

Mutually coupled inductors. Coupling coefficient. Power and energy of mutually coupled inductors. Analysis of circuits with mutually coupled inductor.

6.1. Equivalent circuits of mutually coupled inductors

As was already mentioned in the second topic, when the magnetic field of one coil reaches a second one the two inductors are mutually coupled and are characterized by a coefficient of mutual inductance M . Depending on the connection between inductors there are a number of equivalent circuits which could be used to simplify the circuit analysis.

6.1.1. Mutually coupled inductors in series

Consider there are two inductors L_1 and L_2 in series, which are magnetically coupled and have a mutual inductance M . The magnetic field of the two inductors could be aiding or opposing each other, depending on their orientation (fig 6.1).

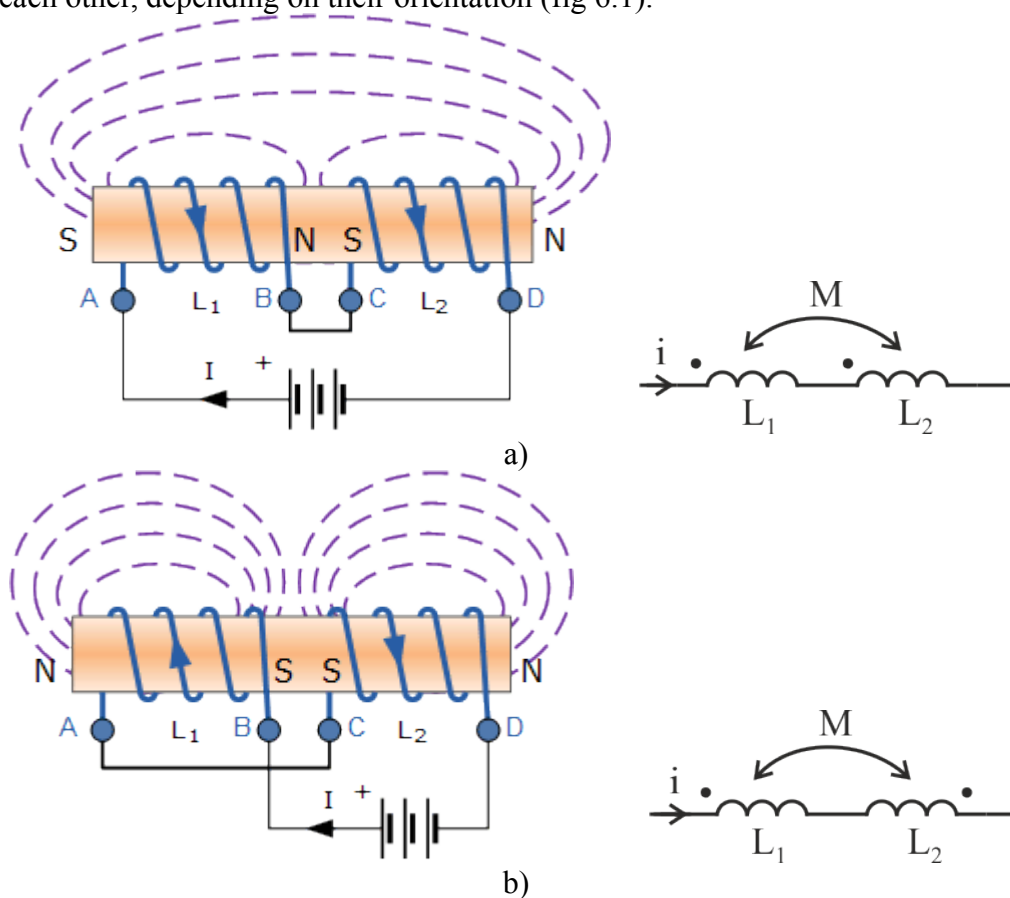


Fig. 6.1. Mutually coupled inductors and dot convention: a) series aiding inductors; b) series opposing inductors.

