is determined from a hydrostatic law, stating that the pressure at the depth y is $\bar{\rho} g(\eta - y)$, where g is the gravitational constant. Thus,

(iii) $$p_x = \bar{\rho} g \eta_x.$$

We multiply equation (i) by u and equation (ii) by h. Adding them together and using (iii) yields the momentum equation for the one-dimensional shallow water in conservative variables

(iv) $$(hu)_t + (hu^2 + \frac{1}{2} g h^2)_x = -ghb_x.$$

Equations (i) and (iv) form the one-dimensional shallow water system.

In more general situations, e.g. oceanography or meteorology, full three-dimensional flows have to be considered. Under the assumption on shallow effects they can be modelled mathematically by the two-dimensional shallow water equations with a source term

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{w})}{\partial x} + \frac{\partial \mathbf{g}(\mathbf{w})}{\partial y} = \mathbf{t}(\mathbf{w}), \quad \mathbf{x} = (x, y)^T \in \mathbb{R}^2,$$

where

$$\mathbf{w} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} \quad \mathbf{f}(\mathbf{w}) = \begin{pmatrix} hu \\ hu^2 + \frac{gh^2}{2} \\ huv \end{pmatrix},$$

$$\mathbf{g}(\mathbf{w}) = \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{gh^2}{2} \end{pmatrix}, \quad \mathbf{t}(\mathbf{w}) = \begin{pmatrix} 0 \\ -gh\frac{\partial b}{\partial x} + fhv - ghS_{f1} \\ -gh\frac{\partial b}{\partial y} - fhu - ghS_{f2} \end{pmatrix}.$$

Here u, v are the x, y components of the depth averaged velocities of the flow, respectively. In general, there are more then just the bottom relief effects. For example, the Coriolis forces f arise from earth's rotation and S_{f1}, S_{f2} are friction slopes resulting from viscosity effects. Due to the bottom friction terms the flow is retarded. Usually it is assumed that the bottom friction stresses depend quadratically on the depth-averaged velocities.

The eigenvalues of this hyperbolic conservation law are $\lambda_1 = u\cos\theta + v\sin\theta - \sqrt{gh}$, $\lambda_2 = u\cos\theta + v\sin\theta$, $\lambda_3 = u\cos\theta + v\sin\theta + \sqrt{gh}$, where $\theta \in [0, 2\pi)$ and $\sqrt{gh} = c$ denotes the wave celerity or the wave speed. The corresponding linearly independent right eigenvectors are

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ u - c\cos\theta \\ v - c\sin\theta \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 0 \\ \sin\theta \\ -\cos\theta \end{pmatrix}, \quad \mathbf{r}_3 = \begin{pmatrix} 1 \\ u + c\cos\theta \\ v + c\sin\theta \end{pmatrix}.$$

Analogously as in the gas dynamics we introduce the so-called Froude number

$$Fr = \frac{|\mathbf{v}|}{c},$$

which plays an important role in the classification of shallow flows. The shallow flow is called supercritical, critical or subcritical for $Fr > 1$, $Fr = 1$, and $Fr < 1$, respectively.

Exercise:
Derive the two-dimensional shallow water equations in primitive variables (h, u, v). Which are the regularity conditions you need to assume? Compute the eigenvalues and the corresponding eigenvectors! Explain why the eigenvalues did not change!

2.1.9 Hyperbolic systems. In this subsection the theory of multi-dimensional hyperbolic conservation laws will be reviewed. The theory gives a good insight into underlying properties that occur in an analytical solution of a set of conservation laws. The aim is to get a better understanding of properties of solutions to hyperbolic systems such that they can be used in construction of an adequate numerical

scheme. To simplify the situation we will consider the Cauchy problem (we eliminate boundary conditions):

$$\frac{\partial \mathbf{w}}{\partial t}+\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\mathbf{f}_j\left(\mathbf{w}\right)=\mathbf{0}\qquad \text{in } \mathbb{R}^d\times\langle 0,\infty),\tag{2.1.10}$$

$$\mathbf{w}\left(\cdot,0\right)=\mathbf{w}_0\qquad \text{in } \mathbb{R}^d,\tag{2.1.11}$$

where $\mathbf{w}:\mathbb{R}^d\times\langle 0,\infty)\to D$, $D\subseteq\mathbb{R}^s$ is an open set, $\mathbf{f}_j\in C^1\left(D;\mathbb{R}^s\right), j=1,2,\ldots,d$. Here s states for the number of equations and d is a dimension.

Repeating Definition 2.1.6 for general case of vector $\mathbf{w}$ and $\mathbf{f}_j,\ j=1,2,\ldots,d$, we get the definition of hyperbolic systems. In what follows we assume that the system (2.1.10) is *hyperbolic.* The concept *hyperbolic conservation laws* is also used in literature. By the definition of hyperbolic system the equivalent form of (2.1.10) is

$$\frac{\partial \mathbf{w}}{\partial t}+\sum_{j=1}^{d}\mathbb{A}_j\left(\mathbf{w}\right)\frac{\partial \mathbf{w}}{\partial x_j}=\mathbf{0},\tag{2.1.12}$$

where $\mathbb{A}_j\left(\mathbf{w}\right)=\dfrac{D\mathbf{f}_j\left(\mathbf{w}\right)}{D\mathbf{w}}$, and the matrix $\mathbb{P}\left(\mathbf{w},\boldsymbol{\nu}\right)=\sum\limits_{j=1}^{d}\nu_j\mathbb{A}_j\left(\mathbf{w}\right);\ \mathbf{w}\in D,\ \boldsymbol{\nu}\in\mathbb{R}^d;$ has s real eigenvalues $\lambda_i=\lambda_i\left(\mathbf{w},\boldsymbol{\nu}\right), i=1,2,\ldots,s.$

Definition 2.1.13. *A vector valued function* $\mathbf{w}\in C^1\left(\mathbb{R}^d\times\langle 0,\infty);D\right)$ *satisfying* (2.1.10), (2.1.11) *pointwise is called a* <u>*classical solution*</u>*.*

We have already mentioned that an important property of hyperbolic conservation law is that it may develop discontinuities in the solution, even if the data $\mathbf{w}_0$ and $\mathbf{f}_j$ are infinitely smooth. This can be shown even for very simple example of the Cauchy problem for inviscid Burgers' equation $(d=s=1)$:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=0,\tag{2.1.14}$$

$$u\left(x,0\right)=\begin{cases}1, & x\le 0,\\ 1-x, & 0\le x\le 1,\\ 0, & 1\le x.\end{cases}$$

The solution of the first order parabolic equation (2.1.14) can be found by the method of characteristics. The characteristics $x=x\left(t\right)$ of (2.1.14) satisfy

$$\frac{d}{dt}x\left(t\right)=u\left(x\left(t\right),t\right),\tag{2.1.15}$$

and along each characteristic u is constant, since

$$\frac{d}{dt}u\left(x\left(t\right),t\right)=\frac{\partial}{\partial t}u\left(x\left(t\right),t\right)+\frac{\partial}{\partial x}u\left(x\left(t\right),t\right)\cdot\frac{d}{dt}x\left(t\right)=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0.$$

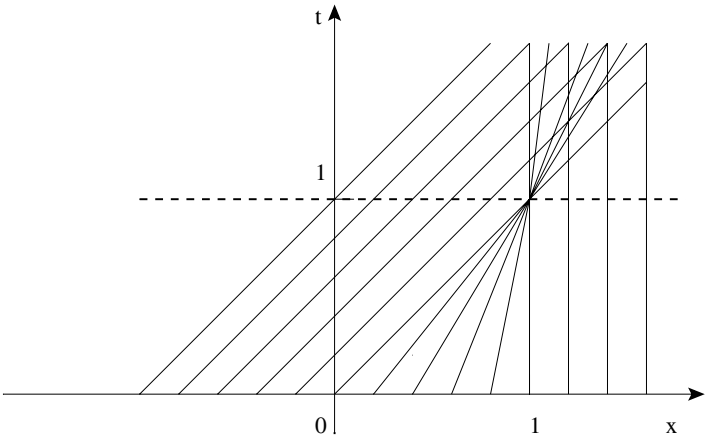

FIG. 2.1. Characteristics for the Burgers equation (2.1.14).

Moreover, since u is constant on each characteristic, the slope $\frac{d}{dt}x(t)$ is constant by (2.1.15). Thus, the characteristics are straight lines, determined by the initial data (see Fig. 2.1).

The characteristics do not intersect for $t < 1$ and the solution, constant along characteristics, is

$$u(x,t) = \begin{cases} 1, & x \le t, \\ (1-x)/(1-t), & t \le x \le 1, \\ 0, & x \ge 1, \end{cases} \qquad \text{for } t < 1. \tag{2.1.16}$$

At the point (1, 1) the characteristics intersect and the discontinuity of the solution develops. It will be shown that for $t \ge 1$ a generalized weak solution with the discontinuity on the line $x = (t+1)/2$ can be defined in the form

$$u(x,t) = \begin{cases} 1, & x < \dfrac{t+1}{2}, \\ 0, & x > \dfrac{t+1}{2}, \end{cases} \qquad \text{for } t \ge 1. \tag{2.1.16$'$}$$

A very suitable technique to describe such phenomena is the theory of distributions. Thus, in order to admit also discontinuous solutions one has to use the weak formulation of conservation laws. This notation is analogous to the concept of weak solution already defined in previous courses of partial differential equations. Note however, that because of less regularity, i.e. "smoothness", of solutions to hyperbolic problems, the Sobolev spaces $W^{k,l}$ used for parabolic and elliptic equations are inappropriate in our case. The only thing we can assume about the solution is that it does not blow up, i.e. it is "essentially" bounded. Thus, we need to work with the class L^∞_{loc} of locally bounded measurable functions, i.e. $L^\infty_{loc}\left(\mathbb{R}^d\right) \equiv \left\{w; \|w\|_{L^\infty(K)} < \infty \ \forall K \subset \mathbb{R}^d,\ K \text{ compact}\right\}$. See also [Kröner], [Feistauer] for more details.

DEFINITION 2.1.17. *Let* $C_0^\infty\left(\mathbb{R}^d \times \langle 0,\infty); \mathbb{R}^s\right) := \left\{\boldsymbol{\varphi} \in C^\infty\left(\mathbb{R}^d \times \langle 0,\infty); \mathbb{R}^s\right);$ *supp* $\boldsymbol{\varphi}$ *is a compact set in* $\mathbb{R}^d \times \langle 0,\infty)\right\}$ *and* $\mathbf{w}_0 \in L^\infty_{loc}\left(\mathbb{R}^d; D\right)$. *A vector valued*

function $\mathbf{w}$*, s.t.* $\mathbf{w} \in L^{\infty}_{loc}\left(\mathbb{R}^d \times \langle 0,\infty); D\right)$*, is called a* <u>*weak solution*</u> *to the problem* (2.1.10), (2.1.11)*, if the following integral identity holds*

(2.1.18)
$$\int_0^{\infty} \int_{\mathbb{R}^d} \left(\mathbf{w} \frac{\partial \boldsymbol{\varphi}}{\partial t} + \sum_{j=1}^{d} \mathbf{f}_j(\mathbf{w}) \frac{\partial \boldsymbol{\varphi}}{\partial x_j} \right)$$
$$+ \int_{\mathbb{R}^d} \mathbf{w}_0 \boldsymbol{\varphi}(\cdot, 0) = 0 \qquad \forall \boldsymbol{\varphi} \in C_0^{\infty}\left(\mathbb{R}^d \times \langle 0,\infty); \mathbb{R}^s\right).$$

Exercise:
Show that the classical solution (2.1.13) is a weak one, and conversely: a weak solution satisfies (2.1.10), (2.1.11) in the sense of distributions on $\mathbb{R}^d \times (0,\infty)$.

Further, it follows from the weak formulation that across discontinuities the so-called Rankine–Hugoniot relation must hold.

Definition 2.1.19. *A function* $\mathbf{w} : \mathbb{R}^d \times \langle 0,\infty) \to \mathbb{R}^s$ *is* <u>*piecewise smooth*</u> *if there is a finite number of smooth hypersurfaces* Γ *in* $\mathbb{R}^d \times \langle 0,\infty)$ *s.t. the function* $\mathbf{w}$ *is smooth in* $\mathbb{R}^d \times \langle 0,\infty) \setminus \Gamma$ *and has one-sided limits* $\mathbf{w}^{\pm}$ *on* Γ*, i.e.* $\mathbf{w}^{\pm}(\mathbf{x},t) := \lim\limits_{\varepsilon \to 0+} \mathbf{w}\left((\mathbf{x},t) \pm \varepsilon \mathbf{n}\right)$*, where* $\mathbf{n} = (n_{x_1}, n_{x_2}, \ldots, n_{x_d}, n_t)$ *is a normal to* Γ.

Theorem 2.1.20. *(Rankine–Hugoniot condition) Let* $\mathbf{w} : \mathbb{R}^d \times \langle 0,\infty) \to D \subset \mathbb{R}^s$ *be a piecewise smooth function. Then* $\mathbf{w}$ *is a solution of* (2.1.10) *in the sense of distributions on* $\mathbb{R}^d \times (0,\infty)$ *if and only if it satisfies the following conditions :*

(1) $\mathbf{w}$ *is a classical solution in any domain where* $\mathbf{w}$ *is* C^1 *function;*
(2) *jump condition:* $(\mathbf{w}^+ - \mathbf{w}^-) n_t + \sum_{j=1}^{d} \left(\mathbf{f}_j(\mathbf{w}^+) - \mathbf{f}_j(\mathbf{w}^-)\right) n_j = 0$ *on any hypersurface of discontinuity* Γ.

Proof. *Do it as an exercise!*

□

Let us come back to the example (2.1.14). We can verify that the Rankine–Hugoniot condition (2.1.20)ii) holds on $\Gamma = \left\{ (x,t)\,;\, x = \dfrac{t+1}{2},\, t > 1 \right\}$. The outer normal to the hypersurface of discontinuity $\Gamma = \{(x,t)\,;\, x = \xi(t)\}$ can be expressed in the form $\mathbf{n} = (n_x, n_t) = (1, -s)$, where $s = \dfrac{d\xi(t)}{dt}$. The Rankine–Hugoniot condition becomes

(2.1.21) $$s\left(u^+ - u^-\right) = f\left(u^+\right) - f\left(u^-\right).$$

From (2.1.16) we get that $u^+ - u^- = 1$, $f(u^+) - f(u^-) = \frac{1}{2}$ at $(x,t) = (1,1)$. Since, by the definition of Γ, $s = \frac{1}{2}$, the condition (2.1.21) is satisfied.

It is easy to verify that outside Γ the function u, define in (2.1.16′), is a classical solution. Thus, it is a weak solution on $\mathbb{R} \times \langle 0,\infty)$.

Unfortunately, there is often more than one weak solution to the conservation law with the same initial data. For example, if we solve Burgers' equation (2.1.14) with the initial data

$$u(x,0) = \begin{cases} 0, & x \le 0, \\ \\ 1, & x > 0, \end{cases} \tag{2.1.22}$$

then there are infinitely many weak solutions. For example,

$$u(x,t) = \begin{cases} 0, & x < s_m t, \\ u_m, & s_m t \le x \le u_m t, \\ \dfrac{x}{t}, & u_m t \le x \le t, \\ 1, & x > t; \end{cases} \tag{2.1.23}$$

is a weak solution for any u_m s.t. $0 \le u_m \le 1$ and $s_m = \dfrac{u_m}{2}$. This can be easily verified by Theorem 2.1.20.

If our conservation law is to model the real world then clearly only one of the solutions is physically relevant. The fact, that the equations have other, spurious, solutions is a result of the fact that our equations are only a model of reality and some physical effects have been ignored. In particular, hyperbolic conservation laws do not include diffusive or viscous effects. Although these effects may be negligible throughout most of the flow, near discontinuities the effect is always strong. In fact, the full Navier–Stokes equations have smooth solutions, for some simple problems that we consider, and the apparent discontinuities are in reality thin regions. What we hope to model with the Euler equations is the limit of the smooth solution to the Navier–Stokes equations as the viscosity parameter approaches zero. This leads to the desired weak solution of the Euler equations.

But there are also other weak solutions even to simple hyperbolic conservation equations. We must use our knowledge of what is being ignored. This helps to pick out the correct weak solution.

The following approach, called vanishing viscosity method, was suggested by P. D. Lax in 1954. We introduce a missing diffusive term into the equation and obtain an equation with a unique smooth solution. Then let the coefficient of this term tends to zero. This idea of vanishing viscosity method is used in analysis of conservation laws and very often for the construction of sufficiently dissipative numerical scheme, which gives physically relevant approximate solutions.

On the other hand, this method is not optimal in order to define physically relevant solution, since it requires study of a more complicated system of equations. This is precisely what we wanted to avoid by introducing an inviscid fluid. But it is a good hint in order to derive other condition that can be imposed directly on the weak solutions of the hyperbolic system to pick out the physically correct solution (cf. Theorem 2.1.27). Taking into account the physical background we see a suitable candidate for such a condition, namely the second law of thermodynamics (cf. (1.3.11)) or the entropy condition. It says that entropy is nondecreasing in time. In fact, as molecules of a gas pass through a shock their entropy increase. Now we approach the mathematical definition of entropy.

DEFINITION 2.1.24. *A convex function $U : \mathbb{R}^s \to \mathbb{R}$ is said to be entropy of (2.1.10), if there are some entropy fluxes $F_1, \dots, F_d : D \subset \mathbb{R}^s \to \mathbb{R}$, s.t.*

$$\left(\operatorname{grad} U(\mathbf{w})\right)^T \cdot \frac{D\mathbf{f}_j(\mathbf{w})}{D\mathbf{w}} = \operatorname{grad} F_j(\mathbf{w}), \qquad j = 1, 2, \dots, d. \tag{2.1.25}$$

It can be verified easily that the entropy $U(\mathbf{w})$ is conserved for smooth solutions of hyperbolic conservation laws (2.1.10), thus

$$\frac{\partial U(\mathbf{w})}{\partial t} + \sum_{j=1}^{d} \frac{\partial F_j(\mathbf{w})}{\partial x_j} = 0 \tag{2.1.26}$$

holds in the sense of distributions on $\mathbb{R}^d \times (0, \infty)$.

But for the discontinuous solutions we cannot perform the same manipulations. Since we are particularly interested in how the entropy behaves for the vanishing viscosity weak solution, we look at the related viscous problem and then let the viscosity tend to zero. This is exactly what the following theorem says.

THEOREM 2.1.27. *Let system (2.1.10) have a convex entropy $U \in C^2(\mathbb{R}^s; \mathbb{R})$ and entropy fluxes $F_j \in C^1(\mathbb{R}^s, \mathbb{R})$, $j = 1, 2, \dots, d$. Let $\{\mathbf{w}^\varepsilon\}$ be a sequence of sufficiently smooth solutions to the pertubated Cauchy problem (2.1.10), (2.1.11), i.e. the R.H.S. of (2.1.10) is equal to $\varepsilon \Delta \mathbf{w}^\varepsilon$. Let the following conditions hold*

(i) $$\|\mathbf{w}^\varepsilon\|_{L^\infty\left(\mathbb{R}^d \times (0,\infty); \mathbb{R}^s\right)} \le c,$$

uniformly with respect to $\varepsilon > 0$,

(ii) $$\mathbf{w}^\varepsilon \to \mathbf{w} \qquad as \qquad \varepsilon \to 0 \qquad a.e.\ in\ \mathbb{R}^d \times \langle 0, \infty),$$

Then $\mathbf{w}$ is a weak solution of (2.1.10), (2.1.11) and

$$\frac{\partial U(\mathbf{w})}{\partial t} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} F_j(\mathbf{w}) \le 0 \tag{2.1.28}$$

holds in the sense of distributions on $\mathbb{R}^d \times (0, \infty)$, i.e.

$$\int_0^\infty \int_{\mathbb{R}^d} U(\mathbf{w}) \frac{\partial \varphi}{\partial t} + \sum_{j=1}^{d} F_j(\mathbf{w}) \frac{\partial \varphi}{\partial x_j} \ge 0 \tag{2.1.29}$$

for all $\varphi \in C_0^\infty\left(\mathbb{R}^d \times (0, \infty)\right), \varphi \ge 0$.

PROOF. (see, e.g., [Kröner]).

□

Now we define an entropy solution on the basis of the result from Theorem 2.1.27.

DEFINITION 2.1.30. *A weak solution* $\mathbf{w}$*, s.t.* $\mathbf{w} \in L^{\infty}_{loc}\left(\mathbb{R}^d \times \langle 0,\infty); \mathbb{R}^s\right)$*, of the Cauchy problem* (2.1.10), (2.1.11)*, is said to be an* entropy weak solution*, if for all entropies* U *and corresponding entropy fluxes* F_j*,* $j = 1, 2, \ldots, d$*,*

$$\frac{\partial U(\mathbf{w})}{\partial t} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} F_j(\mathbf{w}) \leq 0$$

holds in the sense of distributions on $\mathbb{R}^d \times (0,\infty)$*.*

A natural suggestion arises : what is a relation between the mathematical entropy U and the physical entropy η (cf. (1.3.10))? In reference [Feistauer], it was shown that for the Euler equations this relation is

$$U = -\rho\,\eta,\ F_j = -\rho\, v_j\, \eta, \qquad j = 1, 2, \ldots, d. \tag{2.1.31}$$

The inequality (2.1.28) becomes

$$\frac{\partial}{\partial t}(\rho\,\eta) + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}(\rho\, v_j\, \eta) \geq 0 \tag{2.1.32}$$

in the sense of distributions on $\mathbb{R}^d \times (0,\infty)$. Hence, using (2.1.32) and the continuity equation (1.1.3) we obtain

$$\rho \frac{d}{dt}(\eta) \geq 0 \qquad \text{(in the sense of distributions)}. \tag{2.1.33}$$

This inequality is in agreement with the second law of thermodynamics (1.3.11), where now $q = 0$, $\mathbf{q} = \mathbf{0}$. Thus, (2.1.33) is the mathematical formulation of the well-known fact: entropy of the system is not decreasing in time.

Although there is still hope that there exists the weak entropy solution even in the case of a general hyperbolic system and consequently, of the Euler equations, too, till now it has not been proved. As far as the uniqueness concerns, there is a counter example to the uniqueness of entropy solution to a hyperbolic system [Sever]. Nevertheless, it is still believed that there is a unique weak entropy solution of the Euler equations. In the scalar case ($s = 1$) we have a fundamental Kruzhkov's result, which holds in any space dimension d.

