# Multi-Echelon Inventory Optimization: An Overview



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# Outline



- Introduction
- Single-stage models (building blocks)
- Multi-echelon models
  - Network Topology
  - Deterministic Models
  - Stochastic Models
- Decentralized systems

# Introduction

# Factors Influencing Inventory Decisions



- Why hold inventory?
  - Lead times
  - Economies of scale / fixed costs / quantity discounts
  - Service levels
  - Concerns about future availability
  - Sales / promotions
- Why avoid inventory?
  - Cost of capital
  - Shelf space
  - Perishability
  - Risk of theft / fire / etc.

# Classifying Inventory Models



- Deterministic vs. stochastic
- Single- vs. multi-echelon
- Periodic vs. continuous review
- Discrete vs. continuous demand
- Backorders vs. lost sales
- Global vs. local control
- Centralized vs. decentralized optimization
- Fixed cost vs. no fixed cost
- Lead time vs. no lead time

# Costs in Inventory Models



- Holding cost $h$ (\$ / item / unit time)
- Stockout penalty $p$ (\$ / item / unit time)
- Fixed cost $k$ (\$ / order)
- Purchase cost $c$ (\$ / item)
  - Often ignored in optimization models

# A Brief History of Inventory Theory



- Harris (1913): EOQ model
- ??? (19??): newsvendor model
- Wagner and Whitin (1958): time-varying deterministic demands
- Clark and Scarf (1960): serial stochastic systems
- Roundy (1985): serial deterministic systems w/fixed costs, power-of-2 policies
- Graves and Willems (2000): guaranteed-service models

# Single-Stage Models

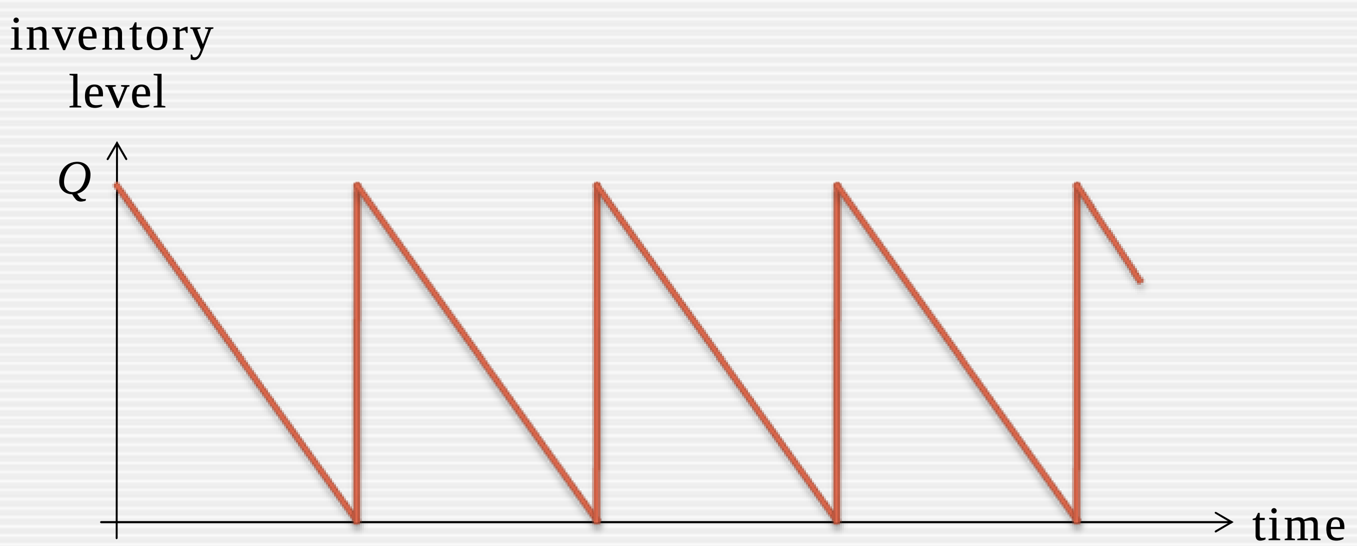


**(BUILDING BLOCKS)**

# The EOQ Model



- Continuous, deterministic demand at rate λ per year
- Fixed cost *k* per order
- Holding cost *h* per item per year
- Stockouts not allowed



# The EOQ Model: Optimization



- Average annual cost:

$$c(Q) = \frac{k\lambda}{Q} + \frac{hQ}{2}$$

- First-order condition:

$$c'(Q) = -\frac{k\lambda}{Q^2} + \frac{h}{2} = 0$$

- Optimal solution:

$$\boxed{Q^* = \sqrt{\frac{2k\lambda}{h}}} \qquad \boxed{c(Q^*) = \sqrt{2k\lambda h} = hQ^*}$$

# The Newsvendor Model



- Each day, newsvendor buys newspapers from publisher for $0.25 each
- Sells newspapers for $0.75 each
- Unsold papers are sold back to publisher for $0.10
- Daily demand is stochastic, $\sim N(50, 10^2)$
- No inventory carryover [perishable inventory]
- No backorder carryover [lost sales]

- **How many newspapers to buy?**
  - Probably >50, but how many?

# A More General Formulation



- Periodic, stochastic demand
  - pdf $f$, cdf $F$
  - We'll assume normal distribution ($\phi$, $\Phi$ = standard normal)
- Inventory carryover allowed [non-perishable] or not
  - Either way, "overage" cost = $h$
  - May include salvage value/cost
- Backorders or lost sales
  - Either way, "underage" cost = $p$
  - May include lost profit, loss of goodwill, admin costs
- Decision variable: base-stock level $y$
  - In each period, order up to $y$

# Expected Cost Function



$$c(y) = h\int_0^y (y-x)f(x)dx + p\int_y^\infty (x-y)f(x)dx$$

- Convex $\Rightarrow$ solve first-order condition (Leibniz's rule)
- Optimal solution:

$$y^* = \mu + \sigma\Phi^{-1}\left(\frac{p}{p+h}\right) = \mu + \sigma z_\alpha$$

where $\alpha = p / (p + h)$ (the *newsvendor ratio*)

# Interpretation of Optimal Solution

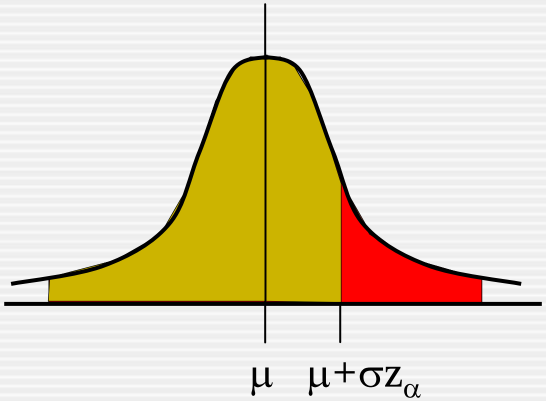


$$y^* = \mu + \sigma z_\alpha$$

cycle stock

safety stock

- No stockouts if demand $\leq \mu + \sigma z_\alpha$
  - Occurs with probability $\alpha$
  - $\alpha$ = optimal service level