Aiding inductors in series as well as the dot convention are presented in fig. 6.1a. Since their magnetic fields aid each other, the KVL for this situation is:

$$v = v_1 + v_M + v_2 + v_M = L_1 \frac{di_L}{dt} + M \frac{di_L}{dt} + L_2 \frac{di_L}{dt} + M \frac{di_L}{dt} = (L_1 + L_2 + 2M) \frac{di_L}{dt}$$

In the above equation v_M is the voltage drop caused by the mutual inductance M . There are two voltage drops v_M : one is produced by the additional magnetic flux coming from L_1 to L_2 and the second one – by the magnetic flux coming from L_2 to L_1 . Then the equivalent inductance is:

$$L_E = L_1 + L_2 + 2M$$

The second situation is when the magnetic field of the two coils oppose each other (fig. 6.1b). Then

the KVL law is:

$$v = v_1 - v_M + v_2 - v_M = L_1 \frac{di_L}{dt} - M \frac{di_L}{dt} + L_2 \frac{di_L}{dt} - M \frac{di_L}{dt} = (L_1 + L_2 - 2M) \frac{di_L}{dt}$$

And the equivalent inductance is:

$$L_E = L_1 + L_2 - 2M$$

6.1.2. Mutually coupled inductors in parallel

Consider two inductors L_1 and L_2 are connected in parallel and are mutually coupled with mutual inductance M . The voltage applied on the inductors is v_s . Once again there are two possibilities: the magnetic fields of the two inductors could be aiding or opposing each other.

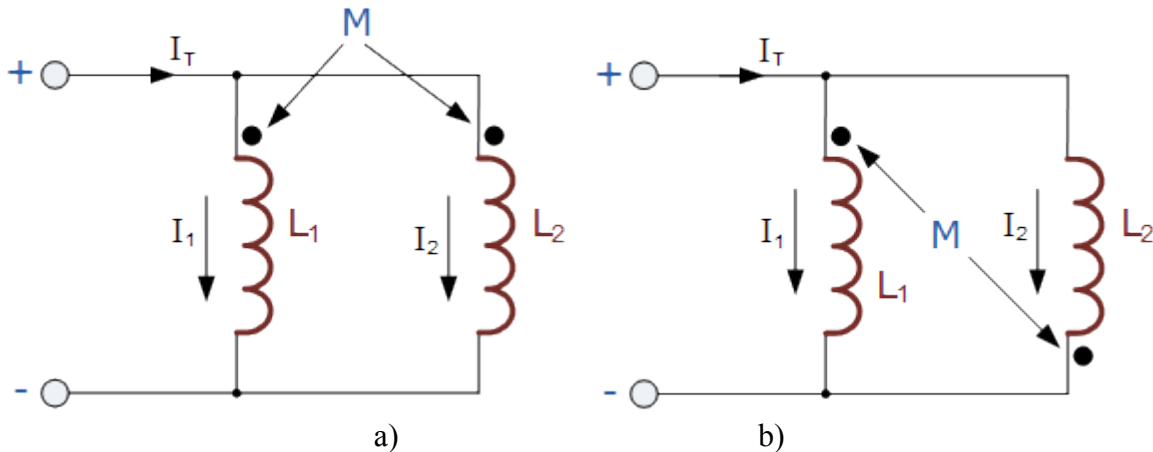


Fig. 6.2. Mutually coupled inductors in parallel: a) aiding inductors; b) opposing inductors.

First, we accept that the inductors are aiding each other (fig. 6.2a) and write down two KVL equations:

$$\begin{cases} v_s = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \\ v_s = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \end{cases}$$

Next, the above system is written in matrix form:

$$\begin{bmatrix} v_s \\ v_s \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \cdot \begin{bmatrix} \frac{di_1}{dt} \\ \frac{di_2}{dt} \end{bmatrix}$$

The determinants are:

$$\begin{aligned} \Delta &= \begin{vmatrix} L_1 & M \\ M & L_2 \end{vmatrix} = L_1 \cdot L_2 - M^2 \\ \Delta_1 &= \begin{vmatrix} v_s & M \\ v_s & L_2 \end{vmatrix} = v_s (L_2 - M) \\ \Delta_2 &= \begin{vmatrix} L_1 & v_s \\ M & v_s \end{vmatrix} = v_s (L_1 - M) \end{aligned}$$

The solutions for the current derivatives are:

$$\frac{di_1}{dt} = \frac{\Delta_1}{\Delta} \frac{di_2}{dt} = \frac{\Delta_2}{\Delta}$$

The KVL for the circuit is:

$$i = i_1 + i_2$$

The above equation is differentiated:

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} = \frac{\Delta_1 + \Delta_2}{\Delta} = \frac{v_S(L_2 - M) + v_S(L_1 - M)}{L_1 \cdot L_2 - M^2} = v_S \frac{L_1 + L_2 - 2 \cdot M}{L_1 \cdot L_2 - M^2}$$