THEOREM 2.1.34. *(Kruzhkov, 1970) If* $w_0 \in L^{\infty}\left(\mathbb{R}^d\right)$*, then the problem* (2.1.10) (2.1.11)*, where* $s = 1$*, has a unique weak entropy solution* $w \in L^{\infty}\left(\mathbb{R}^d \times \langle 0,\infty)\right)$*.*

PROOF. (see [Kruzhkov]).

□

Remark 2.1.35. This Kruzhkov solution can be found by the method of vanishing viscosity. By adding a term $\varepsilon \Delta w^{\varepsilon}$ to the R.H.S. of (2.1.10), we get the parabolic perturbation of (2.1.10), i.e.

$$\frac{\partial w^{\varepsilon}}{\partial t} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} f_j(w^{\varepsilon}) = \varepsilon \Delta w^{\varepsilon} \qquad \text{in } \mathbb{R}^d \times (0,\infty),$$

$$w^{\varepsilon}(\cdot, 0) = w_0 \qquad \text{in } \mathbb{R}^d,\ \varepsilon > 0.$$

The aim is to study the behaviour of the solution w^ε when passing to the limit as $\varepsilon \to 0$. We will not go into the details of the proof, we just note that one possible approach is based on the concept of the *measure-valued solution* , which is a solution taken even in a weaker sense than the concept of the weak solutions. This concept is often used to study the convergence of numerical methods used for approximation of hyperbolic conservation laws; as an example see Theorem 2.2.52.

In the case of general hyperbolic systems of first order there are only particular existence and uniqueness results for the one-dimensional situation. For the rest of this section we assume that $d = 1$ and $s \geq 1$. The existence of weak entropy solution was proved by Glimm in 1965. In order to formulate this result we need to introduce some new concepts.

DEFINITION 2.1.36. *Let* $\mathbf{w} \in L^1_{loc}(\mathbb{R})$. *The total variation of* $\mathbf{w}$ *is given as*

$$TV(\mathbf{w}) \equiv \sup_{h\in\mathbb{R}-\{0\}} \frac{1}{h} \int_{\mathbb{R}} |\mathbf{w}(x+h) - \mathbf{w}(x)|\, dx.$$

The space of all functions from $\mathbb{R}$ *to* $\mathbb{R}^s$ *with bounded total variation is defined as*

$$BV(\mathbb{R}) \equiv \{\mathbf{w} \in L^1_{loc}(\mathbb{R}); TV(\mathbf{w}) < \infty\},$$

and the associated norm is given in the following way

$$\|\mathbf{w}\|_{BV} \equiv \|\mathbf{w}\|_{L^1} + TV(\mathbf{w}).$$

For simplification we are using here and in what follows notations $L^1_{loc}(\mathbb{R})$, $BV(\mathbb{R})$ instead of more precise $L^1_{loc}(\mathbb{R};\mathbb{R}^s)$, $BV(\mathbb{R};\mathbb{R}^s)$, respectively. From the definition above it follows how to understand total variation of sequences approximating discrete functions. Let $w = \{w_j\}_{j\in\mathbb{N}}$ be a sequence of discrete values w_j. Then we have for the total variation of w

$$TV(w) = \sum_{j=0}^{\infty} |w_{j+1} - w_j|.$$

In order to get better understanding of how large the BV space is, let us recall a powerful compactness property, the well-known selection principle of Helly.

THEOREM 2.1.37. *(Selection principle of Helly) Let* $\{u_n\}$ *be a sequence of functions in* $L^1(a,b)$ *such that*

$$\|u_n\|_{L^\infty[a,b]} \leq c, \quad TV_{[a,b]}(u_n) \leq c \qquad n \in \mathbb{N}.$$

Then there exists a subsequence $\{u_{n'}\}$ *and an* $u \in L^1(a,b)$ *such that*

$$u_{n'} \longrightarrow u \qquad \text{in } L^1(a,b).$$

□

In fact, we have for the BV spaces the following imbedding properties $W^{1,1}_{loc}(\mathbb{R}) \hookrightarrow BV(\mathbb{R}) \hookrightarrow\hookrightarrow L^1_{loc}(\mathbb{R})$.

Let $\mathbf{r}_k(\mathbf{w}), k = 1, 2, \ldots, s$, be the eigenvectors of the Jacobian matrix $\mathbb{A}(\mathbf{w}), \mathbf{w} \in D$, and $\lambda_k(\mathbf{w}), k = 1, 2, \ldots, s$, be the corresponding eigenvalues, cf. (2.1.12).

DEFINITION 2.1.38. *An eigenvector* $\mathbf{r}_k$ *is called* genuinely nonlinear, *if*

$$\text{grad } \lambda_k(\mathbf{w}) \cdot \mathbf{r}_k(\mathbf{w}) \neq 0 \qquad \forall \mathbf{w} \in D.$$

We say that $\mathbf{r}_k$ *is* linearly degenerate, *if*

$$\text{grad } \lambda_k(\mathbf{w}) \cdot \mathbf{r}_k(\mathbf{w}) = 0 \qquad \forall \mathbf{w} \in D.$$

Now we can approach to the formulation of the existence result of Glimm.

THEOREM 2.1.39. *(Glimm, 1965) Let the initial data* $\mathbf{w}_0 \in L^\infty_{loc}(\mathbb{R})$ *and let there exist* $\delta > 0$, *such that the total variation* $TV(\mathbf{w}_0) < \delta$. *Further, let the system be strictly hyperbolic, i.e. all eigenvalues* $\lambda_k(\mathbf{w})$ *are distinct, and let each eigenvector is either genuinely nonlinear or linearly degenerate. Then there exists a weak entropy solution* $\mathbf{w} \in L^\infty([0,T], BV(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R}))$ *of* (2.1.10), (2.1.11), $(d = 1)$.

In 1995 Bressan, see e.g. [Bressan], proved the uniqueness of the weak entropy solution in the above class of BV solutions using the semigroup technique. Let us note that the assumption on small total variation is necessary; namely if $TV(\mathbf{w}_0) \to \infty$ there exist at least two weak entropy solutions to the system (2.1.10), (2.1.11), $(d = 1)$.

2.1.40 Riemann problem. Many of nowadays numerical schemes used for hyperbolic systems, cf. Godunov finite volume methods 2.2.54, are based on finding a solution, or at least its approximation, to the Riemann problems solved in the normal directions to the mesh cell interfaces. We will deal with this point more deeply in the next Section 2.2, where the finite volume methods are described. The aim of this section is to show properties of solution of the following one-dimensional Riemann problem

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbb{A}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x} = \mathbf{0} \qquad \text{on } \mathbb{R} \times (0, \infty), \tag{2.1.41}$$

$$\mathbf{w}(x,0) = \begin{cases} \mathbf{w}_L, & x < 0, \\ \\ \mathbf{w}_R, & x > 0 \end{cases}$$

with given constant initial states $\mathbf{w}_L,\ \mathbf{w}_R \in D$. As a physical interpretation of (2.1.41) we can imagine a long tube filled with two types of gas, e.g. rare and dense gas. These are separated with a membrane. We are interested in the gas motion after the membrane is suddenly removed. In Fig. 2.2 solution of the Riemann problem for the Euler equations is plotted. Initial data are taken as follows:

$$\begin{aligned} &\rho = 1,\ p = 1,\ v_1 = v_2 = 0 && x \leq 0 \\ &\rho = 0.125,\ p = 0.1,\ v_1 = v_2 = 0 && x > 0. \end{aligned}$$

We can notice all three fundamental parts of solution: rarefaction wave, contact discontinuity and shock, which we will describe in what follows.

THEOREM 2.1.42. *(Self-similarity form) Let the Riemann problem (2.1.41) has a unique piecewise smooth weak solution* $\mathbf{w}$, *then* $\mathbf{w}$ *can be written in the similarity form* $\mathbf{w}(x,t) = \tilde{\mathbf{w}}(x/t)$, *where* $\tilde{\mathbf{w}} : \mathbb{R} \to \mathbb{R}^s,\ t > 0$.

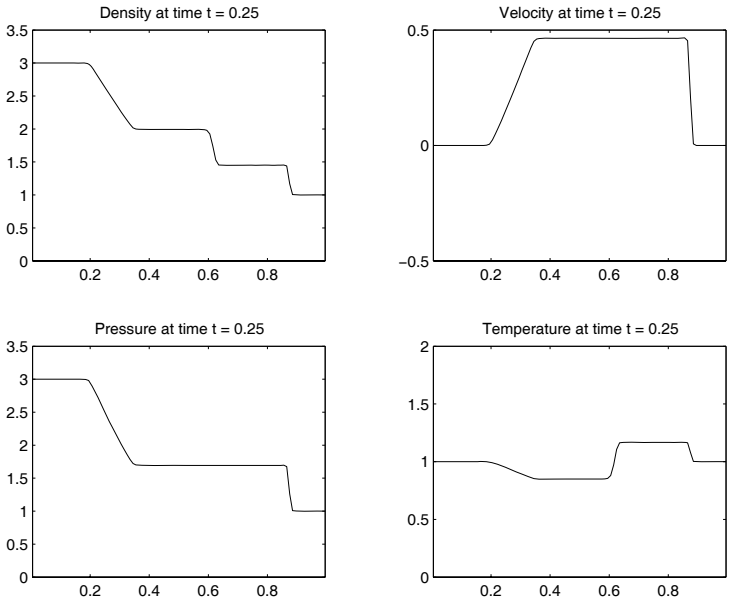

FIG. 2.2. Solution to the Riemann problem of the Euler equations; plots of density, velocity, pressure and temperature at $t = 0.25$.

PROOF. It is easy to realize that $\mathbf{w}(\alpha x, \alpha t)$ is also a solution of (2.1.41) for every $\alpha > 0$. Due to the uniqueness, we have $\mathbf{w}(\alpha x, \alpha t) = \mathbf{w}(x, t)$, which means that $\mathbf{w}$ is homogeneous of order zero. Now, taking $\alpha = 1/t$, we see that $\mathbf{w}(x, t) = \mathbf{w}(x/t, 1) = \tilde{\mathbf{w}}(x/t)$.

□

Solution to the linear Riemann problem. Assume that in (2.1.41) the Jacobian matrix $\mathbb{A} = \text{const}$. The system $\{\mathbf{r}_k\}_{k=1}^s$ of all eigenvectors of $\mathbb{A}$ creates a basis in $\mathbb{R}^s$. Thus,

$$\mathbf{w}_L := \sum_{k=1}^{s} \alpha_k \mathbf{r}_k, \qquad \mathbf{w}_R := \sum_{k=1}^{s} \beta_k \mathbf{r}_k,$$

and we can rewrite the initial data by means of

$$\mathbf{w}_0 = \sum_{k=1}^{s} \left[H(x)\beta_k + (1 - H(x))\alpha_k \right] \mathbf{r}_k.$$

Here $H(x)$ denotes the Heaviside function, i.e. $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ else. Let $\mathbb{T}^{-1}$ denotes the inverse matrix to the matrix $\mathbb{T}$ consisting of all eigenvectors $\mathbf{r}_k$. Further denote by $\mathbf{u}$ the characteristic vector, i.e.

$$\mathbf{u} = \mathbb{T}^{-1}\mathbf{w}.$$

Multiplying system (2.1.41) by the matrix $\mathbb{T}^{-1}$ from the left leads to the diagonal system

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbb{D}\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0} \qquad \text{on } \mathbb{R} \times (0, \infty)\,, \tag{2.1.42}$$

with $\mathbb{D} = \mathbb{T}^{-1}\mathbb{A}\mathbb{T}$. This system consists of s separated linear advection equations for each component k. Thus, analogously we can write

$$\frac{\partial u_k}{\partial t} + \lambda_k \frac{\partial u_k}{\partial x} = 0, \qquad k = 1, \ldots, s,$$

and accompany them with the initial data

$$u_k(x,0) = \begin{cases} \alpha_k & x < 0, \\ \beta_k & x \geq 0. \end{cases}$$

Now similarly as for advection equation (2.1.14) each component u_k is determined by the initial data backwards along the characteristic $dx/dt = \lambda_k$. Thus,

$$u_k(x,t) = u_k(x - \lambda_k t, 0), \qquad k = 1, \ldots, s. \tag{2.1.43}$$

Multiplying (2.1.43) by r_k and summing over each $k = 1, \ldots, s$, we obtain

$$\begin{aligned} \mathbf{w}(x,t) &= \sum_{k=1}^{s} u_k(x,t)\, \mathbf{r}_k = \sum_{k=1}^{s} u_k(x - \lambda_k t, 0)\, \mathbf{r}_k \\ &\sum_{k=1}^{s} \left[\beta_k H(x - \lambda_k t) + \alpha_k (1 - H(x - \lambda_k t))\right] \mathbf{r}_k. \end{aligned} \tag{2.1.44}$$

If $-\infty = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_s < \lambda_{s+1} = \infty$ and for some $k \geq 0$ $\lambda_k < \lambda_{k+1}$ then $\mathbf{w}$ is constant in the domain $\Omega_k = \{(x,t); t > 0, \lambda_k < x/t < \lambda_{k+1}\}$:

$$\begin{aligned} &\mathbf{w}(x,t) = \mathbf{w}_k \qquad (x,t) \in \Omega_k, \\ &\mathbf{w}_k = \sum_{j=1}^{k} \beta_j \mathbf{r}_j + \sum_{j=k+1}^{s} \alpha_j \mathbf{r}_j, \qquad k = 1, \ldots, s. \end{aligned}$$

Exercise:
Show that the function $\mathbf{w}$ defined in (2.1.44) is a weak solution to the linear Riemann problem (2.1.41).
Hint : Use Theorem 2.1.20 and show that on each discontinuity $x/t = \lambda_k$ the Rankine-Hugoniot conditions are satisfied.

Solution to the nonlinear Riemann problem. Above we have shown that solution to the linear Riemann problem consists of constant states, sometimes called *waves*, separated by discontinuities $x/t = \lambda_k$, which are also called *linear shocks*. In what follows we will deal with nonlinear problem (2.1.41). We will see that the structure of solution will be now more complex. First we have to introduce some new concepts.

I. Shock discontinuity. Let the k-th eigenvector $\mathbf{r}_k$, $k = 1, \ldots, s$, be genuinely nonlinear, cf. Definition 2.1.38. Let $\mathbf{w}$ be a piecewise discontinuous solution to (2.1.41), such that

$$\mathbf{w}(x/t) = \begin{cases} \mathbf{w}_L & x/t < \lambda \\ \mathbf{w}_R & x/t > \lambda, \end{cases} \tag{2.1.45}$$

where λ denotes the speed of propagation of discontinuity (direction of discontinuity) corresponding to the eigenvector $\mathbf{r}_k$. The weak entropy solution in the form of (2.1.45) have to satisfy the Rankine-Hugoniot conditions, cf. Theorem 2.1.20, as well as the entropy condition, cf. Definition 2.1.30.
Due to the Rankine-Hugoniot conditions

$$\mathbf{f}(\mathbf{w}_L) - \mathbf{f}(\mathbf{w}_R) = \lambda(\mathbf{w}_L - \mathbf{w}_R). \tag{2.1.46}$$

Moreover we assume that the following so-called *Lax entropy condition* is satisfied

$$\lambda_k(\mathbf{w}_L) > \lambda > \lambda_k(\mathbf{w}_R). \tag{2.1.47}$$

It was proved by Lax [Lax (1)] that for one-dimensional systems condition (2.1.47) is for genuinely nonlinear eigenvectors equivalent to the entropy inequality (2.1.29).

If conditions (2.1.46), (2.1.47) hold, solution (2.1.45) is called *shock discontinuity* or *shock*.

II. Rarefaction wave. Assume again that the k-th eigenvector $\mathbf{r}_k$, $k = 1, \dots, s$, is genuinely nonlinear. Now let $\mathbf{w}$ be a continuous solution to (2.1.41), constructed in such a way that the gap between $\lambda_k(\mathbf{w}_L)$ and $\lambda_k(\mathbf{w}_R)$, $\lambda_k(\mathbf{w}_L) < \lambda_k(\mathbf{w}_R)$, is filled with a fan

$$\lambda_k(\tilde{\mathbf{w}}(x/t)) = \begin{cases} \lambda_k(\mathbf{w}_L), & x/t < \lambda_k(\mathbf{w}_L) \\ x/t, & \lambda_k(\mathbf{w}_L) < x/t < \lambda_k(\mathbf{w}_R) \\ \lambda_k(\mathbf{w}_R), & x/t > \lambda_k(\mathbf{w}_R). \end{cases} \tag{2.1.48}$$

Now, a piecewise smooth weak solution of the Riemann problem (2.1.41) satisfying (2.1.48) can be written in the following form

$$\mathbf{w}(x,t) = \begin{cases} \mathbf{w}_L, & x/t < \lambda_k(\mathbf{w}_L) \\ \tilde{\mathbf{w}}(x/t), & \lambda_k(\mathbf{w}_L) < x/t < \lambda_k(\mathbf{w}_R) \\ \mathbf{w}_R, & x/t > \lambda_k(\mathbf{w}_R) \end{cases} \tag{2.1.49}$$

with C^1- smooth function $\tilde{\mathbf{w}}$. The weak solution given by (2.1.49) is called a *weak rarefaction wave* or a *simple wave*. Note, that since $\tilde{\mathbf{w}}$ is C^1-smooth function the entropy condition is fulfilled as equality (2.1.26).

III. Contact discontinuity. Now, let us assume that the k-th eigenvector $\mathbf{r}_k$ is linearly degenerate, cf. Definition 2.1.38. We look for a piecewise discontinuous solution $\mathbf{w}$ to (2.1.41) in the form (2.1.45). Since $\mathbf{w}$ is a weak solution, the Rankine-Hugoniot conditions (2.1.46) have to be satisfied on the discontinuity $x/t = \lambda$. Further,

$$\lambda_k(\mathbf{w}_L) = \lambda(\mathbf{w}_R) \equiv \lambda, \tag{2.1.50}$$

which follows from the definition of contact discontinuity. Namely,

$$\frac{\lambda_k(\tilde{\mathbf{w}}(\xi))}{d\xi} = \text{grad } \lambda_k(\tilde{\mathbf{w}}(\xi)) \cdot \mathbf{r}_k(\tilde{\mathbf{w}}(\xi)) = 0.$$

The weak entropy solution (2.1.45), which corresponds to the linearly degenerate eigenvector $\mathbf{r}_k$, is called a *contact discontinuity* or a *slip contact*. Note, that this weak discontinuous solution is also an entropy one. It follows from the fact that k-th eigenvalue λ_k stays constant, and thus the problem has in the k-th characteristic field linear-like behaviour, see [Feistauer] for a precise proof.

Now we can approach to the formulation of the existence and uniqueness result for the solution of nonlinear Riemann problem. We present this result without a proof, the reader can find more details e.g. in [Lax (2)].

THEOREM 2.1.51. *(Lax solution to the Riemann problem) Let the hyperbolic system in* (2.1.41) *be strictly hyperbolic and each eigenvector be either genuinely nonlinear or linearly degenerate. Let for the initial conditions of the Riemann problem* (2.1.41) *the difference* $|\mathbf{w}_R - \mathbf{w}_L|$ *be "sufficiently small". Then there exists a unique weak entropy solution to* (2.1.41), *which consists of at most* $s+1$ *constant states separated either by rarefaction waves or shock waves or contact discontinuities.*

Remark 2.1.52. Let us note that the above general considerations for the Riemann problem can be used for particular systems used in practical applications; e.g. the shallow water equations, the Euler equations. For example, for the x-split two-dimensional shallow water equations the vectors $\mathbf{w}$ and $\mathbf{f}(\mathbf{w})$ are

$$\mathbf{w} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \qquad \mathbf{f}(\mathbf{w}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix}.$$

We have three wave families, which are associated with the eigenvalues $\lambda_1 = u - a$, $\lambda_2 = u$, $\lambda_3 = u + a$; $a = \sqrt{gh}$. These three waves separate four constant states $\mathbf{w}_L, \mathbf{w}_R$, which are the left and right given data, and $\mathbf{w}_L^*, \mathbf{w}_R^*$, which are the left and right intermediate states to be determined.

The left and right waves are shocks or rarefactions, while the middle wave is always a shear wave, i.e. the contact discontinuity wave. Across the left and right waves both h and u change, but v remains constant. The tangential velocity v changes across the shear wave discontinuously, other components h and u are constant here. As a result we have for the intermediate star region

$$\mathbf{w}_L^* = \begin{pmatrix} h^* \\ h^*u^* \\ h^*v_L \end{pmatrix}, \qquad \mathbf{w}_R^* = \begin{pmatrix} h^* \\ h^*u^* \\ h^*v_R \end{pmatrix}.$$

The following conditions determine the type of the waves which occur:

$h^* > h_L \ \ldots$	left wave is a shock
$h^* \leq h_L \ \ldots$	left wave is a rarefaction wave;
$h^* > h_R \ \ldots$	right wave is a shock
$h^* \leq h_R \ \ldots$	right wave is a rarefaction wave.

The basic step is to derive two non-linear algebraic equations, which determine h^* and u^*. This follows from the so-called generalised Riemann invariants, which are the ordinary differential equations holding across the corresponding waves, cf., e.g.,

[Toro] for details. In fact for the Riemann problems arising in most practical applications, we can compute exact solution to the one-dimensional Riemann problem. However, due to nonlinearity of the algebraic equations, h^* and u^* are actually computed by some suitable iterative methods, e.g. Newton's iterations.

Remark 2.1.53. In the case of two-dimensional Riemann problem the initial data consists of arbitrary many constant states prescribed in sectors around the origin $(0,0)$. Note that the question of existence and uniqueness of the weak entropy solution to the two-dimensional Riemann problem for nonlinear systems is an open problem. Some particular results have been proven either for linear systems or simplified nonlinear systems, see e.g. Li *et al.* [Li, Zhang, Yang], [Li, Lukáčová, Warnecke].