- If lead time ($L$) > 0:

$$y^* = \mu L + \sigma z_\alpha \sqrt{L}$$

# Multi-Echelon Models

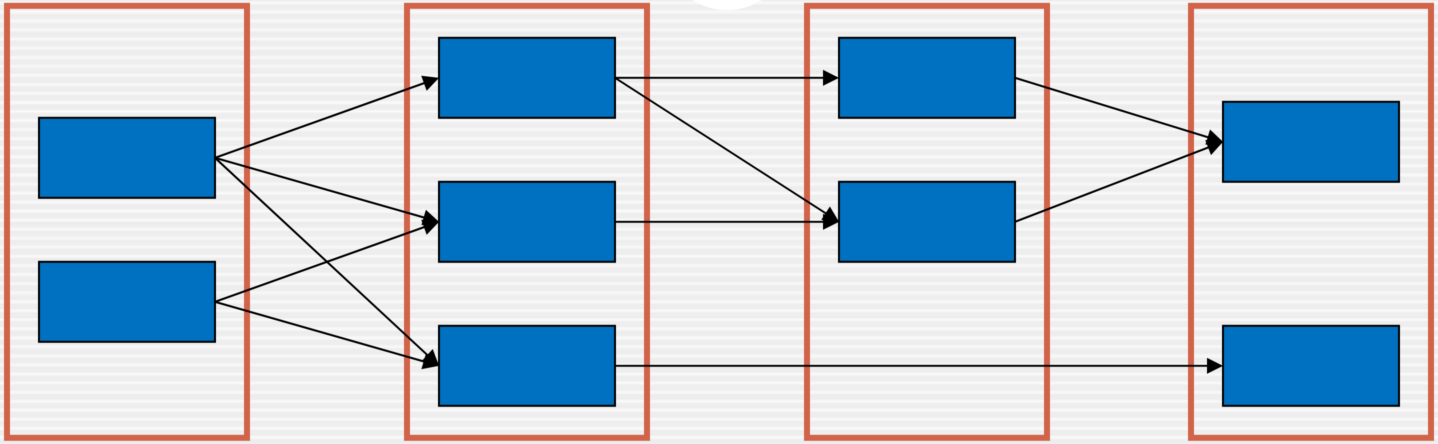


**PART 1:**

**NETWORK TOPOLOGY**

# Network Topology

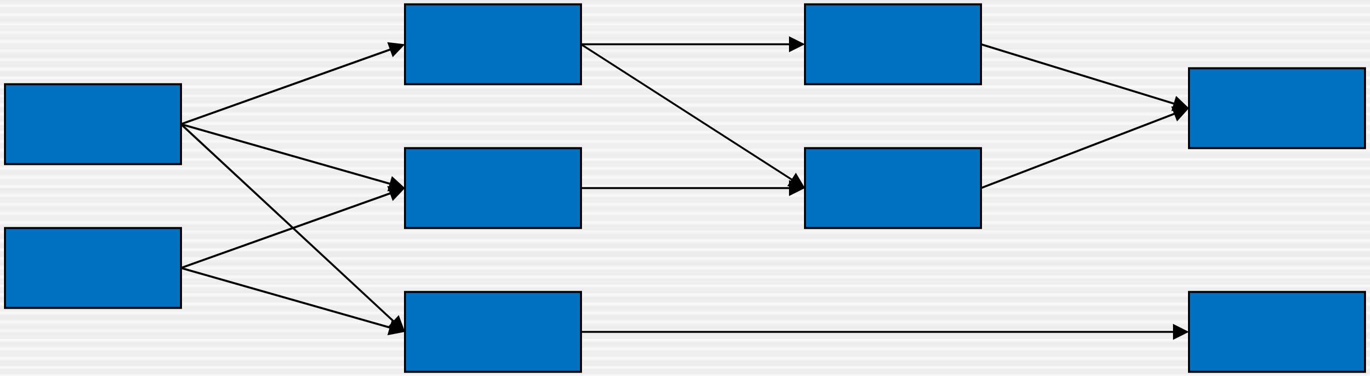




- System is composed of stages (nodes, items, sites…)
- Stages are grouped into echelons
- Stages can represent:
  - Physical locations
  - Items in BOM
  - Processing activities

# Terminology





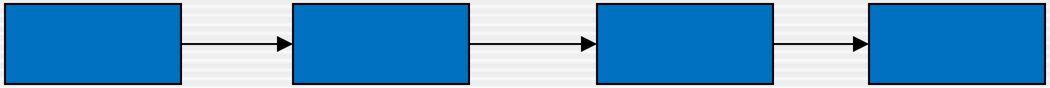
- Stages to the left are *upstream*
- Those to the right are *downstream*
- Downstream stages face customer demand

- Network topologies, in increasing order of complexity:

# Serial System



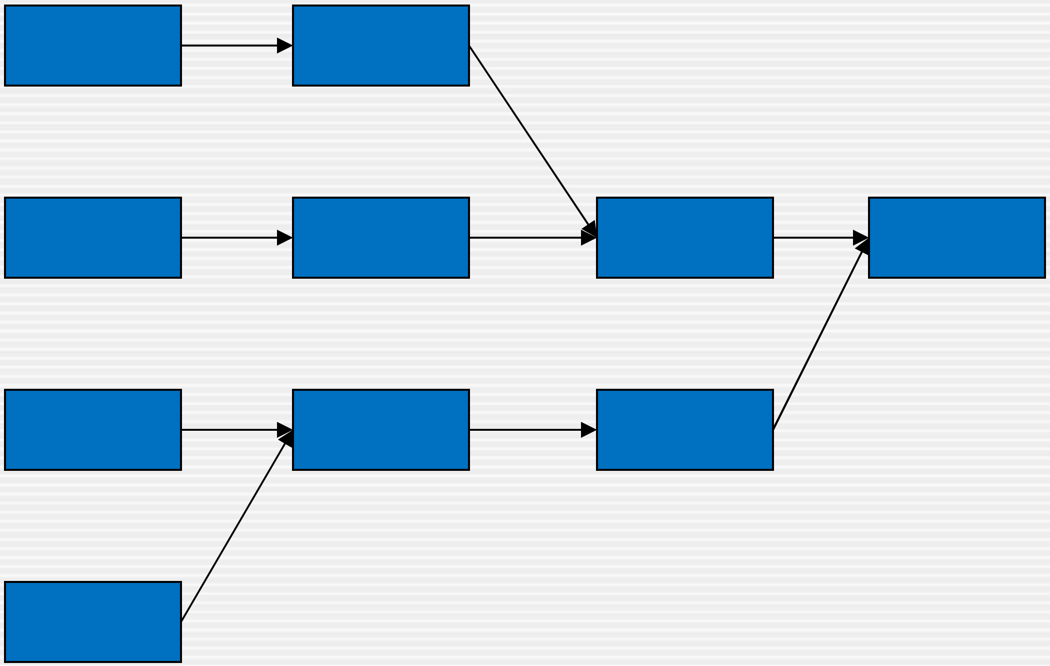
- Each stage has at most one predecessor and at most one successor



