



# Mixed Integer Linear Programming with Python

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# Chapter 1

## Introduction

The Python-MIP package provides tools for modeling and solving [Mixed-Integer Linear Programming Problems \(MIPs\)](#) [[Wols98](#)] in Python. The default installation includes the [COIN-OR Linear Programming Solver - CLP](#), which is currently the [fastest](#) open source linear programming solver and the [COIN-OR Branch-and-Cut solver - CBC](#), a highly configurable MIP solver. It also works with the state-of-the-art [Gurobi](#) MIP solver. Python-MIP was written in modern, [typed Python](#) and works with the fast just-in-time Python compiler [Pypy](#).

In the modeling layer, models can be written very concisely, as in high-level mathematical programming languages such as [MathProg](#). Modeling examples for some applications can be viewed in [Chapter 4](#).

Python-MIP eases the development of high-performance MIP based solvers for custom applications by providing a tight integration with the branch-and-cut algorithms of the supported solvers. Strong formulations with an exponential number of constraints can be handled by the inclusion of [Cut Generators](#) and [Lazy Constraints](#). Heuristics can be integrated for [providing initial feasible solutions](#) to the MIP solver. These features can be used in both solver engines, CBC and GUROBI, without changing a single line of code.

This document is organized as follows: in the [next Chapter](#) installation and configuration instructions for different platforms are presented. In [Chapter 3](#) an overview of some common model creation and optimization code included. Commented examples are included in [Chapter 4](#). [Chapter 5](#) includes some common solver customizations that can be done to improve the performance of application specific solvers. Finally, the detailed reference information for the main classes is included in [Chapter 6](#).

### 1.1 Acknowledgments

We would like to thank for the support of the [Combinatorial Optimization and Decision Support \(CODeS\)](#) research group in [KU Leuven](#) through the senior research fellowship of Prof. Haroldo in 2018-2019, [CNPq](#) “[Produtividade em Pesquisa](#)” grant, [FAPEMIG](#) and the [GOAL](#) research group in the [Computing Department](#) of [UFOP](#).



# Chapter 2

## Installation

Python-MIP requires Python 3.5 or newer. Since Python-MIP is included in the [Python Package Index](#), once you have a [Python installation](#), installing it is as easy as entering in the command prompt:

```
pip install mip
```

If the command fails, it may be due to lack of permission to install globally available Python modules. In this case, use:

```
pip install mip --user
```

The default installation includes pre-compiled libraries of the MIP Solver [CBC](#) for Windows, Linux and MacOS. If you have the commercial solver [Gurobi](#) installed in your computer, Python-MIP will automatically use it as long as it finds the Gurobi dynamic loadable library. Gurobi is free for academic use and has an outstanding performance for solving MIPs. Instructions to make it accessible on different operating systems are included bellow.

### 2.1 Gurobi Installation and Configuration (optional)

For the installation of Gurobi you can look at the [Quickstart guide](#) for your operating system. Python-MIP will automatically find your Gurobi installation as long as you define the `GUROBI_HOME` environment variable indicating where Gurobi was installed.

### 2.2 Pypy installation (optional)

Python-MIP is compatible with the just-in-time Python compiler [Pypy](#). Generally, Python code executes much faster in Pypy. Pypy is also more memory efficient. To install Python-MIP as a Pypy package, just call (add `--user` may be necessary also):

```
pypy3 -m pip install mip
```

## 2.3 Using your own CBC binaries (optional)

Python-MIP provides CBC binaries for 64 bits versions of MacOS, Linux and Windows that run on Intel hardware. These binaries may not be suitable for you in some cases:

- a) if you plan to use Python-MIP in another platform, such as the Raspberry Pi, a 32 bits operating system or FreeBSD, for example;
- b) if you want to build CBC binaries with special optimizations for your hardware, i.e., using the `-march=native` option in GCC, you may also want to enable some optimizations for CLP, such as the use of the parallel AVX2 instructions, available in modern hardware;
- c) if you want use CBC binaries built with debug information, to help elucidating some bug.

In the [CBC page](#) there are instructions on how to build CBC from source on Unix like platforms and on Windows. `Coinbrew` is a script that makes it easier the task of downloading and building CBC and its dependencies. The commands bellow can be used to download and build CBC on Ubuntu Linux, slightly different packages names may be used in different distributions. Comments are included describing some possible customizations.

```
# install dependencies to build
sudo apt-get install gcc g++ gfortran libgfortran-9-dev liblapack-dev libamd2 libcholmod3
↳libmetis-dev libsuitesparse-dev libnauty2-dev git
# directory to download and compile CBC
mkdir -p ~/build ; cd ~/build
# download latest version of coinbrew
wget -nH https://raw.githubusercontent.com/coin-or/coinbrew/master/coinbrew
# download CBC and its dependencies with coinbrew
bash coinbrew fetch Cbc@master --no-prompt
# build, replace prefix with your install directory, add --enable-debug if necessary
bash coinbrew build Cbc@master --no-prompt --prefix=/home/haroldo/prog/ --tests=none --enable-
↳cbc-parallel --enable-relocatable
```

Python-MIP uses the `CbcSolver` shared library to communicate with CBC. In Linux, this file is named `libCbcSolver.so`, in Windows and MacOS the extension should be `.dll` and `.dylp`, respectively. To force Python-MIP to use your freshly compiled CBC binaries, you can set the `PMIP_CBC_LIBRARY` environment variable, indicating the full path to this shared library. In Linux, for example, if you installed your CBC binaries in `/home/haroldo/prog/`, you could use:

```
export PMIP_CBC_LIBRARY="/home/haroldo/prog/lib/libCbcSolver.so"
```

Please note that CBC uses multiple libraries which are installed in the same directory. You may also need to set one additional environment variable specifying that this directory also contains shared libraries that should be accessible. In Linux and MacOS this variable is `LD_LIBRARY_PATH`, on Windows the `PATH` environment variable should be set.

```
export LD_LIBRARY_PATH="/home/haroldo/prog/lib/:$LD_LIBRARY_PATH"
```

In Linux, to make these changes persistent, you may also want to add the `export` lines to your `.bashrc`.



# Chapter 3

## Quick start

This chapter presents the main components needed to build and optimize models using Python-MIP. A full description of the methods and their parameters can be found at [Chapter 4](#).

The first step to enable Python-MIP in your Python code is to add:

```
from mip import *
```

When loaded, Python-MIP will display its installed version:

```
Using Python-MIP package version 1.6.2
```

### 3.1 Creating Models

The model class represents the optimization model. The code below creates an empty Mixed-Integer Linear Programming problem with default settings.

```
m = Model()
```

By default, the optimization sense is set to *Minimize* and the selected solver is set to CBC. If Gurobi is installed and configured, it will be used instead. You can change the model objective sense or force the selection of a specific solver engine using additional parameters for the constructor:

```
m = Model(sense=MAXIMIZE, solver_name=CBC) # use GRB for Gurobi
```

After creating the model, you should include your decision variables, objective function and constraints. These tasks will be discussed in the next sections.

#### 3.1.1 Variables

Decision variables are added to the model using the `add_var()` method. Without parameters, a single variable with domain in  $\mathbb{R}^+$  is created and its reference is returned:

```
x = m.add_var()
```

By using Python list initialization syntax, you can easily create a vector of variables. Let's say that your model will have  $n$  binary decision variables ( $n=10$  in the example below) indicating if each one of 10 items is selected or not. The code below creates 10 binary variables  $y[0], \dots, y[n-1]$  and stores their references in a list.

```
n = 10  
y = [ m.add_var(var_type=BINARY) for i in range(n) ]
```

Additional variable types are `CONTINUOUS` (default) and `INTEGER`. Some additional properties that can be specified for variables are their lower and upper bounds (`lb` and `ub`, respectively), and names (property `name`). Naming a variable is optional and it is particularly useful if you plan to save your model (see *Saving, Loading and Checking Model Properties*) in `.LP` or `.MPS` file formats, for instance. The following code creates an integer variable named `zCost` which is restricted to be in range  $\{-10, \dots, 10\}$ . Note that the variable's reference is stored in a Python variable named `z`.

```
z = m.add_var(name='zCost', var_type=INTEGER, lb=-10, ub=10)
```

You don't need to store references for variables, even though it is usually easier to do so to write constraints. If you do not store these references, you can get them afterwards using the Model function `var_by_name()`. The following code retrieves the reference of a variable named `zCost` and sets its upper bound to 5:

```
vz = m.var_by_name('zCost')
vz.ub = 5
```

### 3.1.2 Constraints

Constraints are linear expressions involving variables, a sense of `==`, `<=` or `>=` for equal, less or equal and greater or equal, respectively, and a constant. The constraint  $x + y \leq 10$  can be easily included within model `m`:

```
m += x + y <= 10
```

Summation expressions can be implemented with the function `xsum()`. If for a knapsack problem with  $n$  items, each one with weight  $w_i$ , we would like to include a constraint to select items with binary variables  $x_i$  respecting the knapsack capacity  $c$ , then the following code could be used to include this constraint within the model `m`:

```
m += xsum(w[i]*x[i] for i in range(n)) <= c
```

Conditional inclusion of variables in the summation is also easy. Let's say that only even indexed items are subjected to the capacity constraint:

```
m += xsum(w[i]*x[i] for i in range(n) if i%2 == 0) <= c
```

Finally, it may be useful to name constraints. To do so is straightforward: include the constraint's name after the linear expression, separating it with a comma. An example is given below:

```
m += xsum(w[i]*x[i] for i in range(n) if i%2 == 0) <= c, 'even_sum'
```

As with variables, reference of constraints can be retrieved by their names. Model function `constr_by_name()` is responsible for this:

```
constraint = m.constr_by_name('even_sum')
```

### 3.1.3 Objective Function

By default a model is created with the *Minimize* sense. The following code alters the objective function to  $\sum_{i=0}^{n-1} c_i x_i$  by setting the `objective` attribute of our example model `m`:

```
m.objective = xsum(c[i]*x[i] for i in range(n))
```

To specify whether the goal is to *Minimize* or *Maximize* the objective function, two useful functions were included: `minimize()` and `maximize()`. Below are two usage examples:

```
m.objective = minimize(xsum(c[i]*x[i] for i in range(n)))
```

```
m.objective = maximize(xsum(c[i]*x[i] for i in range(n)))
```

You can also change the optimization direction by setting the *sense* model property to MINIMIZE or MAXIMIZE.

## 3.2 Saving, Loading and Checking Model Properties

Model methods *write()* and *read()* can be used to save and load, respectively, MIP models. Supported file formats for models are the LP file format, which is more readable and suitable for debugging, and the MPS file format, which is recommended for extended compatibility, since it is an older and more widely adopted format. When calling the *write()* method, the file name extension (.lp or .mps) is used to define the file format. Therefore, to save a model *m* using the lp file format to the file *model.lp* we can use:

```
m.write('model.lp')
```

Likewise, we can read a model, which results in creating variables and constraints from the LP or MPS file read. Once a model is read, all its attributes become available, like the number of variables, constraints and non-zeros in the constraint matrix:

```
m.read('model.lp')
print('model has {} vars, {} constraints and {} nzs'.format(m.num_cols, m.num_rows, m.num_nz))
```

## 3.3 Optimizing and Querying Optimization Results

MIP solvers execute a Branch-&-Cut (BC) algorithm that in *finite time* will provide the optimal solution. This time may be, in many cases, too large for your needs. Fortunately, even when the complete tree search is too expensive, results are often available in the beginning of the search. Sometimes a feasible solution is produced when the first tree nodes are processed and a lot of additional effort is spent improving the *dual bound*, which is a valid estimate for the cost of the optimal solution. When this estimate, the lower bound for minimization, matches exactly the cost of the best solution found, the upper bound, the search is concluded. For practical applications, usually a truncated search is executed. The *optimize()* method, that executes the optimization of a formulation, accepts optionally processing limits as parameters. The following code executes the branch-&-cut algorithm to solve a model *m* for up to 300 seconds.

```
1 m.max_gap = 0.05
2 status = m.optimize(max_seconds=300)
3 if status == OptimizationStatus.OPTIMAL:
4     print('optimal solution cost {} found'.format(m.objective_value))
5 elif status == OptimizationStatus.FEASIBLE:
6     print('sol.cost {} found, best possible: {}'.format(m.objective_value, m.objective_bound))
7 elif status == OptimizationStatus.NO_SOLUTION_FOUND:
8     print('no feasible solution found, lower bound is: {}'.format(m.objective_bound))
9 if status == OptimizationStatus.OPTIMAL or status == OptimizationStatus.FEASIBLE:
10    print('solution:')
11    for v in m.vars:
12        if abs(v.x) > 1e-6: # only printing non-zeros
13            print('{} : {}'.format(v.name, v.x))
```

Additional processing limits may be used: *max\_nodes* restricts the maximum number of explored nodes in the search tree and *max\_solutions* finishes the BC algorithm after a number of feasible solutions are obtained. It is also wise to specify how tight the bounds should be to conclude the search. The model

attribute `max_mip_gap` specifies the allowable percentage deviation of the upper bound from the lower bound for concluding the search. In our example, whenever the distance of the lower and upper bounds is less or equal 5% (see line 1), the search can be finished.

The `optimize()` method returns the status (`OptimizationStatus`) of the BC search: `OPTIMAL` if the search was concluded and the optimal solution was found; `FEASIBLE` if a feasible solution was found but there was no time to prove whether this solution was optimal or not; `NO_SOLUTION_FOUND` if in the truncated search no solution was found; `INFEASIBLE` or `INT_INFEASIBLE` if no feasible solution exists for the model; `UNBOUNDED` if there are missing constraints or `ERROR` if some error occurred during optimization. In the example above, if a feasible solution is available (line 8), variables which have value different from zero are printed. Observe also that even when no feasible solution is available the dual bound (lower bound in the case of minimization) is available (line 8): if a truncated execution was performed, i.e., the solver stopped due to the time limit, you can check this dual bound which is an estimate of the quality of the solution found checking the `gap` property.

During the tree search, it is often the case that many different feasible solutions are found. The solver engine stores this solutions in a solution pool. The following code prints all routes found while optimizing the *Traveling Salesman Problem*.

```
for k in range(model.num_solutions):
    print('route {} with length {}'.format(k, model.objective_values[k]))
    for (i, j) in product(range(n), range(n)):
        if x[i][j].xi(k) >= 0.98:
            print('\tarc ({} , {})' .format(i, j))
```

### 3.3.1 Performance Tuning

Tree search algorithms of MIP solvers deliver a set of improved feasible solutions and lower bounds. Depending on your application you will be more interested in the quick production of feasible solutions than in improved lower bounds that may require expensive computations, even if in the long term these computations prove worthy to prove the optimality of the solution found. The model property `emphasis` provides three different settings:

0. **default setting:** tries to balance between the search of improved feasible solutions and improved lower bounds;
1. **feasibility:** focus on finding improved feasible solutions in the first moments of the search process, activates heuristics;
2. **optimality:** activates procedures that produce improved lower bounds, focusing in pruning the search tree even if the production of the first feasible solutions is delayed.