In Figure 2.3 the two-dimensional circular analogy of the above one-dimensional Riemann problem is modelled by a genuinely multidimensional finite volume method, the so-called evolution Galerkin method, cf. [Lukáčová, Morton, Warnecke], [Lukáčová, Saibertová, Warnecke]. The initial data of the so-called two-dimensional Sod problem are given as

$$\begin{aligned} &\rho = 1, \quad u = 0, \quad v = 0, \quad p = 1, \qquad \|\mathbf{x}\| < 0.4 \\ &\rho = 0.125, \quad u = 0, \quad v = 0, \quad p = 0.1, \qquad \text{else.} \end{aligned}$$

Figure 2.3 shows the isolines of density, $x-$, $y-$ velocity components and pressure computed at time $T = 0.2$ by the second order finite volume evolution Galerkin scheme. The solution exhibits analogous phenomena as in the one-dimensional case. Namely, a circular shock travelling away from the center, a circular contact discontinuity travelling in the same direction and a circular rarefaction wave travelling towards the origin at $(0,0)$.

2.2 Finite Volume Method

In this section we derive commonly used numerical scheme for solving the Euler equations, the so-called **finite volume methods** (FVM). Generally, one has three basic numerical methods for solving PDE's. Namely, the finite difference method, the finite element method and the finite volume method. Advantages of the FVM are the simplicity of the scheme and the automatic control of the conservation, which is a crucial property.

Let us present a derivation of the FVM on the following two-dimensional (i.e. $d = 2$) initial-boundary value problem

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{k=1}^{d} \frac{\partial\, \mathbf{f}_k(\mathbf{w})}{\partial x_k} = \mathbf{0} \qquad \text{in } Q_T \equiv \Omega \times (0,T), \tag{2.2.1}$$

$$\mathbf{w}(\cdot, 0) = \mathbf{w}_0 \qquad \text{in } \Omega, \tag{2.2.2}$$

$$\mathbf{B}(\mathbf{w}) = \mathbf{b} \qquad \text{on } \partial\Omega \times (0,T), \tag{2.2.3}$$

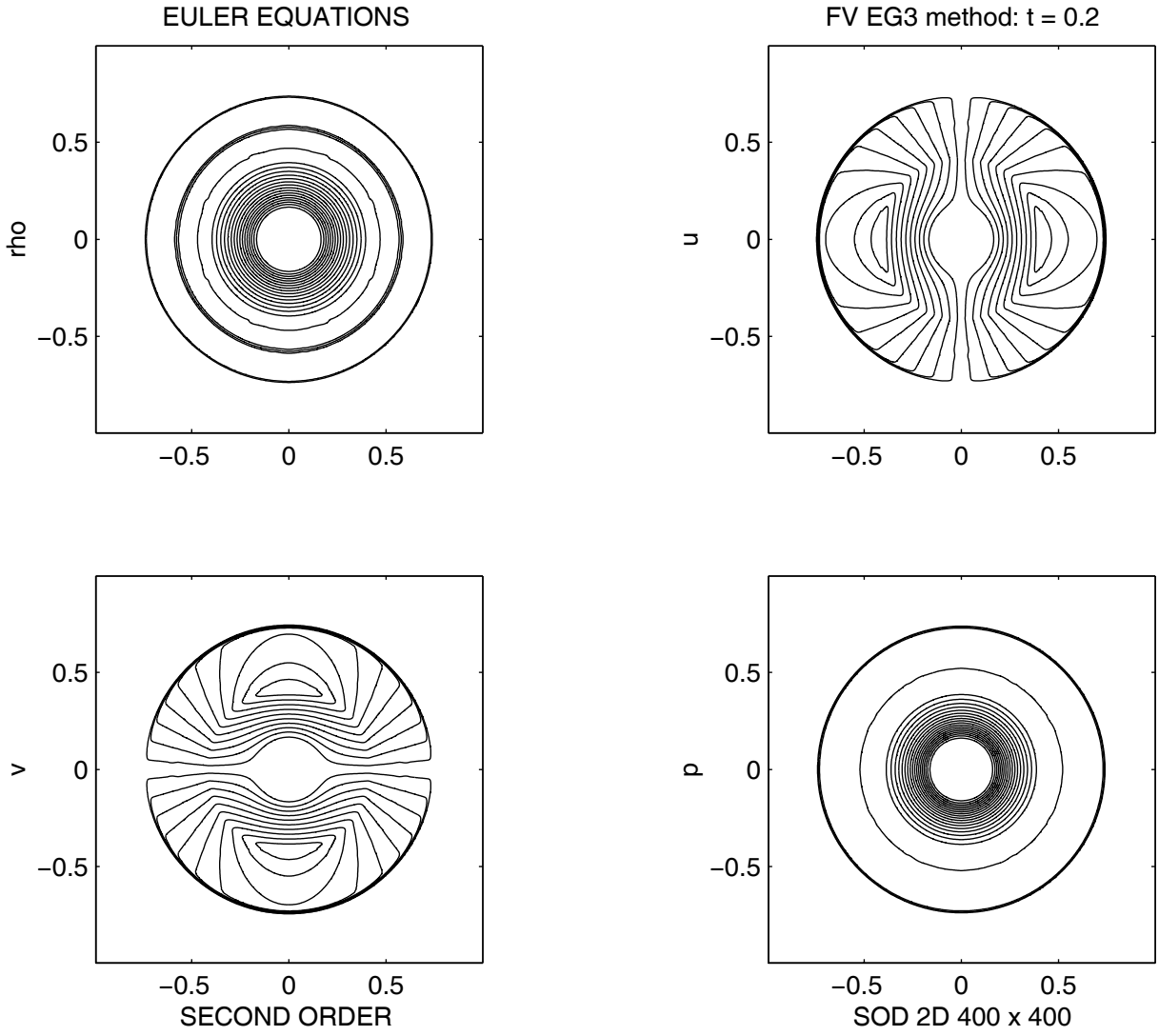

FIG. 2.3. Isolines of the solution to the circular explosion problem for the Euler equations.

We can rewrite (2.2.1) equivalently as

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{f}_1(\mathbf{w})}{\partial x} + \frac{\partial \mathbf{f}_2(\mathbf{w})}{\partial y} = \mathbf{0} \qquad \text{in } Q_T. \tag{2.2.1'}$$

The initial data are denoted by $\mathbf{w}_0$, $\mathbf{B}$ represents the boundary operator and $\mathbf{b}$ is a R.H.S. function. The definition of suitable boundary conditions will be done in Section 2.3. Particularly, we have for the Euler equations (cf. (2.1.1))

$$\mathbf{w} = \begin{pmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ e \end{pmatrix}, \quad \mathbf{f}_j(\mathbf{w}) = \begin{pmatrix} \rho v_j \\ \rho v_1 v_j + p\delta_{1j} \\ \rho v_2 v_j + p\delta_{2j} \\ (e+p)\, v_j \end{pmatrix}, \quad j = 1, 2. \tag{2.2.4}$$

We add the state equation of the form (cf. (1.4.3))

$$p = (\kappa - 1)\left(e - \frac{1}{2}\rho|\mathbf{v}|^2\right). \tag{2.2.5}$$

Let $\mathbb{A}_j(\mathbf{w}) = \dfrac{D\mathbf{f}_j(\mathbf{w})}{D\mathbf{w}}$, $j = 1, 2$, $\mathbf{w} \in D \subset \mathbb{R}^4$, be the Jacobian matrices. It can be shown that

(2.2.6)

$$\mathbb{A}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \dfrac{\kappa-3}{2}v_1^2 + \dfrac{\kappa-1}{2}v_2^2 & (3-\kappa)\,v_1 & (1-\kappa)\,v_2 & \kappa-1 \\ -v_1v_2 & v_2 & v_1 & 0 \\ -\dfrac{\kappa e v_1}{\rho} + (\kappa-1)\,v_1|\mathbf{v}|^2 & \dfrac{\kappa e}{\rho} - \dfrac{\kappa-1}{2}\left(3v_1^2+v_2^2\right) & (1-\kappa)\,v_1v_2 & \kappa v_1 \end{pmatrix}$$

(2.2.7)

$$\mathbb{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -v_1v_2 & v_2 & v_1 & 0 \\ \dfrac{\kappa-3}{2}v_2^2 + \dfrac{\kappa-1}{2}v_1^2 & (1-\kappa)\,v_1 & (3-\kappa)\,v_2 & (\kappa-1) \\ -\dfrac{\kappa e v_2}{\rho} + (\kappa-1)\,v_2|\mathbf{v}|^2 & (1-\kappa)\,v_1v_2 & \dfrac{\kappa e}{\rho} - \dfrac{\kappa-1}{2}\left(3v_2^2+v_1^2\right) & \kappa v_2 \end{pmatrix}$$

Due to hyperbolicity of the Euler equations the matrix

$$\mathbb{P}(\mathbf{w}, \boldsymbol{\nu}) = \nu_1\mathbb{A}_1(\mathbf{w}) + \nu_2\mathbb{A}_2(\mathbf{w}), \tag{2.2.8}$$

where $\mathbf{w} \in D$, $\boldsymbol{\nu} \in \mathbb{R}^2$, is diagonalizable, i.e.

$$\mathbb{P} = \mathbb{T} \cdot \mathbb{D} \cdot \mathbb{T}^{-1}, \tag{2.2.9}$$

$\mathbb{T} = \mathbb{T}(\mathbf{w}, \boldsymbol{\nu})$ is a nonsingular matrix

$$\mathbb{T} = \begin{pmatrix} 1 & 0 & 1/(2a^2) & 1/(2a^2) \\ v_1 & n_2 & (v_1 + an_1)/(2a^2) & (v_1 - an_1)/(2a^2) \\ v_2 & -n_1 & (v_2 + an_2)/(2a^2) & (v_2 - an_2)/(2a^2) \\ |\mathbf{v}|^2/2 & n_2v_1 - n_1v_2 & (H + a\mathbf{v}\cdot\mathbf{n})/(2a^2) & (H - a\mathbf{v}\cdot\mathbf{n})/(2a^2) \end{pmatrix},$$

and

$$\mathbb{D} = \begin{pmatrix} \lambda_1 & \dots & & 0 \\ \vdots & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4 \end{pmatrix}.$$

Here $H = \frac{e+p}{\rho}$ denotes the total specific enthalpy, $n_i = \nu_i/|\boldsymbol{\nu}|$, $i = 1, 2$, $a = \sqrt{\dfrac{\kappa p}{\rho}}$ is a local speed of sound, κ is the Poisson constant, cf. (1.3.6), and the eigenvalues are

$$\begin{aligned} &\lambda_1 = \lambda_2 = \nu_1v_1 + \nu_2v_2, \\ &\lambda_3 = \lambda_1 + a|\boldsymbol{\nu}|, \quad \lambda_4 = \lambda_1 - a|\boldsymbol{\nu}|. \end{aligned} \tag{2.2.10}$$

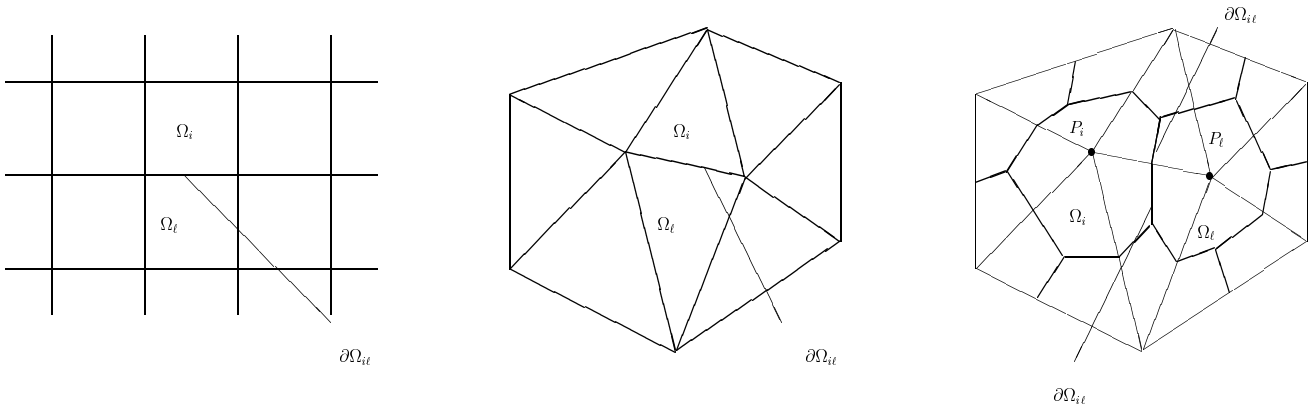

FIG. 2.4. a) Quadrilateral mesh, b) Triangular mesh, c) Dual mesh over a triangular grid.

2.2.11 Unstructured dual finite volume mesh. In what follows we will introduce a discretization of the computational domain, the so-called mesh, which is used for spatial approximation. In fact, we can use arbitrary geometrical objects for mesh cells. In practice, rectangular grids, triangulations or even polygonal discretizations are often used, depending on a problem to be solved; see Fig. 2.4.

In what follows we will work with unstructured, the so-called dual mesh. Having more cell interfaces, i.e. edges, it has an advantage of approximating wave propagation in different directions in a better way. First, we define a basic triangular mesh.

DEFINITION 2.2.12. *Let Ω_h be a polygonal approximation of the given bounded domain $\Omega \subset \mathbb{R}^2$. The set $\mathcal{T}_h = \{T_i\}_{i\in\mathbb{I}}$, where $\mathbb{I} \subseteq \mathbb{N}$ is an index set and $h \in (0, h_0), h_0 > 0$, will be called a basic mesh, if the following properties hold:*

(2.2.13)

(i) *T_i is a closed triangle for all $i \in \mathbb{I}$;*

(ii) $\overline{\Omega}_h = \bigcup_{i\in\mathbb{I}} T_i$, *where* $h = \sup_{i\in\mathbb{I}} \operatorname{diam} T_i$, diam *denotes diameter;*

(iii) *for any $T_i, T_j \in \mathcal{T}_h$, s.t. $T_i \neq T_j$ there are three following possibilities :*
either $T_i \cap T_j = \emptyset$,
or $T_i \cap T_j$ is a common vertex,
or $T_i \cap T_j$ is a common edge;
in this case T_i and T_j are called neighbours.

Now we approach the construction of a dual mesh to the given basic mesh.

2.2.14 Construction of the unstructured dual mesh, which will be denoted by $\mathcal{D}_h = \{D_j\}_{j\in\mathbb{J}}$, $\mathbb{J} \subseteq \mathbb{N}$ is an index set and $h \in (0, h_0)$. Let $\mathcal{P}_h = \{P_j, j \in \mathbb{J}\}$ be a set of all vertices of triangles from $\mathcal{T}_h$. A dual volume D_j that corresponds to the point P_j, $P_j \in \mathcal{P}_h$, will be constructed in the following way.

Join the center of gravity of arbitrary triangle, which contains the vertex P_j, and the center of arbitrary edge of this triangle, which again contains the vertex P_j.

Repeat this procedure for all triangles $T_i \in \mathcal{T}_h$, s.t. $P_j \in T_i$. If the point P_j does not lie on the boundary $\partial\Omega_h$ then the result is a contour of the dual volume D_j, i.e. the boundary ∂D_j. On the other hand, if the point P_j lies on the boundary $\partial\Omega_h$ then there are two edges that lie on $\partial\Omega_h$ and contain point P_j. To get the whole contour of the dual volume D_j, we have to add such segments of these boundary edges, which join point P_j with the centers of the boundary edges. See Fig. 2.5.

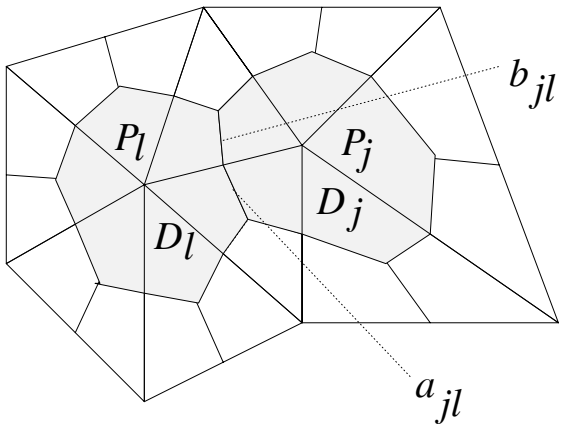

FIG. 2.5. Dual volumes corresponding to the points P_j, P_ℓ $j, \ell \in \mathbb{J}$.

By Definition 2.2.12 of the basic mesh $\mathcal{T}_h$, we obtain the following properties of "our" dual mesh $\mathcal{D}_h$.

(2.2.15)

(i) $D_j,\ j \in \mathbb{J},$ is a closed polygon, not necessarily convex ;

(ii) $\overline{\Omega}_h = \bigcup_{j\in\mathbb{J}} D_j;$

(iii) for any $D_j, D_\ell \in \mathcal{D}_h,$ s.t. $D_j \neq D_\ell$ there are only three possibilities :
either $D_j \cap D_\ell = \emptyset,$
or $D_j \cap D_\ell = \mathbf{a}_{j\ell} \cup \mathbf{b}_{j\ell} \ldots$ two common edges,
or $D_j \cap D_\ell = \mathbf{a}_{j\ell} \ldots$ one common edge; (see Fig. 2.5) .

If $D_j \cap D_\ell \neq \emptyset$, we say that D_j and D_ℓ are neighbours. We will use the following notation.

Notation 2.2.16.

i) $|T_i| := \text{meas}\,(T_i)\,, \quad i \in \mathbb{I}; \quad |D_j| := \text{meas}\,(D_j)\,, \quad j \in \mathbb{J};$
ii) $s\,(j) := \{\ell \in \mathbb{J};\ D_j \text{ and } D_\ell \text{ are neighbours }\}\,,\ j \in \mathbb{J};$
iii) $H := \{j \in \mathbb{J}, \quad \partial D_j \cap \partial\Omega_h \neq \emptyset\} \ldots$ the set of indices of the so-called *boundary volumes* ;
iv) $\partial D_{j\ell} := \partial D_j \cap \partial D_\ell, \quad \ell \in s\,(j)\,,\ j \in \mathbb{J};$
v) if $j \in H$ then we denote $\mathbf{a}_{j,-1} \cup \mathbf{b}_{j,-1} := \partial\Omega_h \cap \partial D_j$, and set $\partial D_{j,-1} := \mathbf{a}_{j,-1} \cup \mathbf{b}_{j,-1}$; the set of indices of the so-called *boundary edges* is defined by

$$\gamma\,(j) := \begin{cases} \{-1\} & j \in H, \\ \emptyset & j \in \mathbb{J} \setminus H; \end{cases}$$

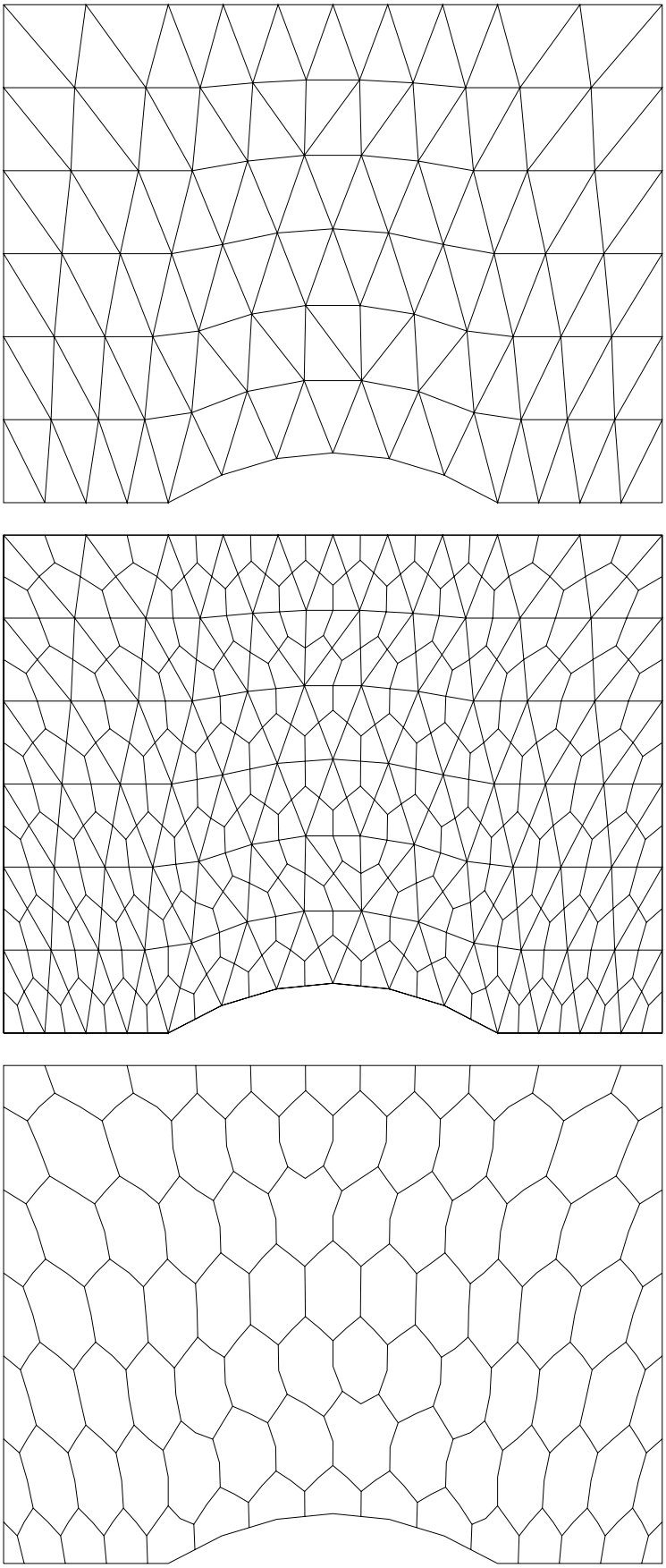

FIG. 2.6. Triangulation of the channel domain, construction of the dual mesh over the original triangulation, and the dual finite volume discretization.

vi) $S(j) := s(j) \cup \gamma(j)\,,\ j \in \mathbb{J}$;

vii) $\mathbf{n}^a_{j\ell} := \left(n^a_{xj\ell}, n^a_{yj\ell}\right),\ j \in \mathbb{J}, \ell \in S(j) \ldots$ a unit outer normal to D_j on the edge $\mathbf{a}_{j\ell} \in \partial D_{j\ell}$; analogously for the edge $\mathbf{b}_{j\ell} \in \partial D_{j\ell}$ (if exists), $\mathbf{n}^b_{j\ell} := \left(n^b_{xj\ell}, n^b_{yj\ell}\right)$;

viii) $\ell_{j\ell} := |\partial D_{j\ell}|, \quad j \in \mathbb{J},\ \ell \in S(j)$;

ix) $d := \sup_{j\in\mathbb{J}} \operatorname{diam} D_j$.