# Assembly System



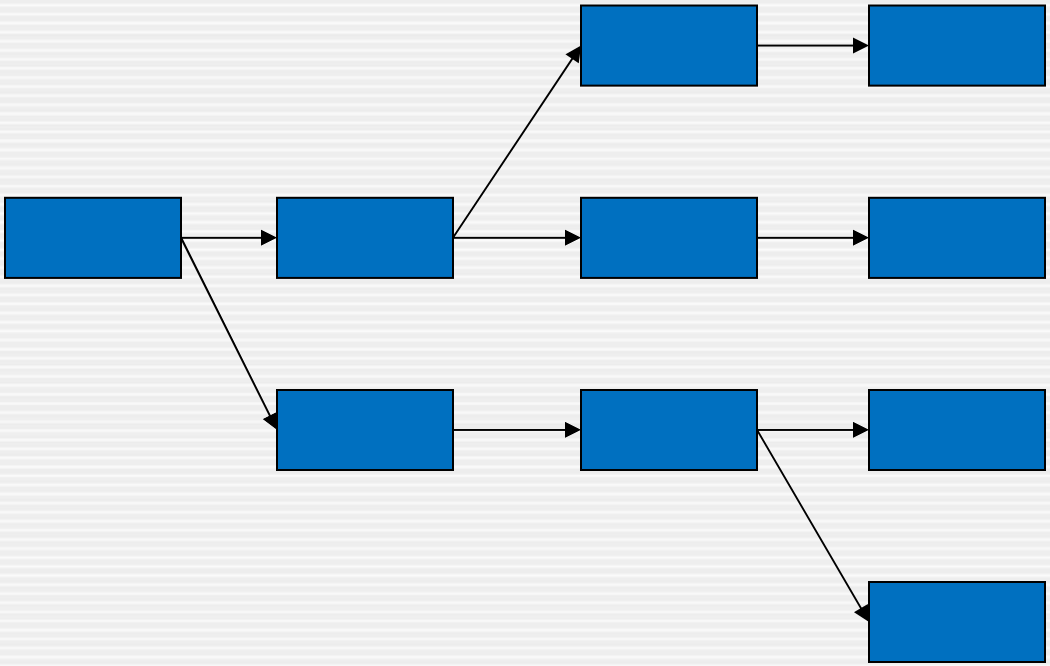
- Each stage has at most one successor



# Distribution System



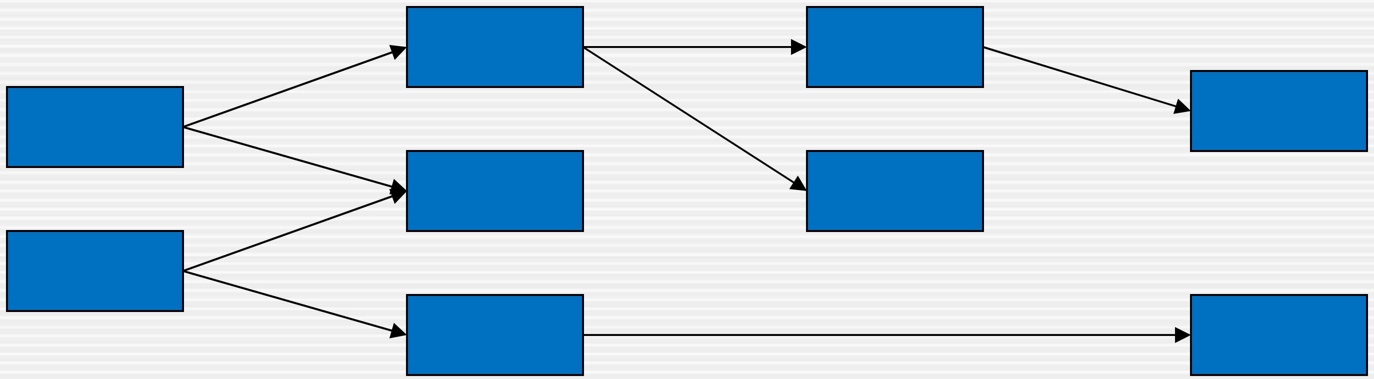
- Each stage has at most one predecessor



# Tree System



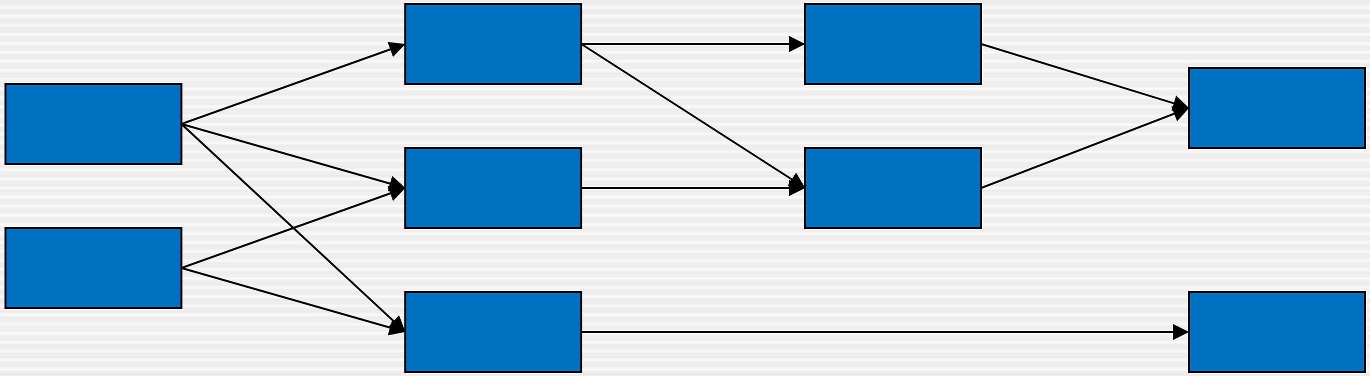
- No restrictions on neighbors, but no cycles



# General System



- No restrictions on cycles



# Multi-Echelon Models



**PART 2:**

**DETERMINISTIC SYSTEMS**

**(WITH FIXED COSTS)**

# Assumptions

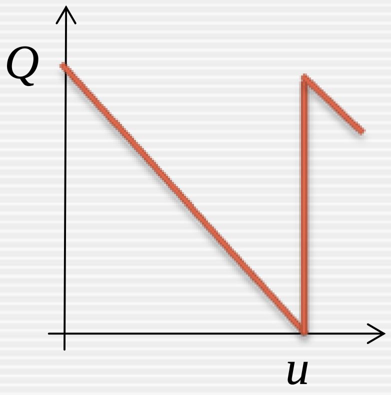


- Each stage functions like an EOQ system:
  - Continuous, deterministic demand (last stage only)
  - Fixed ordering cost
  - No stockouts allowed
- We'll consider serial systems only

# The Optimization Problem



- Need to choose $Q$ at all stages simultaneously
- Properties of optimal solutions:
  - **Zero-inventory ordering (ZIO):** order only when inventory = 0
  - **Stationary:** same $Q$ for every order
    - (but different for different stages)
  - **Nested:** whenever one stage orders, so does its customer
- Instead of optimizing over $Q$, we optimize over $u$ (reorder interval)
  - $u = Q / \lambda$



# NLIP Formulation



$$\min C(\mathbf{u}) = \sum_j \left( \frac{k_j}{u_j} + \frac{h_j \lambda u_j}{2} \right)$$

$$\text{s.t.} \quad u_j = \theta_j u_{j+1}$$

$$u_j \geq 0$$

$$\theta_j \in \{1,2,3,\ldots\}$$

- Non-convex mixed-integer NLP
- Optimal solution **u*** is not known
  - In fact, no guarantee an optimal solution exists, except in limit
- Therefore, get lower bound by solving relaxed problem
- And upper bound by rounding relaxed solution to feasible solution

