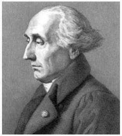
Non-Euclidean Geometry and Group Theory

Fall 2021 - R. L. Herman








Euclidean Geometry

- 300 BCE - Euclid's *Elements*
- Five Postulates.
- 5th Postulate - not needed in first 28 propositions.

If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

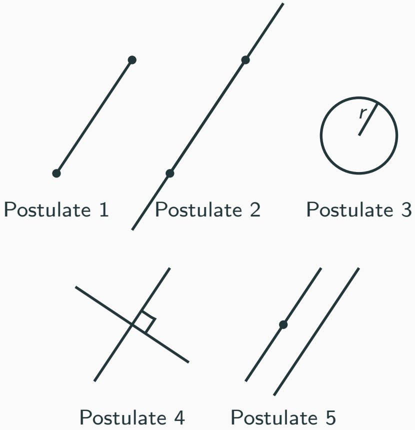


Figure 1: Euclid's 5 Postulates.

Statements of Parallel Axiom in Text

$\mathbf{P}_1$ For each straight line L and point P outside L there is exactly one line through P that does not meet L.

Equivalent statements

The angle sum of a triangle $= \pi$. - Euclid.

The locus of points equidistant from a straight line is a straight line. - al-Haytham.

Similar triangles of different sizes exist. - Wallis

Saccheri (1733) - provided two alternatives to arrive at proof by contradiction.

$\mathbf{P}_0$ There is not line through P that does not meet L.

$\mathbf{P}_2$ There are at least two lines through P that do not meet L.

Parallel Postulate

- Proclus (410-485) Equivalent postulate. Revived as Playfair axiom.
- William Ludlam (1785):
  Two straight lines, meeting at a point, are not both parallel to a third line.
- John Playfair, *Elements of Geometry* (1795):
  Playfair's axiom: Two straight lines which intersect one another cannot be both parallel to the same straight line.
- Many false attempts to prove based on other four postulates.

- 1663 John Wallis "To each triangle, there exists a similar triangle of arbitrary magnitude."
- Giralomo Saccheri (1667-1733) Assume 5th postulate false and get contradiction.
- Used assumption - lines are infinite. Led to contradiction of $\mathbf{P}_1$, almost $\mathbf{P}_2$.
- d'Alembert, 1767 - "The scandal of elementary geometry."

Spherical Geometry

- Lines = geodesics,
  Lie on great circles.
- Euclidean triangles, $a + b + c = \pi$.
- Spherical triangles, $a + b + c > \pi$.
- Thomas Harriot (1560-1621), astronomy, mathematics, and navigation
- Johann Heinrich Lambert (1726-1777)
  - General properties of map projections.
  - hyperbolic functions
  - π is irrational
  - optics

$$a + b + c = \pi + \frac{A}{R^2}.$$

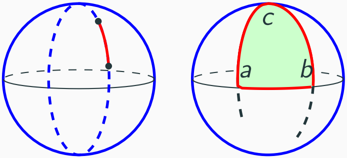


Figure 2: Harriot and Lambert.

Other Geometries

- Ferdinand Karl Schweikart (1780) Astral geometry, sum of three angles of a triangle is less than two right angles.
- Wrote to Gauss, 1818, via student Christian Ludwig Gerling (1788-1864).
- Franz Taurinus (1784-1854), Schweikart's nephew. Proposed geometry on a sphere of imaginary radius, logarithmic-spherical geometry.
- 1826, hyperbolic law of cosines in *Geometriae prima elementa.*
- Wrote to Gauss. after being encouraged, he sent copies of his works with no reply.
- Later he burned copies of his book.

Figure 3: Gerling and Schweikart

Parallel Postulate Revisited

- Carl Friedrich Gauss (1777-1885) started on it in 1799; was convinced it was independent of first 4.
- Discussed with Farkas Bolyai (1775 - 1856) - told his son no to waste his time.
- János Bolyai (1802-1860) - Believed a non-Euclidean geometry existed.
- Nikolai Lobachevsky (1792-1856) - independently 1840 new 5th postulate: There exists two lines parallel to a given line through a given point not on the line.
  Developed trig identities, hyperbolic geometry.