Clearly,

$$\partial D_j = \bigcup_{\ell \in S(j)} \partial D_{j\ell}, \quad j \in \mathbb{J}. \tag{2.2.17}$$

If we use suitable notation, i.e. $\mathbf{a}_{j\ell} = \mathbf{a}_{\ell j}$, and $\mathbf{b}_{j\ell} = \mathbf{b}_{\ell j}$, then

$$\mathbf{n}^a_{j\ell} = -\mathbf{n}^a_{\ell j}, \quad \mathbf{n}^b_{j\ell} = -\mathbf{n}^b_{\ell j}, \qquad j \in \mathbb{J},\ \ell \in s(j)\,.$$

From the construction 2.2.14 it follows that

(2.2.18)

(i) if $j \in \mathbb{J}, \quad \ell \in s(j)$, then

$$\partial D_{j\ell} = \begin{cases} \mathbf{a}_{j\ell} \cup \mathbf{b}_{j\ell}, & \text{if} \quad j \notin H \text{ or } \ell \notin H, \\ \mathbf{a}_{j\ell}, & \text{if} \quad j \in H \text{ and } \ell \in H; \end{cases}$$

(ii) if $\Omega = \mathbb{R}^2$ then $H = \emptyset$, $\gamma(j) = \emptyset$, $S(j) = s(j)$ for all $j \in \mathbb{J}$.

Further, we will define a discretization of time.

Definition 2.2.19. *Let $\tau \in (0, \tau_0)$, $\tau_0 > 0$. The set $\{t_k;\ t_k = k \cdot \tau,\ k = 0, \ldots, N\}$, $N\tau = T$, $0 < T \leq \infty$ is called a time mesh and τ is called a time step.*

To obtain a stable and convergent finite volume scheme we need to assume some *regularity properties* of the mesh, see e.g. [Ciarlet] for analogous assumptions in the framework of the FEM.

Assumptions 2.2.20. The parameter of a mesh $h \to 0$, whenever $\tau \to 0$, i.e.

$$\exists\, c_1, c_2 > 0 : \qquad 0 < c_1 \leq \frac{\tau}{h} \leq c_2, \qquad \text{if } \tau, h \to 0. \tag{2.2.21}$$

The triangles do not degenerate when $h \to 0$, i.e.

$$\exists\, c_3 > 0 : \qquad \forall i \in \mathbb{I} \qquad \frac{h}{\rho_i} \leq c_3, \qquad \text{if } h \to 0, \tag{2.2.22}$$

where $\rho_i = \operatorname{diam} B_i$, B_i is the largest ball contained in T_i.

Remark 2.2.23. The regularity assumption (2.2.22) implies the following assumption,

$$\exists\, c_V > 0 : \qquad \sup_{i\in\mathbb{I}} \frac{h^2}{|T_i|} \le c_V, \qquad \text{if } h \to 0.$$

This can be easily verified by the fact that

$$|T_i| \ge \pi \rho_i^2.$$

The following remark says that the regularity properties of the basic triangular mesh are preserved for the dual one, too. *Show it as an exercise!*

Remark 2.2.24. If Assumptions 2.2.20 hold for the basic mesh $\mathcal{T}_h$, then the following conditions are satisfied for the dual mesh $\mathcal{D}_h$.

$$\exists\, c_1^*, c_2^* > 0 : \qquad 0 < c_1^* \le \frac{\tau}{d} \le c_2^*, \tag{2.2.25}$$

if $\tau \to 0$ and $d = \sup_{j\in\mathbb{J}} \text{diam } D_j \to 0$.

$$\exists\, c_V^* > 0 : \qquad \sup_{j\in\mathbb{J}} \frac{d^2}{|D_j|} \le c_V^*, \qquad \text{if } d \to 0. \tag{2.2.26}$$

2.2.27 Finite volume (FV) discretization with a cell-centered control volumes. The basic principle underlying a conservation law is that the total quantity of conservation variable in any region can be changed only due to flux through the boundaries. Mathematically, we integrate (2.2.1) over $D_j \times (t_k, t_{k+1}), D_j$ is a dual control volume and (t_k, t_{k+1}) is a chosen time interval. By Green's theorem we obtain

(2.2.28)
$$\int_{D_j} \mathbf{w}\,(x, y, t_{k+1}) = \int_{D_j} \mathbf{w}\,(x, y, t_k) - \sum_{\ell\in S(j)} \int_{t_k}^{t_{k+1}} \int_{\partial D_{j\ell}} \mathbf{f}_1\,(\mathbf{w})\, n_x + \mathbf{f}_2\,(\mathbf{w})\, n_y.$$

In the cell-centered approach the discrete variables are associated with the grid points $P_j \in \mathcal{P}_h$, or in fact, with the dual control volumes $D_j \in \mathcal{D}_h$. We approximate

$$\mathbf{w}\,(x, y, t_k) \approx \mathbf{w}_j^k = \text{ const.} \qquad \text{for any } (x, y) \in D_j. \tag{2.2.29}$$

In the literature, see e.g. [Morton, Mayers], one can also meet a concept of cell-vertex finite volumes. Here the discrete variables are associated with the values at vertices of the computational grid. Actually, the cell-vertex FVM can be reformulated as the cell-centered FVM for the corresponding dual grid.
Further, we need to approximate time integrals along the cell interfaces $\partial D_{j\ell}$. In many practical problems explicit numerical schemes are used, i.e.

$$\int_{t_k}^{t_{k+1}} \int_{\partial D_{j\ell}} \mathbf{f}_1\,(\mathbf{w})\, n_x + \mathbf{f}_2\,(\mathbf{w})\, n_y \approx \tau \int_{\partial D_{j\ell}} \mathbf{f}_1\left(\mathbf{w}^k\right) n_x + \mathbf{f}_2\left(\mathbf{w}^k\right) n_y. \tag{2.2.30}$$

Note that the implementation of explicit schemes is much easier than that of implicit ones, which would lead to a solution of algebraic systems at each time step t_k. However, there is still a question how to approximate $\mathbf{f}_1\left(\mathbf{w}^k\right), \mathbf{f}_2\left(\mathbf{w}^k\right)$ on the boundary $\partial D_{j\ell}$ of the control volume $D_j, j \in \mathbb{J}$. To this end, a *numerical flux function* $\mathbf{H}\left(\mathbf{w}_j^k, \mathbf{w}_\ell^k, \mathbf{n}_{j\ell}\right)$, which approximates the amount of a quantity passing through a unit of the area of $\partial D_{j\ell}$ per a unit of time, is introduced. An important thing is to choose this function in an appropriate way. Some properties of a suitable numerical flux will be specified later. Now we introduce the following notation. If $\partial D_{j\ell} = \mathbf{a}_{j\ell} \cup \mathbf{b}_{j\ell}$, then the *total flux* through $\partial D_{j\ell}, j \in \mathbb{J}, \ell \in S(j)$, will be written as

$$\mathbf{g}_{j\ell}(\mathbf{u}, \mathbf{v}) = \mathbf{H}\left(\mathbf{u}, \mathbf{v}, \mathbf{n}_{j\ell}^a\right)|\mathbf{a}_{j\ell}| + \mathbf{H}\left(\mathbf{u}, \mathbf{v}, \mathbf{n}_{j\ell}^b\right)|\mathbf{b}_{j\ell}|. \tag{2.2.31}$$

If $\partial D_{j\ell} = \mathbf{a}_{j\ell}, j \in \mathbb{J}, \ell \in s(j)$, which situation appears when $j, \ell \in H$, then

$$\mathbf{g}_{j\ell}(\mathbf{u}, \mathbf{v}) = \mathbf{H}\left(\mathbf{u}, \mathbf{v}, \mathbf{n}_{j\ell}^a\right)|\mathbf{a}_{j\ell}|. \tag{2.2.32}$$

The above discussion leads to the following approximations of the terms in (2.2.28) .

(2.2.33)

$$\int_{D_j} \mathbf{w}(x, y, t_{k+1}) \approx |D_j|\mathbf{w}_j^{k+1},$$

$$\int_{D_j} \mathbf{w}(x, y, t_k) \approx |D_j|\mathbf{w}_j^k,$$

$$\sum_{\ell \in S(j)} \int_{t_k}^{t_{k+1}} \int_{\partial D_{j\ell}} \mathbf{f}_1(\mathbf{w})\, n_x + \mathbf{f}_2(\mathbf{w})\, n_y \approx \tau \sum_{\ell \in S(j)} \mathbf{g}_{j\ell}\left(\mathbf{w}_j^k, \mathbf{w}_\ell^k\right).$$

Consequently, we derive the *finite volume numerical scheme.*

$$\mathbf{w}_j^{k+1} := \mathbf{w}_j^k - \frac{\tau}{|D_j|} \sum_{\ell \in S(j)} \mathbf{g}_{j\ell}\left(\mathbf{w}_j^k, \mathbf{w}_\ell^k\right), \tag{2.2.34}$$

$j \in \mathbb{J}, k = 0, 1, \ldots,$ to which the initial conditions are added

$$\mathbf{w}_j^0 := \frac{1}{|D_j|} \int_{D_j} \mathbf{w}_0(x, y), \qquad j \in \mathbb{J}. \tag{2.2.35}$$

The scheme (2.2.34) can be rewritten as any explicit numerical scheme in the form

$$\mathbf{w}_j^{k+1} := \mathbf{N}\left(\mathbf{w}_j^k, \left\{\mathbf{w}_\ell^k; \ell \in S(j)\right\}\right). \tag{2.2.36}$$

The numerical scheme (2.2.36) is said to be *conservative* , if there is a continuous numerical flux function $\mathbf{H}$, s.t. (2.2.36) can be written in the form (2.2.34) (cf. (2.2.31), (2.2.32)).

As we have already pointed out the numerical flux should have some suitable properties. We will assume the following.

(2.2.37)

$$\begin{aligned}&\mathbf{H}(\mathbf{u},\mathbf{v},\mathbf{n}) : D\times D\times S_1 \to \mathbb{R}^4 \text{ is } \textit{locally Lipschitz continuous :}\\ &\big(\forall\, R>0\big)\big(\exists\, c=c(R)>0\big)\big(\forall\, \mathbf{u},\mathbf{v},\mathbf{u}^*,\mathbf{v}^*\in\mathcal{B}_R\big)\\ &|\mathbf{H}(\mathbf{u},\mathbf{v},\mathbf{n})-\mathbf{H}(\mathbf{u}^*,\mathbf{v}^*,\mathbf{n})|\le c(R)\left(|\mathbf{u}-\mathbf{u}^*|+|\mathbf{v}-\mathbf{v}^*|\right),\end{aligned}$$

where S_1 is a unit sphere in $\mathbb{R}^2$ and $\mathcal{B}_R=\{\mathbf{x}\in D;\ |\mathbf{x}|\le R\}$.

(2.2.38)

$$\begin{aligned}&\mathbf{H} \text{ is } \textit{consistent} \text{ with an original equation (2.2.1) :}\\ &\mathbf{H}(\mathbf{u},\mathbf{u},\mathbf{n})=\mathbf{f}_1(\mathbf{u})\,n_x+\mathbf{f}_2(\mathbf{u})\,n_y,\qquad \mathbf{u}\in D,\mathbf{n}\in S_1.\end{aligned}$$

(2.2.39)

$$\begin{aligned}&\mathbf{H} \text{ is } \textit{conservative} :\\ &\mathbf{H}(\mathbf{u},\mathbf{v},\mathbf{n})=-\mathbf{H}(\mathbf{v},\mathbf{u},-\mathbf{n}),\qquad \mathbf{u},\mathbf{v},\in D,\mathbf{n}\in S_1.\end{aligned}$$

One can easily realize the following result.

LEMMA 2.2.40. *If (2.2.37) – (2.2.39) are fulfilled for the numerical flux function* $\mathbf{H}$*, then the same properties hold for the numerical flux function* $\mathbf{g}_{j\ell}$*,* $j\in\mathbb{J},\ell\in S(j)$*. It means that for any* $j\in\mathbb{J},\ell\in S(j)$*, we have*

(2.2.41)

(i) $\mathbf{g}_{j\ell}: D\times D\to\mathbb{R}^4$ *is locally Lipschitz continuous :*

$$\begin{aligned}&\big(\forall\, R>0\big)\big(\exists\, c_1=c_1(R)>0\big)\big(\forall\, \mathbf{u},\mathbf{v},\mathbf{u}^*,\mathbf{v}^*\in\mathcal{B}_R\big)\\ &|\mathbf{g}_{j\ell}(\mathbf{u},\mathbf{v})-\mathbf{g}_{j\ell}(\mathbf{u}^*,\mathbf{v}^*)|\le c_1(R)\,\ell_{j\ell}\big(|\mathbf{u}-\mathbf{u}^*|+|\mathbf{v}-\mathbf{v}^*|\big).\end{aligned}$$

(ii) $\mathbf{g}_{j\ell}$ *is consistent :*

$$\begin{aligned}&\mathbf{g}_{j\ell}(\mathbf{u},\mathbf{u})=\mathbf{f}_1(\mathbf{u})\,\nu_{xj\ell}+\mathbf{f}_2(\mathbf{u})\,\nu_{y\ell j},\qquad \textit{where } \mathbf{u}\in D,\\ &\nu_{xj\ell}=n^a_{xj\ell}|\mathbf{a}_{j\ell}|+n^b_{xj\ell}|\mathbf{b}_{j\ell}|\qquad \textit{for } j\notin H \textit{ or } \ell\notin H;\\ &\nu_{xj\ell}=n^a_{xj\ell}|\mathbf{a}_{j\ell}|\qquad \textit{for } j,\ell\in H. \textit{ Analogously for } \nu_y.\end{aligned}$$

(iii) $\mathbf{g}_{j\ell}$ *is conservative:*

$$\mathbf{g}_{j\ell}(\mathbf{u},\mathbf{v})=-\mathbf{g}_{\ell j}(\mathbf{v},\mathbf{u}),\qquad \mathbf{u},\mathbf{v}\in D.$$

□

Theoretical results for the FVM. In the investigation of a numerical scheme we try to answer the following questions:

a) consistency of the method with the equation (2.2.1)
b) stability
c) convergence and order of the method
d) computational costs

In what follows we give an overview of some basic theoretical results for the FVM. Let us consider a Cauchy problem for (2.2.1) with the initial data (2.2.2), i.e. $\Omega = \mathbb{R}^d$.

We will deal first with the one-dimensional hyperbolic system (2.2.1), i.e. $s \geq 1$, $d = 1$. Note that if $d = 1$ finite volume cells D_j reduce to simple one-dimensional intervals $(x_{j-1/2}, x_{j+1/2})$.

Given an approximate solution $\mathbf{w}_j^k$, $j \in \mathbb{Z}$, $k = 0, 1, \ldots$ obtained by (2.2.34), we introduce the piecewise constant function $\mathbf{w}_{h\tau}(x,t)$, such that

$$\mathbf{w}_{h\tau}(x,t) = \mathbf{w}_j^k \quad x \in D_j,\ t \in [t_k, t_{k+1}). \tag{2.2.42}$$

The convergence result of the finite volume method (2.2.34), (2.2.35) is established in the well-known Lax-Wendroff convergence theorem.

THEOREM 2.2.43. *(Lax, Wendroff (1960)) Let the numerical flux* $\mathbf{g}_{\ell j}$ *in* (2.2.34) *be conservative and consistent. Moreover, let* $\mathbf{w}_0 \in L^\infty(\mathbb{R})$.[1] *Let us consider sequences* $\{h_k\}_k$, $\{\tau_k\}_k$*; such that* $h_k \to 0+$, $\tau_k \to 0+$ *as* $k \to \infty$. *Let us assume that for numerical solution* $\mathbf{w}^k \equiv \mathbf{w}_{h_k,\tau_k}$ *the following conditions hold:*

$$\begin{aligned} &\|\mathbf{w}^k\|_{L^\infty(\mathbb{R}\times(0,\infty))} \leq c, \qquad k = 1, 2, \ldots \\ &\mathbf{w}^k \longrightarrow \mathbf{w} \quad \text{a.e. in } \mathbb{R} \times (0,\infty). \end{aligned} \tag{2.2.44}$$

Then $\mathbf{w}$ *is a weak solution of the Cauchy problem* (2.2.1), (2.2.2) *on* $\mathbb{R} \times [0,\infty)$.

PROOF. See, e.g. [Lax, Wendroff].

□

In fact, condition (2.2.44) yields an assumption on the stability of the scheme. More precisely, we say that the numerical scheme (2.2.36) is *(strongly) stable* iff the following inequality holds in appropriate norms

$$\|\mathbf{w}^{k+1}\| \leq \|\mathbf{w}^k\|.$$

Thus, it implies that $\|\mathbf{w}^k\| \leq \|\mathbf{w}^0\|$ for each $k = 1, 2, \ldots$. Note that for linear problems L^1 or L^2-norms are suitable, whereas for nonlinear hyperbolic problems stronger assumptions are needed, and the L^∞ stability is required in order to get convergence.

It should be pointed out here that due to the fact that we work with explicit methods some additional assumption on the mesh size parameter $h \in (0, h_0)$ and time step $\tau \in (0, \tau_0)$ has to be required. This is the so-called CFL (Courant-Friedrichs-Lewy) stability condition. It has to be taken in such a way that numerical domain of dependence lies inside the physical one. We will explain this in the following example.

Example: Consider the scalar one-dimensional equation

$$w_t + a w_x = 0 \qquad \mathbb{R} \times [0,\infty).$$

[1] Remind that we are using for the vector-valued functions simplified notation $L^\infty(\mathbb{R})$ instead of more precise $L^\infty(\mathbb{R};\mathbb{R}^s)$.

Let us use for the evaluation of numerical fluxes on the interfaces $x_{j\pm1/2}$ the left sided values of the approximate solution. We obtain the one-sided method

$$w_j^{k+1} = w_j^k - \frac{a\tau}{h}(w_j^k - w_{j-1}^k), \quad j \in \mathbb{Z},\ k = 0, 1, \dots .$$

Hence we have the following estimates in the discrete L^1 norm

$$\|w^{k+1}\| = h\sum_j |w_j^{k+1}| \le h\sum_j \left|\left(1 - \frac{a\tau}{h}\right) w_j^k\right| + h\sum_j \left|\frac{a\tau}{h} w_{j-1}^k\right|.$$

If we now assume that the *Courant-Friedrichs-Lewy number* $\dfrac{a\tau}{h}$ satisfies

$$0 \le \frac{a\tau}{h} \le 1 \tag{CFL}$$

the coefficients of w_j^k and w_{j-1}^k are both nonnegative, and we obtain the stability bound $\|w^{k+1}\| \le \|w^k\|$. Note that the condition (CFL) requires in particular that $a \ge 0$, since $\tau, h > 0$. Thus, the one-sided method can only be used when $a \ge 0$. We will recall this point also in the next subsection, where the notion of the upwind method is introduced, cf. (2.2.55). For a system of equations we would get

$$0 \le \frac{\lambda_k \tau}{h} \le 1$$

for all eigenvalues λ_k of $\mathbb{A} = \dfrac{D\mathbf{f}(\mathbf{w})}{D\mathbf{w}}$. This result is easily obtained from the characteristic decomposition of the corresponding system, cf. (2.1.42).

Consider again the Cauchy problem (2.2.1), (2.2.2), i.e. $\Omega = \mathbb{R},\ s \ge 1,\ d = 1$. Let $\mathbf{w}$ be its classical solution. Substituting into (2.2.36) we obtain

(2.2.45)
$$\mathbf{w}(x_j, t_{k+1}) = \mathbf{N}\big(\mathbf{w}(x_j, t_k), \{\mathbf{w}(x_\ell, t_k); \ell \in S(j)\}\big) + \tau\boldsymbol{\varepsilon}_j^k, \qquad j \in \mathbb{N}, k = 1, 2, \dots .$$

The quantity $\boldsymbol{\varepsilon}_j^k$ is called a <u>local truncation error</u> of the scheme. We say that the accuracy of the numerical scheme (2.2.36) is <u>of order p in time</u> and <u>q in space</u>, if under the assumption that the exact solution $\mathbf{w}$ of (2.2.1), (2.2.2) and the flux $\mathbf{f} \equiv \mathbf{f}_1$ are sufficiently smooth, there exist constants M, h_0, τ_0 such that

$$|\boldsymbol{\varepsilon}_j^k| \le M(\tau^p + h^q), \qquad j \in \mathbb{N},\ k = 1, 2, \dots; \tau \in (0, \tau_0),\ h \in (0, h_0). \tag{2.2.46}$$

We simply write $\boldsymbol{\varepsilon}_j^k = O(\tau^p + h^q)$. The investigation of the local truncation error and the order of accuracy is usually carried out by the Taylor expansion.

In Theorem 2.2.43 we have assumed that we already have a convergent subsequence. Now we shall show how we can get it. Assume that the system (2.2.1) is one-dimensional and linear. Thus, we consider the following situation

$$\begin{aligned} &\frac{\partial \mathbf{w}}{\partial t} + \mathbb{A}\frac{\partial \mathbf{w}}{\partial x} = \mathbf{0} && \text{on } \mathbb{R} \times (0, \infty), \\ &\mathbf{w}(x, 0) = \mathbf{w}_0 && \text{on } \mathbb{R}. \end{aligned}$$

The following fundamental Lax-Equivalence Theorem says that in the linear case stability and the local truncation error gives the global error.

THEOREM 2.2.47. *(Lax-Equivalence Theorem) Let us consider linear hyperbolic system* (2.2.1). *A linear numerical scheme* (2.2.36) *is stable and its local truncation error is of order q in space and p in time if and only if the global error of numerical scheme is of order q and p in space and time, respectively, i.e.*

$$\|\mathbf{w}(t_k) - \mathbf{w}^k\|_{L^2(\mathbb{R})} \leq c(h^q + \tau^p) \qquad \textit{uniformly for } k = 1, 2, \ldots.$$

PROOF. See, e.g. [Richtmeyr, Morton].

□

The study of convergence of the FVM (2.2.34), (2.2.35) for nonlinear hyperbolic problems is much more complicated. In fact, for general nonlinear hyperbolic systems this question is still open. Consider nonlinear hyperbolic equation (2.2.1) with $s = 1$. Since we have to approximate discontinuous solutions, the choice of the function space in which we have to construct the converging sequence is the most important problem. Following theoretical results of Glimm and Bressan, cf. Theorem 2.1.39, we find out that the space BV of functions with bounded total variation TV should be appropriate also in order to study convergence of discrete solutions. First we need to fix some additional notations.