# Relaxed Problem



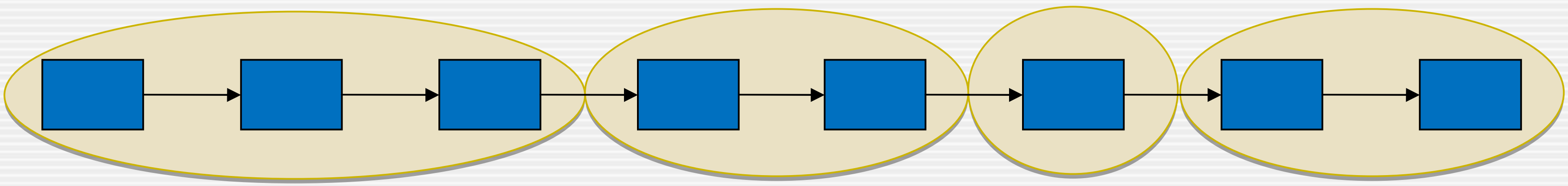
$$\begin{aligned} &\min C(\mathbf{u}) \\ &\text{s.t.} \quad u_j \geq u_{j+1} \\ &\qquad\quad u_j \geq 0 \end{aligned}$$

- Convex NLP
- Could solve using NLP solver
- But there's a better way…

# Solving the Relaxed Problem



- Partition the stages:



- In each partition, require every stage to have the same $u_j = u$
  - Find $u$ by solving EOQ—easy!
- If we use the "correct" partition, we solve the relaxed problem
  - Find correct partition by finding upper concave envelope of set of points in 2D—easy!

# Power-of-2 Policies



- Let $\hat{u}$ be a fixed base period
  - e.g., 1 week, 3 days, etc.
- Power-of-2 policy: each $u_j$ is an integer-power-of-2 multiple of $\hat{u}$
- To get feasible solution, round solution to relaxed problem to *nearest power-of-2 policy*
- Power-of-2 policies are simple to implement and intuitive
  - (Stage 1 orders every 2 weeks, stage 2 orders every week, etc.)

# Worst-Case Error Bound



- Let $\mathbf{u}^*$ be the (unknown) optimal policy
- Let $\mathbf{u}^+$ be the power-of-2 policy

- **Theorem (Roundy 1985):** For any $\hat{u}$,

$$\boxed{\frac{C(\mathbf{u}^+)}{C(\mathbf{u}^*)} \le \frac{3}{2\sqrt{2}} \approx 1.06}$$

- If we can choose $\hat{u}$, then the bound reduces to 1.02

# Multi-Echelon Models



**PART 3:**

**STOCHASTIC SYSTEMS**

**(WITHOUT FIXED COSTS)**

# Assumptions



- Each stage functions like a newsvendor system:
  - Periodic, stochastic demand (last stage only)
  - No fixed ordering cost
  - Inventory carryover and backorders
- Each stage follows base-stock policy
- Lead time ($L$) = deterministic transit time between stages
- Waiting time ($W$) = stochastic time between when stage places an order and when it receives it
  - Includes $L$ plus delay due to stockouts at supplier

# Stochastic- vs. Guaranteed-Service Models



- Two main modeling approaches
- Stochastic-service models:
  - Each stage meets demands from stock whenever possible ($W=L$)
  - Excess demands are backordered and incur $W>L$
- Guaranteed-service models:
  - Each stage sets a *committed service time* (CST) and guarantees that $W$ = CST for every demand
  - Demand is assumed to be bounded
- Let $\alpha$ = service level (% with $W \leq$ CST)
  - Stochastic service: CST = 0, $\alpha < 1$
  - Guaranteed service: CST > 0, $\alpha = 1$

# Stochastic-Service Models

# Serial Systems: The Clark-Scarf Algorithm



- Objective function:

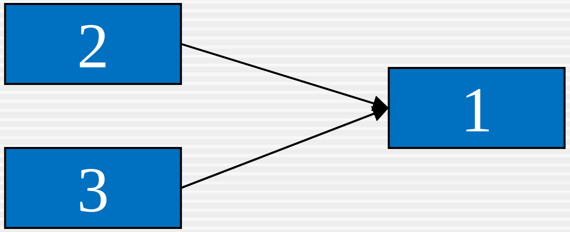
$$c(\mathbf{y}) = \sum_j \left[ hE[\text{on-hand inventory}] + pE[\text{backorders}] \right]$$

- $E$[on-hand] and $E$[backorders] at stage $j$ depend on $y$ at $j$ and upstream
- Clark and Scarf (1960) rewrite $c(\mathbf{y})$ so that system decomposes by stage
  - $y_j$ can be determined at each stage in sequence
  - Use decisions from downstream stages but ignore upstream ones
  - At each stage, solve 1-variable convex minimization problem
  - (At last stage, it's a newsvendor problem)
- Easy computationally but cumbersome to implement
- Good heuristics exist: e.g., Shang and Song (1993)

# Assembly Systems



- **Theorem (Rosling 1989):** Every assembly system can be reduced to an equivalent serial system
  - Solve using Clark-Scarf algorithm
- Based on inventory balance principle:



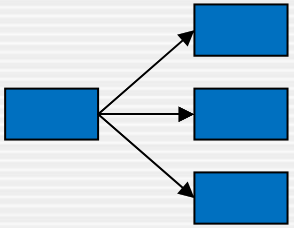

  - If inventory of 2 > inventory of 3, the extra is useless
  - Therefore, attempt to keep $I_2 = I_3$ at all times

# Distribution Systems



- Inventory balance principle does not apply
- Allocation rule becomes critical factor

- The one-warehouse, multiple retailer (OWMR) system
  - Famous special case
  - Exact algorithm: Axsäter 1993
  - Heuristics:
    - Sherbrooke 1968 (METRIC): approximate waiting time with its mean
    - Graves 1985: 2-moment approximation of backorder levels
    - Gallego, Özer, and Zipkin 2007: newsvendor approximation
    - Rong, Bulut, and Snyder 2008: decompose into serial systems



# Extensions



- Fixed ordering costs
- Stochastic lead times
- Limited capacity
- Imperfect quality

- Some are hard, some are not
  - Tractability of standard problems is somewhat “fragile”

# Guaranteed-Service Models

# Guaranteed-Service Models: Overview

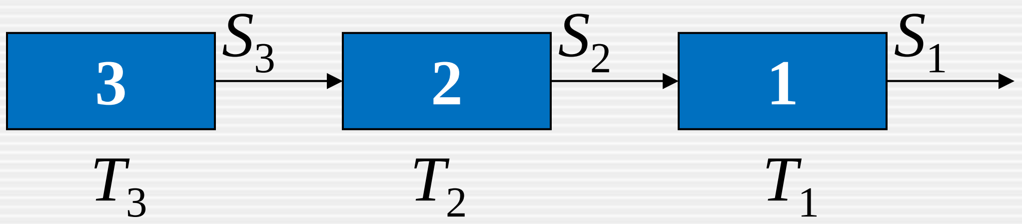


- Each stage promises to deliver *every* item within a fixed number of periods
  - Called the committed service time (CST)
- Requires assumption that demand is *bounded*
  - e.g., $D \leq \mu + \sigma z_\alpha$
  - Equivalently, ignore excess demand when $D$ exceeds bound
- CST assumption allows us to treat waiting time ($W$) as *deterministic*

- References: Kimball 1955, Simpson 1958, Graves 1988, Graves and Willems 2000, 2003