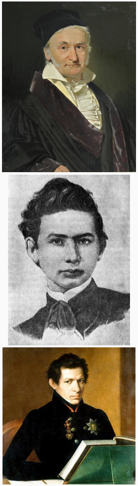
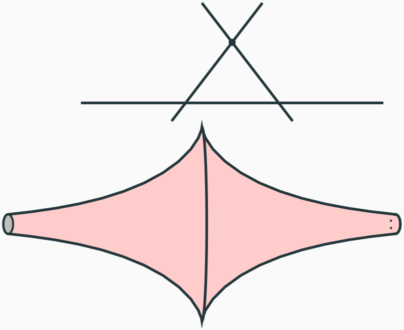


Figure 4: Gauss, Bolyai, Lobachevsky

Riemannian Geometry

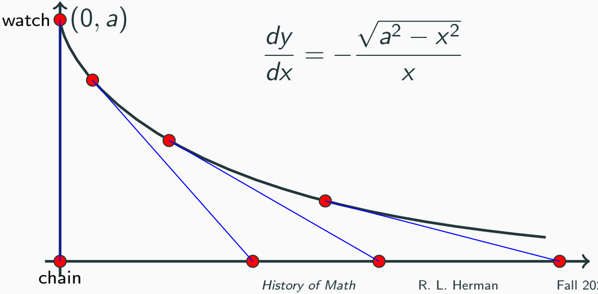
- Georg Friedrich Bernhard Riemann (1826-1866)
  - Published in 1868 Lecture
  - Spherical geometry
  - Riemannian geometry $\rightarrow$ differential geometry
  - Every line through a point not on a given line meets the line.
- Eugenio Beltrami (1835-1900) Published interpretations of non-Euclidean geometry - introduced pseudosphere in 1868 using a **tractrix**.



tractrix $(a(t - \tanh t), a\operatorname{sech} t)$

Aside: The Tractrix

- Claude Perrault [brother Charles author of *Cinderella*, *Puss-in-Boots*] in 1693, Paris, placed a watch in the middle of a table and pulled its chain along the edge of the table. What was the curve traced out ?
- Studied by Newton (1676), Huygens (1692) and Leibniz (1693). Euler gave complete theory in 1788. [Am. Math. Monthly, 72(10) (1965), 1065-1071.]
- Huygens coined name from Latin, *tractus*.



$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}$$

Curvature

- $k = 0,\ k > 0,\ k < 0.$
- sums of angles of triangles $a + b + c - \pi = kA$.

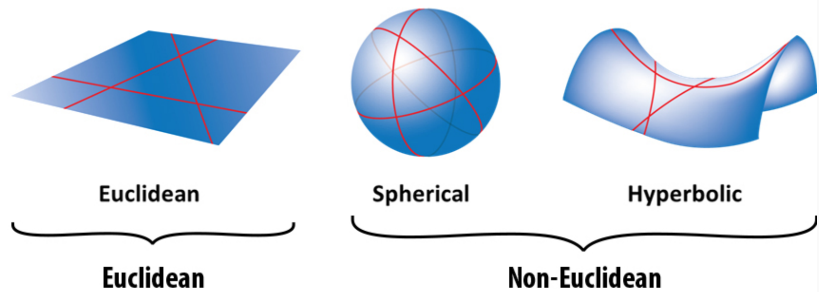


Figure 5: Surfaces of Constant Curvature.