As we have already shown in Section 2.1, the hyperbolic conservation laws admit non-physical weak solutions. The entropy inequality (cf. Definition 2.1.30) must be satisfied in order to obtain a physically relevant solution. Similarly, in the discrete case the discrete entropy condition is required to guarantee a physically correct numerical solution. To this end the entropy inequality is discretize. For simplicity, let us do this for a scalar equation (2.2.1) with $s = 1$.

Let U be the entropy function and $G_{j\ell}$ be the discrete analogy of entropy flux functions F_1, F_2, such that $G_{j\ell}(u, u) = F_1(u)\nu_{xj\ell} + F_2(u)\nu_{yj\ell}$, $u \in D$. Function $G_{j\ell}$ is the so-called *numerical entropy flux* . Discrete entropy inequality reads:

$$U\left(w_j^{k+1}\right) \leq U\left(w_j^k\right) - \frac{\tau}{|D_j|} \sum_{\ell \in S(j)} G_{j\ell}\left(w_j^k, w_\ell^k\right). \tag{2.2.48}$$

Note that the above inequality is only a discrete finite volume version of the "continuous" entropy inequality (2.1.28). If we regard the numerical flux $g_{j\ell}$ as a function of the flux vector $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and the solution w, i.e.

$$g_{j\ell} = g\left(w, \mathbf{f}\right),$$

then the numerical entropy flux can be chosen in the form

$$G_{j\ell} = g\left(U, \mathbf{F}\right),$$

where $\mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ denotes the entropy flux vector.

In [Sonar (2)] it was shown that the above choice of discrete entropy flux leads to a solution consistent with the Lax entropy condition (2.1.47). The general case of systems ($s > 1$) is more complicated and the reader is referred to [Van der Burg]. Practically control of discrete entropy inequality gives a very suitable tool to detect physically incorrect features in a numerical solution, such as expansion shocks and wiggles near shock (see, e.g., [Van der Burg]).

Now we can approach formulation of the convergence result for nonlinear equation (2.2.1) with $s = d = 1$. Its proof can be found e.g. in [Kröner].

THEOREM 2.2.49. *(sufficient conditions for convergence)*
Let $w_0 \in L^1_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $\{w^k \equiv w_{h_k,\tau_k}\}_k$ be a sequence of discrete solutions, cf. (2.2.42)*, such that the following hold*

(1) *numerical flux $g_{j\ell}$ is consistent and conservative*

(2)
$$\|w^k\|_{L^\infty(\mathbb{R}\times(0,\infty))} \le c, \qquad k = 1, 2, \dots$$

(3)
$$TV(w^k) \le c \qquad k = 1, 2, \dots$$

then there exists a $w \in L^1_{loc}(\mathbb{R} \times [0,\infty))$ and a subsequence $\{w^{k'}\}$ such that

$$w^{k'} \longrightarrow w \quad \text{in } L^1_{loc}(\mathbb{R} \times [0,\infty)) \qquad \text{as } k' \longrightarrow \infty$$

and w is a solution to (2.2.1), (2.2.2)*, $s = d = 1$, in the sense of distributions.*

Additionally if w^k_j, $j \in \mathbb{N}$, $k = 0, 1, \dots$ satisfies the discrete entropy inequality (2.2.48) *then w satisfies the entropy condition* (2.1.29)*. This solution is then uniquely defined.*

In the case of <u>two-dimensional nonlinear scalar equation</u>, i.e. (2.2.1) with $s = 1$, $d = 2$ the situation is more complicated. Convergence of the FVM for meshes, which consist of convex $q-$polygonals, was proved by Kröner and Rokyta [Kröner, Rokyta]. They used techniques of the measure valued solutions, the Young measures and the compensated compactness arguments based on [DiPerna(1), (2)], [Tartar], etc. Its generalization to dual meshes was studied in [Lukáčová]. In what follows we formulate the convergence result of the FVM for dual meshes. First, we need some preliminary stability considerations.

We will suppose that the flux function $g = g_{j\ell}$ has the consistency, conservativity and Lipschitz continuity property (cf. (2.2.41)). However, it is still not enough for the proof of stability and convergence of the FVM. We introduce the concept of *monotonicity,* which will play an important role.

DEFINITION 2.2.50. *(monotonicity) The scheme* (2.2.34) *is called <u>monotone</u>, if the functions*

$$H_j(w) := w_j - \frac{\tau}{|D_j|} \sum_{\ell \in S(j)} g_{j\ell}(w_j, w_\ell) \qquad j \in \mathbb{J},$$

where $w = \{w_i\}_{i\in\mathbb{J}}$, are monotonously nondecreasing on

$\mathcal{M}_M = \{w = \{w_i\}_{i\in\mathbb{J}}, \|w\|_{L^\infty(\mathbb{R}^2)} \le M\}$ for some $M > 0$. More precisely, if $w = \{w_i\}_{i\in\mathbb{J}}, w^* = \{w^*_i\}_{i\in\mathbb{J}} \in \mathcal{M}_M$ and $w \le w^*$ (i.e. $w_i \le w^*_i$ for all $i \in \mathbb{J}$), then $H_j(w) \le H_j(w^*)$.

Note that we have measured a sequence $\{w_i\}$ in the L^∞ norm, instead of ℓ^∞. This is can be done since $w = \{w_i\}$ can naturally be associated with a function $w \in L^\infty(\mathbb{R}^2)$, such that $w\Big|_{D_j} = w_j$.

Clearly, monotonicity is stronger property than the boundedness of the total variation TV. For example, if f be a Lipschitz-continuous function on $[a, b]$, which is monotonously increasing then $TV(f) = f(b) - f(a)$. Thus $f \in BV([a, b])$.

We now approach to the proof of the L^∞-stability property or the discrete maximum principle.

THEOREM 2.2.51. *(maximum principle) Let the approximate solution $w^k, k = 0, 1, \ldots,$ be defined by the numerical scheme* (2.2.34), (2.2.35) *for the scalar conservation law* ($s = 1$). *Let $w^0 = \{w_j^0\}_{j\in\mathbb{J}} \in \mathcal{M}_M$, where $M > 0$. Let the scheme be consistent (cf.* (2.2.41) ii)) *and monotone. Then we have for all $k \in \mathbb{N}$:*

$$\|w^k\|_{L^\infty(\mathbb{R}^2)} \le \|w^0\|_{L^\infty(\mathbb{R}^2)}.$$

PROOF will be carried out by the mathematical induction. Let the desired inequality holds for some $k \in \mathbb{N}$. Obviously, it holds for $k = 0$. We denote $m = \|w^k\|_{L^\infty(\mathbb{R}^2)}$ and construct $w^* = \{w_j^*\}_{j\in\mathbb{J}}$ s.t. $w_j^* = m$ for each $j \in \mathbb{J}$. Then $w^* \in \mathcal{M}_M$ and $w^k \le w^*$. Due to the monotonicity property we obtain $H_j(w^k) \le H_j(w^*)$ for each $j \in \mathbb{J}$. Thus,

$$\begin{aligned} w_j^{k+1} = H_j(w^k) \le H_j(w^*) &= w_j^* - \frac{\tau}{|D_j|}\sum_{\ell\in S(j)} g_{j\ell}(w_j^*, w_\ell^*) \\ &= (\text{using the consistency}) = m - \frac{\tau}{|D_j|}\sum_{\ell\in S(j)} f_1(m)\nu_{xj\ell} + f_2(m)\nu_{yj\ell} \\ &= m - \frac{\tau}{|D_j|}(f_1(m)\int_{\partial D_j} n_x dS + f_2(m)\int_{\partial D_j} n_y dS) \\ &= (\text{using Green's theorem}) = m - \frac{\tau}{|D_j|}(f_1(m)\cdot 0 + f_2(m)\cdot 0) = m. \end{aligned}$$

We showed that $w_j^{k+1} \le m$ for each $j \in \mathbb{J}$, and we can analogously prove that $w_j^{k+1} \ge -m$ by setting $w_j^* = -m$, $j \in \mathbb{J}$. This leads to

$$|w_j^{k+1}| \le m = \|w^k\|_{L^\infty(\mathbb{R}^2)} \qquad \text{for each } j \in \mathbb{J},$$

and finally

$$\|w^{k+1}\|_{L^\infty(\mathbb{R}^2)} \le \|w^k\|_{L^\infty(\mathbb{R}^2)}.$$

□

Now we can prove the L^1-stability property.

THEOREM 2.2.52. *(L^1-estimates) Let us assume that the scheme* (2.2.34), (2.2.35), ($s = 1$), *is conservative, consistent, monotone, with locally Lipschitz continuous numerical flux (cf.* (2.2.41)). *Let $w^0 \in \mathcal{M}_M$ and $\|w^0\|_{L^1(\mathbb{R}^2)} < \infty$. We will suppose that the following CFL-like stability condition holds:*

$$\text{(CFL)} \qquad \frac{\tau c(M)|\partial D_j|}{|D_j|} \le 1, \qquad j \in \mathbb{J},$$

where $c(M)$ is the constant of Lipschitz continuity of flux function. Then the scheme is L^1-stable, i.e.

$$\|w^k\|_{L^1(\mathbb{R}^2)} \le \|w^0\|_{L^1(\mathbb{R}^2)}, \quad k \in \mathbb{N}.$$

PROOF.

Let us denote $w^* = \{w_j^*\}_{j\in\mathbb{J}},\ w_j^* = 0$. Then clearly, due to the consistency property we have

$$H_j(w^*) = 0 \qquad \text{for all} \quad j \in \mathbb{J}.$$

Further, if we use the usual notation $a^+ = \max(a,0),\ a^- = \min(a,0)$, then

$$(H_j(w^k))^+ \le H_j((w^k)^+)$$
$$\text{and } (H_j(w^k))^- \ge H_j((w^k)^-), \quad j \in \mathbb{J}.$$

Here $(w^k)^\pm = \{(w_j^k)^\pm\}_{j\in\mathbb{J}}$. We obtain

$$|w_j^{k+1}| = |H_j(w^k)| = (H_j(w^k))^+ - (H_j(w^k))^- \le H_j((w^k))^+) - H_j((w^k))^-) =$$
$$(*) \qquad = |w_j^k| - \frac{\tau}{|D_j|}\sum_{\ell\in S(j)} g_{j\ell}((w_j^k)^+,(w_\ell^k)^+) + \frac{\tau}{|D_j|}\sum_{\ell\in S(j)} g_{j\ell}((w_j^k)^-,(w_\ell^k)^-).$$

Our aim will be to show that $\tau\sum_{j\in\mathbb{J}}\sum_{\ell\in S(j)} g_{j\ell}(v_j,v_\ell)$ is absolutely convergent for $v = (w^0)^+$ or $v = (w^0)^-$. Let us use Lipschitz continuity of the flux function:

$$\begin{aligned}\tau\sum_{j\in\mathbb{J}}\sum_{\ell\in S(j)} |g_{j\ell}(v_j,v_\ell)| &\le \tau\sum_{j\in\mathbb{J}}\sum_{\ell\in S(j)} |g_{j\ell}(v_j,v_\ell) - g_{j\ell}(w_j^*,w_\ell^*)| \\ &\le \tau\sum_{j\in\mathbb{J}}\sum_{\ell\in S(j)} c(M)\ell_{j\ell}(|v_j|+|v_\ell|) \\ &\le \tau c(M)\sum_{j\in\mathbb{J}}|\partial D_j||v_j| + \tau c(M)\sum_{j\in\mathbb{J}}\sum_{\ell\in S(j)}|\partial D_\ell||v_\ell|.\end{aligned}$$

Now we use the stability condition (CFL). Due to the regularity of basic triangulation there exist only finitely many, say q^*, neighbours of each dual volume D_j. Thus $|S(j)| \le q^*$ and the R.H.S. of the above inequality can be estimated by

$$\sum_{j\in\mathbb{J}}|D_j||v_j| + q^*\sum_{j\in\mathbb{J}}|D_j||v_j| < \infty,$$

because $w^0 \in L^1(\mathbb{R}^2)$.

By the conservativity property of $g_{j\ell}$ we find that

$$\sum_{j\in\mathbb{J}}\sum_{\ell\in S(j)} g_{j\ell}((w_j^0)^+,(w_\ell^0)^+) = \sum_{j\in\mathbb{J}}\sum_{\ell\in S(j)} g_{j\ell}((w_j^0)^-,(w_\ell^0)^-) = 0.$$

Hence, multiplying (*) by $|D_j|$ and summing up over $j \in \mathbb{J}$, leads to

$$\|w^1\|_{L^1(\mathbb{R}^2)} \le \|w^0\|_{L^1(\mathbb{R}^2)}.$$

The proof is finished by the mathematical induction.

□

In the following theorem we formulate convergence result of the FVM for nonlinear scalar conservation law (2.2.1), (2.2.2), $s = 1, d = 2$. We do not present its proof since it would lead us behind the framework of this scriptum. Details can be found in [Lukacova].

THEOREM 2.2.53. *(convergence of the FVM on dual meshes) Let us assume that an unstructured dual grid is constructed over general triangulation satisfying Assumptions* 2.2.20. *Let* $w_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $w_0 \in \mathcal{M}_M$, *the CFL-like condition* (CFL) *hold and* w_j^k, $j \in \mathbb{J}$, $k = 0, 1, \dots$, *be defined by the monotone finite volume scheme* (2.2.34), (2.2.35), $(s = 1)$. *Let* $T > 0$ *be an arbitrary final time and* N *such that* $N\tau = T$. *We suppose that for the numerical flux function the properties* (2.2.41) *hold and that analogous conditions to (2.2.41) hold also for the numerical entropy flux functions* $G_{j\ell}$, $j \in \mathbb{J}$, $\ell \in S(j)$.

Moreover, we suppose that there exists $\beta \in \langle 0, 1)$ *such that for all compact sets* $K \subset \mathbb{R}^2$ *uniformly in* τ, h *the numerical and entropy fluxes are controlled in the following way*

$$\tau^\beta h \sum_{k=0}^{N} \sum_{\substack{j \in \mathbb{J}: \\ D_j \cap K \neq 0}} \sum_{\ell \in S(j)} |g_{j\ell}(w_j^k, w_\ell^k) - g_{j\ell}(w_j^k, w_j^k)| \leq c(K),$$

$$\tau^\beta h \sum_{k=0}^{N} \sum_{\substack{j \in \mathbb{J}: \\ D_j \cap K \neq 0}} \sum_{\ell \in S(j)} |G_{j\ell}(w_j^k, w_\ell^k) - G_{j\ell}(w_j^k, w_j^k)| \leq c(K).$$

Finally, let the discrete entropy inequality hold

$$U(w_j^{k+1}) - U(w_j^k) \leq -\frac{\tau}{|D_j|} \sum_{\ell \in S(j)} G_{j\ell}(w_j^k, w_\ell^k)$$

for all entropies U *and corresponding numerical entropy fluxes* $G_{j\ell}$. *Then*

$$w^k \rightharpoonup^* w \quad {}^*\text{-weakly in} \quad L^\infty(\mathbb{R}^2 \times (0, T)) \qquad \text{as } k \longrightarrow \infty$$

and w *is a Kruzhkov weak entropy solution of the problem* (2.2.1), (2.2.2), $s = 1$, $d = 2$.

In the next subsection we will introduce several numerical flux functions $g_{j\ell}$ which are used for practical computations. For example, the Engquist–Osher numerical scheme, cf. (2.2.59), satisfies all conditions of Theorem 2.2.53.

2.2.54 Godunov's method. It was observed that a good numerical flux function should include information propagated along characteristics, which leads to the so-called upwinding. To describe this phenomenon let us consider a linear convection equation in 1D.

$$\frac{\partial w}{\partial t} + a\frac{\partial w}{\partial x} = 0, \qquad x \in \mathbb{R}, \quad t > 0,$$

where a =const. is the speed of propagation of disturbances.

Recall that in the case $d = 1$ the finite volumes reduce to $D_j = \left(x_{j-1/2}, x_{j+1/2}\right)$; $x_j = jh$, $j \in \mathbb{Z}$, $h > 0$, are mesh points, and $x_{j\pm1/2} = \dfrac{x_j + x_{j\pm1}}{2}$. The upwind scheme reads:

$$w_j^{k+1} = w_j^k - \frac{\tau}{h} a \cdot \begin{cases} (w_j^k - w_{j-1}^k), & a > 0, \\ \\ (w_{j+1}^k - w_j^k), & a < 0. \end{cases} \tag{2.2.55}$$

We can see that for approximation of $\frac{\partial w}{\partial x}(x_j, t_k)$ we used finite difference approximation oriented against the direction of propagation of disturbances.

For the nonlinear case $a = a(w)$ is not constant and the same approach does not work directly. Then the local characteristic structure, obtained by solving a Riemann problem (cf. (2.1.41)), is used to define a natural upwind method. This was firstly used in 1959 by Godunov, who proposed a way to make use of the characteristic information within the framework of conservative method. Rather than attempting to follow characteristics backward in time, he suggested solving the following Riemann problem forward in time.

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = 0, \qquad x \in \mathbb{R}, \quad t > 0. \tag{2.2.56}$$
$$w(x,0) = \begin{cases} w_L, & x < 0, \\ \\ w_R, & x > 0. \end{cases}$$

Here $w_L, w_R \in \mathbb{R}$ are given initial conditions. We know from Theorems 2.1.42, 2.1.51 that under some assumptions on w_L, w_R, the Riemann problem (2.2.56) has a unique entropy solution $w(x,t) = \widetilde{w}_R(x/t; w_L, w_R)$. This result holds even in the case of systems in 1D. *Godunov numerical flux* is defined in the following way

$$f_G(u,v) = f\big(\widetilde{w}_R(0; u, v)\big)$$

and leads to the so-called Godunov's method:

$$w_j^{k+1} = w_j^k - \frac{\tau}{h}\big[f\big(\widetilde{w}_R\left(0; w_j^k, w_{j+1}^k\right)\big) - f\big(\widetilde{w}_R\left(0; w_{j-1}^k, w_j^k\right)\big)\big]. \tag{2.2.57}$$

Although in some cases we know the exact solution $\widetilde{w}_R$ of the Riemann problem (2.2.56), it is not the thruth in general. Thus, instead of $\widetilde{w}_R$ we use an approximate solution of (2.2.56), which leads us to the introduction of *Riemann numerical flux* or *Riemann solver* $f_R(u,v)$. The Godunov finite volume method can be then written in the following way

$$w_j^{k+1} = w_j^k - \frac{\tau}{h}\big[f_R\left(w_j^k, w_{j+1}^k\right) - f_R\left(w_{j-1}^k, w_j^k\right)\big].$$

The aim is to construct a good numerical flux function, which is now denoted by $f_R(u,v)$. Using the method of characteristics it can be shown that if $f'(w) = a(w) \geq 0$ for all $w \in \mathbb{R}$, then $\widetilde{w}_R(x/t; w_L, w_R) = w_L$, and if $f'(w) = a(w) < 0$ for all $w \in \mathbb{R}$, then $\widetilde{w}_R(x/t; w_L, w_R) = w_R$. Consequently, the Riemann numerical flux is

$$f_R(u,v) = f(u) \qquad \text{if} \quad a \geq 0, \tag{2.2.58}$$
$$f_R(u,v) = f(v) \qquad \text{if} \quad a < 0.$$

This observation gives us a hint how to define Riemann numerical flux in general nonlinear case. We split the flux function $f(u)$ s.t.

$$f(u) = f^+(u) + f^-(u), \quad \left(f^+\right)' \geq 0, \quad \left(f^-\right)' \leq 0,$$

and define

$$f_R(u,v) := f^+(u) + f^-(v).$$

Finally, we obtain the following Godunov finite volume scheme:

$$w_j^{k+1} = w_j^k - \frac{\tau}{h}\left[f^+\left(w_j^k\right) + f^-(w_{j+1}^k) - f^+(w_{j-1}^k) - f^-\left(w_j^k\right)\right].$$

The numerical flux can be rewritten in the form

$$f_{EO}(u,v) = \frac{1}{2}\Big(f(u) + f(v) - \int_u^v |a(q)|\,dq\Big), \tag{2.2.59}$$

which is referred to as the Engquist–Osher numerical flux, see e.g. [Osher(1)], [Osher-Solomon].

Now we approach the generalization of these ideas to the case of Euler equations in two space dimensions. Let $\mathcal{P} = \mathcal{P}(\mathbf{w}, \mathbf{n})$ be the flux function in 2D, i.e.

$$\mathcal{P}(\mathbf{w}, \mathbf{n}) := \mathbf{f}_1(\mathbf{w})\, n_x + \mathbf{f}_2(\mathbf{w})\, n_y, \tag{2.2.60}$$

$\mathbf{w} \in D, \mathbf{n} \in S_1$, and the Jacobi matrix of $\mathcal{P}$ will be denoted by $\mathbb{P} = \mathbb{P}(\mathbf{w}, \mathbf{n})$ (cf. (2.2.8)).

One of the most useful properties of the Euler equations is the rotational invariance of their flux functions, i.e. if the matrix Λ is given by

$$\Lambda(\mathbf{n}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_x & n_y & 0 \\ 0 & -n_y & n_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then $\mathcal{P}(\mathbf{w}, \mathbf{n}) = \Lambda^{-1}(\mathbf{n})\, \mathbf{f}_1\big(\Lambda(\mathbf{n})\, \mathbf{w}\big)$ for all $\mathbf{w} \in D, \mathbf{n} \in S_1$. The proof of this statement follows from straightforward calculations. Due to this property the two-dimensional Euler equations can be transformed to a one-dimensional system in the direction of the normal vector to a control volume $D_j, j \in \mathbb{J}$. It is this property that allows us to use of one-dimensional Riemann solvers to introduce two-dimensional flux functions. Thus, in analogy to (2.2.59) one obtains that the numerical flux function is

$$\mathbf{H}_{EO}(\mathbf{u}, \mathbf{v}, \mathbf{n}) := \frac{1}{2}\Big(\mathcal{P}(\mathbf{u}, \mathbf{n}) + \mathcal{P}(\mathbf{v}, \mathbf{n}) - \int_{\mathbf{u}}^{\mathbf{v}} |\mathbb{P}(\mathbf{w}, \mathbf{n})|\, d\mathbf{w}\Big). \tag{2.2.61}$$

Of course, this formula has sense only if the integral $\int_{\mathbf{u}}^{\mathbf{v}}$ exists, but on the other hand this gives us a good hint to derive numerical methods by suitable approximation of the integral $\int_{\mathbf{u}}^{\mathbf{v}}$.