# Net Lead Time






- Each stage has:
  - Processing time $T$
  - CST $S$
- *Net lead time* (NLT) at stage $i$ = $\underbrace{S_{i+1} + T_i}_{\text{“bad” LT}} \underbrace{- S_i}_{\text{“good” LT}}$

# Net Lead Time vs. Inventory



- Suppose $S_i = S_{i+1} + T_i$
  - e.g., inbound CST = 4, proc time = 2, outbound CST = 6
  - Don't need to hold any inventory
  - Operate entirely as pull (make-to-order, JIT) system
- Suppose $S_i = 0$
  - Promise immediate order fulfillment
  - Make-to-stock system

# Net Lead Time vs. Inventory



- In general:

$$y^* = \mu \times NLT + \sigma z_\alpha \sqrt{NLT}$$

- NLT replaces LT in earlier formula
- Choosing inventory levels $\Leftrightarrow$ choosing NLTs, i.e., choosing $S$ at each stage

# Optimization



- Objective:
  - Find optimal *S* values (CSTs)
  - To minimize expected holding cost
  - Subject to end-customer service requirement
- Solution methods:
  - Serial systems: dynamic programming (Graves 1988)
  - Tree systems: dynamic programming (Graves and Willems 2000)
  - General systems: piecewise-linear approximation + CPLEX (Magnanti et al., 2006)

# Key Insight

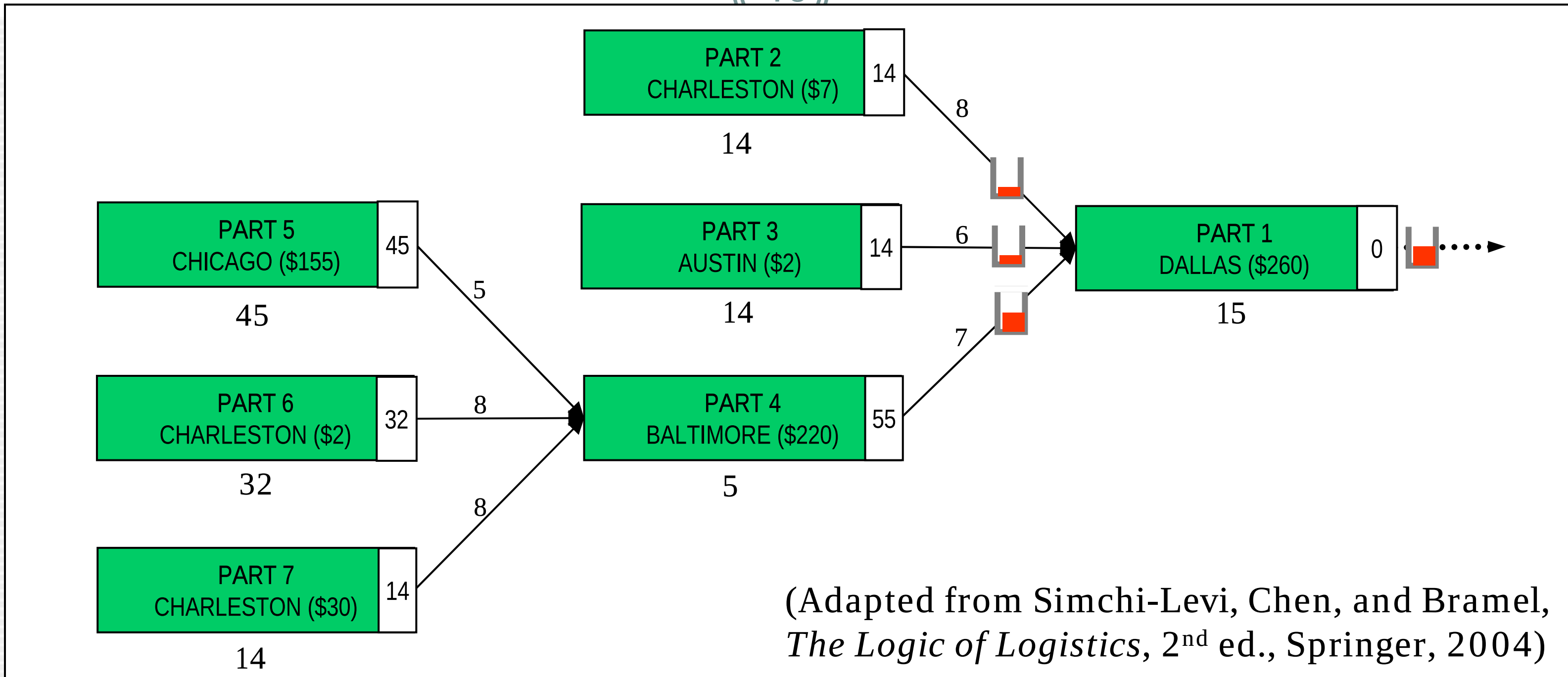


- It is usually optimal for only a few stages to hold inventory
  - Other stages operate as pull systems

- **Theorem (Graves 1988):** In a serial system, every stage either:
  - holds zero inventory (and quotes maximum CST)
  - or quotes CST of zero (and holds maximum inventory)