Then the equivalent inductance is:

$$v_S(t) = L_E \frac{di(t)}{dt} = \frac{L_1 \cdot L_2 - M^2}{L_1 + L_2 - 2 \cdot M} \cdot \frac{di(t)}{dt}$$

or

$$L_E = \frac{L_1 \cdot L_2 - M^2}{L_1 + L_2 - 2 \cdot M}$$

In a similar manner can be proven that the equivalent inductance of parallel inductors whose magnetic fields are oppose each other, is:

$$L_E = \frac{L_1 \cdot L_2 - M^2}{L_1 + L_2 + 2 \cdot M}$$

6.1.3. Elimination of mutual inductance

Consider the case when two coils are mutually coupled (fig. 6.3). They could be replaced with an equivalent circuit without mutual inductance and three coils as shown in the figure. To prove this we write the system of equations for the original circuit:

$$\begin{cases} v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \\ v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \end{cases}$$

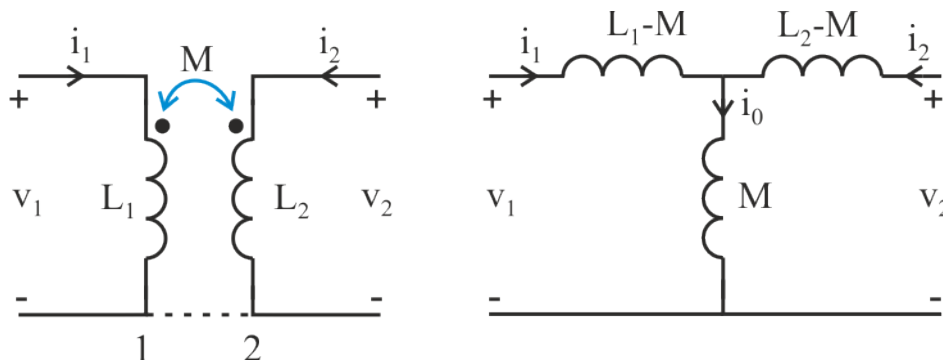
Then we write the system of equations for the equivalent circuit:

$$\begin{cases} v_1 = (L_1 - M) \frac{di_1}{dt} + M \frac{di_0}{dt} \\ v_2 = (L_2 - M) \frac{di_2}{dt} + M \frac{di_0}{dt} \end{cases}$$

Considering $\frac{di_0}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$ we obtain the same system of equations as for the original circuit:

$$\begin{cases} v_1 = (L_1 - M) \frac{di_1}{dt} + M \frac{di_1}{dt} + M \frac{di_2}{dt} = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \\ v_2 = (L_2 - M) \frac{di_2}{dt} + M \frac{di_1}{dt} + M \frac{di_2}{dt} = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \end{cases}$$

In a similar manner we can prove the circuit for the mutually opposing inductors.



a)

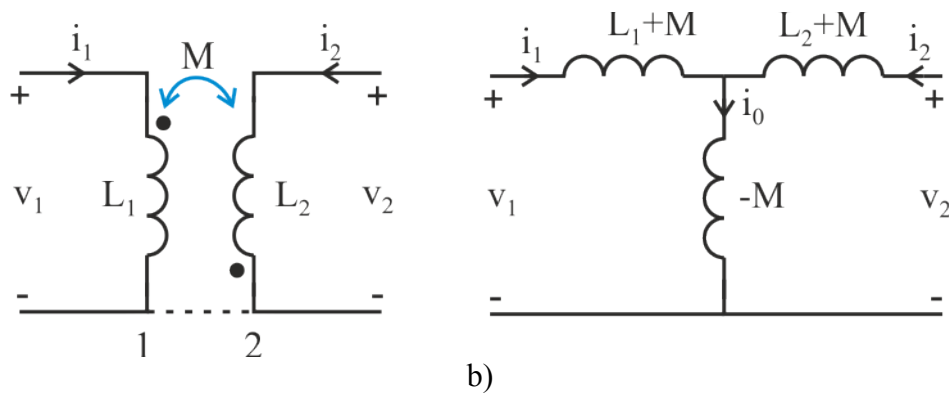


Fig. 6.3. Equivalent circuit for: a) aiding inductors; b) opposing inductors.

Note that the nodes 1 and 2 of the original circuit may or may not be connected and that doesn't change the equivalent circuit.

6.1.4. General case

Consider two inductors with mutual inductance which may or may not have an electric connection. If the two inductors are aiding each other, the equivalent circuit is presented in fig. 6.4. The mutual inductance is replaced by two dependent sources $M \frac{di_2}{dt}$ and $M \frac{di_1}{dt}$, which are opposing the direction of the currents.