Changing this setting to 1 or 2 triggers the activation/deactivation of several algorithms that are processed at each node of the search tree that impact the solver performance. Even though in average these settings change the solver performance as described previously, depending on your formulation the impact of these changes may be very different and it is usually worth to check the solver behavior with these different settings in your application.

Another parameter that may be worth tuning is the `cuts` attribute, that controls how much computational effort should be spent in generating cutting planes.

# Chapter 4

## Modeling Examples

This chapter includes commented examples on modeling and solving optimization problems with Python-MIP.

### 4.1 The 0/1 Knapsack Problem

As a first example, consider the solution of the 0/1 knapsack problem: given a set  $I$  of items, each one with a weight  $w_i$  and estimated profit  $p_i$ , one wants to select a subset with maximum profit such that the summation of the weights of the selected items is less or equal to the knapsack capacity  $c$ . Considering a set of decision binary variables  $x_i$  that receive value 1 if the  $i$ -th item is selected, or 0 if not, the resulting mathematical programming formulation is:

Maximize:

$$\sum_{i \in I} p_i \cdot x_i$$

Subject to:

$$\sum_{i \in I} w_i \cdot x_i \leq c$$

$$x_i \in \{0, 1\} \quad \forall i \in I$$

The following python code creates, optimizes and prints the optimal solution for the 0/1 knapsack problem

Listing 1: Solves the 0/1 knapsack problem: knapsack.py

```
1 from mip import Model, xsum, maximize, BINARY
2
3 p = [10, 13, 18, 31, 7, 15]
4 w = [11, 15, 20, 35, 10, 33]
5 c, I = 47, range(len(w))
6
7 m = Model("knapsack")
8
9 x = [m.add_var(var_type=BINARY) for i in I]
10
11 m.objective = maximize(xsum(p[i] * x[i] for i in I))
12
13 m += xsum(w[i] * x[i] for i in I) <= c
14
15 m.optimize()
16
```

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```

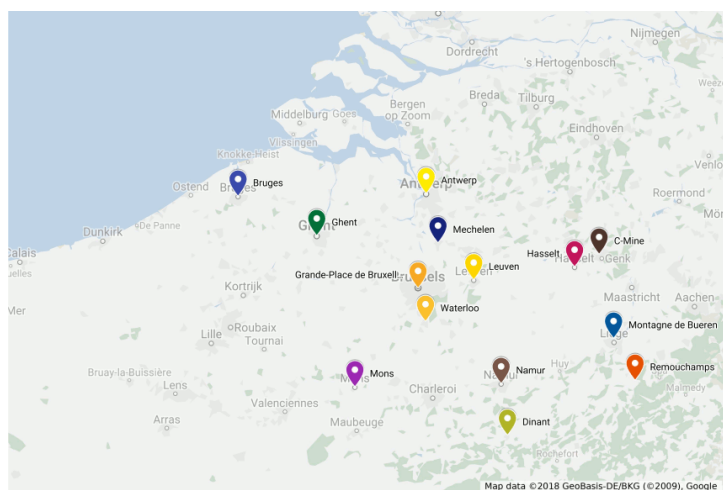
17 selected = [i for i in I if x[i].x >= 0.99]
18 print("selected items: {}".format(selected))
19

```

Line 3 imports the required classes and definitions from Python-MIP. Lines 5-8 define the problem data. Line 10 creates an empty maximization problem  $m$  with the (optional) name of “knapsack”. Line 12 adds the binary decision variables to model  $m$  and stores their references in a list  $x$ . Line 14 defines the objective function of this model and line 16 adds the capacity constraint. The model is optimized in line 18 and the solution, a list of the selected items, is computed at line 20.

## 4.2 The Traveling Salesman Problem

The traveling salesman problem (TSP) is one of the most studied combinatorial optimization problems, with the first computational studies dating back to the 50s [Dantz54], [Appleg06]. To illustrate this problem, consider that you will spend some time in Belgium and wish to visit some of its main tourist attractions, depicted in the map below:



You want to find the shortest possible tour to visit all these places. More formally, considering  $n$  points  $V = \{0, \dots, n-1\}$  and a distance matrix  $D_{n \times n}$  with elements  $c_{i,j} \in \mathbb{R}^+$ , a solution consists in a set of exactly  $n$  (origin, destination) pairs indicating the itinerary of your trip, resulting in the following formulation:

Minimize:

$$\sum_{i \in I, j \in I} c_{i,j} \cdot x_{i,j}$$

Subject to:

$$\sum_{j \in V \setminus \{i\}} x_{i,j} = 1 \quad \forall i \in V$$

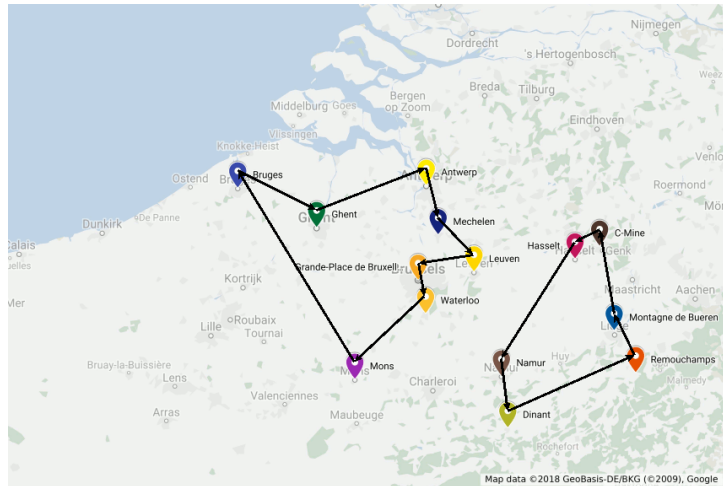
$$\sum_{i \in V \setminus \{j\}} x_{i,j} = 1 \quad \forall j \in V$$

$$y_i - (n+1) \cdot x_{i,j} \geq y_j - n \quad \forall i \in V \setminus \{0\}, j \in V \setminus \{0, i\}$$

$$x_{i,j} \in \{0, 1\} \quad \forall i \in V, j \in V$$

$$y_i \geq 0 \quad \forall i \in V$$

The first two sets of constraints enforce that we leave and arrive only once at each point. The optimal solution for the problem including only these constraints could result in a solution with sub-tours, such as the one below.



To enforce the production of connected routes, additional variables  $y_i \geq 0$  are included in the model indicating the sequential order of each point in the produced route. Point zero is arbitrarily selected as the initial point and conditional constraints linking variables  $x_{i,j}$ ,  $y_i$  and  $y_j$  are created for all nodes except the the initial one to ensure that the selection of the arc  $x_{i,j}$  implies that  $y_j \geq y_i + 1$ .

The Python code to create, optimize and print the optimal route for the TSP is included below:

Listing 2: Traveling salesman problem solver with compact formulation: tsp-compact.py

```

1 from itertools import product
2 from sys import stdout as out
3 from mip import Model, xsum, minimize, BINARY
4
5 # names of places to visit
6 places = ['Antwerp', 'Bruges', 'C-Mine', 'Dinant', 'Ghent',
7           'Grand-Place de Bruxelles', 'Hasselt', 'Leuven',
8           'Mechelen', 'Mons', 'Montagne de Bueren', 'Namur',
9           'Remouchamps', 'Waterloo']
10
11 # distances in an upper triangular matrix
12 dists = [[83, 81, 113, 52, 42, 73, 44, 23, 91, 105, 90, 124, 57],
13          [161, 160, 39, 89, 151, 110, 90, 99, 177, 143, 193, 100],
14          [90, 125, 82, 13, 57, 71, 123, 38, 72, 59, 82],
15          [123, 77, 81, 71, 91, 72, 64, 24, 62, 63],
16          [51, 114, 72, 54, 69, 139, 105, 155, 62],
17          [70, 25, 22, 52, 90, 56, 105, 16],
18          [45, 61, 111, 36, 61, 57, 70],
19          [23, 71, 67, 48, 85, 29],
20          [74, 89, 69, 107, 36],
21          [117, 65, 125, 43],
22          [54, 22, 84],
23          [60, 44],
24          [97],
25          []]
26
27 # number of nodes and list of vertices
28 n, V = len(dists), set(range(len(dists)))
29
30 # distances matrix
31 c = [[0 if i == j
32       else dists[i][j-i-1] if j > i
33       else dists[j][i-j-1]
34       for j in V] for i in V]
35

```

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```

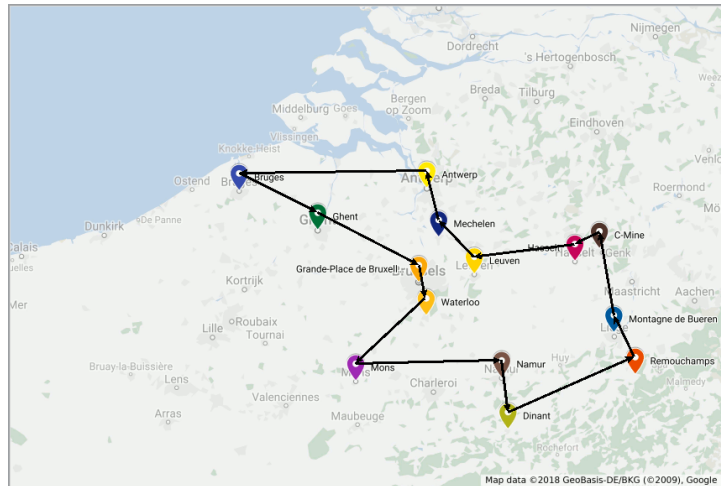
36 model = Model()
37
38 # binary variables indicating if arc (i,j) is used on the route or not
39 x = [[model.add_var(var_type=BINARY) for j in V] for i in V]
40
41 # continuous variable to prevent subtours: each city will have a
42 # different sequential id in the planned route except the first one
43 y = [model.add_var() for i in V]
44
45 # objective function: minimize the distance
46 model.objective = minimize(xsum(c[i][j]*x[i][j] for i in V for j in V))
47
48 # constraint : leave each city only once
49 for i in V:
50     model += xsum(x[i][j] for j in V - {i}) == 1
51
52 # constraint : enter each city only once
53 for i in V:
54     model += xsum(x[j][i] for j in V - {i}) == 1
55
56 # subtour elimination
57 for (i, j) in product(V - {0}, V - {0}):
58     if i != j:
59         model += y[i] - (n+1)*x[i][j] >= y[j]-n
60
61 # optimizing
62 model.optimize()
63
64 # checking if a solution was found
65 if model.num_solutions:
66     out.write('route with total distance %g found: %s'
67              % (model.objective_value, places[0]))
68     nc = 0
69     while True:
70         nc = [i for i in V if x[nc][i].x >= 0.99][0]
71         out.write(' -> %s' % places[nc])
72         if nc == 0:
73             break
74     out.write('\n')

```

In line 10 names of the places to visit are informed. In line 17 distances are informed in an upper triangular matrix. Line 33 stores the number of nodes and a list with nodes sequential ids starting from 0. In line 36 a full  $n \times n$  distance matrix is filled. Line 41 creates an empty MIP model. In line 44 all binary decision variables for the selection of arcs are created and their references are stored a  $n \times n$  matrix named  $x$ . Differently from the  $x$  variables,  $y$  variables (line 48) are not required to be binary or integral, they can be declared just as continuous variables, the default variable type. In this case, the parameter `var_type` can be omitted from the `add_var` call.

Line 51 sets the total traveled distance as objective function and lines 54-62 include the constraints. In line 66 we call the optimizer specifying a time limit of 30 seconds. This will surely not be necessary for our Belgium example, which will be solved instantly, but may be important for larger problems: even though high quality solutions may be found very quickly by the MIP solver, the time required to *prove* that the current solution is optimal may be very large. With a time limit, the search is truncated and the best solution found during the search is reported. In line 69 we check for the availability of a feasible solution. To repeatedly check for the next node in the route we check for the solution value (`.x` attribute) of all variables of outgoing arcs of the current node in the route (line 73). The optimal solution for our trip has length 547 and is depicted below:





### 4.3 n-Queens

In the  $n$ -queens puzzle  $n$  chess queens should to be placed in a board with  $n \times n$  cells in a way that no queen can attack another, i.e., there must be at most one queen per row, column and diagonal. This is a constraint satisfaction problem: any feasible solution is acceptable and no objective function is defined. The following binary programming formulation can be used to solve this problem:

$$\sum_{j=1}^n x_{ij} = 1 \quad \forall i \in \{1, \dots, n\}$$

$$\sum_{i=1}^n x_{ij} = 1 \quad \forall j \in \{1, \dots, n\}$$

$$\sum_{i=1}^n \sum_{j=1: i-j=k}^n x_{i,j} \leq 1 \quad \forall i \in \{1, \dots, n\}, k \in \{2-n, \dots, n-2\}$$

$$\sum_{i=1}^n \sum_{j=1: i+j=k}^n x_{i,j} \leq 1 \quad \forall i \in \{1, \dots, n\}, k \in \{3, \dots, n+n-1\}$$

$$x_{i,j} \in \{0, 1\} \quad \forall i \in \{1, \dots, n\}, j \in \{1, \dots, n\}$$

The following code builds the previous model, solves it and prints the queen placements:

Listing 3: Solver for the n-queens problem: queens.py

```

1 from sys import stdout
2 from mip import Model, xsum, BINARY
3
4 # number of queens
5 n = 40
6
7 queens = Model()
8
9 x = [[queens.add_var('x({}, {})'.format(i, j), var_type=BINARY)
10        for j in range(n)] for i in range(n)]
11
12 # one per row
13 for i in range(n):
14     queens += xsum(x[i][j] for j in range(n)) == 1, 'row({})'.format(i)
15
16 # one per column
17 for j in range(n):

```

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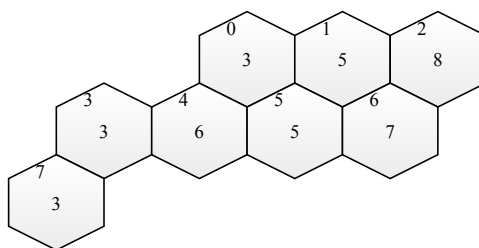
```

18     queens += xsum(x[i][j] for i in range(n)) == 1, 'col({})'.format(j)
19
20     # diagonal \
21     for p, k in enumerate(range(2 - n, n - 2 + 1)):
22         queens += xsum(x[i][i - k] for i in range(n)
23                        if 0 <= i - k < n) <= 1, 'diag1({})'.format(p)
24
25     # diagonal /
26     for p, k in enumerate(range(3, n + n)):
27         queens += xsum(x[i][k - i] for i in range(n)
28                        if 0 <= k - i < n) <= 1, 'diag2({})'.format(p)
29
30     queens.optimize()
31
32     if queens.num_solutions:
33         stdout.write('\n')
34         for i, v in enumerate(queens.vars):
35             stdout.write('0 ' if v.x >= 0.99 else '. ')
36             if i % n == n-1:
37                 stdout.write('\n')

```

## 4.4 Frequency Assignment

The design of wireless networks, such as cell phone networks, involves assigning communication frequencies to devices. These communication frequencies can be separated into channels. The geographical area covered by a network can be divided into hexagonal cells, where each cell has a base station that covers a given area. Each cell requires a different number of channels, based on usage statistics and each cell has a set of neighbor cells, based on the geographical distances. The design of an efficient mobile network involves selecting subsets of channels for each cell, avoiding interference between calls in the same cell and in neighboring cells. Also, for economical reasons, the total bandwidth in use must be minimized, i.e., the total number of different channels used. One of the first real cases discussed in literature are the Philadelphia [Ande73] instances, with the structure depicted below:



Each cell has a demand with the required number of channels drawn at the center of the hexagon, and a sequential id at the top left corner. Also, in this example, each cell has a set of at most 6 adjacent neighboring cells (distance 1). The largest demand (8) occurs on cell 2. This cell has the following adjacent cells, with distance 1: (1, 6). The minimum distances between channels in the same cell in this example is 3 and channels in neighbor cells should differ by at least 2 units.