Hyperbolic Geometry

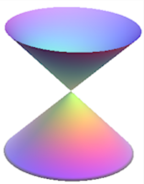
- Sphere

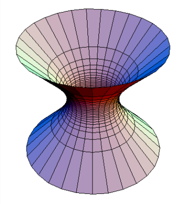
$$x^2 + y^2 + z^2 = \text{const}$$

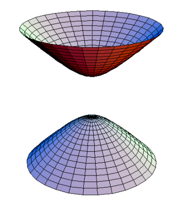
- Modify

$$x^2 + y^2 - z^2 = K$$

- $K = 0$, $z^2 = x^2 + y^2$. Cones.
- $K = 1$, $x^2 + y^2 - z^2 = 1$. Hyperboloid of one sheet
- $K = 1$, $z^2 - x^2 - y^2 = 1$. Hyperboloid of two sheets.





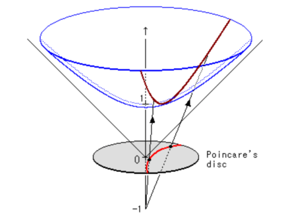


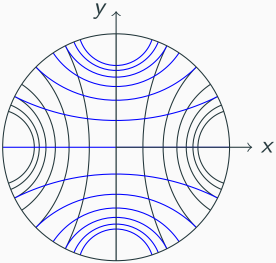
Beltrami-Poincaré Model

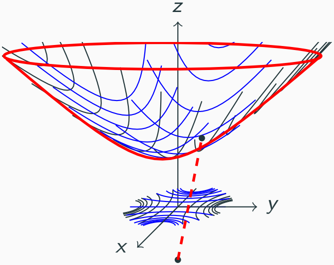
- Poincaré's Disks

  $$(x, y, z) = (c \cosh t, \sinh t, \sqrt{1+c^2} \cosh t)$$

- Stereographic Projection thru $(0,0,-1)$ to $z=0$: $(x,y,z) \to \frac{(x,y)}{1+z}$.
- Hyperbolic geometry.



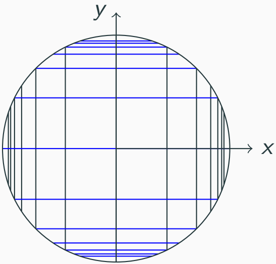


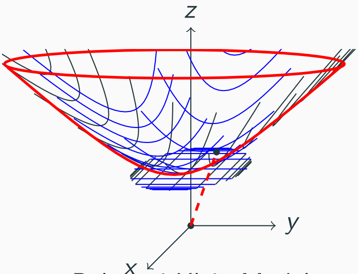


Beltrami-Poincaré Model

Beltrami-Klein Model

- Stereographic Projection thru $(0,0,0)$ to $z=1:(x,y,z)\to\frac{(x,y)}{z}$.
- Klein's Disks
  Projection to $(0,0,1)$

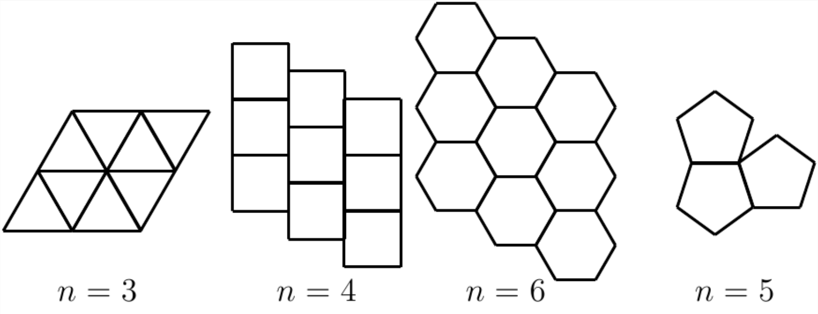




Beltrami-Klein Model

Tiling the Plane

One can tile the plane with a single polygon with sides 3, 4, and 6. However, one cannot fit pentagons together. As seen below, the angles do not allow for a fit. For large *n*, the interior angles are too small.