# Case Study



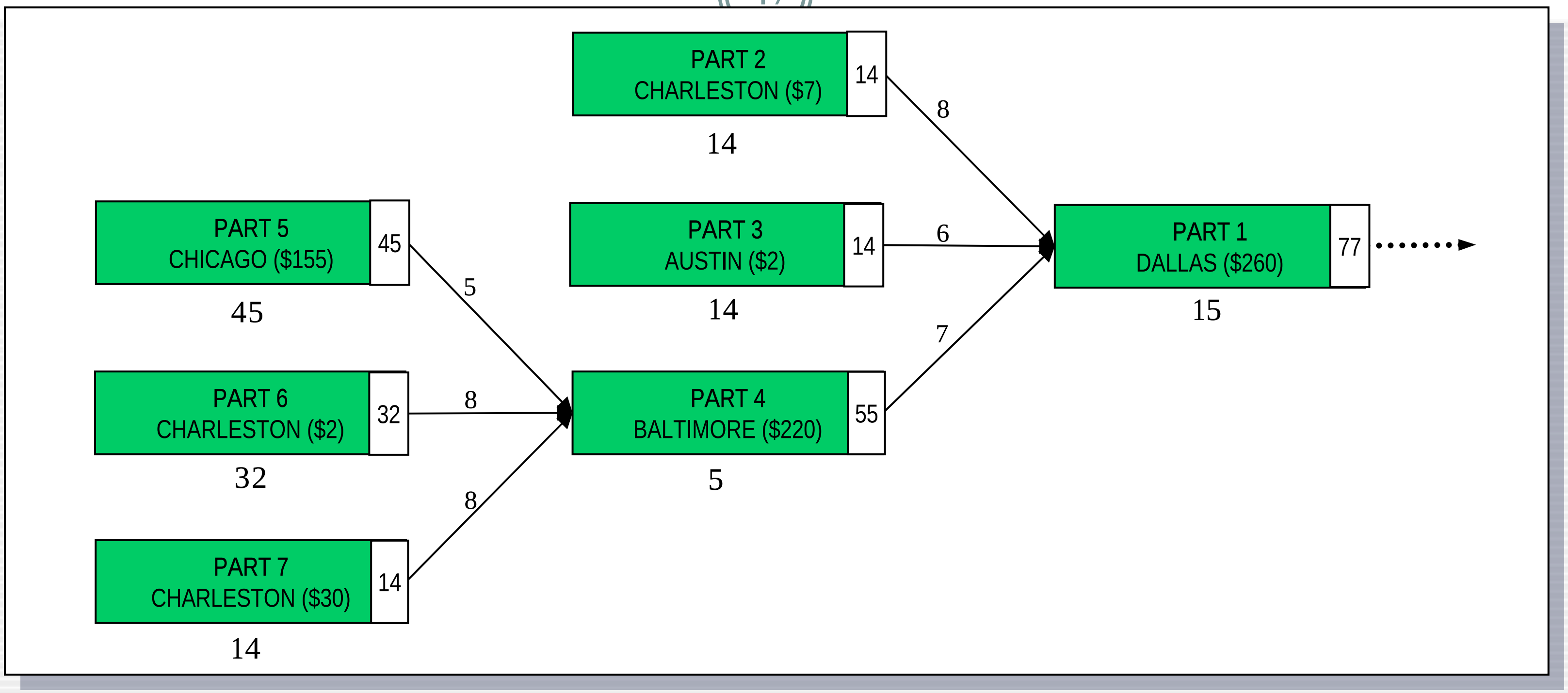



(Adapted from Simchi-Levi, Chen, and Bramel, *The Logic of Logistics*, 2nd ed., Springer, 2004)

- # below stage = processing time
- # in white box = CST
- In this solution, inventory is held of finished product and its raw materials