A generalization of this problem (not restricted to the hexagonal topology), is the Bandwidth Multicoloring Problem (BMCP), which has the following input data:

$N$ : set of cells, numbered from 1 to  $n$ ;

$r_i \in \mathbb{Z}^+$ : demand of cell  $i \in N$ , i.e., the required number of channels;

$d_{i,j} \in \mathbb{Z}^+$ : minimum distance between channels assigned to nodes  $i$  and  $j$ ,  $d_{i,i}$  indicates the minimum distance between different channels allocated to the same cell.

Given an upper limit  $\bar{u}$  on the maximum number of channels  $U = \{1, \dots, \bar{u}\}$  used, which can be obtained using a simple greedy heuristic, the BMPC can be formally stated as the combinatorial optimization

problem of defining subsets of channels  $C_1, \dots, C_n$  while minimizing the used bandwidth and avoiding interference:

Minimize:

$$\max_{c \in C_1 \cup C_2, \dots, C_n} c$$

Subject to:

$$\begin{aligned} |c_1 - c_2| &\geq d_{i,j} \quad \forall (i,j) \in N \times N, (c_1, c_2) \in C_i \times C_j \\ C_i &\subseteq U \quad \forall i \in N \\ |C_i| &= r_i \quad \forall i \in N \end{aligned}$$

This problem can be formulated as a mixed integer program with binary variables indicating the composition of the subsets: binary variables  $x_{(i,c)}$  indicate if for a given cell  $i$  channel  $c$  is selected ( $x_{(i,c)} = 1$ ) or not ( $x_{(i,c)} = 0$ ). The BMCP can be modeled with the following MIP formulation:

Minimize:

$$z$$

Subject to:

$$\begin{aligned} \sum_{c=1}^{\bar{u}} x_{(i,c)} &= r_i \quad \forall i \in N \\ z &\geq c \cdot x_{(i,c)} \quad \forall i \in N, c \in U \\ x_{(i,c)} + x_{(j,c')} &\leq 1 \quad \forall (i,j,c,c') \in N \times N \times U \times U : i \neq j \wedge |c - c'| < d_{(i,j)} \\ x_{(i,c)} + x_{(i,c')} &\leq 1 \quad \forall i, c \in N \times U, c' \in \{c, +1, \dots, \min(c + d_{i,i}, \bar{u})\} \\ x_{(i,c)} &\in \{0, 1\} \quad \forall i \in N, c \in U \\ z &\geq 0 \end{aligned}$$

Follows the example of a solver for the BMCP using the previous MIP formulation:

Listing 4: Solver for the bandwidth multi coloring problem:  
bmc.py

```

1 from itertools import product
2 from mip import Model, xsum, minimize, BINARY
3
4 # number of channels per node
5 r = [3, 5, 8, 3, 6, 5, 7, 3]
6
7 # distance between channels in the same node (i, i) and in adjacent nodes
8 #   0 1 2 3 4 5 6 7
9 d = [[3, 2, 0, 0, 2, 2, 0, 0], # 0
10      [2, 3, 2, 0, 0, 2, 2, 0], # 1
11      [0, 2, 3, 0, 0, 0, 3, 0], # 2
12      [0, 0, 0, 3, 2, 0, 0, 2], # 3
13      [2, 0, 0, 2, 3, 2, 0, 0], # 4
14      [2, 2, 0, 0, 2, 3, 2, 0], # 5
15      [0, 2, 2, 0, 0, 2, 3, 0], # 6
16      [0, 0, 0, 2, 0, 0, 0, 3]] # 7
17
18 N = range(len(r))
19
20 # in complete applications this upper bound should be obtained from a feasible
21 # solution produced with some heuristic
22 U = range(sum(d[i][j] for (i, j) in product(N, N)) + sum(e1 for e1 in r))
23
24 m = Model()
25
26 x = [[m.add_var('x({}, {})'.format(i, c), var_type=BINARY)
27        for c in U] for i in N]
```

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```

28
29 z = m.add_var('z')
30 m.objective = minimize(z)
31
32 for i in N:
33     m += xsum(x[i][c] for c in U) == r[i]
34
35 for i, j, c1, c2 in product(N, N, U, U):
36     if i != j and c1 <= c2 < c1+d[i][j]:
37         m += x[i][c1] + x[j][c2] <= 1
38
39 for i, c1, c2 in product(N, U, U):
40     if c1 < c2 < c1+d[i][i]:
41         m += x[i][c1] + x[i][c2] <= 1
42
43 for i, c in product(N, U):
44     m += z >= (c+1)*x[i][c]
45
46 m.optimize(max_nodes=30)
47
48 if m.num_solutions:
49     for i in N:
50         print('Channels of node %d: %s' % (i, [c for c in U if x[i][c].x >=
51             0.99]))

```

## 4.5 Resource Constrained Project Scheduling

The Resource-Constrained Project Scheduling Problem (RCPSP) is a combinatorial optimization problem that consists of finding a feasible scheduling for a set of  $n$  jobs subject to resource and precedence constraints. Each job has a processing time, a set of successors jobs and a required amount of different resources. Resources may be scarce but are renewable at each time period. Precedence constraints between jobs mean that no jobs may start before all its predecessors are completed. The jobs must be scheduled non-preemptively, i.e., once started, their processing cannot be interrupted.

The RCPSP has the following input data:

$\mathcal{J}$  jobs set

$\mathcal{R}$  renewable resources set

$\mathcal{S}$  set of precedences between jobs  $(i, j) \in \mathcal{J} \times \mathcal{J}$

$\mathcal{T}$  planning horizon: set of possible processing times for jobs

$p_j$  processing time of job  $j$

$u_{(j,r)}$  amount of resource  $r$  required for processing job  $j$

$c_r$  capacity of renewable resource  $r$

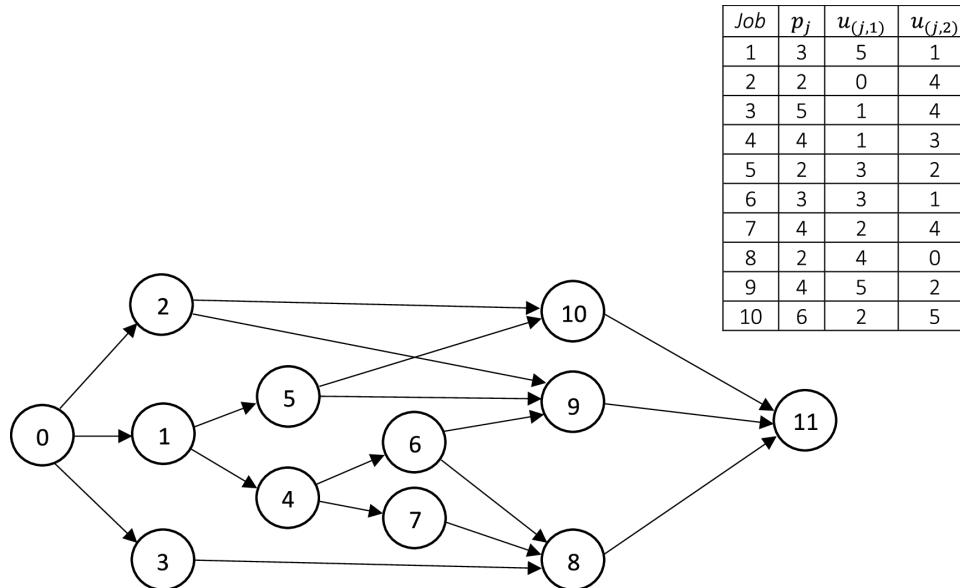
In addition to the jobs that belong to the project, the set  $\mathcal{J}$  contains jobs 0 and  $n+1$ , which are dummy jobs that represent the beginning and the end of the planning, respectively. The processing time for the dummy jobs is always zero and these jobs do not consume resources.

A binary programming formulation was proposed by Pritsker et al. [PWW69]. In this formulation, decision variables  $x_{jt} = 1$  if job  $j$  is assigned to begin at time  $t$ ; otherwise,  $x_{jt} = 0$ . All jobs must finish in a single instant of time without violating precedence constraints while respecting the amount of

available resources. The model proposed by Pristker can be stated as follows:

$$\begin{aligned}
 & \text{Minimize} \\
 & \quad \sum_{t \in \mathcal{T}} t \cdot x_{(n+1,t)} \\
 & \text{Subject to:} \\
 & \quad \sum_{t \in \mathcal{T}} x_{(j,t)} = 1 \quad \forall j \in J \\
 & \quad \sum_{j \in J} \sum_{t_2=t-p_j+1}^t u_{(j,r)} x_{(j,t_2)} \leq c_r \quad \forall t \in \mathcal{T}, r \in R \\
 & \quad \sum_{t \in \mathcal{T}} t \cdot x_{(s,t)} - \sum_{t \in \mathcal{T}} t \cdot x_{(j,t)} \geq p_j \quad \forall (j,s) \in S \\
 & \quad x_{(j,t)} \in \{0,1\} \quad \forall j \in J, t \in \mathcal{T}
 \end{aligned}$$

An instance is shown below. The figure shows a graph where jobs in  $\mathcal{J}$  are represented by nodes and precedence relations  $\mathcal{S}$  are represented by directed edges. The time-consumption  $p_j$  and all information concerning resource consumption  $u_{(j,r)}$  are included next to the graph. This instance contains 10 jobs and 2 renewable resources,  $\mathcal{R} = \{r_1, r_2\}$ , where  $c_1 = 6$  and  $c_2 = 8$ . Finally, a valid (but weak) upper bound on the time horizon  $\mathcal{T}$  can be estimated by summing the duration of all jobs.



The Python code for creating the binary programming model, optimize it and print the optimal scheduling for RCPSP is included below:

Listing 5: Solves the Resource Constrained Project Scheduling Problem: rcpsp.py

```

1 from itertools import product
2 from mip import Model, xsum, BINARY
3
4 n = 10 # note there will be exactly 12 jobs (n=10 jobs plus the two 'dummy' ones)
5
6 p = [0, 3, 2, 5, 4, 2, 3, 4, 2, 4, 6, 0]
7
8 u = [[0, 0], [5, 1], [0, 4], [1, 4], [1, 3], [3, 2], [3, 1], [2, 4],
9       [4, 0], [5, 2], [2, 5], [0, 0]]
10
11 c = [6, 8]
12
    
```

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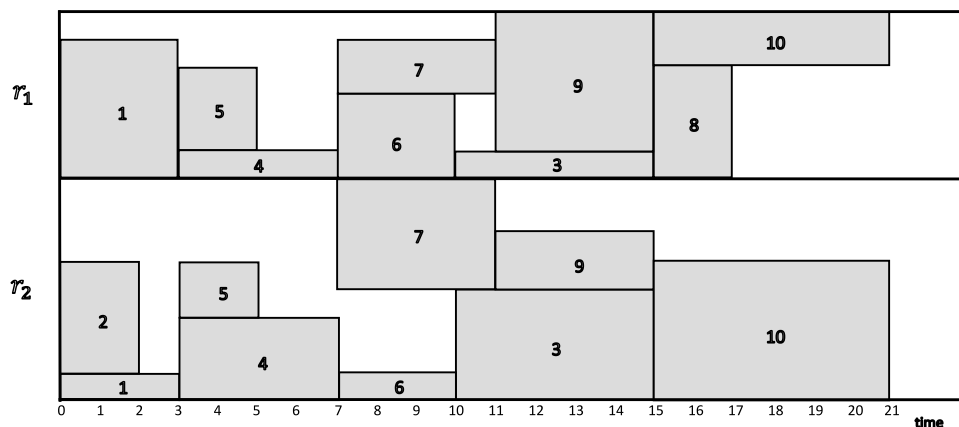
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```

13 S = [[0, 1], [0, 2], [0, 3], [1, 4], [1, 5], [2, 9], [2, 10], [3, 8], [4, 6],
14      [4, 7], [5, 9], [5, 10], [6, 8], [6, 9], [7, 8], [8, 11], [9, 11], [10, 11]]
15
16 (R, J, T) = (range(len(c)), range(len(p)), range(sum(p)))
17
18 model = Model()
19
20 x = [[model.add_var(name="x({}, {})".format(j, t), var_type=BINARY) for t in T] for j in J]
21
22 model.objective = xsum(t * x[n + 1][t] for t in T)
23
24 for j in J:
25     model += xsum(x[j][t] for t in T) == 1
26
27 for (r, t) in product(R, T):
28     model += (
29         xsum(u[j][r] * x[j][t2] for j in J for t2 in range(max(0, t - p[j] + 1), t + 1))
30         <= c[r])
31
32 for (j, s) in S:
33     model += xsum(t * x[s][t] - t * x[j][t] for t in T) >= p[j]
34
35 model.optimize()
36
37 print("Schedule: ")
38 for (j, t) in product(J, T):
39     if x[j][t].x >= 0.99:
40         print("Job {}: begins at t={} and finishes at t={}".format(j, t, t+p[j]))
41 print("Makespan = {}".format(model.objective_value))

```

One optimum solution is shown bellow, from the viewpoint of resource consumption.