Before developing some methods, we state the following result.

LEMMA 2.2.62. *Let $\mathbf{F} \in C^1(\mathbb{R}^s; \mathbb{R}^s)$ be a homogeneous function of order one, i.e. $\mathbf{F}(\alpha\, \mathbf{u}) = \alpha\, \mathbf{F}(\mathbf{u})$ for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then*

$$\mathbf{F}(\mathbf{u}) = \mathbb{A}(\mathbf{u}) \cdot \mathbf{u},$$

where $\mathbb{A}(\mathbf{u}) = \dfrac{D\mathbf{F}(\mathbf{u})}{D\mathbf{u}}$.

PROOF.

$$\mathbb{A}(\mathbf{u})\cdot\mathbf{u} = \frac{D\mathbf{F}(\mathbf{u})}{D\mathbf{u}}\cdot\mathbf{u} = \lim_{\alpha\to 0}\frac{\mathbf{F}(\mathbf{u}+\alpha\,\mathbf{u})-\mathbf{F}(\mathbf{u})}{\alpha} = \\ = \lim_{\alpha\to 0}\frac{(1+\alpha)\,\mathbf{F}(\mathbf{u})-\mathbf{F}(\mathbf{u})}{\alpha} = \mathbf{F}(\mathbf{u}).$$

□

It can be verified by straightforward calculations that for the Euler equations, the flux functions $\mathbf{f}_1, \mathbf{f}_2$, and consequently $\mathcal{P}$ are homogeneous of order one. Hence,

$$\begin{aligned}\mathbf{f}_1(\mathbf{u}) &= \mathbb{A}_1(\mathbf{u})\cdot\mathbf{u},\\ \mathbf{f}_2(\mathbf{u}) &= \mathbb{A}_2(\mathbf{u})\cdot\mathbf{u},\\ \mathcal{P}(\mathbf{u},\mathbf{n}) &= \mathbb{P}(\mathbf{u},\mathbf{n})\cdot\mathbf{u}.\end{aligned} \tag{2.2.63}$$

Let us denote

$$\mathbb{D}^{\pm} := \operatorname{diag}\left(\lambda_1^{\pm},\ldots,\lambda_4^{\pm}\right), \tag{2.2.64}$$

where $\lambda_i^+ = \max(\lambda_i, 0)$, $\lambda_i^- = \min(\lambda_i, 0)$.

$$|\mathbb{D}| = \operatorname{diag}\big(|\lambda_1|,\ldots,|\lambda_4|\big). \tag{2.2.65}$$

$$\mathbb{P}^{\pm} := \mathbb{T}\cdot\mathbb{D}^{\pm}\cdot\mathbb{T}^{-1}, \qquad |\mathbb{P}| = \mathbb{T}\cdot|\mathbb{D}|\cdot\mathbb{T}^{-1}. \tag{2.2.66}$$

Obviously, it holds

$$\mathbb{P} = \mathbb{P}^+ + \mathbb{P}^-, \quad |\mathbb{P}| = \mathbb{P}^+ - \mathbb{P}^-. \tag{2.2.67}$$

Now we approach approximation of the R. H. S. of (2.2.61).

$$\begin{aligned}&\text{a)} \quad \int_{\mathbf{u}}^{\mathbf{v}} |\mathbb{P}(\mathbf{w},\mathbf{n})|\,d\mathbf{w} \approx \left|\mathbb{P}\left(\frac{\mathbf{u}+\mathbf{v}}{2},\mathbf{n}\right)\right|(\mathbf{v}-\mathbf{u})\\ &\text{b)} \quad \frac{1}{2}\big(\mathcal{P}(\mathbf{u},\mathbf{n})+\mathcal{P}(\mathbf{v},\mathbf{n})\big) \approx \mathcal{P}\big(\frac{\mathbf{u}+\mathbf{v}}{2},\mathbf{n}\big) = \\ &\quad = \big(\text{using (2.2.63)}\big) = \mathbb{P}\big(\frac{\mathbf{u}+\mathbf{v}}{2},\mathbf{n}\big)\frac{\mathbf{u}+\mathbf{v}}{2}.\end{aligned} \tag{2.2.68}$$

These approximations give the Vijayasundaram numerical flux; [Vijayasundaram]:

$$\mathbf{H}_V(\mathbf{u},\mathbf{v},\mathbf{n}) = \mathbb{P}^+\big(\frac{\mathbf{u}+\mathbf{v}}{2},\mathbf{n}\big)\mathbf{u} + \mathbb{P}^-\big(\frac{\mathbf{u}+\mathbf{v}}{2},\mathbf{n}\big)\mathbf{v}. \tag{2.2.69}$$

Using (2.2.58) and analogy with the one-dimensional case, one can derive the Steger–Warming numerical flux:

$$\mathbf{H}_{SW}(\mathbf{u},\mathbf{v},\mathbf{n}) = \mathbb{P}^+(\mathbf{u},\mathbf{n})\,\mathbf{u} + \mathbb{P}^-(\mathbf{v},\mathbf{n})\,\mathbf{v}. \tag{2.2.70}$$

If we use (2.2.68) a) and (2.2.61) we obtain the Van Leer numerical flux :

$$\mathbf{H}_{VL}(\mathbf{u},\mathbf{v},\mathbf{n}) = \frac{1}{2}\Big(\mathcal{P}(\mathbf{u},\mathbf{n}) + \mathcal{P}(\mathbf{v},\mathbf{n}) - \left|\mathbb{P}\left(\frac{\mathbf{u}+\mathbf{v}}{2},\mathbf{n}\right)\right|\cdot(\mathbf{v}-\mathbf{u})\Big). \tag{2.2.71}$$

Remarks 2.2.72.

a) Let us note that in the Vijayasundaram scheme (cf. (2.2.69)) a *partial upwinding* is included. The numerical flux between the volumes D_j and D_ℓ is computed according to the sign of local eigenvalues $\lambda_i,\ i=1,\dots,4$, of the matrix $\mathbb{P}$ and either $\mathbf{w}_j^k$ or $\mathbf{w}_\ell^k$ is chosen for approximation.

More precisely, we take this one, which lies against the propagation of disturbances. This upwinding is only partial, since the local eigenvalues are evaluated at the central points : $\lambda_i = \lambda_i\left(\frac{\mathbf{w}_j^k+\mathbf{w}_\ell^k}{2}\right),\ j\in\mathbb{J}, \ell\in S(j)$.

The *fully upwind* scheme is obtained by the Steger–Warming numerical flux function (2.2.70). It is known that the full upwinding leads to the so-called numerical diffusion, i.e. the leading order truncation error in the terms with second order derivatives. As a result, discontinuities and shocks are smeared out.

b) It is easy to realize that all these numerical fluxes (2.2.69) – (2.2.71) have the properties (2.2.37) – (2.2.39).

c) To derive the Vijayasundaram and the Steger–Warming numerical flux one needs the homogenity property (2.2.63) of the flux functions $\mathbf{f}_1, \mathbf{f}_2$. However, in the case of a scalar conservation law ($s=1$) this implies

$$f_1'(u)\cdot u = f_1(u), \quad f_2'(u)\cdot u = f_2(u). \tag{2.2.73}$$

But in a scalar case this is fulfilled only for linear functions. We conclude that there makes no sense to consider these schemes for nonlinear scalar conservation law. This restriction does not include the Van Leer numerical flux.

d) Discrete entropy condition. Numerical as well as theoretical results, obtained at least for scalar conservation law, show that the numerical solution obtained by the method of Godunov's type can violate entropy condition only in the case when

$$\lambda_k(\mathbf{w}_L) < 0 < \lambda_k(\mathbf{w}_R)$$

for some $k=1,2,\dots,s$.

Clearly, the Lax entropy inequality (2.1.47) is not satisfied. Hence, discontinuity, i.e. shock, is not an entropy solution. If the FVM approximates the Riemann problem between $\mathbf{w}_L$ and $\mathbf{w}_R$ by means of the discontinuity it leads to an entropy violating unphysical approximation. In this situation an exact solution of the Riemann problem (2.1.41) is a continuous *sonic rarefaction wave.*

To obtain the numerical solution, which satisfies the entropy inequality, we need to cure this case. This is referred to as *entropy fix* in literature. One approach, proposed in [Harten], [Osher (1)] is outlined here.

If one of the eigenvalues of the flux Jacobian matrix equals zero then approximate solution can violate entropy condition. In order to prevent entropy violating a correction to the eigenvalue is made. For the case of the Euler equations in 2D we have the following eigenvalues (cf. 2.2.10)

$$\lambda_1 = \lambda_2 = \mathbf{v}\cdot\mathbf{n},\ \lambda_3 = \mathbf{v}\cdot\mathbf{n}+a,\ \lambda_4 = \mathbf{v}\cdot\mathbf{n}-a.$$

Let $\lambda := |\mathbf{v}\cdot\mathbf{n}| + a$ be the maximum absolute eigenvalue. Then the entropy fix correction determines to put

$$|\lambda_k| := \max\{|\lambda_k|, \delta\lambda\}, \quad k = 1, \ldots, 4;$$

where $\delta > 0$ is an entropy fix parameter, commonly chosen s.t. $\delta \in (0,01;1)$. Sign of λ_k is unchanged.

e) The present schemes fall to the wide class of flux vector splitting methods. In literature one can meet other numerical schemes like Osher, Lax–Friedrichs, Roe, Mac Cormarck, Jameson ones, etc.

2.3. Boundary Conditions

In this section we want to derive suitable boundary conditions for solving the Euler equations in a bounded domain $\Omega \subset \mathbb{R}^2$. In literature various approaches can be found. We present here one, which is based on the method of characteristics. The general finite volume scheme reads (cf.(2.2.43)):

$$\mathbf{w}_j^{k+1} = \mathbf{w}_j^k - \frac{\tau}{|D_j|}\sum_{\ell\in S(j)} \mathbf{g}_{j\ell}\left(\mathbf{w}_j^k, \mathbf{w}_\ell^k\right), \qquad j \in \mathbb{J}, k = 0, 1, \ldots. \tag{2.3.1}$$

If $j \in H$ and $\ell \in \gamma(j)$ (i.e. $\ell = -1$, cf. (2.2.16)) then there is a problem how to compute $\mathbf{g}_{j\ell}\left(\mathbf{w}_j^k, \mathbf{w}_\ell^k\right)$, since there is no dual finite volume D_ℓ outside the domain Ω_h and the question is what is $\mathbf{w}_\ell^k$?

a) At first, if the edge $S_\ell \subset \partial D_j \cap \partial\Omega_h$, $\ell \in \gamma(j)$, (i.e. S_ℓ is either $\mathbf{a}_{j\ell}$ or $\mathbf{b}_{j\ell}$ (see Fig. 2.5)) is a part of the *solid, impermeable wall,* then the so-called slip boundary conditions are imposed (cf. (1.4.7)).

$$\mathbf{v}\cdot\mathbf{n}_{j\ell} = 0 \qquad \text{on } S_\ell, \tag{2.3.2}$$

where $\mathbf{n}_{j\ell}$ is an outer normal to D_j on S_ℓ.

Then in numerical calculations we do not use numerical flux function $\mathbf{g}_{j\ell}$, but we approximate the flux through S_ℓ in the following way

$$\int_{S_\ell}\left(\mathbf{f}_1(\mathbf{w})\,n_x + \mathbf{f}_2(\mathbf{w})\,n_y\right)dS = \tag{2.3.3}$$

$$= \int_{S_\ell}\left\{(v_1\,n_x + v_2\,n_y)\,\mathbf{w} + p\cdot\begin{pmatrix}0\\ n_x\\ n_y\\ 0\end{pmatrix}\right\}dS \approx |S_\ell|\,p_j\begin{pmatrix}0\\ n_x\\ n_y\\ 0\end{pmatrix}.$$

Hence, we extrapolate the value of the pressure from the point P_j to the whole boundary edge S_ℓ. The numerical flux $\mathbf{H}$ on the solid part of the boundary $S_\ell \subset$

$\partial\Omega_h \cap \partial D_j$ is then defined by the formula

$$\mathbf{H}\left(\mathbf{w}_j^k, \mathbf{w}_\ell^k, \mathbf{n}_{j\ell}\right) := p_j^k \begin{pmatrix} 0 \\ n_{xj\ell} \\ n_{yj\ell} \\ 0 \end{pmatrix}, \tag{2.3.4}$$

and the numerical flux function $\mathbf{g}_{j\ell}$ is computed in the usual way

$$\mathbf{g}_{j\ell}\left(\mathbf{w}_j^k, \mathbf{w}_\ell^k\right) = \mathbf{H}\left(\mathbf{w}_j^k, \mathbf{w}_\ell^k, \mathbf{n}_{j\ell}^a\right)|\mathbf{a}_{j\ell}| + \mathbf{H}\left(\mathbf{w}_j^k, \mathbf{w}_\ell^k, \mathbf{n}_{j\ell}^b\right)|\mathbf{b}_{j\ell}|. \tag{2.3.5}$$

b) If the edge $S_\ell \subset \partial D_j \cap \partial\Omega_h$, $\ell \in \gamma(j)$, lies on *inlet* or *outlet part* of the boundary, we use the heuristical approach proposed in [Feistauer]. Let a computational domain Ω be represented as $(-\infty, 0)$ and solve the linearized one-dimensional conservation system

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbb{A}\frac{\partial \mathbf{w}}{\partial x} = 0 \qquad \text{on } (-\infty, 0) \times (0, \infty). \tag{2.3.6}$$

By the hyperbolicity property one gets

$$\mathbb{A} = \mathbb{T} \cdot \mathbb{D} \cdot \mathbb{T}^{-1}, \qquad \mathbb{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_s). \tag{2.3.7}$$

From (2.3.6) and (2.3.7) we find the system of s independent equations

$$\frac{\partial z_i}{\partial t} + \lambda_i \frac{\partial z_i}{\partial x} = 0, \qquad i = 1, 2, \ldots, s, \text{ on } (-\infty, 0) \times (0, \infty).$$

These equations can be solved by the method of characteristics. If $\lambda_i > 0$, then the initial value, which is prescribed at $\{t = 0, x < 0\}$, is propagated along the characteristics with the slope $\lambda_i > 0$ to the half-line $\{x = 0, t > 0\}$ and the value of the solution is uniquely determined here.

On the other hand, if $\lambda_i < 0$, then the characteristics starting at $\{t = 0, x < 0\}$ do not intersect the half line $\{x = 0, t > 0\}$ and the boundary conditions should be prescribed here.

These ideas are heuristically extended to the case of the Euler equations. Hence on $S_\ell \subset \partial D_j \cap \partial\Omega_h$, a part of inlet or outlet, we prescribe n_p quantities characterizing the state vector $\mathbf{w}_\ell^k$, which corresponds to the boundary edge S_ℓ; n_p is a number of negative local eigenvalues of the matrix $\mathbb{P}\left(\mathbf{w}_j^k\right)$. Further, we extrapolate n_e quantities from the point P_j to the edge S_ℓ, n_e is a number of positive local eigenvalues of $\mathbb{P}\left(\mathbf{w}_j^k\right)$.

Now, there is a question : Which quantities should be prescribed and which ones are extrapolated? There is the following classical approach.

On inlet, i.e. $\mathbf{v}_j^k \cdot \mathbf{n}_{j\ell} < 0$, for the case of *supersonic flow* $\left(-\mathbf{v}_j^k \cdot \mathbf{n}_{j\ell} > a_j^k\right)$, a_j^k is a local speed of sound computed from $\mathbf{w}_j^k$, we have (cf. (2.2.10) for eigenvalues of the Euler equations):

$$\lambda_1 = \lambda_2 < 0,\ \lambda_3 < 0,\ \lambda_4 < 0 \Longrightarrow n_p = 4,\ n_e = 0, \tag{2.3.8, i}$$

and we prescribe boundary condition for ρ, v_1, v_2, p to characterize $\mathbf{w}_\ell^k$.

For the case of *subsonic flow* $\left(-\mathbf{v}_j^k \cdot \mathbf{n}_{j\ell} < a_j^k\right)$:

$$\lambda_1 = \lambda_2 < 0,\ \lambda_3 > 0,\ \lambda_4 < 0 \Longrightarrow n_p = 3,\ n_e = 1. \tag{2.3.8, ii}$$

We prescribe ρ, v_1, v_2 and extrapolate p from the point P_j.

On outlet, i.e. $\mathbf{v}_j^k \cdot \mathbf{n}_{j\ell} > 0$, for the case of *supersonic flow* $\left(\mathbf{v}_j^k \cdot \mathbf{n}_{j\ell} > a_j^k\right)$ we have :

$$\lambda_1 = \lambda_2 > 0,\ \lambda_3 > 0,\ \lambda_4 > 0 \Longrightarrow n_p = 0,\ n_e = 4. \tag{2.3.8, iii}$$

We extrapolate ρ, v_1, v_2, p from P_j. For the case of *subsonic flow* $\left(\mathbf{v}_j^k \cdot \mathbf{n}_{j\ell} < a_j^k\right)$:

$$\lambda_1 = \lambda_2 > 0,\ \lambda_3 > 0,\ \lambda_4 < 0 \Longrightarrow n_p = 1,\ n_e = 3, \tag{2.3.8, iv}$$

and we extrapolate ρ, v_1, v_2 from P_j and prescribe p.

This discussion leads to the definition of $\mathbf{w}_\ell^k$, which is the state vector corresponding to the boundary edge S_ℓ. Now we can evaluate $\mathbf{H}\left(\mathbf{w}_j^k, \mathbf{w}_\ell^k, \mathbf{n}_{j\ell}\right)$, i.e. the flux through the edge S_ℓ, and consequently we compute $\mathbf{g}_{j\ell}\left(\mathbf{w}_j^k, \mathbf{w}_\ell^k\right)$ by (2.3.5).

2.4 Second Order Method

In practise it is often important to derive a good approximation of the exact, often unknown, solution. Thus, the numerical scheme should be enough accurate and posse an adequate rate of dissipation to give physically relevant entropy solution and has a *higher order accuracy* to obtain sharp shock waves in the correct solution. We first discuss the generalized **MUSCL** methodology, which is used to increase spatial accuracy of the basic finite volume scheme (2.2.34) , (2.2.35). Thus, instead of (2.2.34) we use the following scheme

$$\mathbf{w}_j^{k+1} := \mathbf{w}_j^k - \frac{\tau}{|D_j|} \sum_{\ell \in S(j)} \mathbf{g}_{j\ell}\left(\widehat{\mathbf{w}}_{j\ell}^k, \widehat{\mathbf{w}}_{\ell j}^k\right), \quad j \in \mathbb{J}, \tag{2.4.1}$$

where $\widehat{\mathbf{w}}_{j\ell}^k, \widehat{\mathbf{w}}_{\ell j}^k$ are values of a new higher order function $\widehat{\mathbf{w}}_h$ at the boundary $\partial D_{j\ell}$ at time step t_k.

MUSCL approach requires to replace the piecewise constant function $\mathbf{w}_h$, which is defined on dual volumes by a higher order function. Since we want to get the second order scheme (in the sense of truncation error), we choose a piecewise linear recovery function $\widehat{\mathbf{w}}_h$, which will be linear on dual volumes and discontinuous at the boundaries of dual volumes. Thus, $\widehat{\mathbf{w}}_h$ will be recovered from $\mathbf{w}_h$ in such a way that we have

$$\mathbf{w}_h\Big|_{D_j} = \mathbf{w}_h\left(P_j\right) = \frac{1}{|D_j|} \int_{D_j} \widehat{\mathbf{w}}_h\left(x, y\right). \tag{2.4.2}$$

Now we show one possibility how to construct this linear recovery not on the conservative variables, but separately on the physical variables ρ, v_1, v_2, p. Let $q \in \{\rho, v_1, v_2, p\}$, and q_h be a piecewise constant function on the dual volumes, which corresponds to some physical variable q. We will define $\widehat{q}_h$, i.e. a piecewise linear recovery function on the dual volumes, in the following way:

$$\widehat{q}_h(x,y) := q_h\Big|_{D_j} + (\mathrm{grad}_h q_h)\Big|_{D_j} \cdot (x - x_j, y - y_j)^T, \tag{2.4.3}$$

$(x,y) \in D_j, j \in \mathbb{J}, (x_j, y_j)$ are coordinates of the point $P_j; P_j \in \mathcal{P}_h$. Since $q_h\Big|_{D_j} = q_h(P_j)$, the piecewise linear recovery function $\widehat{q}_h$ will be determined by (2.4.3), if we define $(\mathrm{grad}_h q_h)\Big|_{D_j}$.

Now we realize that from piecewise constant function on the dual volumes q_h one can construct a continuous, piecewise linear function on triangles. More precisely, let

$$X_h = \left\{ \omega_h \in C\left(\bar{\Omega}; \mathbb{R}\right); \quad \omega_h\Big|_{T_i} \in P_i(T_i),\ i \in \mathbb{I} \right\} \tag{2.4.4}$$

be the space of linear finite element functions, and

$$D_h = \left\{ q_h : \bar{\Omega} \to \mathbb{R}; \quad q_h\Big|_{D_j} \in P_0(D_j),\ j \in \mathbb{J} \right\} \tag{2.4.5}$$

be the space of piecewise constant finite volume functions. Here $P_k(\cdot)$ denotes the space of polynomials of order not greater than $k, k = 0, 1$.

The basis of X_h is formed by the functions $w_j, j \in \mathbb{J}$, s.t. $w_j(P_\ell) = \delta_{j\ell}, w_j \in X_h$. Further, the basis of D_h is formed by the characteristic functions of dual finite volumes D_j, denoted by $d_j, j \in \mathbb{J}$.