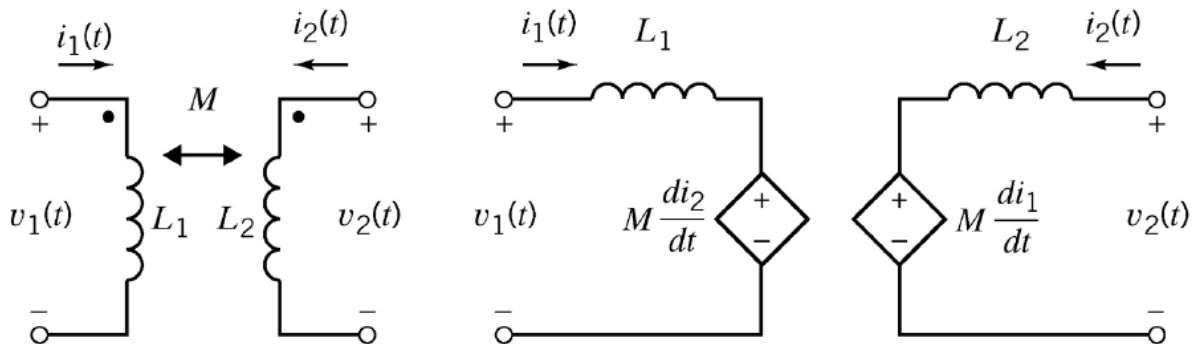


Fig. 6.4.

The KVL equations for both the original and the equivalent circuit is:

$$\begin{cases} v_1(t) = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \\ v_2(t) = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \end{cases}$$

Next consider the two inductors are opposing each other (fig. 6.5). The equivalent circuit is the same however this time dependent sources are with a negative sign: $-M \frac{di_2}{dt}$ and $-M \frac{di_1}{dt}$.

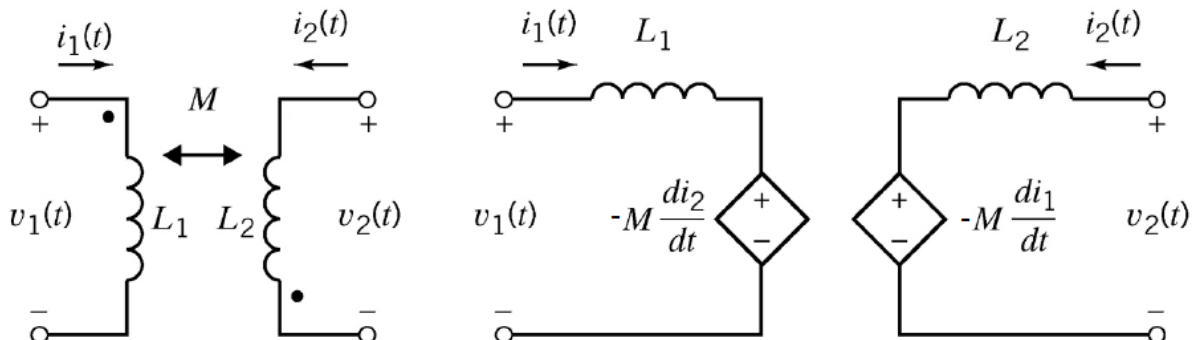


Fig. 6.5.

The KVL equations for both the original and the equivalent circuit is:

$$\begin{cases} v_1(t) = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \\ v_2(t) = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} \end{cases}$$

If the above time domain variables are replaced with phasors, the dependent sources in the equivalent circuits becomes $j\omega M \hat{I}_1$ and $j\omega M \hat{I}_2$ for aiding inductors and $-j\omega M \hat{I}_1$ and $-j\omega M \hat{I}_2$ for opposing inductors (fig. 6.6).