Other Tilings

- Johannes Kepler (1571-1630)
    - Studied Tilings
    - *Harmonicae Mundi* (Harmony of the World).
    - Planned in 1599.
    - Published 1619 - delay by Tycho Brahe to look as orbit of Mars.
- Roger Penrose (1931-)
    - 2020 Nobel Prize
    - 70's Inspired by Tilings - Penrose tilings. In 80's found in nature.
    - and M. C. Escher (1889-1972)
    - Circle Limit - Tiling Hyperbolic Plane.
- Others - Polyominoes and Pentominoes.

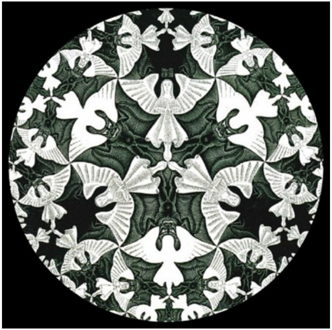


Figure 6: Circle Limit IV

Hyperbolic Tessellations

- Tessellation = cover plane by tiles, no tiles overlap and no space empty.
- Schläfli symbol: $\{n, m\}$,
  $n =$ number of sides on the tile,
  $m =$ number of tiles that meet at a vertex.
  Euclidean: $\frac{1}{n} + \frac{1}{m} = \frac{1}{2}$,
  Hyperbolic: $\frac{1}{n} + \frac{1}{m} < \frac{1}{2}$.

Figure 7: Circle Limits I-IV.

Group Theory

1843 - Joseph Louiville (1809-1882) reviewed Galois' delayed manuscript, published 1846. - introduction of groups and fields.

- Multiplicative group modulo n.
- Euler - Fermat's Little Theorem
  p prime, $(a, p) = 1$,
  $$a^{p-1} \equiv 1 (\text{mod } p).$$
- Euler's ϕ function:
  $$\phi(n) = \#\{k \in \{1, 2, \ldots, n-1\} | (k, n) = 1\}.$$
  $\phi(5) = 4$, $\{1, 2, 3, 4\}$,
  $\phi(8) = 4$, $\{1, 3, 5, 7\}$.
- Group Properties:

  closed, identity, Inverse, associative

Examples: Mod 5 and 8.

x	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

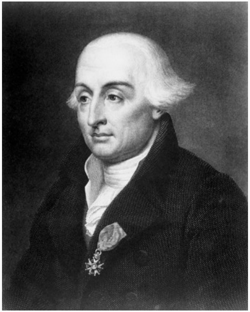
x	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Carl Friedrich Gauss (1777-1855)

- *Disquisitions Arithmeticae* - 1801
- Summary and Extension on Number Theory.
- Initiated finite Abelian groups.
- Proved Fermat's Little Theorem. Represented integers as quadratic forms, like Fermat Primes ($4n+1=x^2+y^2$.) for x and y integers.
- Binary quadratic forms - $ax^2+bxy+cy^2$ - for a, b, c integers.
  - composition has properties of an abelian group.
- Did not have a general theory of groups.

Joseph Louis Lagrange (1736-1813)

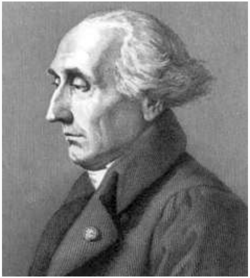
- Born in Turin, Italy.
- Professor at 19 (artillery school).
- 1766 Frederick the Great sought great mathematician.
- Lagrange replaced Euler in Berlin for 20 yrs.
- Invited by Louis XVI to Paris.
- 1795 - established dept. at École Normal.
- 1797 - established dept. at École Polytechnique.
- Napoleon made him senator, count, and he received many other honors.
- Sought solution of quintic by studying cubic and quartic.
- Made many other contributions.