It is noteworthy that this particular problem instance has multiple optimal solutions. Keep in the mind that the solver may obtain a different optimum solution.

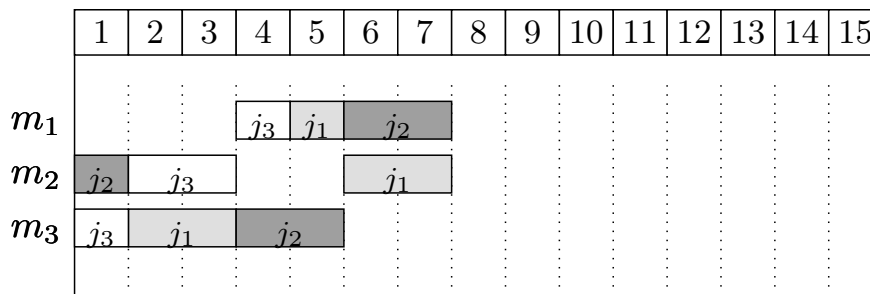
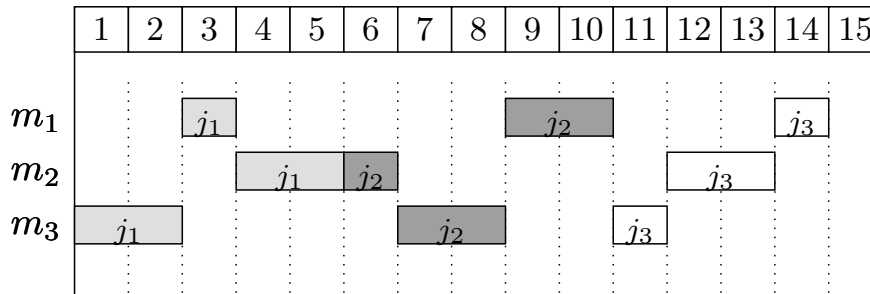
## 4.6 Job Shop Scheduling Problem

The Job Shop Scheduling Problem (JSSP) is an NP-hard problem defined by a set of jobs that must be executed by a set of machines in a specific order for each job. Each job has a defined execution time for each machine and a defined processing order of machines. Also, each job must use each machine only once. The machines can only execute a job at a time and once started, the machine cannot be interrupted until the completion of the assigned job. The objective is to minimize the makespan, i.e. the maximum completion time among all jobs.

For instance, suppose we have 3 machines and 3 jobs. The processing order for each job is as follows (the processing time of each job in each machine is between parenthesis):

- Job  $j_1$ :  $m_3$  (2)  $\rightarrow$   $m_1$  (1)  $\rightarrow$   $m_2$  (2)
- Job  $j_2$ :  $m_2$  (1)  $\rightarrow$   $m_3$  (2)  $\rightarrow$   $m_1$  (2)
- Job  $j_3$ :  $m_3$  (1)  $\rightarrow$   $m_2$  (2)  $\rightarrow$   $m_1$  (1)

Bellow there are two feasible schedules:



The first schedule shows a naive solution: jobs are processed in a sequence and machines stay idle quite often. The second solution is the optimal one, where jobs execute in parallel.

The JSSP has the following input data:

$\mathcal{J}$  set of jobs,  $\mathcal{J} = \{1, \dots, n\}$ ,

$\mathcal{M}$  set of machines,  $\mathcal{M} = \{1, \dots, m\}$ ,

$\sigma_r^j$  the machine that processes the  $r$ -th operation of job  $j$ , the sequence without repetition  $O^j = (\sigma_1^j, \sigma_2^j, \dots, \sigma_m^j)$  is the processing order of  $j$ ,

$p_{ij}$  non-negative integer processing time of job  $j$  in machine  $i$ .

A JSSP solution must respect the following constraints:

- All jobs  $j$  must be executed following the sequence of machines given by  $O^j$ ,
- Each machine can process only one job at a time,
- Once a machine starts a job, it must be completed without interruptions.

The objective is to minimize the makespan, the end of the last job to be executed. The JSSP is NP-hard for any fixed  $n \geq 3$  and also for any fixed  $m \geq 3$ .

The decision variables are defined by:

$x_{ij}$  starting time of job  $j \in J$  on machine  $i \in M$

$$y_{ijk} = \begin{cases} 1, & \text{if job } j \text{ precedes job } k \text{ on machine } i, \\ & i \in \mathcal{M}, j, k \in \mathcal{J}, j \neq k \\ 0, & \text{otherwise} \end{cases}$$

$C$  variable for the makespan

Follows a MIP formulation [Mann60] for the JSSP. The objective function is computed in the auxiliary variable  $C$ . The first set of constraints are the precedence constraints, that ensure that a job on a machine only starts after the processing of the previous machine concluded. The second and third set of disjunctive constraints ensure that only one job is processing at a given time in a given machine. The  $M$  constant must be large enough to ensure the correctness of these constraints. A valid (but weak) estimate for this value can be the summation of all processing times. The fourth set of constraints ensure that the makespan value is computed correctly and the last constraints indicate variable domains.

min:

$C$

s.t.:

$$\begin{aligned} x_{o_r^j} &\geq x_{o_{r-1}^j} + p_{o_{r-1}^j} \quad \forall r \in \{2, \dots, m\}, j \in \mathcal{J} \\ x_{ij} &\geq x_{ik} + p_{ik} - M \cdot y_{ijk} \quad \forall j, k \in \mathcal{J}, j \neq k, i \in \mathcal{M} \\ x_{ik} &\geq x_{ij} + p_{ij} - M \cdot (1 - y_{ijk}) \quad \forall j, k \in \mathcal{J}, j \neq k, i \in \mathcal{M} \\ C &\geq x_{o_m^j} + p_{o_m^j} \quad \forall j \in \mathcal{J} \\ x_{ij} &\geq 0 \quad \forall i \in \mathcal{J}, i \in \mathcal{M} \\ y_{ijk} &\in \{0, 1\} \quad \forall j, k \in \mathcal{J}, i \in \mathcal{M} \\ C &\geq 0 \end{aligned}$$

The following Python-MIP code creates the previous formulation, optimizes it and prints the optimal solution found:

Listing 6: Solves the Job Shop Scheduling Problem (examples/jssp.py)

```

1 from itertools import product
2 from mip import Model, BINARY
3
4 n = m = 3
5
6 times = [[2, 1, 2],
7           [1, 2, 2],
8           [1, 2, 1]]
9
10 M = sum(times[i][j] for i in range(n) for j in range(m))
11
12 machines = [[2, 0, 1],
13             [1, 2, 0],
14             [2, 1, 0]]
15
16 model = Model('JSSP')
17
18 c = model.add_var(name="C")
19 x = [[model.add_var(name='x({}, {})'.format(j+1, i+1))
20       for i in range(m)] for j in range(n)]
21 y = [[[model.add_var(var_type=BINARY, name='y({}, {}, {})'.format(j+1, k+1, i+1))

```

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```

22     for i in range(m)] for k in range(n)] for j in range(n)]
23
24 model.objective = c
25
26 for (j, i) in product(range(n), range(1, m)):
27     model += x[j][machines[j][i]] - x[j][machines[j][i-1]] >= \
28         times[j][machines[j][i-1]]
29
30 for (j, k) in product(range(n), range(n)):
31     if k != j:
32         for i in range(m):
33             model += x[j][i] - x[k][i] + M*y[j][k][i] >= times[k][i]
34             model += -x[j][i] + x[k][i] - M*y[j][k][i] >= times[j][i] - M
35
36 for j in range(n):
37     model += c - x[j][machines[j][m - 1]] >= times[j][machines[j][m - 1]]
38
39 model.optimize()
40
41 print("Completion time: ", c.x)
42 for (j, i) in product(range(n), range(m)):
43     print("task %d starts on machine %d at time %g " % (j+1, i+1, x[j][i].x))

```

## 4.7 Cutting Stock / One-dimensional Bin Packing Problem

The One-dimensional Cutting Stock Problem (also often referred to as One-dimensional Bin Packing Problem) is an NP-hard problem first studied by Kantorovich in 1939 [Kan60]. The problem consists of deciding how to cut a set of pieces out of a set of stock materials (paper rolls, metals, etc.) in a way that minimizes the number of stock materials used.

[Kan60] proposed an integer programming formulation for the problem, given below:

$$\begin{aligned}
 \min: & \sum_{j=1}^n y_j \\
 \text{s.t.}: & \sum_{j=1}^n x_{i,j} \geq b_i \quad \forall i \in \{1 \dots m\} \\
 & \sum_{i=1}^m w_i x_{i,j} \leq L y_j \quad \forall j \in \{1 \dots n\} \\
 & y_j \in \{0, 1\} \quad \forall j \in \{1 \dots n\} \\
 & x_{i,j} \in \mathbb{Z}^+ \quad \forall i \in \{1 \dots m\}, \forall j \in \{1 \dots n\}
 \end{aligned}$$

This formulation can be improved by including symmetry reducing constraints, such as:

$$y_{j-1} \geq y_j \quad \forall j \in \{2 \dots n\}$$

The following Python-MIP code creates the formulation proposed by [Kan60], optimizes it and prints the optimal solution found.

Listing 7: Formulation for the One-dimensional Cutting Stock Problem (examples/cuttingstock\_kantorovich.py)

```

1 from mip import Model, xsum, BINARY, INTEGER
2
3 n = 10 # maximum number of bars
4 L = 250 # bar length

```

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```

5 m = 4 # number of requests
6 w = [187, 119, 74, 90] # size of each item
7 b = [1, 2, 2, 1] # demand for each item
8
9 # creating the model
10 model = Model()
11 x = {(i, j): model.add_var(obj=0, var_type=INTEGER, name="x[%d,%d]" % (i, j))
12       for i in range(m) for j in range(n)}
13 y = {j: model.add_var(obj=1, var_type=BINARY, name="y[%d]" % j)
14       for j in range(n)}
15
16 # constraints
17 for i in range(m):
18     model.add_constr(xsum(x[i, j] for j in range(n)) >= b[i])
19 for j in range(n):
20     model.add_constr(xsum(w[i] * x[i, j] for i in range(m)) <= L * y[j])
21
22 # additional constraints to reduce symmetry
23 for j in range(1, n):
24     model.add_constr(y[j - 1] >= y[j])
25
26 # optimizing the model
27 model.optimize()
28
29 # printing the solution
30 print('')
31 print('Objective value: {model.objective_value:.3}'.format(**locals()))
32 print('Solution: ', end='')
33 for v in model.vars:
34     if v.x > 1e-5:
35         print('{v.name} = {v.x}'.format(**locals()))
36         print(' ', end='')

```

Note in the code above that argument `obj` was employed to create the variables (see lines 11 and 13). By setting `obj` to a value different than zero, the created variable is automatically added to the objective function with coefficient equal to `obj`'s value.

## 4.8 Two-Dimensional Level Packing

In some industries, raw material must be cut in several pieces of specified size. Here we consider the case where these pieces are *rectangular* [LMM02]. Also, due to machine operation constraints, pieces should be grouped horizontally such that firstly, horizontal layers are cut with the height of the largest item in the group and secondly, these horizontal layers are then cut according to items widths. Raw material is provided in rolls with large height. To minimize waste, a given batch of items must be cut using the minimum possible total height to minimize waste.

Formally, the following input data defines an instance of the Two Dimensional Level Packing Problem (TDLPP):

$W$  raw material width

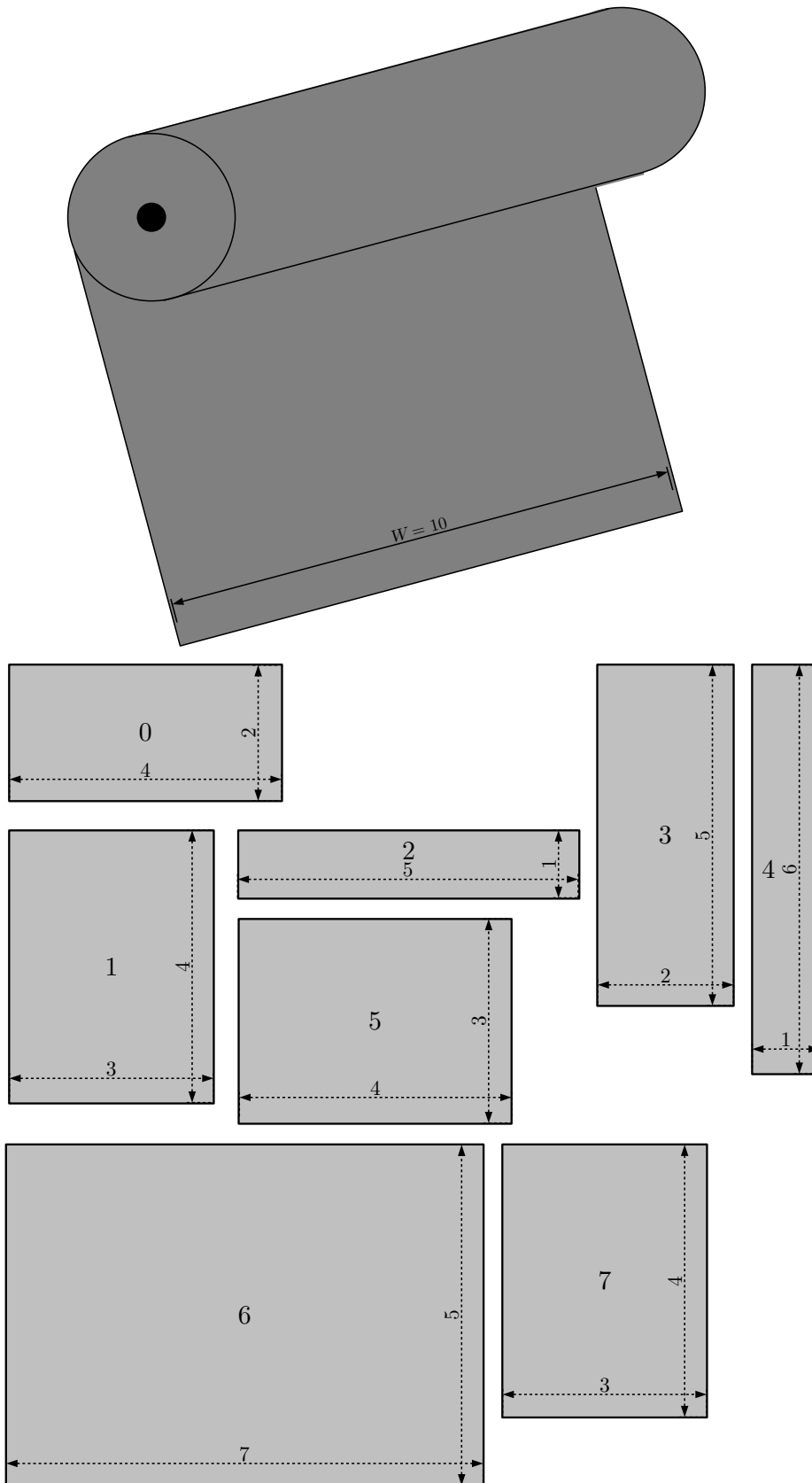
$n$  number of items

$I$  set of items =  $\{0, \dots, n - 1\}$

$w_i$  width of item  $i$

$h_i$  height of item  $i$

The following image illustrate a sample instance of the two dimensional level packing problem.