# A Pure Pull System

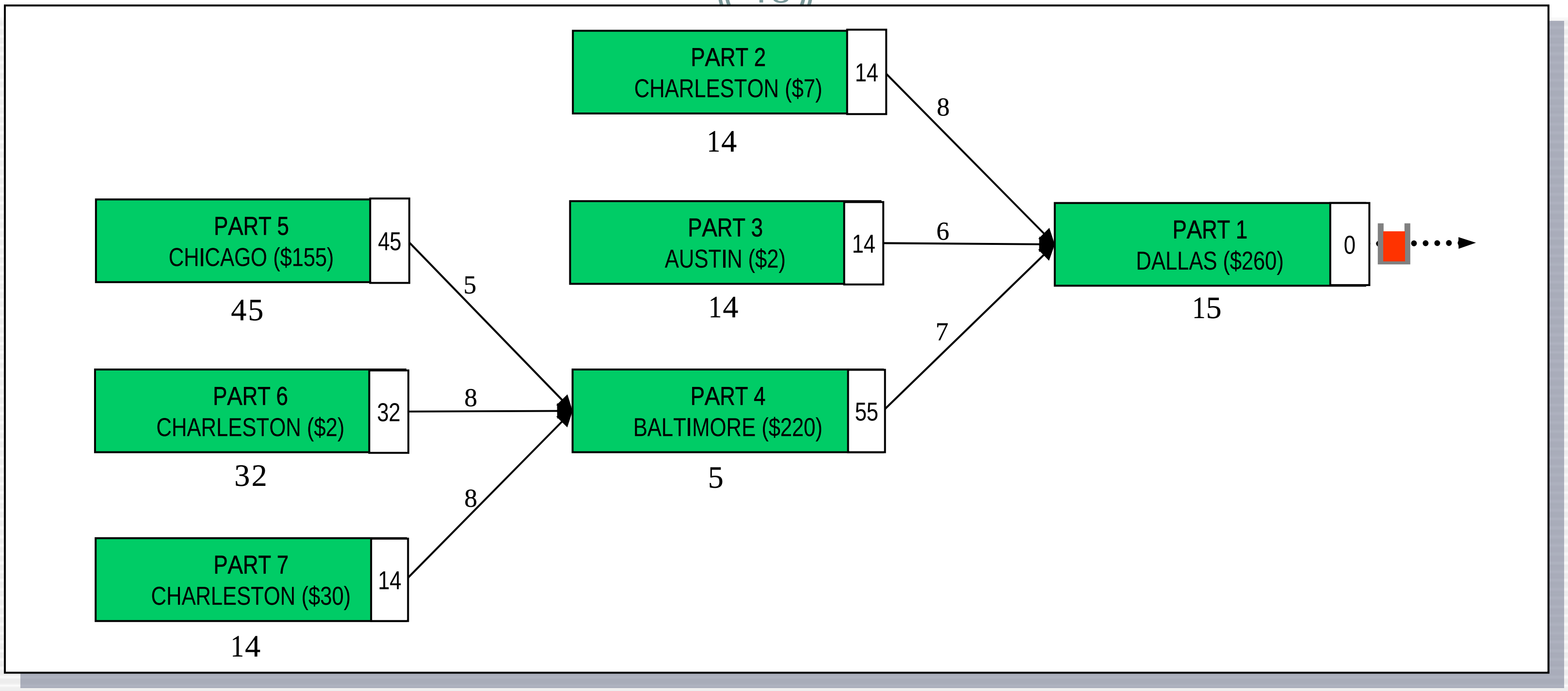





- Produce to order
- Long CST to customer
- No inventory held in system

# A Pure Push System

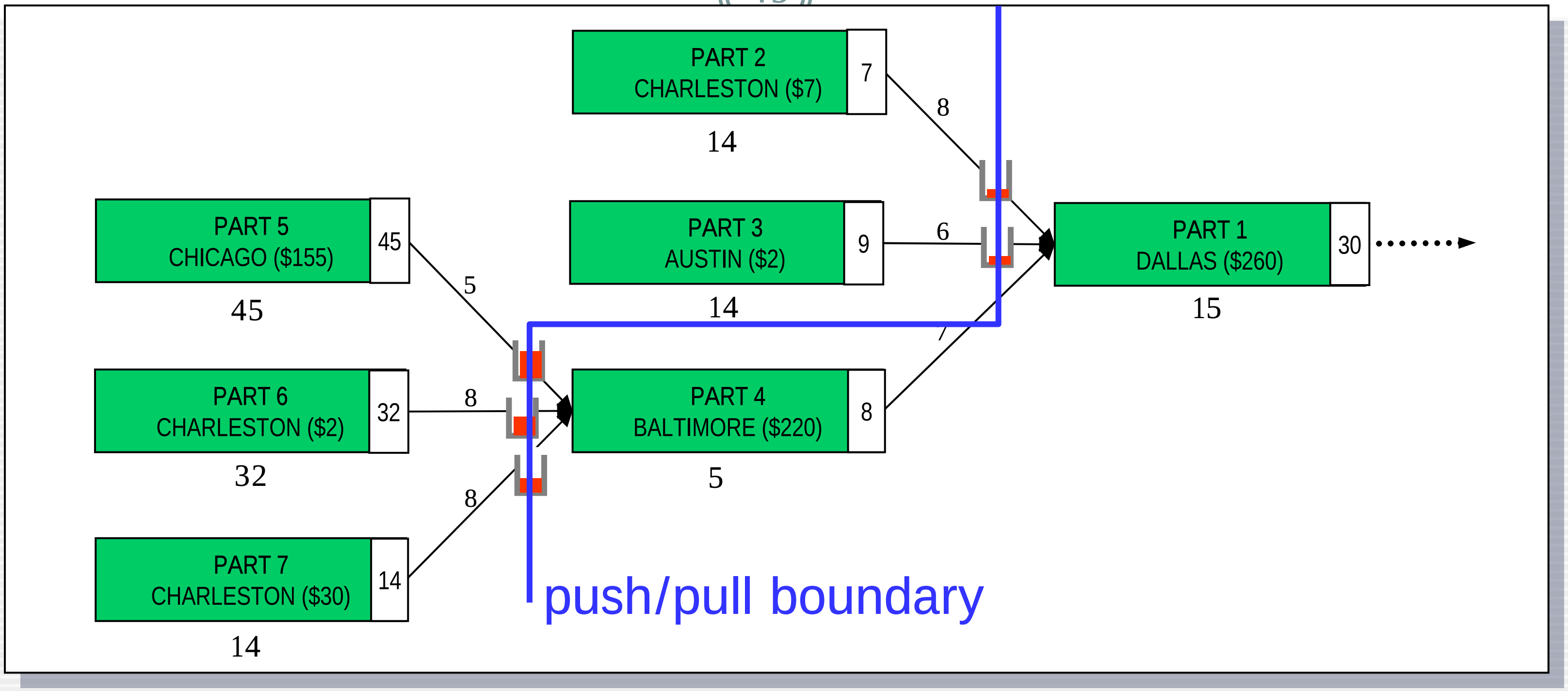





- Produce to forecast
- Zero CST to customer
- Hold lots of finished goods inventory

# A Hybrid Push-Pull System

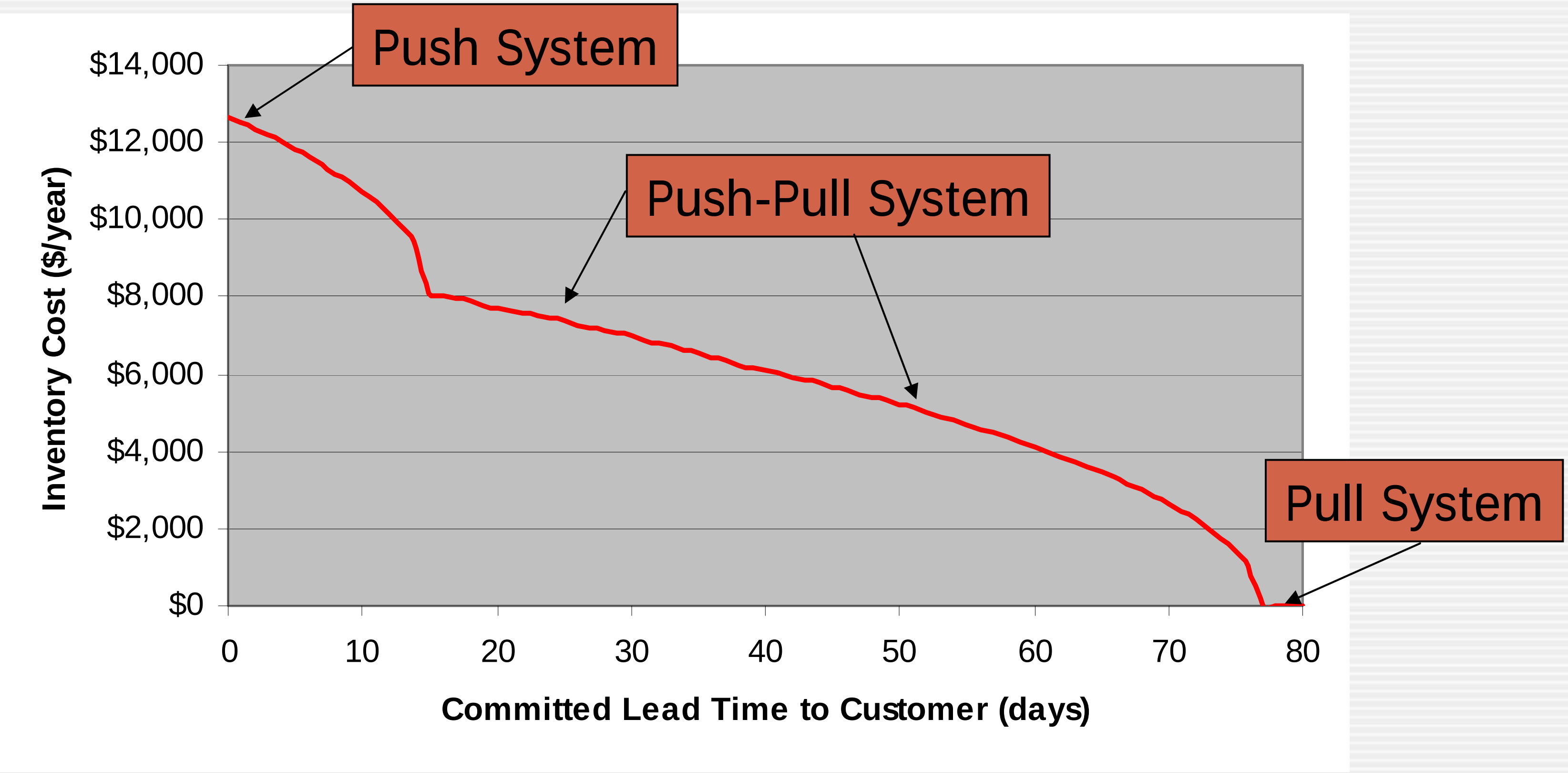





- Part of system operated produce-to-stock, part produce-to-order
- Moderate lead time to customer