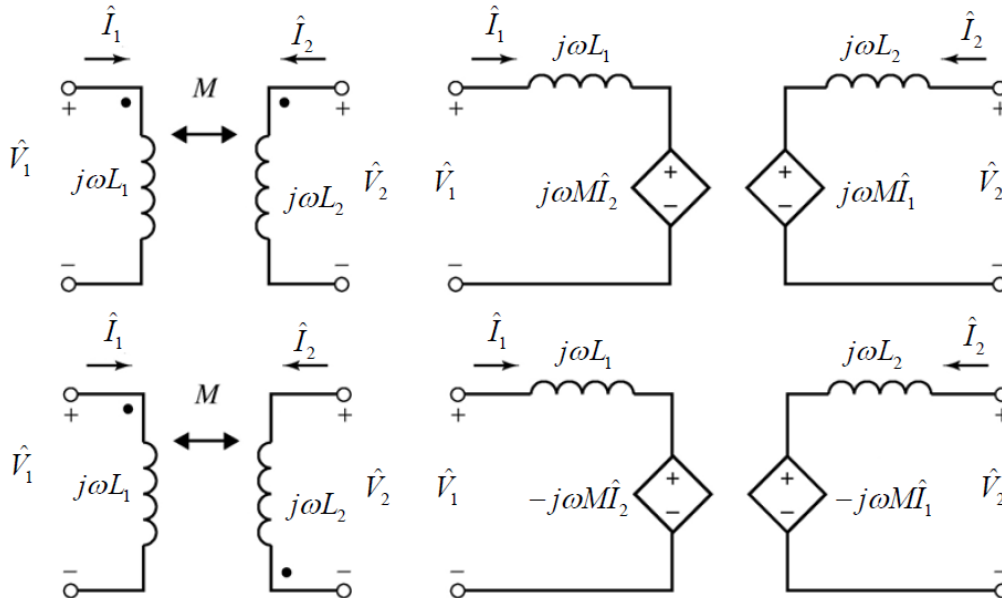


Fig. 6.6.

The KVL equations in complex form for the circuits are:

$$\begin{cases} \dot{V}_1 = j\omega L_1 \dot{I}_1 \pm j\omega M \dot{I}_2 \\ \dot{V}_2 = j\omega L_2 \dot{I}_2 \pm j\omega M \dot{I}_1 \end{cases}$$

where the plus sign corresponds to aiding inductors and the minus sign – to opposing inductors.

6.2. Energy in mutually coupled inductors

It was already demonstrated in the second topic that the energy stored in an inductor is:

$$W_L = \frac{1}{2} \cdot L \cdot i^2$$

Let's consider two mutually coupled inductors (fig. 6.7). The power transferred from the first to the second coil is:

$$p_{M12}(t) = i_2 \cdot v_M = i_2 \cdot M \frac{di_1}{dt}$$

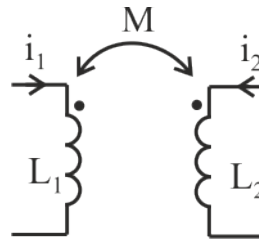


Fig. 6.7.

If we integrate the power from t_1 to t_2 the energy is:

$$W_{M12} = \int_{t_1}^{t_2} p_{M12}(t) dt = \int_{t_1}^{t_2} i_2 \cdot M \frac{di_1}{dt} dt = \int_0^I M \cdot i_2 \cdot di_1 = \frac{1}{2} \cdot M \cdot i_1 i_2$$

The power transferred from the second to the first coil is:

$$p_{M21}(t) = i_1 \cdot v_M = i_1 \cdot M \frac{di_2}{dt}$$

Then the transferred energy is:

$$W_{M12} = \int_{t_1}^{t_2} p_{M21}(t) dt = \int_{t_1}^{t_2} i_1 \cdot M \frac{di_2}{dt} dt = \int_0^I M \cdot i_1 \cdot di_2 = \frac{1}{2} \cdot M \cdot i_1 i_2$$

The energy stored in the two coils due to their self-inductance is:

$$W_{L1} = \frac{1}{2} \cdot L_1 \cdot i_1^2, W_{L2} = \frac{1}{2} \cdot L_2 \cdot i_2^2$$

Then the total energy stored in two mutually coupled inductors is:

$$W = \frac{1}{2} \cdot L_1 \cdot i_1^2 + \frac{1}{2} \cdot L_2 \cdot i_2^2 \pm M \cdot i_1 i_2$$

The plus sign corresponds to aiding inductors and the minus – to opposing ones.

Since energy usage is always positive we can rewrite the above equation (use equation for opposing inductors) as:

$$\frac{1}{2} \cdot L_1 \cdot i_1^2 + \frac{1}{2} \cdot L_2 \cdot i_2^2 - M \cdot i_1 i_2 \geq 0$$

or

$$\frac{1}{2} (\sqrt{L_1} \cdot i_1 - \sqrt{L_2} \cdot i_2)^2 + i_1 i_2 (\sqrt{L_1 L_2} - M) \geq 0$$

Considering $(\sqrt{L_1} \cdot i_1 - \sqrt{L_2} \cdot i_2)^2$ is always positive or zero, then the second term is also greater or equal to zero:

$$\sqrt{L_1 L_2} - M \geq 0$$

This way is defined the coupling coefficient k :

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

The coupling coefficient takes values in the range $0 \leq k \leq 1$ and shows how good the coupling between the two coils is. For coils which are not coupled, $k=0$ and in case of ideal coupling (only possible in theory) $k=1$.