This problem can be formulated using binary variables  $x_{i,j} \in \{0,1\}$ , that indicate if item  $j$  should be grouped with item  $i$  ( $x_{i,j} = 1$ ) or not ( $x_{i,j} = 0$ ). Inside the same group, all elements should be linked to the largest element of the group, the *representative* of the group. If element  $i$  is the representative of the group, then  $x_{i,i} = 1$ .

Before presenting the complete formulation, we introduce two sets to simplify the notation.  $S_i$  is the set of items with width equal or smaller to item  $i$ , i.e., items for which item  $i$  can be the representative item. Conversely,  $G_i$  is the set of items with width greater or equal to the width of  $i$ , i.e., items which can be the representative of item  $i$  in a solution. More formally,  $S_i = \{j \in I : h_j \leq h_i\}$  and  $G_i = \{j \in I : h_j \geq h_i\}$ . Note that both sets include the item itself.

$$\begin{aligned} \min: & \sum_{i \in I} x_{i,i} \\ \text{s.t.}: & \sum_{j \in G_i} x_{i,j} = 1 \quad \forall i \in I \\ & \sum_{j \in S_i: j \neq i} x_{i,j} \leq (W - w_i) \cdot x_{i,i} \quad \forall i \in I \\ & x_{i,j} \in \{0, 1\} \quad \forall (i, j) \in I^2 \end{aligned}$$

The first constraints enforce that each item needs to be packed as the largest item of the set or to be included in the set of another item with width at least as large. The second set of constraints indicates that if an item is chosen as representative of a set, then the total width of the items packed within this same set should not exceed the width of the roll.

The following Python-MIP code creates and optimizes a model to solve the two-dimensional level packing problem illustrated in the previous figure.

Listing 8: Formulation for two-dimensional level packing packing  
(examples/two-dim-pack.py)

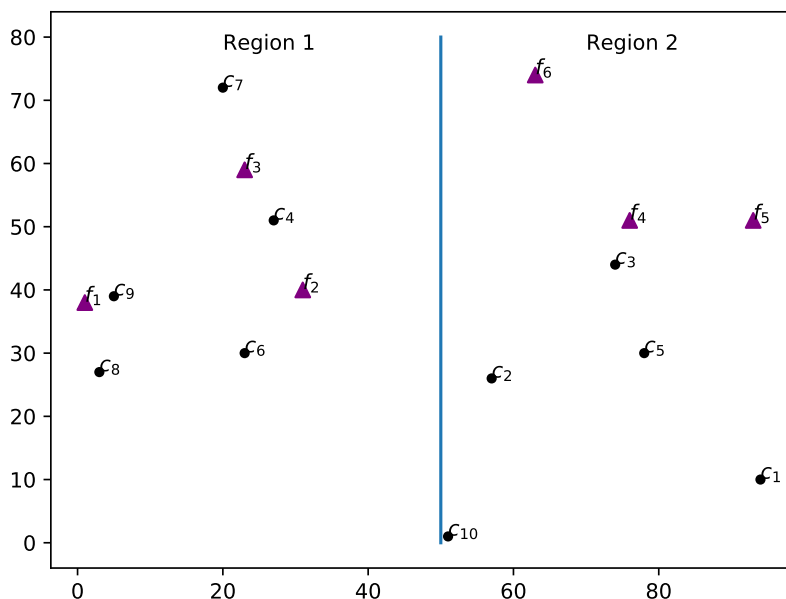
```

1 from mip import Model, BINARY, minimize, xsum
2
3 # 0 1 2 3 4 5 6 7
4 w = [4, 3, 5, 2, 1, 4, 7, 3] # widths
5 h = [2, 4, 1, 5, 6, 3, 5, 4] # heights
6 n = len(w)
7 I = set(range(n))
8 S = [[j for j in I if h[j] <= h[i]] for i in I]
9 G = [[j for j in I if h[j] >= h[i]] for i in I]
10
11 # raw material width
12 W = 10
13
14 m = Model()
15
16 x = [{j: m.add_var(var_type=BINARY) for j in S[i]} for i in I]
17
18 m.objective = minimize(xsum(h[i] * x[i][i] for i in I))
19
20 # each item should appear as larger item of the level
21 # or as an item which belongs to the level of another item
22 for i in I:
23     m += xsum(x[j][i] for j in G[i]) == 1
24
25 # represented items should respect remaining width
26 for i in I:
27     m += xsum(w[j] * x[i][j] for j in S[i] if j != i) <= (W - w[i]) * x[i][i]
28
29 m.optimize()
30
31 for i in [j for j in I if x[j][j].x >= 0.99]:
32     print(
33         "Items grouped with {} : {}".format(
34             i, [j for j in S[i] if i != j and x[i][j].x >= 0.99]
35         )
36     )

```

## 4.9 Plant Location with Non-Linear Costs

One industry plans to install two plants, one to the west (region 1) and another to the east (region 2). It must decide also the production capacity of each plant and allocate clients with different demands to plants in order to minimize shipping costs, which depend on the distance to the selected plant. Clients can be served by facilities of both regions. The cost of installing a plant with capacity  $z$  is  $f(z) = 1520 \log z$ . The Figure below shows the distribution of clients in circles and possible plant locations as triangles.



This example illustrates the use of *Special Ordered Sets* (SOS). We'll use Type 1 SOS to ensure that only one of the plants in each region has a non-zero production capacity. The cost  $f(z)$  of building a plant with capacity  $z$  grows according to the non-linear function  $f(z) = 1520 \log z$ . Type 2 SOS will be used to model the cost of installing each one of the plants in auxiliary variables  $y$ .

Listing 9: Plant location problem with non-linear costs handled with Special Ordered Sets

```

1 import matplotlib.pyplot as plt
2 from math import sqrt, log
3 from itertools import product
4 from mip import Model, xsum, minimize, OptimizationStatus
5
6 # possible plants
7 F = [1, 2, 3, 4, 5, 6]
8
9 # possible plant installation positions
10 pf = {1: (1, 38), 2: (31, 40), 3: (23, 59), 4: (76, 51), 5: (93, 51), 6: (63, 74)}
11
12 # maximum plant capacity
13 c = {1: 1955, 2: 1932, 3: 1987, 4: 1823, 5: 1718, 6: 1742}
14
15 # clients
16 C = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
17
18 # position of clients
19 pc = {1: (94, 10), 2: (57, 26), 3: (74, 44), 4: (27, 51), 5: (78, 30), 6: (23, 30),
20       7: (20, 72), 8: (3, 27), 9: (5, 39), 10: (51, 1)}

```

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```

21
22 # demands
23 d = {1: 302, 2: 273, 3: 275, 4: 266, 5: 287, 6: 296, 7: 297, 8: 310, 9: 302, 10: 309}
24
25 # plotting possible plant locations
26 for i, p in pf.items():
27     plt.scatter((p[0]), (p[1]), marker="^", color="purple", s=50)
28     plt.text((p[0]), (p[1]), "$f_%d$" % i)
29
30 # plotting location of clients
31 for i, p in pc.items():
32     plt.scatter((p[0]), (p[1]), marker="o", color="black", s=15)
33     plt.text((p[0]), (p[1]), "$c_%d$" % i)
34
35 plt.text((20), (78), "Region 1")
36 plt.text((70), (78), "Region 2")
37 plt.plot((50, 50), (0, 80))
38
39 dist = {(f, c): round(sqrt((pf[f][0] - pc[c][0]) ** 2 + (pf[f][1] - pc[c][1]) ** 2), 1)
40         for (f, c) in product(F, C) }
41
42 m = Model()
43
44 z = {i: m.add_var(ub=c[i]) for i in F} # plant capacity
45
46 # Type 1 SOS: only one plant per region
47 for r in [0, 1]:
48     # set of plants in region r
49     Fr = [i for i in F if r * 50 <= pf[i][0] <= 50 + r * 50]
50     m.add_sos([(z[i], i - 1) for i in Fr], 1)
51
52 # amount that plant i will supply to client j
53 x = {(i, j): m.add_var() for (i, j) in product(F, C)}
54
55 # satisfy demand
56 for j in C:
57     m += xsum(x[(i, j)] for i in F) == d[j]
58
59 # SOS type 2 to model installation costs for each installed plant
60 y = {i: m.add_var() for i in F}
61 for f in F:
62     D = 6 # nr. of discretization points, increase for more precision
63     v = [c[f] * (v / (D - 1)) for v in range(D)] # points
64     # non-linear function values for points in v
65     vn = [0 if k == 0 else 1520 * log(v[k]) for k in range(D)]
66     # w variables
67     w = [m.add_var() for v in range(D)]
68     m += xsum(w) == 1 # convexification
69     # link to z vars
70     m += z[f] == xsum(v[k] * w[k] for k in range(D))
71     # link to y vars associated with non-linear cost
72     m += y[f] == xsum(vn[k] * w[k] for k in range(D))
73     m.add_sos([(w[k], v[k]) for k in range(D)], 2)
74
75 # plant capacity
76 for i in F:
77     m += z[i] >= xsum(x[(i, j)] for j in C)
78
79 # objective function
80 m.objective = minimize(
81     xsum(dist[i, j] * x[i, j] for (i, j) in product(F, C)) + xsum(y[i] for i in F) )

```

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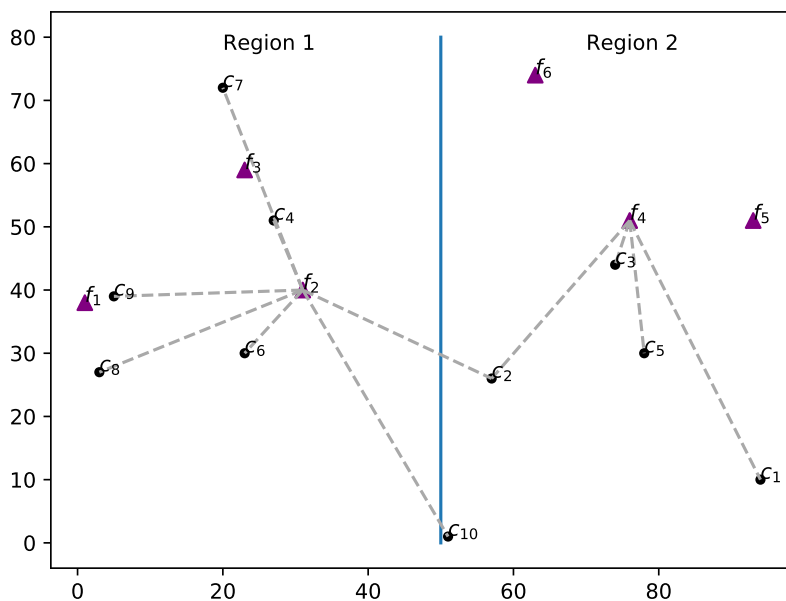
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```

82
83 m.optimize()
84
85 plt.savefig("location.pdf")
86
87 if m.num_solutions:
88     print("Solution with cost {} found.".format(m.objective_value))
89     print("Facilities capacities: {}".format([z[f].x for f in F]))
90     print("Facilities cost: {}".format([y[f].x for f in F]))
91
92     # plotting allocations
93     for (i, j) in [(i, j) for (i, j) in product(F, C) if x[(i, j)].x >= 1e-6]:
94         plt.plot(
95             (pf[i][0], pc[j][0]), (pf[i][1], pc[j][1]), linestyle="--", color="darkgray"
96         )
97
98     plt.savefig("location-sol.pdf")

```

The allocation of clients and plants in the optimal solution is shown below. This example uses Matplotlib to draw the Figures.







# Chapter 5

## Special Ordered Sets

Special Ordered Sets (SOSs) are ordered sets of variables, where only one/two contiguous variables in this set can assume non-zero values. Introduced in [BeTo70], they provide powerful means of modeling nonconvex functions [BeFo76] and can improve the performance of the branch-and-bound algorithm.

**Type 1 SOS (S1):** In this case, only one variable of the set can assume a non zero value. This variable may indicate, for example the site where a plant should be build. As the value of this non-zero variable would not have to be necessarily its upper bound, its value may also indicate the size of the plant.

**Type 2 SOS (S2):** In this case, up to two consecutive variables in the set may assume non-zero values. S2 are specially useful to model piecewise linear approximations of non-linear functions.

Given nonlinear function  $f(x)$ , a linear approximation can be computed for a set of  $k$  points  $x_1, x_2, \dots, x_k$ , using continuous variables  $w_1, w_2, \dots, w_k$ , with the following constraints:

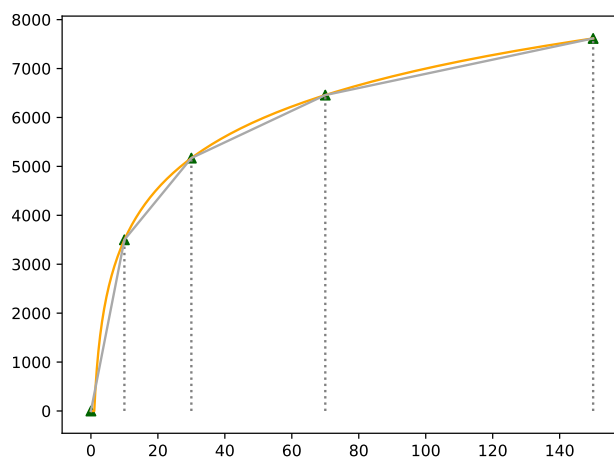
$$\sum_{i=1}^k w_i = 1$$
$$\sum_{i=1}^k x_i \cdot w_i = x$$

Thus, the result of  $f(x)$  can be approximate in  $z$ :

$$z = \sum_{i=1}^k f(x_i) \cdot w_i$$

Provided that at most two of the  $w_i$  variables are allowed to be non-zero and they are adjacent, which can be ensured by adding the pairs (variables, weight)  $\{(w_i, x_i) \forall i \in \{1, \dots, k\}\}$  to the model as a S2 set, using function `add_sos()`. The approximation is exact at the selected points and is adequately approximated by linear interpolation between them.