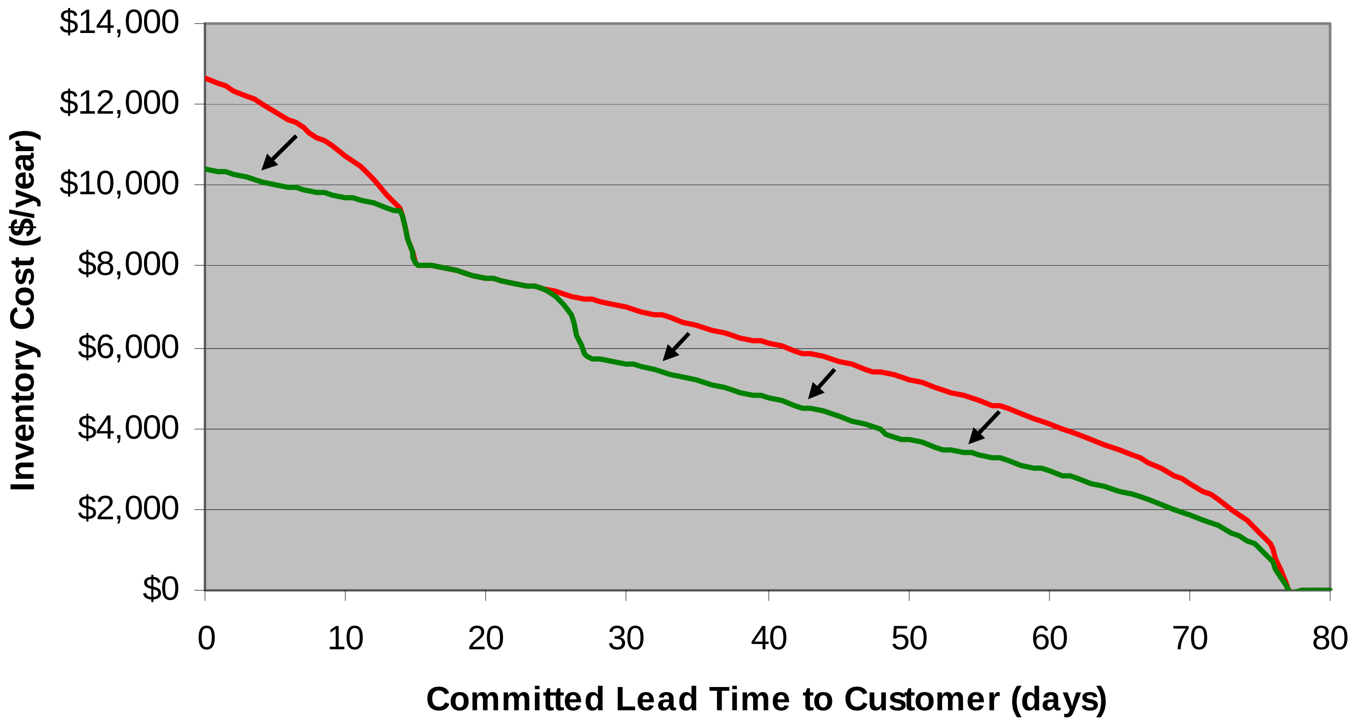
# CST vs. Inventory Cost





# Optimization Shifts the Tradeoff Curve





# Decentralized Systems

# Decentralized Systems



- We have assumed the system is centralized
  - Can optimize at all stages globally
  - One stage may incur higher costs to benefit the system as a whole
- What if each stage acts independently to minimize its own cost / maximize its own profit?

# Suboptimality



- Optimizing locally results in suboptimality
- Example: upstream stages want to operate make-to-order
  - Results in too much inventory downstream
- Another example:
  - Wholesaler chooses wholesale price
  - Retailer chooses order quantity
  - Optimizing independently, the two parties will always leave money on the table

# Supply Chain Contracts / Coordination

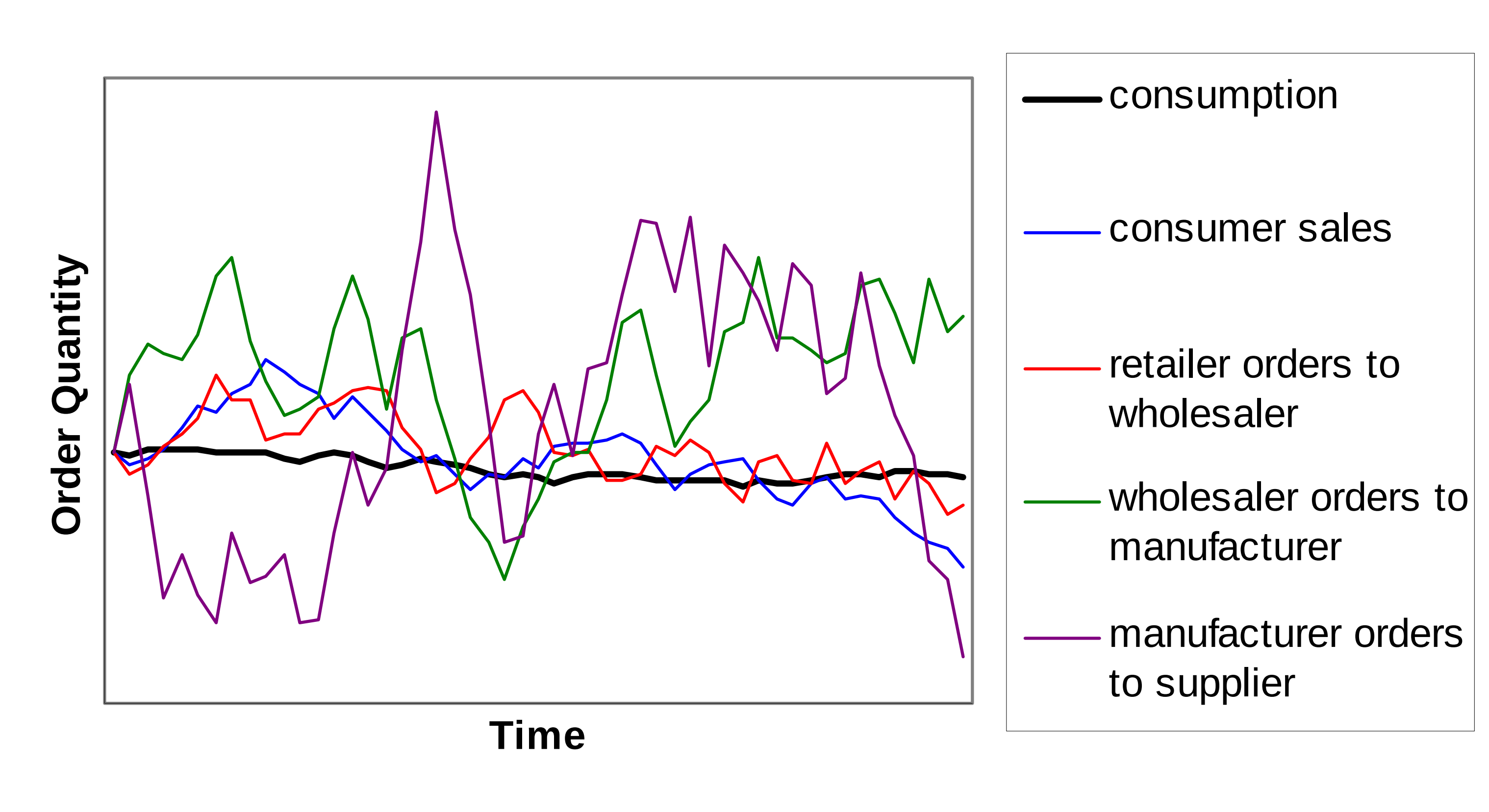


- One solution is for the parties to impose a *contracting mechanism*
  - Splits the costs / profits / risks / rewards
  - Still allows each party to act in its own best interest
  - If structured correctly, system achieves optimal cost / profit, even with parties acting selfishly
- There is a large body of literature on contracting
  - Review: Cachon 2003
  - Based on game theory
  - In practice, idea is commonly used
  - Actual OR models rarely implemented

# Bullwhip Effect (BWE)



- Demand for diapers:



# Irrational Behavior Causes BWE



- Firms over-react to demand signals
  - Order too much when they perceive an upward demand trend
  - Then back off when they accumulate too much inventory
- Firms under-weight the supply line
- Both are irrational behaviors
- Demonstrated by “beer game”

- Sterman 1989

# Rational Behavior Causes BWE



- BWE can be caused by rational behavior
  - i.e., by acting in “optimal” ways according to OR inventory models
- Four causes:
  - Demand forecast updating
  - Batch ordering
  - Rationing game
  - Price variations

- Lee, Padmanabhan, and Whang 1997

# Further Reading



- Single-stage and multi-echelon stochastic-service models:
  - Undergrad / MBA textbooks:
    - Simchi-Levi, Kaminsky, and Simchi-Levi, 3rd ed., 2007
    - Chopra and Meindl, 3rd ed., 2006
    - Nahmias, 5th ed., 2004
  - Graduate textbooks:
    - Zipkin, 2000
    - Axsäter, 2nd ed., 2006
    - Porteus, 2002
    - Simchi-Levi, Chen, and Bramel, 2nd ed., 2004
    - Silver, Pyke, and Peterson, 3rd ed., 1998
- Guaranteed-service models:
  - Graves and Willems 2003 (book chapter)

# Questions?



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