6.3. Analysis of circuits with magnetically coupled inductors

6.3.1. Kirchhoff's laws analysis

The analysis of circuits with magnetically coupled inductors could be achieved using an equivalent circuit without magnetic couples. The analysis using the Kirchhoff's laws includes creating the equivalent circuit and analyzing it by writing a system of equations whose number is equal to the number of unknown currents.

Example: Estimate the currents for the circuit in fig. 6.8 using the Kirchoff's laws.

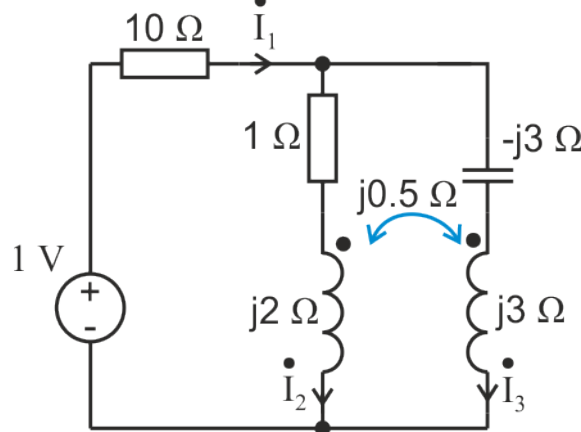


Fig. 6.8.

It can be seen that the mutual inductance $j\omega M$ is given as mutual resistance:

$$j\omega M = j0.5 \Omega$$

First we are going to create an equivalent circuit by replacing the mutual inductance with dependent source (fig. 6.9). Since both currents \dot{I}_1 and \dot{I}_2 enter the dots, the dependent sources are with plus sign.

Next we write the system of equations:

$$\begin{aligned} \dot{I}_1 &= \dot{I}_2 + \dot{I}_3 \\ 1 - j0.5 \dot{I}_3 &= 10 \dot{I}_1 + (1 + j2) \dot{I}_2 \\ j0.5 \dot{I}_3 - j0.5 \dot{I}_2 &= -(1 + j2) \dot{I}_2 + 0 \dot{I}_3 \end{aligned}$$

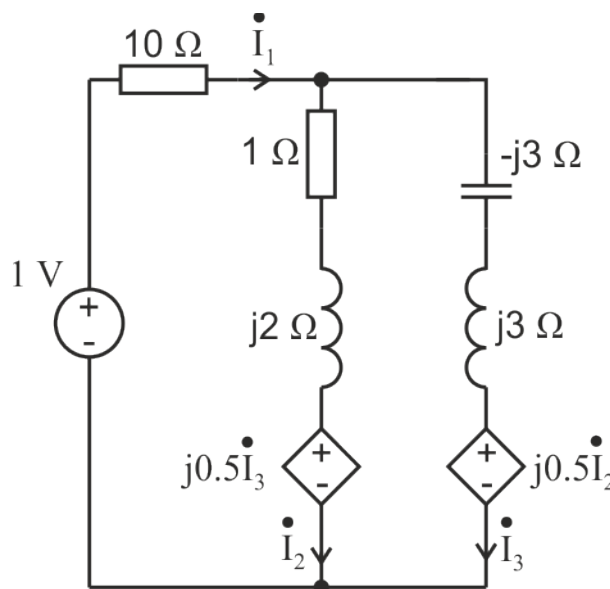


Fig. 6.9.

In matrix form it becomes:

$$\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 10 & 1+j2 & j0.5 \\ 0 & 1+j1.5 & j0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The determinants are:

$$\Delta = \begin{bmatrix} -1 & 1 & 1 \\ 10 & 1+j2 & j0.5 \\ 0 & 1+j1.5 & j0.5 \end{bmatrix} = -j0.5(1+j2) + 10(1+j1.5) + j0.5(1+j1.5) - j0.5 \cdot 10 =$$

$$= -j0.5 + 1 + 10 + j15 + j0.5 - 0.75 - j5 = 10.25 + j10$$

$$\Delta_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1+j2 & j0.5 \\ 0 & 1+j1.5 & j0.5 \end{bmatrix} = 1 + j1.5 - j0.5 = 1 + j$$

$$\Delta_2 = \begin{bmatrix} -1 & 0 & 1 \\ 10 & 1 & j0.5 \\ 0 & 0 & j0.5 \end{bmatrix} = -j0.5$$

$$\Delta_3 = \begin{bmatrix} -1 & 1 & 0 \\ 10 & 1+j2 & 1 \\ 0 & 1+j1.5 & 0 \end{bmatrix} = 1 + j1.5$$