As an example, consider that the production cost of some product that due to some economy of scale phenomenon, is  $f(x) = 1520 * \log x$ . The graph bellow depicts the growing of  $f(x)$  for  $x \in [0, 150]$ . Triangles indicate selected discretization points for  $x$ . Observe that, in this case, the approximation (straight lines connecting the triangles) remains pretty close to the real curve using only 5 discretization points. Additional discretization points can be included, not necessarily evenly distributed, for an improved precision.



In this example, the approximation of  $z = 1520 \log x$  for points  $x = (0, 10, 30, 70, 150)$ , which correspond to  $z = (0, 3499.929, 5169.82, 6457.713, 7616.166)$  could be computed with the following constraints over  $x, z$  and  $w_1, \dots, w_5$  :

$$\begin{aligned}
 w_1 + w_2 + w_3 + w_4 + w_5 &= 1 \\
 x &= 0w_1 + 10w_2 + 30w_3 + 70w_4 + 150w_5 \\
 z &= 0w_1 + 3499.929w_2 + 5169.82w_3 + 6457.713w_4 + 7616.166w_5
 \end{aligned}$$

provided that  $\{(w_1, 0), (w_2, 10), (w_3, 30), (w_4, 70), (w_5, 150)\}$  is included as S2.

For a complete example showing the use of Type 1 and Type 2 SOS see [this example](#).

## Chapter 6

# Developing Customized Branch-&-Cut algorithms

This chapter discusses some features of Python-MIP that allow the development of improved Branch-&-Cut algorithms by linking application specific routines to the generic algorithm included in the solver engine. We start providing an introduction to cutting planes and cut separation routines in the next section, following with a section describing how these routines can be embedded in the Branch-&-Cut solver engine using the generic cut callbacks of Python-MIP.

### 6.1 Cutting Planes

In many applications there are **strong formulations** that may include an exponential number of constraints. These formulations cannot be direct handled by the MIP Solver: entering all these constraints at once is usually not practical, except for very small instances. In the **Cutting Planes** [Dantz54] method the LP relaxation is solved and only constraints which are *violated* are inserted. The model is re-optimized and at each iteration a stronger formulation is obtained until no more violated inequalities are found. The problem of discovering which are the missing violated constraints is also an optimization problem (finding *the most* violated inequality) and it is called the *Separation Problem*.

As an example, consider the Traveling Salesman Problem. The compact formulation (Section 4.2) is a *weak* formulation: dual bounds produced at the root node of the search tree are distant from the optimal solution cost and improving these bounds requires a potentially intractable number of branchings. In this case, the culprit are the sub-tour elimination constraints linking variables  $x$  and  $y$ . A much stronger TSP formulation can be written as follows: consider a graph  $G = (N, A)$  where  $N$  is the set of nodes and  $A$  is the set of directed edges with associated traveling costs  $c_a \in A$ . Selection of arcs is done with binary variables  $x_a \forall a \in A$ . Consider also that edges arriving and leaving a node  $n$  are indicated in  $A_n^+$  and  $A_n^-$ , respectively. The complete formulation follows:

Minimize:

$$\sum_{a \in A} c_a \cdot x_a$$

Subject to:

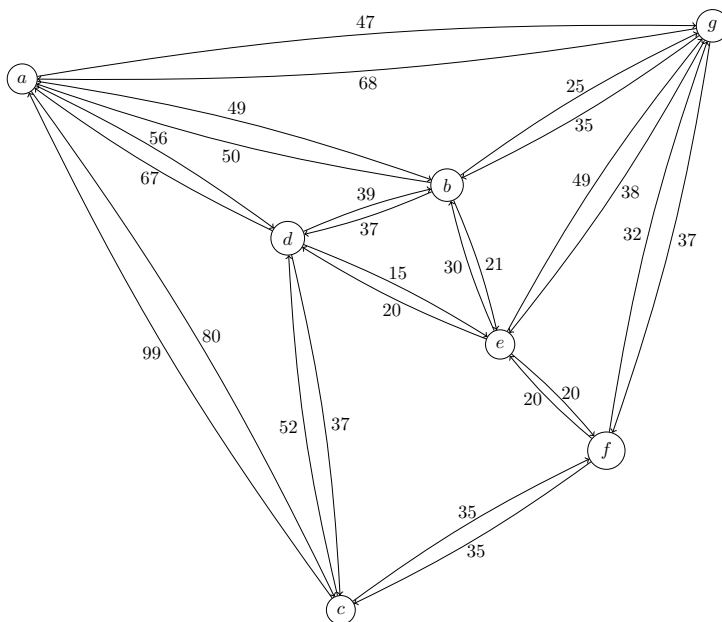
$$\sum_{a \in A_n^+} x_a = 1 \quad \forall n \in N$$

$$\sum_{a \in A_n^-} x_a = 1 \quad \forall n \in N$$

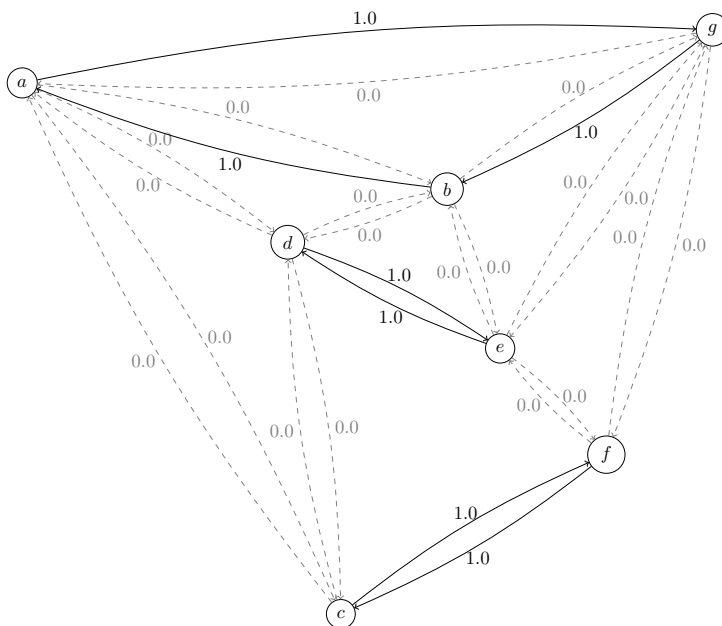
$$\sum_{(i,j) \in A: i \in S \wedge j \in S} x_{(i,j)} \leq |S| - 1 \quad \forall S \subset I$$

$$x_a \in \{0, 1\} \quad \forall a \in A$$

The third constraints are sub-tour elimination constraints. Since these constraints are stated for *every subset* of nodes, the number of these constraints is  $O(2^{|N|})$ . These are the constraints that will be separated by our cutting plane algorithm. As an example, consider the following graph:

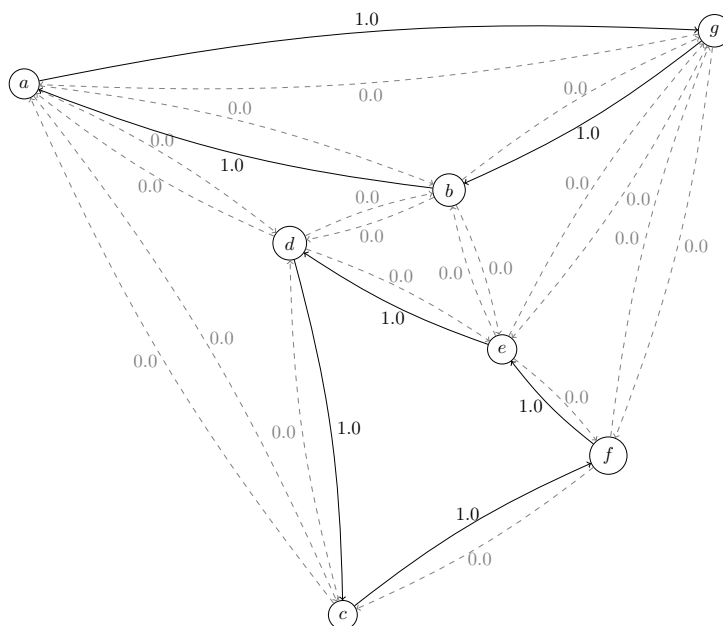


The optimal LP relaxation of the previous formulation without the sub-tour elimination constraints has cost 237:



As it can be seen, there are tree disconnected sub-tours. Two of these include only two nodes. Forbidding sub-tours of size 2 is quite easy: in this case we only need to include the additional constraints:  $x_{(d,e)} + x_{(e,d)} \leq 1$  and  $x_{(c,f)} + x_{(f,c)} \leq 1$ .

Optimizing with these two additional constraints the objective value increases to 244 and the following new solution is generated:



Now there are sub-tours of size 3 and 4. Let's consider the sub-tour defined by nodes  $S = \{a, b, g\}$ . The valid inequality for  $S$  is:  $x_{(a,g)} + x_{(g,a)} + x_{(a,b)} + x_{(b,a)} + x_{(b,g)} + x_{(g,b)} \leq 2$ . Adding this cut to our model increases the objective value to 261, a significant improvement. In our example, the visual identification of the isolated subset is easy, but how to automatically identify these subsets efficiently in the general case? Isolated subsets can be identified when a *cut* is found in the graph defined by arcs active in the unfeasible solution. To identify the most isolated subsets we just have to solve the [Minimum cut problem in graphs](#). In python you can use the [networkx min-cut module](#). The following code implements a cutting plane algorithm for the asymmetric traveling salesman problem:

Listing 1: A pure cutting-planes approach for the Traveling Salesman Problem (examples/cutting\_planes.py)

```

1
2 from itertools import product
3 from networkx import minimum_cut, DiGraph
4 from mip import Model, xsum, BINARY, OptimizationStatus, CutType
5
6 N = ["a", "b", "c", "d", "e", "f", "g"]
7 A = { ("a", "d"): 56, ("d", "a"): 67, ("a", "b"): 49, ("b", "a"): 50,
8       ("f", "c"): 35, ("g", "b"): 35, ("g", "b"): 35, ("b", "g"): 25,
9       ("a", "c"): 80, ("c", "a"): 99, ("e", "f"): 20, ("f", "e"): 20,
10      ("g", "e"): 38, ("e", "g"): 49, ("g", "f"): 37, ("f", "g"): 32,
11      ("b", "e"): 21, ("e", "b"): 30, ("a", "g"): 47, ("g", "a"): 68,
12      ("d", "c"): 37, ("c", "d"): 52, ("d", "e"): 15, ("e", "d"): 20,
13      ("d", "b"): 39, ("b", "d"): 37, ("c", "f"): 35, }
14 Aout = {n: [a for a in A if a[0] == n] for n in N}
15 Ain = {n: [a for a in A if a[1] == n] for n in N}
16
17 m = Model()
18 x = {a: m.add_var(name="x({}, {})".format(a[0], a[1]), var_type=BINARY) for a in A}
19
20 m.objective = xsum(c * x[a] for a, c in A.items())
21
22 for n in N:
23     m += xsum(x[a] for a in Aout[n]) == 1, "out({})".format(n)
24     m += xsum(x[a] for a in Ain[n]) == 1, "in({})".format(n)
25
26 newConstraints = True
27
28 while newConstraints:

```

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```

29 m.optimize(relax=True)
30 print("status: {} objective value : {}".format(m.status, m.objective_value))
31
32 G = DiGraph()
33 for a in A:
34     G.add_edge(a[0], a[1], capacity=x[a].x)
35
36 newConstraints = False
37 for (n1, n2) in [(i, j) for (i, j) in product(N, N) if i != j]:
38     cut_value, (S, NS) = minimum_cut(G, n1, n2)
39     if cut_value <= 0.99:
40         m += (xsum(x[a] for a in A if (a[0] in S and a[1] in S)) <= len(S) - 1)
41         newConstraints = True
42 if not newConstraints and m.solver_name.lower() == "cbc":
43     cp = m.generate_cuts([CutType.GOMORY, CutType.MIR,
44                          CutType.ZERO_HALF, CutType.KNAPSACK_COVER])
45     if cp.cuts:
46         m += cp
47     newConstraints = True

```

Lines 6-13 are the input data. Nodes are labeled with letters in a list  $N$  and a dictionary  $A$  is used to store the weighted directed graph. Lines 14 and 15 store output and input arcs per node. The mapping of binary variables  $x_a$  to arcs is made also using a dictionary in line 18. Line 20 sets the objective function and the following tree lines include constraints enforcing one entering and one leaving arc to be selected for each node. Line 29 will only solve the LP relaxation and the separation routine can be executed. Our separation routine is executed for each pair of nodes at line 38 and whenever a disconnected subset is found the violated inequality is generated and included at line 40. The process repeats while new violated inequalities are generated.

Python-MIP also supports the automatic generation of cutting planes, i.e., cutting planes that can be generated for any model just considering integrality constraints. Line 43 triggers the generation of these cutting planes with the method `generate_cuts()` when our sub-tour elimination procedure does not find violated sub-tour elimination inequalities anymore.

## 6.2 Cut Callback

The cutting plane method has some limitations: even though the first rounds of cuts improve significantly the lower bound, the overall number of iterations needed to obtain the optimal integer solution may be too large. Better results can be obtained with the [Branch-&-Cut algorithm](#), where cut generation is *combined* with branching. If you have an algorithm like the one included in the previous Section to separate inequalities for your application you can combine it with the complete BC algorithm implemented in the solver engine using *callbacks*. Cut generation callbacks (CGC) are called at each node of the search tree where a fractional solution is found. Cuts are generated in the callback and returned to the MIP solver engine which adds these cuts to the *Cut Pool*. These cuts are merged with the cuts generated with the solver builtin cut generators and a *subset* of these cuts is included to the LP relaxation model. Please note that in the Branch-&-Cut algorithm context cuts are *optional* components and only those that are classified as *good* cuts by the solver engine will be accepted, i.e., cuts that are too dense and/or have a small violation could be discarded, since the cost of solving a much larger linear program may not be worth the resulting bound improvement.

When using cut callbacks be sure that cuts are used only to *improve* the LP relaxation but not to *define* feasible solutions, which need to be defined by the initial formulation. In other words, the initial model without cuts may be *weak* but needs to be *complete*<sup>1</sup>. In the case of TSP, we can include the weak sub-tour elimination constraints presented in [Section 4.2](#) in the initial model and then add the stronger sub-tour elimination constraints presented in the previous section as cuts.

<sup>1</sup> If you want to initially enter an incomplete formulation than see the next sub-section on Lazy-Constraints.

In Python-MIP, CGC are implemented extending the *ConstrsGenerator* class. The following example implements the previous cut separation algorithm as a *ConstrsGenerator* class and includes it as a cut generator for the branch-and-cut solver engine. The method that needs to be implemented in this class is the *generate\_constrs()* procedure. This method receives as parameter the object *model* of type *Model*. This object must be used to query the fractional values of the model *vars*, using the *x* property. Other model properties can be queried, such as the problem constraints (*constrs*). Please note that, depending on which solver engine you use, some variables/constraints from the original model may have been removed in the pre-processing phase. Thus, direct references to the original problem variables may be invalid. The method *translate()* (line 15) translates references of variables from the original model to references of variables in the model received in the callback procedure. Whenever a violated inequality is discovered, it can be added to the model using the += operator (line 31). In our example, we temporarily store the generated cuts in a *CutPool* object (line 25) to discard repeated cuts that eventually are found.

Listing 2: Branch-and-cut for the traveling salesman problem  
(examples/tsp-cuts.py)