Resolvents

- Consider $x^3 + nx + p = 0$. Let $x = y - \frac{n}{3y}$.
- Yields 6^{th} degree polynomial, $y^6 + py^3 - \frac{n^3}{27} = 0$, the resolvent.
- Let $r = y^3$, $r^2 + pr - \frac{n^3}{27} = 0$.
- Has roots r_1, r_2, where $r_2 = -\left(\frac{n}{3}\right)^3 \frac{1}{r_1}$.
- Then, $x = \sqrt[3]{r_1} + \sqrt[3]{r_2}$
- Cardano got this real root but did not seek complex solutions.
- Lagrange knew there should be 3 roots. $\sqrt[3]{r}, \omega\sqrt[3]{r}, \omega^2\sqrt[3]{r}$, where ω is a cube root of unity, $\omega^3 = 1$. Then,

$$\begin{aligned} x_1 &= \sqrt[3]{r_1} + \sqrt[3]{r_2} \\ x_2 &= \omega\sqrt[3]{r_1} + \omega\sqrt[3]{r_2} \\ x_2 &= \omega^2\sqrt[3]{r_1} + \omega^2\sqrt[3]{r_2} \end{aligned}$$



Permutation of Roots

- Lagrange then wrote roots of the resolvent
  $y = x_i + \omega x_j + \omega^2 x_k, \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k.$
- 3!=6 permutations of cubic roots.
- In $y^6 + py^3 - \frac{n^3}{27} = 0$, the coefficients of y^5, y^4, y^2, y are
  $x_1 + x_2 + x_3$, $p = x_1 x_2 x_3$, and $\dfrac{n^3}{27} = \dfrac{(x_1 x_2 + x_1 x_3 + x_2 x_3)^3}{27}$.
- Resolvent coefficients are rational functions of the cubic roots.
- Lagrange obtained similar results for the quartic.
- Lagrange sought solutions of higher order equations using symmetric functions of the roots and permutations.
- Paola Ruffini (1765 – 1822) - 1802, 1805, 1813 - gave proofs that quintic can't be solved. Proofs not understood.

Niels Henrick Abel (1802-1829)

- Born in Norway into poverty and had a pulmonary condition.
- Mathematical ability discovered by his teacher.
- Toured Europe after college and published 5 papers in *Journal für die reine und angewandte Mathematik*.
- Studied convergence of infinite series, the theory of doubly periodic functions, elliptic functions, elliptic functions and the theory of equations.
- Could not get employment, so tutored.
- At university, thought he had solution of quintic. Then, proved no solution existed.
- Died of tuberculosis before completing his work.

Évariste Galois (1811-1832)

- Born Oct 25, 1811
- Interest in math at 14.
- Read Adrien-Marie Legendre (1752-1833).
- 1828 Failed to get into École Polytechnique.
- 1829 Paper on continued fractions.
- Studied polynomial equations.
- Wrote two papers.
  Reviewed by Arthur Cayley (1821-1895)
  Entered competition.
- 1830 Submitted to Joseph Fourier (1768–1830) - got lost.
  Winners - Niels Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851).
- Published 3 papers.

Figure 8: Évariste Galois

Évariste Galois (cont'd)

- Political turmoil in France.
- Student uprising - Galois left school.
- He was arrested and acquitted.
- Arrested Jul 1831 - April 29, 1832.
- Siméon Denis Poisson (1781-1840) asked him to submit work 1831.
- July 4 - declared work incomprehensible.
- Galois found out in October.
- Stayed up all night; wrote letters and note to Auguste Chevalier.
- On May 30, fought in duel and lost.
- Chevalier forwarded papers for publication by Joseph Liouville.

Figure 9: Legendre, Cayley, Fourier, Jacobi, Poisson, Liouville

Symmetry Groups

- Levi ben Gorshun (1321) Number of permutations of n objects $= n!$
- Leads to Symmetric Group.
- Felix Klein (1872) extended groups to geometry - studied invariants of groups of transformations.
- Sophus Lie (1842-1899) continuous groups of transformations, applied to differential equations.
- Emmy Noether (1882-1935) related symmetries to constants of motion in physics.

Figure 10: Sophus Lie and Emmy Noether.