And the currents are:

$$\dot{I}_1 = \frac{\Delta_1}{\Delta} = \frac{1+j}{10.25+j10} = 0.099 + j0.001 \text{ A}$$

$$\dot{I}_2 = \frac{\Delta_2}{\Delta} = \frac{-j0.5}{10.25+j10} = -0.024 - j0.025 \text{ A}$$

$$\dot{I}_3 = \frac{\Delta_3}{\Delta} = \frac{1+j1.5}{10.25+j10} = 0.123 + j0.026 \text{ A}$$

6.3.2. Nodal analysis

The nodal analysis cannot be applied if dependent sources are used. However in certain situations where the mutual inductance could be eliminated, the nodal analysis can be applied.

Example: Estimate the currents for the circuit in fig. 6.8 using nodal analysis.

Considering the two coils have a common end, the mutual inductance could be replaced with an equivalent circuit with three coils (fig. 6.10). Their resistances are:

$$j\omega L_1 = j2 - j0.5 = j1.5 \Omega$$

$$j\omega L_2 = j3 - j0.5 = j2.5 \Omega$$

$$j\omega L_3 = j0.5 \Omega$$

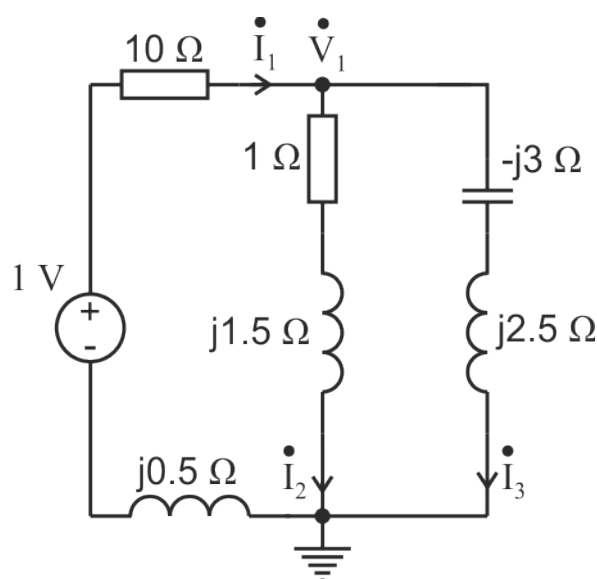


Fig. 6.10.

Now we can apply nodal voltage analysis. The KCL is:

$$\dot{I}_1 = \dot{I}_2 + \dot{I}_3 = \dot{I} \frac{1 - \dot{V}_1}{10 + j0.5} = \frac{\dot{V}_1}{1 + j1.5} + \frac{\dot{V}_1}{-j0.5}$$

The node voltage then becomes:

$$\dot{V}_1 = \frac{\frac{1}{10 + j0.5}}{\frac{1}{1 + j1.5} - \frac{1}{j0.5} + \frac{1}{10 + j0.5}} = \frac{0.100 - j0.005}{0.407 + j1.533} = 0.013 - j0.062$$

And the currents are:

$$\begin{aligned}\dot{I}_1 &= \frac{0.987 + j0.062}{10 + j0.5} = 0.099 + j0.001 \text{ A} \\ \dot{I}_2 &= \frac{0.013 - j0.062}{1 + j1.5} = -0.025 - j0.025 \text{ A} \\ \dot{I}_3 &= \frac{0.013 - j0.062}{-j0.5} = 0.124 + j0.026 \text{ A}\end{aligned}$$

6.3.3. Mesh analysis

The mesh analysis method can be applied for analysis of circuits with mutual inductance. However in order to do that the dependent sources should be expressed with the mesh currents. Once this is done the equations are written according to KVL.

Example: Estimate the currents for the circuit in fig. 6.8 using mesh analysis.