There exists a bijection $L_h : X_h \to D_h$, called the mass-lumping operator and defined in the following way

$$L_h \omega_h = \sum_{j \in \mathbb{J}} \omega_h(P_j)\, d_j, \tag{2.4.6}$$

where $\omega_h = \sum_{j \in \mathbb{J}} \left(\omega_h(P_j) \cdot w_j\right) \in X_h$. Let $L(j) = \{k \in \mathbb{I};\ \exists T_k \in \mathcal{T}_h : P_j \in T_k\}$. Using the inverse mapping to L_h, i.e. $\Pi_h = L_h^{-1}$, we obtain a function $\Pi_h q_h = \sum\limits_{j \in \mathbb{J}} q_h\Big|_{D_j} w_j$, which is piecewise linear on triangles. This allows us to compute the so-called *averaged gradient :*

$$(\mathrm{grad}_h\, q_h)\Big|_{D_j} := \frac{1}{|D_j|} \sum_{k \in L(j)} \int_{T_k \cap D_j} \mathrm{grad}\ \Pi_h q_h = \tag{2.4.7}$$

$$= \frac{1}{|D_j|} \sum_{k \in L(j)} (\mathrm{grad}\ \Pi_h q_h)\Big|_{T_k} \cdot |T_k \cap D_j|.$$

Now we can use (2.4.3) to define the values of $\widehat{q}_h$ at the boundary $\partial D_{j\ell}, \ell \in S(j)$, of the dual volume D_j. Let $z^a_{j\ell} = \left(x^a_{j\ell}, y^a_{j\ell}\right), z^b_{j\ell} = \left(x^b_{j\ell}, y^b_{j\ell}\right)$ be the midpoints of

the edges $\mathbf{a}_{j\ell}, \mathbf{b}_{j\ell}$, respectively. Recall that $\mathbf{a}_{j\ell} \cup \mathbf{b}_{j\ell} = \partial D_{j\ell}$, if $j \notin H$ or $\ell \notin H$, and $\mathbf{a}_{j\ell} = \partial D_{j\ell}$, if $j, \ell \in H$ (cf. (2.2.18)i)). The extrapolation of $\widehat{q}_h$ at $\partial D_{j\ell}$, i.e. to the midpoints $z^a_{j\ell}, z^b_{j\ell}$, is given in the following way

$$\begin{aligned} \widehat{q}^a_{j\ell} &:= \widehat{q}_h\left(z^a_{j\ell}\right) = q_h\Big|_{D_j} + \left(\operatorname{grad}_h q_h\right)\Big|_{D_j} \cdot \left(x^a_{j\ell} - x_j, y^a_{j\ell} - y_j\right)^T, \\ \widehat{q}^b_{j\ell} &:= \widehat{q}_h\left(z^b_{j\ell}\right) = q_h\Big|_{D_j} + \left(\operatorname{grad}_h q_h\right)\Big|_{D_j} \cdot \left(x^b_{j\ell} - x_j, y^b_{j\ell} - y_j\right)^T. \end{aligned} \tag{2.4.8}$$

From (2.4.8) a new state vector $\widehat{\mathbf{w}}_h$ at $z^a_{j\ell}, z^b_{j\ell}$ can be computed. Let us denote

$$\left(\widehat{\mathbf{w}}^a_{j\ell}\right)^k := \widehat{\mathbf{w}}^k_h\left(z^a_{j\ell}\right), \quad \left(\widehat{\mathbf{w}}^b_{j\ell}\right)^k := \widehat{\mathbf{w}}^k_h\left(z^b_{j\ell}\right),$$

then

$$\begin{aligned} \mathbf{g}_{j\ell}\left(\widehat{\mathbf{w}}^k_{j\ell}, \widehat{\mathbf{w}}^k_{\ell j}\right) &= \mathbf{H}\big(\left(\widehat{\mathbf{w}}^a_{j\ell}\right)^k, \left(\widehat{\mathbf{w}}^a_{\ell j}\right)^k, \mathbf{n}^a_{j\ell}\big)|\mathbf{a}_{j\ell}| + \\ &\mathbf{H}\big(\left(\widehat{\mathbf{w}}^b_{j\ell}\right)^k, \left(\widehat{\mathbf{w}}^b_{\ell j}\right)^k, \mathbf{n}^b_{j\ell}\big)|\mathbf{b}_{j\ell}| \qquad j \notin H \text{ or } \ell \notin H. \\ \mathbf{g}_{j\ell}\left(\widehat{\mathbf{w}}^k_{j\ell}, \widehat{\mathbf{w}}^k_{\ell j}\right) &= \mathbf{H}\big(\left(\widehat{\mathbf{w}}^a_{j\ell}\right)^k, \left(\widehat{\mathbf{w}}^a_{\ell j}\right)^k, \mathbf{n}^a_{j\ell}\big)|\mathbf{a}_{j\ell}| \quad j, \ell \in H. \end{aligned} \tag{2.4.9}$$

In case that $\mathbf{a}_{j\ell}, \mathbf{b}_{j\ell} \subset \partial D_j \cap \partial\Omega_h$, i.e. $j \in H, \ell \in \gamma(j)$, we compute linear recovery only for $\widehat{\mathbf{w}}^k_{j\ell}$ in the same way as before, and $\widehat{\mathbf{w}}^k_{\ell j} = \mathbf{w}^k_\ell$ is defined by boundary conditions as in the first order method.

It is a well-known fact that higher order methods often suffer from the lack of dissipation which reflects in spurious oscillations near discontinuities. A possibility how to improve this result is to use the *slope limiter* approach.

In [Barth, Jespersen] the following slope limiter was proposed. The idea is to limit slopes in such a way that oscillations will be suppressed.

$$\Phi_j := \min_{\ell \in S(j)} \big(\min \Phi\left(\mathbf{a}_{j\ell}\right), \Phi\left(\mathbf{b}_{j\ell}\right)\big), \tag{2.4.10}$$

and

$$\Phi\left(\mathbf{a}_{j\ell}\right) := \begin{cases} 1, & \widehat{q}^a_{j\ell} - q_j = 0, \\ \min\Big(1, \dfrac{q^{\max}_j - q_j}{\widehat{q}^a_{j\ell} - q_j}\Big), & \widehat{q}^a_{j\ell} - q_j > 0, \\ \min\Big(1, \dfrac{q^{\min}_j - q_j}{\widehat{q}^a_{j\ell} - q_j}\Big), & \widehat{q}^a_{j\ell} - q_j < 0; \end{cases} \tag{2.4.11}$$

where $q^{\max}_j := \max\left(q_j, \max\limits_{\ell \in S(j)} q_\ell\right); q^{\min}_j := \min\left(q_j, \min\limits_{\ell \in S(j)} q_\ell\right)$. Analogously, $\Phi\left(\mathbf{b}_{j\ell}\right)$ is defined. Finally, the linear recovery (2.4.8) will be rewritten in the form

$$\widehat{q}^a_{j\ell} = q_h\Big|_{D_j} + \Phi_j\left(\operatorname{grad}_h q_h\right)\Big|_{D_j} \cdot \left(x^a_{j\ell} - x_j, y^a_{j\ell} - y_j\right)^T, \tag{2.4.12}$$

and in the same way for $\widehat{q}^b_{j\ell}$.

A discrete scheme is said to be TVD (total variation diminishing), if for the sequence $w^k = \left\{w_j^k\right\}_{j\in\mathbb{J}}$, $k = 0, 1, \dots$, the total variation, i.e. $\mathrm{TV}\left(w^k\right) = \sum\limits_{\substack{j\in\mathbb{J}\\ \ell\in S(j)}} |w_j^k - w_\ell^k|$, satisfies the condition

$$\mathrm{TV}\left(w^k\right) \geq \mathrm{TV}\left(w^{k+1}\right), \quad k = 0, 1, \dots. \tag{2.4.13}$$

The described construction of second order method is based on a one-dimensional second order TVD extension of the first order scheme, it can be considered as a second order TVD-MUSCL scheme, see e.g. [Osher(2)], [Rostard, Stoufflet] for further results.

Although it can be even impossible to prove for fully nonlinear systems that our multidimensional scheme is the second order, the number of numerical experiments confirm the reliability of the method and the higher order behaviour of numerical solutions, see e.g. [Kröner, Noelle, Rokyta] for some convergence results of the higher order FVM. In Fig. 2.7 the comparison of the first and the second order FVM and the corresponding exact solution is given. We have chosen the two-dimensional Riemann problem, cf. Fig. 2.3, and plot the $y = 0$-cut. It can be observed that the second order method approximates extrema of the solution in a better way and resolves discontinuity more sharply than the first order scheme.

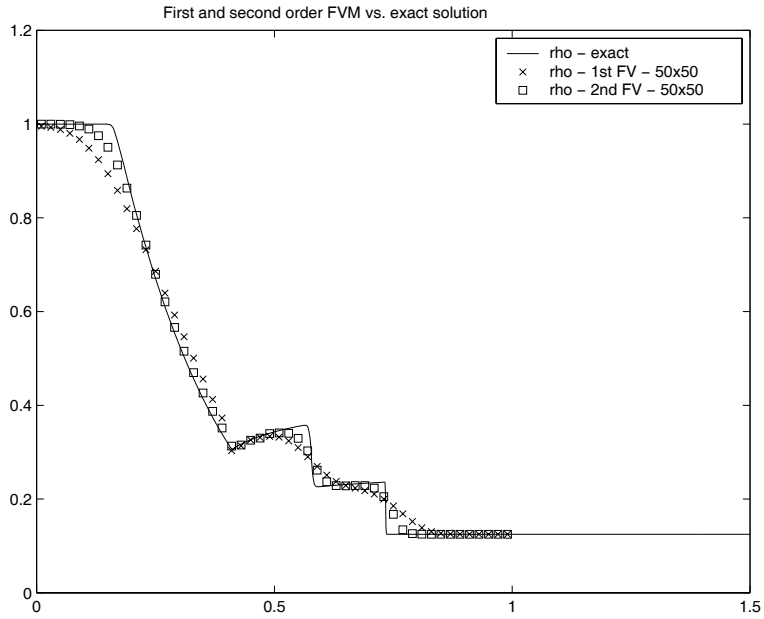

FIG. 2.7. Comparison of the first and second order FVM.

In order to obtain fully higher order FVM it is further necessary to consider better approximation in time. In fact, we apply the so-called method of lines and treat separately spatial and time approximation. Thus, let us consider the following system of ordinary differential equations with respect to time

$$\frac{d\mathbf{w}(t)}{dt} = \mathcal{F}(\mathbf{w}(t)), \tag{2.4.14}$$

where the operator $\mathcal{F}$ represents the FV approximation of the divergence of physical flux function, i.e. $\partial_x \mathbf{f}_1(\mathbf{w}) + \partial_y \mathbf{f}_2(\mathbf{w})$.

Generally, one can use any numerical scheme for ordinary differential equations in order to approximate (2.4.14). Thus, the backward Euler method naturally leads to the first order explicit FVM (2.2.34). In order to get the second order method in time, we can apply two variants of the second order Runge-Kutta scheme; using the midpoint rule or the trapezoidal rule. It was shown, e.g., in [Shu, Osher] that the later yields to the second order FVM, which is TVD

$$\mathbf{w}_j^* := \mathbf{w}_j^k - \frac{\tau}{|D_j|} \sum_{\ell \in S(j)} \mathbf{g}_{j\ell} \left(\widehat{\mathbf{w}}_{j\ell}^k, \widehat{\mathbf{w}}_{\ell j}^k \right),$$

$$\mathbf{w}_j^{k+1} := \mathbf{w}_j^k - \frac{\tau}{2|D_j|} \sum_{\ell \in S(j)} \left[\mathbf{g}_{j\ell} \left(\widehat{\mathbf{w}}_{j\ell}^k, \widehat{\mathbf{w}}_{\ell j}^k \right) + \mathbf{g}_{j\ell} \left(\widehat{\mathbf{w}}_{j\ell}^*, \widehat{\mathbf{w}}_{\ell j}^* \right) \right], \quad j \in \mathbb{J}.$$

Some further higher order Runge-Kutta approximations leading to the FVM, which are TVD, are presented in [Shu, Osher], [Sonar(1)].

In the following we simulated two-dimensional flow through the channel with a circular cap (see Fig. 2.8). The length of the channel is 2 m, the height is 1 m, the circular cap is centered at the point $(0., 0.)$ and its height is 0.1 m. The fluid (in our case air) flows through the channel in the x - direction. The left boundary is inflow, the right one is outflow boundary. Rest parts of the boundary are solid walls.

The Euler equations are approximated by the second order Vijayasundaram finite volume method. The initial data are taken to be

$$\begin{aligned} \rho_0 &= 1.5 \text{ kg m}^{-3} \\ v_{0,1} &= 206.835 \text{ m s}^{-1} \\ v_{0,2} &= 0 \text{ m s}^{-1} \\ p_0 &= 101000 \text{ Pa} \end{aligned}$$

The inflow Mach number $M := \dfrac{|\mathbf{v}_0|}{a_0} = 0.67$. Here a_0 stands for the local speed of sound at the inflow.

Fig. 2.9 shows the Mach number distribution, on the walls and the Mach number isolines, respectively. We can notice that the discontinuity, i.e. shock, is resolved sharply. This is due to the use of the second order approximation as well as due to the fact that the mesh was refined appropriately. Despite the second order method no spurious oscillations near shocks are obtained, which is due to the use of limiter. The flow is transonic since the Mach number varies between 0.35 and 1.45. In Fig. 2.10 the entropy isolines are plotted. We can notice that entropy is increasing after the shock, which is consistent with the second law of thermodynamics or, equivalently, with the entropy inequality. The solution shown in Fig. 2.9 is the physically unique weak entropy solution.

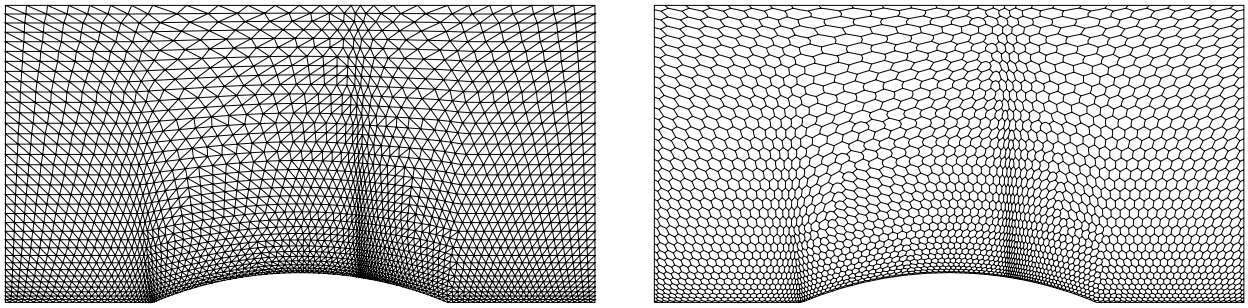

FIG. 2.8. Triangular mesh for the channel and the dual mesh.

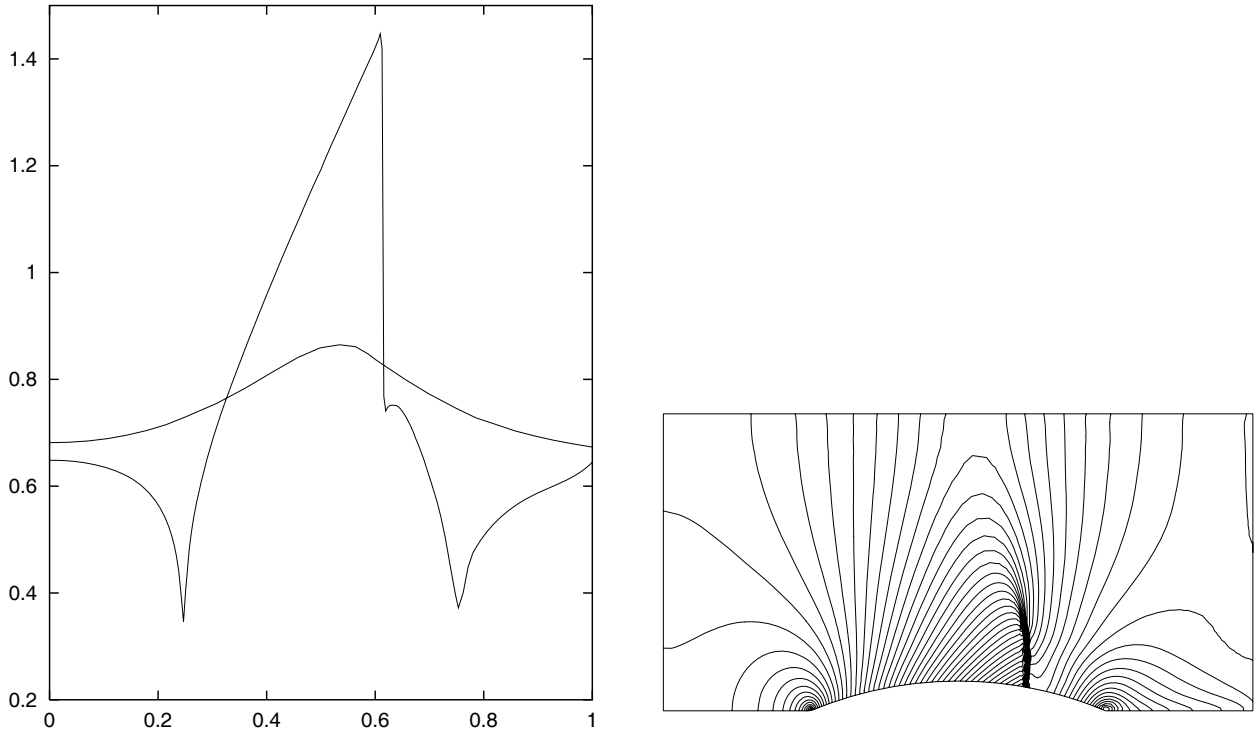

FIG. 2.9. Mach number distribution on the walls of the channel and Mach number isolines.

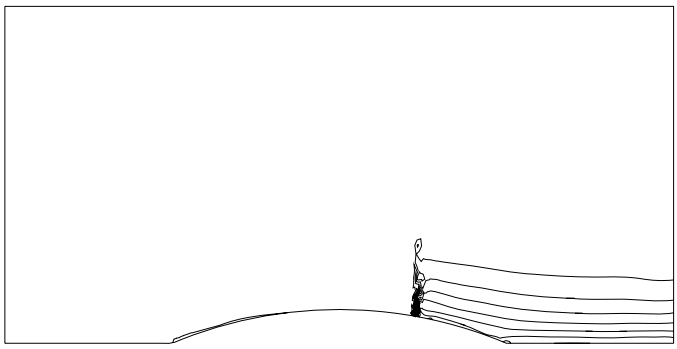

FIG. 2.10. Entropy isolines.

CHAPTER III

TUTORIALS

In this chapter we give a list of open problems for students' individual work or for the use in seminars.

Exercise 1.

Show that in $\mathbb{R}^3$ the following equality holds

$$\text{rot rot} = \text{grad div} - \text{div grad}.$$

Exercise 2. (**1D Euler equations in conservative and primitive variables**)

Consider the system of the Euler equations in 1D in conservative variables $(\rho, \rho u, e)$

$$\begin{aligned}
&\partial_t \rho + \partial_x(\rho u) = 0,\\
&\partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0,\\
&\partial_t e + \partial_x((e+p)u) = 0
\end{aligned}$$

with the so-called state equation for ideal polytropic gas

$$p = (\kappa - 1)(e - \rho u^2/2) \qquad \kappa \neq 1.$$

Let $(\rho, \rho u, e)$ be a solution to the Euler equations in conservative variables. Show that under some regularity assumptions, i.e. smoothness of the solution, it is also a solution to the Euler equations in the so-called primitive variables (ρ, u, p)

$$\begin{aligned}
&\partial_t \rho + \partial_x(\rho u) = 0,\\
&\partial_t u + u\partial_x u + \frac{\partial_x p}{\rho} = 0,\\
&\partial_t p + \rho a^2 \partial_x u + u\partial_x p = 0,
\end{aligned}$$

where

$$a := \sqrt{\frac{\kappa p}{\rho}}$$

denotes the speed of sound. Which regularity conditions have to be assumed?

Exercise 3. (**Euler equations)**
Compute for the above one-dimensional Euler equations Jacobian $\mathbb{A}$, all eigenvalues and corresponding eigenvectors. Compare the results with those given in Chapter 2.2 for the full two-dimensional system. How it will be for the three-dimensional system?

Programming exercises:

Exercise 4. (**One-dimensional convection-diffusion equation**)
The so-called **convection-diffusion** equation is a simple mathematical model of fluid dynamics which describes fluid flowing in a given direction and take into account viscous (or diffusive) effects. In one space dimension we have

$$-\varepsilon u''(x) + bu'(x) = f(x) \qquad x \in (0,1) \tag{3.1}$$

$$u(0) = 0, \quad u(1) = 0. \tag{3.2}$$

Here ε is a small parameter (physically $\varepsilon \approx 1/\mathrm{Re}$), b is a given velocity of the flow (transport or advection parameter). Taking $b > 0$ means that fluid flows from left to right. By f we denote some outer forces which cause fluid movement, e.g. heat sources.

Mathematically (3.1), (3.2) gives a boundary value problem for linear ODE of the second order. That is we need two boundary conditions. Now taking $\varepsilon \to 0$, e.g. $\varepsilon = 10^{-10}$, we get as an approximate model to equation (3.1) the following equation

$$bu'(x) = f(x) \qquad x \in (0,1). \tag{3.3}$$

This is now a linear ordinary differential equation of the first order and we can prescribe just one boundary condition. Actually, if $b > 0$ we should take the left boundary value of (3.2) and conversely, if $b < 0$ we take the right value of (3.2).

Equation (3.1) with $0 < \varepsilon \ll 1$ is called **singularly perturbed** equation. It is a small perturbation of (3.3), but this small perturbation changes character of the equation completely (from the first order to the second order). Singular perturbation problems cause numerical difficulties because the solution changes rapidly over a very small interval in space. In this region derivatives of $u(x)$ are large, giving rise to large errors in our finite difference approximations.