First we should estimate the branch currents with the mesh currents:

$$\begin{aligned}\dot{I}_1 &= \dot{I}_1^{(k)} \\ \dot{I}_2 &= \dot{I}_1^{(k)} - \dot{I}_2^{(k)} \\ \dot{I}_3 &= \dot{I}_2^{(k)}\end{aligned}$$

Then the equivalent dependent sources become (fig. 6.11):

$$\begin{aligned}j0.5 \dot{I}_2 &= j0.5 (\dot{I}_1^{(k)} - \dot{I}_2^{(k)}) \\ j0.5 \dot{I}_3 &= j0.5 \dot{I}_2^{(k)}\end{aligned}$$

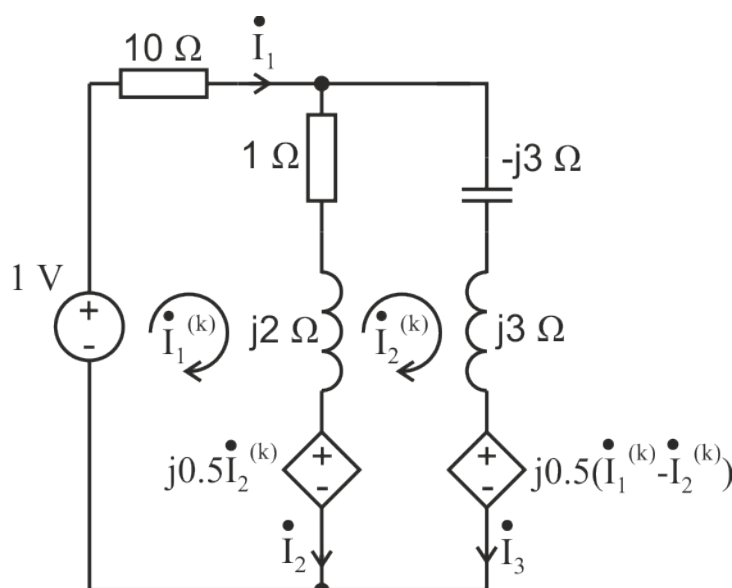


Fig. 6.11.

The two KVL equations are:

$$1 - j0.5 \dot{I}_2^{(k)} = \dot{I}_1^{(k)}(10 + 1 + j2) - \dot{I}_2^{(k)}(1 + j2)$$

$$1 = \dot{I}_1^{(k)}(11 + j2) - \dot{I}_2^{(k)}(1 + j1.5)$$

and

$$j0.5 \dot{I}_2^{(k)} - j0.5 \dot{I}_1^{(k)} + j0.5 \dot{I}_2^{(k)} = \dot{I}_2^{(k)}(1 + j2 + j3 - j3) - \dot{I}_1^{(k)}(1 + j2)$$

$$0 = -\dot{I}_1^{(k)}(1 + j1.5) + \dot{I}_2^{(k)}(1 + j)$$

In matrix form the equations are:

$$\begin{bmatrix} \dot{I}_1^{(k)} \\ \dot{I}_2^{(k)} \end{bmatrix} \begin{bmatrix} 11 + j2 & -(1 + j1.5) \\ -(1 + j1.5) & 1 + j \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The determinants are:

$$\Delta = \begin{vmatrix} 11 + j2 & -(1 + j1.5) \\ -(1 + j1.5) & 1 + j \end{vmatrix} = 11 + j11 + j2 - 2 - 1 - j1.5 - j1.5 + 2.25 = 10.25 + j10$$

$$\Delta_1 = \begin{vmatrix} 1 & -(1 + j1.5) \\ 0 & 1 + j \end{vmatrix} = 1 + j$$

$$\Delta_2 = \begin{vmatrix} 11 + j2 & 1 \\ -(1 + j1.5) & 0 \end{vmatrix} = 1 + j1.5$$

And the mesh currents are:

$$\dot{I}_1^{(k)} = \frac{\Delta_1}{\Delta} = \frac{1 + j}{10.25 + j10} = 0.099 + j0.001$$

$$\dot{I}_2^{(k)} = \frac{\Delta_2}{\Delta} = \frac{1 + j1.5}{10.25 + j10} = 0.123 + j0.026$$

Then the branch currents are:

$$\dot{I}_1 = \dot{I}_1^{(k)} = 0.099 + j0.001 \text{ A}$$

$$\dot{I}_2 = 0.099 + j0.001 - 0.123 - j0.026 = -0.024 - j0.025 \text{ A}$$

$$\dot{I}_3 = 0.123 + j0.026 \text{ A}$$

References

1. Alexander Ch., Sadiku M. Fundamentals of electric circuits. Fifth edition. McGraw-Hill. 2013. Chapter 13.
2. Hayt W., Buck J. Engineering electromagnetics. Sixth edition. McGraw-Hill. 2001. Chapters 5 and 9.
3. <http://www.electronics-tutorials.ws/category/inductor>