In what follows let us consider, for simplicity, the following equation

$$-\varepsilon u''(x) + u'(x) = 1 \qquad x \in (0,1) \tag{3.4}$$

$$u(0) = 0, \quad u(1) = 0. \tag{3.5}$$

The exact solution to this problem is

$$u(x) = x + \frac{1 - e^{x/\varepsilon}}{e^{1/\varepsilon} - 1}.$$

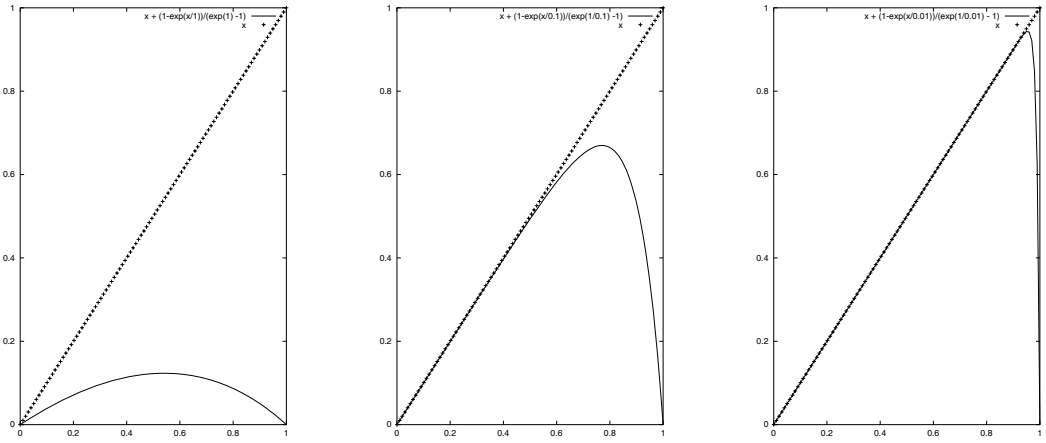

FIG. 3.1. Exact solution for various values of $\varepsilon = 1, 0.1, 0.01$ and the solution for $\varepsilon = 0$.

This solution is plotted for different values of ε in the Fig. 3.1.

Discretize (3.4) by means of the finite difference method. Take an equidistant mesh on the interval $(0, 1)$ of the mesh size h. Take the boundary conditions (3.5) at $U_0 \equiv u(0) = 0$ and $U_N \equiv u(1) = 0$, where $N = 1/h$. You will obtain a linear algebraic system of the following type

$$\mathbb{A}\,\mathbf{U} = \mathbf{R},$$

with a sparse three-diagonal matrix $\mathbb{A}$ and $\mathbf{U} = (U_1, \ldots, U_N)$, $U_i \approx u(ih)$. Solve this linear system for $\mathbf{U}$! (You can use, e.g., some appropriate MATLAB subroutines.) Note that you can use the sparsity of the matrix $\mathbb{A}$ and do not need to save all of $(N-1) \times (N-1)$ entries into memory.

(1) Compare three possibilities to approximate the first derivative of $u'(x)$. Take a fixed ε (e.g. $\varepsilon = 0.1, 0.01$) and plot the exact and approximate solutions for several values of h.
(2) Plot logarithm of the error $\ln\|u - \mathbf{U}\|$ over $\ln(h)$; (e.g. take $h = 1/10, h = 1/25, h = 1/100$). Here $\|u - \mathbf{U}\| = \max_{i=1,\ldots,N} |u(ih) - U_i|$.
(3) What is the leading order local truncation error in finite difference approximation of (3.4)? Where can you see its influence in the graphs of numerical solution?

The following exercises 5.- 8. have to be solved using the finite volume method of the wave propagation algorithm, the so-called CLAWPACK (Version 4.0) (Conservation Law Package). See: http://www.amath.washington.edu/ claw

Exercise 5. **(Viscous Burgers equation)**
Solve the one-dimensional viscous Burgers equation

$$q_t + \left(\frac{1}{2}q^2\right)_x = \varepsilon q_{xx}, \qquad x \in [-2, 3].$$

The finite volume solver of Roe ("Roe solver") is based on the linearization $q_t + A_i q_x = 0$ with $A_i = \frac{1}{2}(q_{i-1} + q_i)$. Note that for this scalar problem $A_i = \lambda_i^1$ and this is the correct shock speed in the case that the solution is a shock wave. For the wave propagation algorithm we set

$$\begin{aligned} \mathcal{W}_i^1 &:= q_i - q_{i-1}, \\ \lambda_i^1 &:= \frac{1}{2}(q_{i-1} + q_i), \\ \mathcal{A}^- \Delta q_i &:= \min(\lambda_i^1, 0)\mathcal{W}_i^1 = -\frac{1}{2} q_{i-1}^2, \\ \mathcal{A}^+ \Delta q_i &:= \max(\lambda_i^1, 0)\mathcal{W}_i^1 = \frac{1}{2} q_i^2. \end{aligned}$$

In this case we have $\mathcal{A}^- \Delta q_i + \mathcal{A}^+ \Delta q_i = \frac{1}{2}\left(q_i^2 - q_{i-1}^2\right) = f(q_i) - f(q_{i-1})$.

In order to solve the viscous part the so-called operator splitting is implemented in the CLAWPACK and the scr1 subroutine solves the diffusion equation

$$q_t = \varepsilon q_{xx}$$

over a time step. This is done using the Crank-Nicolson method, which requires solving a tridiagonal linear system of equations for the whole update of q_i.

Solve the Burgers equation with the extrapolation boundary conditions and the following initial data

$$\begin{aligned} q(x,0) &= 1 \qquad x < 0 \\ q(x,0) &= 0 \qquad x > 0 \end{aligned}$$

or

$$\begin{aligned} q(x,0) &= 0 \qquad x < 0 \\ q(x,0) &= 1 \qquad x > 0 \end{aligned}$$

Test the behavior of solution for several viscosity coefficients $\varepsilon = 0, 1, 0.1, 0.01$, 0.001. Plot solutions at several time steps!

Exercise 6. **(Acoustics equation)**

The two-dimensional acoustics equation for the pressure perturbation p and velocities u and v can be written as

$$\mathbf{q}_t + \mathbb{A}\mathbf{q}_x + \mathbb{B}\mathbf{q}_y = \mathbf{0},$$

where

$$\mathbf{q} = \begin{pmatrix} p \\ u \\ v \end{pmatrix}, \ \mathbb{A} = \begin{pmatrix} 0 & K & 0 \\ 1/\rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbb{B} = \begin{pmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 1/\rho & 0 & 0 \end{pmatrix}.$$

Solve the acoustics equation with the radially symmetric initial data $p(x, y, 0) = P_0(r), u = v = 0$ where $r = \sqrt{x^2 + y^2}$. Set $P_0(r) = 1 - 0.1(\cos(4\pi r) - 1)$. The

solution should remain radially symmetric. Moreover you can test the accuracy of the two-dimensional scheme by computing solution on a very fine grid by solving the one-dimensional equations

$$P_t + KU_r = -a\,U/r$$
$$\rho U_t + P_r = 0,$$

where U is the radial velocity and $a = \sqrt{K/\rho}$. Solve this system on a very fine grid using 1D solver and then solve 2D equations on a sequence of $N \times N$ grids for $N = 10, 20, 40, 80$ on $[-1, 1] \times [-1, 1]$. The end time is set to be $t = 0.4$. Plot the comparisons between the "exact" 1D solution and scatter plot of the 2D numerical solutions. These latter are obtained by the first as well as the second order FV schemes. Compute numerical errors between the reference and numerical solutions in L^1 as well as in L^∞-norms for all $N \times N$ grids.

Exercise 7. **(Acoustics in heterogeneous medium)**

Now we suppose that sound speed varies in x, y due to variations in the material parameters ρ and K. We have then the system

$$\mathbf{q}_t + \mathbb{A}(x, y)\mathbf{q}_x + \mathbb{B}(x, y)\mathbf{q}_y = \mathbf{0},$$

where

$$\mathbf{q} = \begin{pmatrix} p \\ u \\ v \end{pmatrix},\ \mathbb{A} = \begin{pmatrix} 0 & K(x,y) & 0 \\ 1/\rho(x,y) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \mathbb{B} = \begin{pmatrix} 0 & 0 & K(x,y) \\ 0 & 0 & 0 \\ 1/\rho(x,y) & 0 & 0 \end{pmatrix}.$$

In this example we model the propagation of a plane wave when striking an interface of two materials. As an initial data take a planar wave propagating in some direction at an angle to the interface

$$p(x, y, 0) = 0.5 + \cos(\pi r/0.1) \ \text{ if } r = \sqrt{(x - 0.25)^2 + (y + 0.4)^2} < 0.1,\ 0 \text{ else}$$
$$u(x, y, 0) = 1.5,\ v(x, y, 0) = 1.$$

Set the interface variables as follows

$$\rho(x, y) = \rho_L = 1.0,\ a(x, y) = \sqrt{K(x, y)/\rho(x, y)} = a_L = 1.0 \quad x < 0.5,$$
$$\rho(x, y) = \rho_R = 4.0,\ a(x, y) = \sqrt{K(x, y)/\rho(x, y)} = a_R = 0.5 \quad x > 0.5$$

or

$$\rho(x, y) = \rho_L = 1.0,\ a(x, y) = a_L = 1.0 \quad x < y/5 + 0.4,$$
$$\rho(x, y) = \rho_R = 4.0,\ a(x, y) = a_R = 0.5 \quad x > y/5 + 0.4.$$

The end time is set to be $t = 0.64$ and the computational domain is $[0, 1.5] \times [-1, 1]$. Use 300×200 grid. Compare pressure isolines of the numerical solutions as well as the graphs along cross sections at $x = 0.6$ and $x = 1.0$ for both types of interfaces.

Note that in order to plot graph along the cross sections you have to modify slightly procedure claw/clawpack/2d/lib/out2.f in order to get only output data along the specified cross-section. Use extrapolated boundary conditions. Compare first and second order methods.

Exercise 8. **(Euler equations with shock in entropy)**

In this example we model a shock wave hitting sinusoidal variation in entropy (i.e. variation in density with constant pressure). The system of the Euler equations reads

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = \mathbf{0},$$

where the vector of conservative variables and the fluxes are

$$\mathbf{q} := \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}, \quad \mathbf{f}(\mathbf{q}) := \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (e+p)u \end{pmatrix}, \ x \in [-5,5], \ t \in [0,2].$$

In order to close the system the state equation of ideal gas is used. Take the following initial data

$$\begin{aligned} &\rho(x,0) = 3.857, \ u(x,0) = 2.629, \ p(x,0) = 10.333, \qquad x < -4 \\ &\rho(x,0) = 1 + 0.2\sin(5x), \ u(x,0) = 0, \ p(x,0) = 1, \qquad x \geq -4. \end{aligned}$$

Plot ρ, u and p on fine and coarse grid using 200 and 4000 grid cells. Compare results for the first and second order methods. Use extrapolated boundary conditions on the left and the right boundary, i.e. $x = \pm 5$.

Exercise 9. (**Two-dimensional nonlinear convection-diffusion equation**) **Combined finite volume - finite element method**

Solve the following two-dimensional convection-diffusion equation

$$\begin{aligned} &u_t + uu_x + uu_y = \varepsilon \Delta u, \qquad (x,y) \in [-1,1] \times [-2,2], \\ &u(x,y,0) = 10 \qquad (x^2+y^2)^{1/2} < 0.4 \\ &\qquad\quad = 1 \qquad \text{else} \\ &\frac{\partial u}{\partial n} = 0 \qquad \text{Neumann B.C.} \end{aligned} \tag{3.6}$$

Use regular mesh consisting of square cells

$$\begin{aligned} \Omega_{ij} &\equiv [(i-1/2)h, (i+1/2)h] \times [(j-1/2)h, (j+1/2)h] \\ &= [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}], \end{aligned}$$

where indices $i, j \in \mathbb{Z}$, and $h > 0$ is the mesh size parameter.

Approximate equation (3.6) with the so-called operator splitting finite element-finite volume method. It means to divide time step $[t_n, t_{n+1}]$ into two half time steps. Solve in the first time step $[t_n, t_{n+1/2}]$ a pure advection equation

$$u_t + uu_x + uu_y = 0$$

with the finite volume method

$$U_{ij}^{n+1/2} := U_{ij}^{n} - \frac{\Delta t}{2h^2}\left[H_{i+1/2,j} - H_{i-1/2,j} + H_{i,j+1/2} - H_{i,j-1/2}\right].$$

Here $H_{i\pm 1/2,j}$ denotes a numerical flux function on the right/left cell interface. Analogous notation holds for y-direction. Try the following methods: second order Lax-Wendroff scheme as well as first order Lax-Friedrichs and Vijayasundaram methods.

In the second step $[t_{n+1/2}, t_n]$ solve the rest parabolic equation

$$u_t = \varepsilon \Delta u \tag{3.7}$$

with linear conforming finite element method using the so-called dual mesh to the original mesh $\{\Omega_{ij}\}_{i,j\in\mathbb{Z}}$. Dual mesh is a staggered mesh, i.e. "shifted mesh". It consists of squares $\{\Omega_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{Z}}$, where (α,β) are vertices of the original mesh. Let φ^{ij} be a finite element test function corresponding to the vertex (i,j) of the dual mesh $\{\Omega_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{Z}}$. It is a bilinear function, which equals 1 at the vertex (i,j) and zero at other vertices (k,l). Show that the finite element approximation of (3.7) leads to the following scheme

$$U_{ij}^{n+1} := U_{ij}^{n+1/2} - \frac{\Delta t}{4h^2}\varepsilon(U_{i-1j}^{n+1/2} + U_{i+1j}^{n+1/2} + U_{ij-1}^{n+1/2} + U_{ij+1}^{n+1/2} - 4U_{ij}^{n+1/2}).$$

In order to approximate integrals of any continuous function $f(x,y)$ over one mesh cell the trapezoidal quadrature is used, i.e.

$$\int_{\Omega_{\alpha,\beta}} f(x,y)dx = \frac{h^2}{4} \sum_{k,\ell:(x_k,y_\ell)\in\Omega_{\alpha,\beta}} f_{k\ell}.$$

Note that gradients of test functions are constant on each dual mesh cell $\Omega_{\alpha,\beta}$.

Plot three-dimensional representations of the solution u at three different time instances, e.g. $T = 0.1, 0.5, 1$. Try several viscosity parameters, e.g. $\varepsilon = 10, 1, 0.01$, 0.001. Which stability condition, i.e. condition on relationship between Δt and h, has to be fulfilled?

REFERENCES

Barth T.J., Jespersen D.C., *The design and application of upwind schemes on unstructured meshes,* AIAA paper **366**, 1989.

Bressan, A., *Hyperbolic Systems of Conservation Laws, The One-Dimensional Cauchy Problem,* Oxford University Press 2000.

Ciarlet P.G., *The Finite Element Method for Elliptic Problems,* North - Holland, Amsterdam, 1979.

Courant R., Friedrichs K.O., *Supersonic Flow and Shock Waves,* Springer - Verlag, New York - Heidelberg - Berlin, 1976.

Di Perna R. J. (1), *Measure-valued solutions to conservation laws,* Arch. Rational Mech. Anal. **88**, 1985, pp. 223 - 270.

Di Perna R. J. (2), *Convergence of approximate solutions to conservation laws,* Arch. Rational Mech. Anal. **82**, 1983, pp. 27 - 70.

Feistauer M., *Mathematical Methods in Fluid Dynamics,* Longman Scientific & Technical , Harlow, 1993.

Glimm J. *Solutions in the large for nonlinear hyperbolic systems of equations,* Comm. Pure Appl. Math., **18**, 1965, pp. 95 - 105.

Godlewski E., Raviart P.A.(1), *Hyperbolic Systems of Conservation Laws,* Mathematiques & Applications, S.M.A.I., Ellipses, Paris , 1991.

Godlewski E., Raviart P.A.(2), *Numerical Approximation of Hyperbolic Systems of Conservation Laws,* Springer Verlag, New York, 1996.

Gurtin M.E., *An Introduction to Continuum Mechanics,* Academic Press, New York, 1981.

Hirsch C., *Numerical Computation of Internal and External Flows, Vols 1.,2., John Wiley & Sons Ltd., Chichester, 1984.*

Harten A., On a class of high resolution total-variation-stable finite difference schemes, SIAM J. Numer. Anal. **21**, 1984, pp. 1 - 23.

Kröner D., *Numerical Schemes for Conservations Laws,* Wiley Teubner, Stuttgart 1997.

Kröner D., Rokyta M., *Convergence of upwind finite volume schemes for scalar conservation laws in 2D ,* SIAM J. Numer. Anal. **31(2)**, 1994, pp. 324-343.

Kröner D., Noelle S., Rokyta M., *Convergence of higher order upwind finite volume schemes on unstructured grids for scalar conservation laws in several space dimensions.* Numer. Math. **71 (4)**, 1995, pp. 527-560.

Kruzhkov S.N., *First order quasilinear equations in several independent variables,* Math. USSR Sbornik **10**, 1970, pp. 217 - 243.

Kufner A., John O., Fučík S., *Function Spaces,* Academia, Prague, 1977.

Ladyzhenskaya O.A., *The Mathematical Theory of Viscous Incompressible Flow,* Nauka, Moscow, 1970, (in Russian).

Lax P.D. (1), *Shock waves and entropy,* In: Contributions to Nonlinear Funct. Analysis, (Zarontonello E.A. - ed.), Academic Press, 1971, pp. 603 - 634.

Lax P.D. (2), *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves,* SIAM, 1973.

Lax P.D., Wendroff B., *Systems of conservation laws,* Comm. Pure Appl. Math. **13**, 1960, pp. 217 - 237.

LeVeque R.J. (1), *Numerical Methods for Conservation Laws,* Birkhäuser, Basel - Boston - Berlin, 1990.

LeVeque R.J. (2), *Finite Volume Methods for Hyperbolic Problems,* Cambridge University Press, Cambridge, 2002.

Li J., Zhang T., Yang S., *The Two–dimensional Riemann Problem in Gas Dynamics,* Pitman Monographs and Surveys in Pure and Applied Mathematics 98, Longman, 1998.

Li J., Lukáčová - Medviďová M., Warnecke G., *Evolution Galerkin schemes applied to two-dimensional Riemann problems for the wave equation system,* Discrete Contin. Dyn. Syst. **9(3)**, 2003, pp. 559-576.

Lukáčová - Medviďová M., *Numerical Solution of Compressible Flows,* PhD Dissertation, Charles University Prague, 1994.

Lukáčová - Medviďová M., Morton K.W., Warnecke G., *Finite Volume Evolution Galerkin Methods for Euler Equations of Gas Dynamics*, Int. J. Numer. Meth. Fluids **40 (3-4)**, 2002, pp. 425-434.

Lukáčová-Medviďová M., Saibertová J., Warnecke G., *Finite Volume Evolution Galerkin Methods for Nonlinear Hyperbolic Systems*, J. Comput. Phys. **183(2)**, 2002, pp. 533-562.

Málek J., Nečas J., Rokyta M., Růžička M., *Weak and Measure-valued Solutions to Evolutionary Partial Differential Equations,* Applied Mathematics and Mathematical Computation, 13, Chapman & Hall, London, 1996.

Morton K.W., Mayers D.*Numerical Solution of Partial Differential Equations,* Cambridge University Press, Cambridge, 1994.

Nečas J., *Les méthodes directes en théorie des équations elliptiques,* Academia, Prague, 1967.

Osher S. (1), *Riemann solvers, the entropy condition, and difference approximations,* SIAM J. Numer. Anal. **21 (2)**, 1984, pp. 217 - 235.

Osher S. (2), *Convergence of generalized MUSCL schemes,* SIAM J. Num. Anal. **22**, 1985, pp. 947 - 961.

Osher S., Solomon F., *Upwind difference schemes for hyperbolic systems of conservation laws,* Math. Comp. **38**, 1982, pp. 339 - 374.

Richtmeyr R.D., Morton K.W., *Difference Methods for Initial-value Problems,* Wiley-Interscience, 1967.

Rostand Ph., Stoufflet B., *TVD schemes to compute compressible viscous flows on unstructured grids,* In: Nonlinear Hyperbolic Equations - Theory, Computation Methods and Applications (Ballman J., Jeltsch R. - eds.), Notes on Numerical Fluid Mechanics **24**, Vieweg, Braunschweig, 1989.

Schochet S., *The compressible Euler equations in a bounded domain: existence of solutions and incompressible limit,* Comm. Math. Phys. **104**, 1986, pp. 49 - 75.

Shu, C.-W., Osher S., *Efficient implementation of essentially non-oscillatory shock capturing schemes II,* J. Comput. Phys. **83**, 1989, pp. 32 - 78.

Sever M., *Uniqueness failure for entropy solutions of hyperbolic systems of conservation laws,* Comm. Pure Appl. Math. **42**, 1989, pp. 173 - 183.

Smoller J., *Shock Waves and Reaction-Diffusion Equations,* Springer, 1983.

Sonar Th. (1), *Mehrdimensionale ENO-Verfahren,* Teubner Stuttgart, 1997.

Sonar Th. (2), *Nichtlineare Dissipationsmodelle und Entropie - Production in finiten Differenzenverfahren,* PhD Dissertation, Univ. Stuttgart, 1991.

Tartar L.C., *The compensated compactness method applied to systems of conservation laws,* In: Systems of Nonlinear P.D.E.'s (J. Ball - ed.), D. Reidel Publishing Company, 1983, pp. 263 - 285.

Temam R., *Navier–Stokes Equations,* North - Hollad, Amsterdam - New York - Oxford, 1977.

Toro E.F., *Riemann Solvers and Numerical Methods for Fluid Dynamics,* Springer-Verlag, 1999.

Valli A., *Mathematical results for compressible flows,* Preprint Univ. degli Studi di Trento, 1992.

Van der Burg J.W., *Numerical Techniques for Transonic Flow Calculations,* PhD Dissertation, University of Twente, 1992.

Vijayasundaram G., *Transonic flow simulation using an upstream centered scheme of Godunov in finite elements,* J. Comp. Phys. **63**, 1986, pp. 416 - 433.

Wada Y., Kubota H., Ishiguro T., Ogawa S., *A fully implicit high-resolution scheme for chemically reacting compressible flows,* In: Nonlinear Hyperbolic Equations - Theory, Computation Methods, and Applications (Ballman J., Jeltsch R. - eds.), Notes on Numerical Fluid Mechanics **24**, Vieweg, Braunschweig, 1989.

Warnecke G., *Analytische Methoden in der Theorie der Erhatungsgleichungen,* Teubner, Stuttgart-Leipzig, 1999.

Wesseling P., *Principles of Computational Fluid Dynamics,* Springer Verlag Berlin-Heidelberg, 2001.

Yosida K., *Nonlinear Analysis,* Springer - Verlag, New York, 1974.

Zeidler E., *Nonlinear Functional Analysis and its Applications 1. - 4.,* Springer - Verlag, New York, 1985 - 1989.

Zienkiewicz O.C., Taylor R.L., *Finite Element Method, Vol 3. Fluid Dynamics,* Butterworth-Heinemann, Bristol, 2000.

Ženíšek A., *Nonlinear elliptic and evolution problems and their finite element approximations,* Computational Mathematics and Applications, Academic Press, London, 1990.