```

1 from typing import List, Tuple
2 from random import seed, randint
3 from itertools import product
4 from math import sqrt
5 import networkx as nx
6 from mip import Model, xsum, BINARY, minimize, ConstrsGenerator, CutPool
7
8
9 class SubTourCutGenerator(ConstrsGenerator):
10     """Class to generate cutting planes for the TSP"""
11     def __init__(self, F1: List[Tuple[int, int]], x_, V_):
12         self.F, self.x, self.V = F1, x_, V_
13
14     def generate_constrs(self, model: Model, depth: int = 0, npass: int = 0):
15         xf, V_, cp, G = model.translate(self.x, self.V, CutPool(), nx.DiGraph())
16         for (u, v) in [(k, l) for (k, l) in product(V_, V_) if k != l and xf[k][l]]:
17             G.add_edge(u, v, capacity=xf[u][v].x)
18         for (u, v) in F:
19             val, (S, NS) = nx.minimum_cut(G, u, v)
20             if val <= 0.99:
21                 aInS = [(xf[i][j], xf[i][j].x)
22                        for (i, j) in product(V_, V_) if i != j and xf[i][j] and i in S and j
↪in S]
23                 if sum(f for v, f in aInS) >= (len(S)-1)+1e-4:
24                     cut = xsum(1.0*v for v, fm in aInS) <= len(S)-1
25                     cp.add(cut)
26                     if len(cp.cuts) > 256:
27                         for cut in cp.cuts:
28                             model += cut
29                 return
30         for cut in cp.cuts:
31             model += cut
32
33
34 n = 30 # number of points
35 V = set(range(n))
36 seed(0)
37 p = [(randint(1, 100), randint(1, 100)) for i in V] # coordinates
38 Arcs = [(i, j) for (i, j) in product(V, V) if i != j]
39
40 # distance matrix
41 c = [[round(sqrt((p[i][0]-p[j][0])**2 + (p[i][1]-p[j][1])**2)) for j in V] for i in V]
42
43 model = Model()

```

(continues on next page)

(continued from previous page)

```

44
45 # binary variables indicating if arc (i,j) is used on the route or not
46 x = [[model.add_var(var_type=BINARY) for j in V] for i in V]
47
48 # continuous variable to prevent subtours: each city will have a
49 # different sequential id in the planned route except the first one
50 y = [model.add_var() for i in V]
51
52 # objective function: minimize the distance
53 model.objective = minimize(xsum(c[i][j]*x[i][j] for (i, j) in Arcs))
54
55 # constraint : leave each city only once
56 for i in V:
57     model += xsum(x[i][j] for j in V - {i}) == 1
58
59 # constraint : enter each city only once
60 for i in V:
61     model += xsum(x[j][i] for j in V - {i}) == 1
62
63 # (weak) subtour elimination
64 # subtour elimination
65 for (i, j) in product(V - {0}, V - {0}):
66     if i != j:
67         model += y[i] - (n+1)*x[i][j] >= y[j]-n
68
69 # no subtours of size 2
70 for (i, j) in Arcs:
71     model += x[i][j] + x[j][i] <= 1
72
73 # computing farthest point for each point, these will be checked first for
74 # isolated subtours
75 F, G = [], nx.DiGraph()
76 for (i, j) in Arcs:
77     G.add_edge(i, j, weight=c[i][j])
78 for i in V:
79     P, D = nx.dijkstra_predecessor_and_distance(G, source=i)
80     DS = list(D.items())
81     DS.sort(key=lambda x: x[1])
82     F.append((i, DS[-1][0]))
83
84 model.cuts_generator = SubTourCutGenerator(F, x, V)
85 model.optimize()
86
87 print('status: %s route length %g' % (model.status, model.objective_value))
88

```

## 6.3 Lazy Constraints

Python-MIP also supports the use of constraint generators to produce *lazy constraints*. Lazy constraints are dynamically generated, just as cutting planes, with the difference that lazy constraints are also applied to *integer solutions*. They should be used when the initial formulation is *incomplete*. In the case of our previous TSP example, this approach allow us to use in the initial formulation only the degree constraints and add all required sub-tour elimination constraints on demand. Auxiliary variables  $y$  would also not be necessary. The lazy constraints TSP example is exactly as the cut generator callback example with the difference that, besides starting with a smaller formulation, we have to inform that the constraint generator will be used to generate lazy constraints using the model property *lazy\_constrs\_generator*.



```

...
model.cuts_generator = SubTourCutGenerator(F, x, V)
model.lazy_constrs_generator = SubTourCutGenerator(F, x, V)
model.optimize()
...

```

## 6.4 Providing initial feasible solutions

The Branch-&-Cut algorithm usually executes faster with the availability of an integer feasible solution: an upper bound for the solution cost improves its ability of pruning branches in the search tree and this solution is also used in local search MIP heuristics. MIP solvers employ several heuristics for the automatic production of these solutions but they do not always succeed.

If you have some problem specific heuristic which can produce an initial feasible solution for your application then you can inform this solution to the MIP solver using the `start` model property. Let's consider our TSP application (Section 4.2). If the graph is complete, i.e. distances are available for each pair of cities, then *any* permutation  $\Pi = (\pi_1, \dots, \pi_n)$  of the cities  $N$  can be used as an initial feasible solution. This solution has exactly  $|N|$   $x$  variables equal to one indicating the selected arcs:  $((\pi_1, \pi_2), (\pi_2, \pi_3), \dots, (\pi_{n-1}, \pi_n), (\pi_n, \pi_1))$ . Even though this solution is obvious for the modeler, which knows that binary variables of this model refer to arcs in a TSP graph, this solution is not obvious for the MIP solver, which only sees variables and a constraint matrix. The following example enters an initial random permutation of cities as initial feasible solution for our TSP example, considering an instance with  $n$  cities, and a model `model` with references to variables stored in a matrix `x[0, ..., n-1][0, ..., n-1]`:

```

1 from random import shuffle
2 S=[i for i in range(n)]
3 shuffle(S)
4 model.start = [(x[S[k-1]][S[k]], 1.0) for k in range(n)]

```

The previous example can be integrated in our TSP example (Section 4.2) by inserting these lines before the `model.optimize()` call. Initial feasible solutions are informed in a list (line 4) of (`var`, `value`) pairs. Please note that only the original non-zero problem variables need to be informed, i.e., the solver will automatically compute the values of the auxiliary  $y$  variables which are used only to eliminate sub-tours.



# Chapter 7

## Benchmarks

This section presents computational experiments on the creation of Integer Programming models using different mathematical modelling packages. Gurobi is the default Gurobi<sup>®</sup> Python interface, which currently only supports the Python interpreter CPython. Pulp supports both CPython and also the just-in-time compiler Pypy. MIP also supports both. JuMP [DHL17] is the Mathematical Programming package of the Julia programming language. Both Jump and Pulp use intermediate data structures to store the mathematical programming model before flushing it to the solver, so that the selected solver does not impact on the model creation times. MIP does not store the model itself, directly calling problem creation/modification routines on the solver engine.

Since MIP communicates every problem modification directly to the solver engine, the engine must handle efficiently many small modification requests to avoid potentially expensive resize/move operations in the constraint matrix. Gurobi automatically buffers problem modification requests and has an update method to flush these requests. CBC did not have an equivalent mechanism, so we implemented an automatic buffering/flushing mechanism in the CBC C Interface. Our computational results already consider this updated CBC version.

Computational experiments executed on a ThinkPad<sup>®</sup> X1 notebook with an Intel<sup>®</sup> Core™ i7-7600U processor and 8 Gb of RAM using the Linux operating system. The following software releases were used: CPython 3.7.3, Pypy 7.1.1, Julia 1.1.1, JuMP 0.19 and Gurobi 8.1 and CBC svn 2563.

### 7.1 n-Queens

These are binary programming models. The largest model has 1 million variables and roughly 6000 constraints and 4 million of non-zero entries in the constraint matrix.

$n$	Gurobi CPython	Pulp		Python-MIP				JuMP
				Gurobi		CBC		
		CPython	Pypy	CPython	Pypy	CPython	Pypy	
100	0.24	0.27	<b>0.18</b>	0.65	0.97	0.30	0.45	2.38
200	1.43	1.73	0.43	1.60	<b>0.18</b>	1.47	0.19	0.25
300	4.97	5.80	0.92	5.48	<b>0.37</b>	5.12	0.38	0.47
400	12.37	14.29	1.55	13.58	<b>0.72</b>	13.24	0.74	1.11
500	24.70	32.62	2.62	27.25	1.25	26.30	<b>1.23</b>	2.09
600	43.88	49.92	4.10	47.75	2.02	46.23	<b>1.99</b>	2.86
700	69.77	79.57	5.84	75.97	3.04	74.47	<b>2.94</b>	3.64
800	105.04	119.07	8.19	114.86	4.33	112.10	<b>4.26</b>	5.58
900	150.89	169.92	10.84	163.36	5.95	160.67	<b>5.83</b>	8.08
1000	206.63	232.32	14.26	220.56	8.02	222.09	<b>7.76</b>	10.02

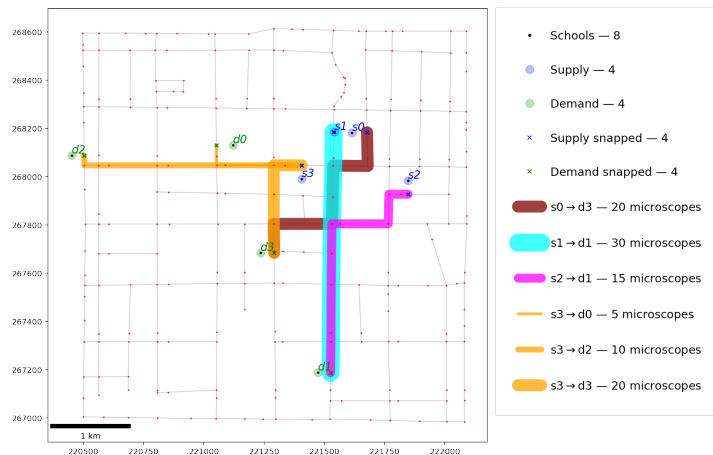


# Chapter 8

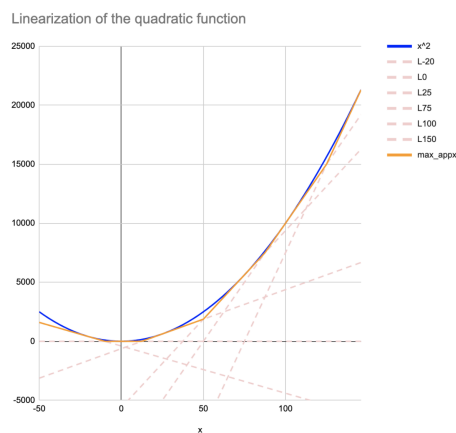
## External Documentation/Examples

This section includes links for external documentation and examples. Some documents provide information on how to integrate Python-MIP with other Python tools.

- [Network-constrained Transportation Problem.](#): provides a notebook for solving a transportation problem and integrating with Pysal, by James D. Gaboardi.



- [How to choose stocks to invest in with Python](#), by Khuyen Tran, includes an example of building an optimal multi-year investment plan in Python-MIP.
- [Solving a Quadratic Problem](#), by pabloazurduy.





# Chapter 9

## Classes

Classes used in solver callbacks, for a bi-directional communication with the solver engine

### 9.1 Model

```
class Model(name='', sense='MIN', solver_name='', solver=None)
    Mixed Integer Programming Model
```

This is the main class, providing methods for building, optimizing, querying optimization results and re-optimizing Mixed-Integer Programming Models.

To check how models are created please see the *examples* included.

**vars**  
list of problem variables (*Var*)  
**Type** *mip.VarList*

**constrs**  
list of constraints (*Constr*)  
**Type** *mip.ConstrList*

#### Examples

```
>>> from mip import Model, MAXIMIZE, CBC, INTEGER, OptimizationStatus
>>> model = Model(sense=MAXIMIZE, solver_name=CBC)
>>> x = model.add_var(name='x', var_type=INTEGER, lb=0, ub=10)
>>> y = model.add_var(name='y', var_type=INTEGER, lb=0, ub=10)
>>> model += x + y <= 10
>>> model.objective = x + y
>>> status = model.optimize(max_seconds=2)
>>> status == OptimizationStatus.OPTIMAL
True
```

```
add_constr(lin_expr, name='', priority=None)
    Creates a new constraint (row).
```

Adds a new constraint to the model, returning its reference.

#### Parameters

- `lin_expr` (*mip.LinearExpr*) – linear expression
- `name` (*str*) – optional constraint name, used when saving model to lp or mps files

- `priority` (*mip.constants.ConstraintPriority*) – optional constraint priority

Examples:

The following code adds the constraint  $x_1 + x_2 \leq 1$  (`x1` and `x2` should be created first using `add_var()`):

```
m += x1 + x2 <= 1
```

Which is equivalent to:

```
m.add_constr( x1 + x2 <= 1 )
```

Summation expressions can be used also, to add the constraint  $\sum_{i=0}^{n-1} x_i = y$  and name this constraint `cons1`:

```
m += xsum(x[i] for i in range(n)) == y, "cons1"
```

Which is equivalent to:

```
m.add_constr( xsum(x[i] for i in range(n)) == y, "cons1" )
```

**Return type** *mip.Constr*

`add_cut(cut)`

Adds a violated inequality (cutting plane) to the linear programming model. If called outside the cut callback performs exactly as `add_constr()`. When called inside the cut callback the cut is included in the solver's cut pool, which will later decide if this cut should be added or not to the model. Repeated cuts, or cuts which will probably be less effective, e.g. with a very small violation, can be discarded.

**Parameters** `cut` (*mip.LinExpr*) – violated inequality

`add_lazy_constr(expr)`

Adds a lazy constraint

A lazy constraint is a constraint that is only inserted into the model after the first integer solution that violates it is found. When lazy constraints are used a restricted pre-processing is executed since the complete model is not available at the beginning. If the number of lazy constraints is too large then they can be added during the search process by implementing a *ConstrsGenerator* and setting the property *lazy\_constrs\_generator* of *Model*.

**Parameters** `expr` (*mip.LinExpr*) – the linear constraint

`add_sos(sos, sos_type)`

Adds an Special Ordered Set (SOS) to the model

An explanation on Special Ordered Sets is provided [here](#).

**Parameters**

- `sos` (*List [ Tuple [ Var, numbers.Real ] ]*) – list including variables (not necessarily binary) and respective weights in the model
- `sos_type` (*int*) – 1 for Type 1 SOS, where at most one of the binary variables can be set to one and 2 for Type 2 SOS, where at most two variables from the list may be selected. In type 2 SOS the two selected variables will be consecutive in the list.

`add_var(name='', lb=0.0, ub=inf, obj=0.0, var_type='C', column=None)`

Creates a new variable in the model, returning its reference

**Parameters**



- `name` (*str*) – variable name (optional)
- `lb` (*numbers.Real*) – variable lower bound, default 0.0
- `ub` (*numbers.Real*) – variable upper bound, default infinity
- `obj` (*numbers.Real*) – coefficient of this variable in the objective function, default 0
- `var_type` (*str*) – CONTINUOUS (“C”), BINARY (“B”) or INTEGER (“I”)
- `column` (*mip.Column*) – constraints where this variable will appear, necessary only when constraints are already created in the model and a new variable will be created.

### Examples

To add a variable `x` which is continuous and greater or equal to zero to model `m`:

```
x = m.add_var()
```

The following code adds a vector of binary variables `x[0]`, ..., `x[n-1]` to the model `m`:

```
x = [m.add_var(var_type=BINARY) for i in range(n)]
```

**Return type** *mip.Var*

`add_var_tensor(shape, name, **kwargs)`

Creates new variables in the model, arranging them in a numpy tensor and returning its reference

#### Parameters

- `shape` (*Tuple[int, ..]*) – shape of the numpy tensor
- `name` (*str*) – variable name
- `**kwargs` – all other named arguments will be used as `add_var()` arguments

### Examples

To add a tensor of variables `x` with shape (3, 5) and which is continuous in any variable and have all values greater or equal to zero to model `m`:

```
x = m.add_var_tensor((3, 5), "x")
```

**Return type** *mip.LinExprTensor*

`check_optimization_results()`

Checks the consistency of the optimization results, i.e., if the solution(s) produced by the MIP solver respect all constraints and variable values are within acceptable bounds and are integral when requested.

`clear()`

Clears the model

All variables, constraints and parameters will be reset. In addition, a new solver instance will be instantiated to implement the formulation.

`property clique`

Controls the generation of clique cuts. -1 means automatic, 0 disables it, 1 enables it and 2 enables more aggressive clique generation.

**Return type** `int`

`clique_merge(constrs=None)`

This procedure searches for constraints with conflicting variables and attempts to group these constraints in larger constraints with all conflicts merged.

For example, if your model has the following constraints:

$$x_1 + x_2 \leq 1$$

$$x_2 + x_3 \leq 1$$

$$x_1 + x_3 \leq 1$$

Then they can all be removed and replaced by the stronger inequality:

$$x_1 + x_2 + x_3 \leq 1$$

**Parameters** `constrs` (*Optional* [*List* [*mip.Constr*]]) – constraints that should be checked for merging. All constraints will be checked if `constrs` is `None`.

property `conflict_graph`

Returns the *ConflictGraph* of a MIP model.

**Return type** *mip.ConflictGraph*

`constr_by_name(name)`

Queries a constraint by its name

**Parameters** `name` (*str*) – constraint name

**Return type** *Optional*[*mip.Constr*]

**Returns** constraint or `None` if not found

`copy(solver_name='')`

Creates a copy of the current model

**Parameters** `solver_name` (*str*) – solver name (optional)

**Return type** *Model*

**Returns** clone of current model

property `cut_passes`

Maximum number of rounds of cutting planes. You may set this parameter to low values if you see that a significant amount of time is being spent generating cuts without any improvement in the lower bound. -1 means automatic, values greater than zero specify the maximum number of rounds.

**Return type** `int`

property `cutoff`

upper limit for the solution cost, solutions with `cost > cutoff` will be removed from the search space, a small cutoff value may significantly speedup the search, but if cutoff is set to a value too low the model will become infeasible

**Return type** *Real*

property `cuts`

Controls the generation of cutting planes, -1 means automatic, 0 disables completely, 1 (default) generates cutting planes in a moderate way, 2 generates cutting planes aggressively and 3 generates even more cutting planes. Cutting planes usually improve the LP relaxation bound but also make the solution time of the LP relaxation larger, so the overall effect is hard to predict and experimenting different values for this parameter may be beneficial.

**Return type** `int`

**property cuts\_generator**

A cuts generator is an *ConstrsGenerator* object that receives a fractional solution and tries to generate one or more constraints (cuts) to remove it. The cuts generator is called in every node of the branch-and-cut tree where a solution that violates the integrality constraint of one or more variables is found.

**Return type** Optional[*mip.ConstrsGenerator*]

**property emphasis**

defines the main objective of the search, if set to 1 (FEASIBILITY) then the search process will focus on try to find quickly feasible solutions and improving them; if set to 2 (OPTIMALITY) then the search process will try to find a provable optimal solution, procedures to further improve the lower bounds will be activated in this setting, this may increase the time to produce the first feasible solutions but will probably pay off in longer runs; the default option is 0, where a balance between optimality and feasibility is sought.

**Return type** *mip.SearchEmphasis*

**property gap**

The optimality gap considering the cost of the best solution found (*objective\_value*)  $b$  and the best objective bound (*objective\_bound*)  $g$  is computed as:  $g = \frac{frac}{|b - l|} |b|$ . If no solution was found or if  $b = 0$  then  $g = \infty$ . If the optimal solution was found then  $g = 0$ .

**Return type** float

**generate\_cuts(*cut\_types=None, depth=0, npass=0, max\_cuts=8192, min\_viol=0.0001*)**

Tries to generate cutting planes for the current fractional solution. To optimize only the linear programming relaxation and not discard integrality information from variables you must call `first_model.optimize(relax=True)`.

This method only works with the CBC mip solver, as Gurobi does not supports calling only cut generators.

**Parameters**

- **cut\_types** (*List [CutType]*) – types of cuts that can be generated, if an empty list is specified then all available cut generators will be called.
- **depth** (*int*) – depth of the search tree, when informed the cut generator may decide to generate more/less cuts depending on the depth.
- **max\_cuts** (*int*) – cut separation will stop when at least `max_cuts` violated cuts were found.
- **min\_viol** (*float*) – cuts which are not violated by at least `min_viol` will be discarded.

**Return type** *mip.CutPool*

**property infeas\_tol**

Maximum allowed violation for constraints.

Default value: 1e-6. Tightening this value can increase the numerical precision but also probably increase the running time. As floating point computations always involve some loss of precision, values too close to zero will likely render some models impossible to optimize.

**Return type** float

**property integer\_tol**

Maximum distance to the nearest integer for a variable to be considered with an integer value. Default value: 1e-6. Tightening this value can increase the numerical precision but also probably increase the running time. As floating point computations always involve some loss of precision, values too close to zero will likely render some models impossible to optimize.

**Return type** float

property `lazy_constrs_generator`

A lazy constraints generator is an *ConstrsGenerator* object that receives an integer solution and checks its feasibility. If the solution is not feasible then one or more constraints can be generated to remove it. When a lazy constraints generator is informed it is assumed that the initial formulation is incomplete. Thus, a restricted pre-processing routine may be applied. If the initial formulation is incomplete, it may be interesting to use the same *ConstrsGenerator* to generate cuts *and* lazy constraints. The use of *only* lazy constraints may be useful then integer solutions rarely violate these constraints.

**Return type** Optional[*mip.ConstrsGenerator*]

property `lp_method`

Which method should be used to solve the linear programming problem. If the problem has integer variables that this affects only the solution of the first linear programming relaxation.

**Return type** *mip.LP\_Method*

property `max_mip_gap`

value indicating the tolerance for the maximum percentage deviation from the optimal solution cost, if a solution with cost  $c$  and a lower bound  $l$  are available and  $(c - l)/l < \text{max\_mip\_gap}$  the search will be concluded. Default value: 1e-4.

**Return type** float

property `max_mip_gap_abs`

Tolerance for the quality of the optimal solution, if a solution with cost  $c$  and a lower bound  $l$  are available and  $c - l < \text{mip\_gap\_abs}$ , the search will be concluded, see *max\_mip\_gap* to determine a percentage value. Default value: 1e-10.

**Return type** float

property `max_nodes`

maximum number of nodes to be explored in the search tree

**Return type** int

property `max_seconds`

time limit in seconds for search

**Return type** float

property `max_solutions`

solution limit, search will be stopped when `max_solutions` were found

**Return type** int

property `name`

The problem (instance) name.

This name should be used to identify the instance that this model refers, e.g.: production-PlanningMay19. This name is stored when saving (*write()*) the model in .LP or .MPS file formats.

**Return type** str

property `num_cols`

number of columns (variables) in the model

**Return type** int

property `num_int`

number of integer variables in the model

**Return type** int

property `num_nz`

number of non-zeros in the constraint matrix

**Return type** int

property `num_rows`  
 number of rows (constraints) in the model

**Return type** `int`

property `num_solutions`  
 Number of solutions found during the MIP search

**Return type** `int`

**Returns** number of solutions stored in the solution pool

property `objective`  
 The objective function of the problem as a linear expression.

### Examples

The following code adds all `x` variables `x[0]`, ..., `x[n-1]`, to the objective function of model `m` with the same cost `w`:

```
m.objective = xsum(w*x[i] for i in range(n))
```

A simpler way to define the objective function is the use of the model operator `+=`

```
m += xsum(w*x[i] for i in range(n))
```

Note that the only difference of adding a constraint is the lack of a sense and a rhs.

**Return type** `mip.LinExpr`

property `objective_bound`  
 A valid estimate computed for the optimal solution cost, lower bound in the case of minimization, equals to `objective_value` if the optimal solution was found.

**Return type** `Optional[Real]`

property `objective_const`  
 Returns the constant part of the objective function

**Return type** `float`

property `objective_value`  
 Objective function value of the solution found or `None` if model was not optimized

**Return type** `Optional[Real]`

property `objective_values`  
 List of costs of all solutions in the solution pool

**Return type** `List[Real]`

**Returns** costs of all solutions stored in the solution pool as an array from 0 (the best solution) to `num_solutions-1`.

property `opt_tol`  
 Maximum reduced cost value for a solution of the LP relaxation to be considered optimal. Default value: `1e-6`. Tightening this value can increase the numerical precision but also probably increase the running time. As floating point computations always involve some loss of precision, values too close to zero will likely render some models impossible to optimize.

**Return type** `float`

```
optimize(max_seconds=inf, max_nodes=1073741824, max_solutions=1073741824,
         max_seconds_same_incumbent=inf, max_nodes_same_incumbent=1073741824,
         relax=False)
```

Optimizes current model

Optimizes current model, optionally specifying processing limits.

To optimize model `m` within a processing time limit of 300 seconds:

```
m.optimize(max_seconds=300)
```

#### Parameters

- `max_seconds` (*numbers.Real*) – Maximum runtime in seconds (default: `inf`)
- `max_nodes` (*int*) – Maximum number of nodes (default: `inf`)
- `max_solutions` (*int*) – Maximum number of solutions (default: `inf`)
- `max_seconds_same_incumbent` (*numbers.Real*) – Maximum time in seconds that the search can go on if a feasible solution is available and it is not being improved
- `max_nodes_same_incumbent` (*int*) – Maximum number of nodes that the search can go on if a feasible solution is available and it is not being improved
- `relax` (*bool*) – if true only the linear programming relaxation will be solved, i.e. integrality constraints will be temporarily discarded.

**Returns** optimization status, which can be `OPTIMAL(0)`, `ERROR(-1)`, `INFEASIBLE(1)`, `UNBOUNDED(2)`. When optimizing problems with integer variables some additional cases may happen, `FEASIBLE(3)` for the case when a feasible solution was found but optimality was not proved, `INT_INFEASIBLE(4)` for the case when the lp relaxation is feasible but no feasible integer solution exists and `NO_SOLUTION_FOUND(5)` for the case when an integer solution was not found in the optimization.

**Return type** *mip.OptimizationStatus*

property `preprocess`

Enables/disables pre-processing. Pre-processing tries to improve your MIP formulation. -1 means automatic, 0 means off and 1 means on.

**Return type** *int*

property `pump_passes`

Number of passes of the Feasibility Pump [FGL05] heuristic. You may increase this value if you are not getting feasible solutions.

**Return type** *int*

method `read(path)`

Reads a MIP model or an initial feasible solution.

One of the following file name extensions should be used to define the contents of what will be loaded:

`.lp` mip model stored in the [LP file format](#)

`.mps` mip model stored in the [MPS file format](#)

`.sol` initial integer feasible solution

`.bas` [optimal basis](#) for the linear programming relaxation.

Note: if a new problem is readed, all variables, constraints and parameters from the current model will be cleared.

**Parameters** `path` (*str*) – file name

method `relax()`

Relax integrality constraints of variables

Changes the type of all integer and binary variables to continuous. Bounds are preserved.

`remove(objects)`

removes variable(s) and/or constraint(s) from the model

**Parameters** `objects` (`Union[mip.Var, mip.Constr, List[Union[mip.Var, mip.Constr]]]`) – can be a *Var*, a *Constr* or a list of these objects

property `round_int_vars`

MIP solvers perform computations using *limited precision* arithmetic. Thus a variable with value 0 may appear in the solution as 0.000000000001. Thus, comparing this var to zero would return false. The safest approach would be to use something like `abs(v.x) < 1e-7`. To simplify code the solution value of integer variables can be automatically rounded to the nearest integer and then, comparisons like `v.x == 0` would work. Rounding is not always a good idea specially in models with numerical instability, since it can increase the infeasibilities.

**Return type** `bool`

property `search_progress_log`

Log of bound improvements in the search. The output of MIP solvers is a sequence of improving incumbent solutions (primal bound) and estimates for the optimal cost (dual bound). When the costs of these two bounds match the search is concluded. In truncated searches, the most common situation for hard problems, at the end of the search there is a *gap* between these bounds. This property stores the detailed events of improving these bounds during the search process. Analyzing the evolution of these bounds you can see if you need to improve your solver w.r.t. the production of feasible solutions, by including an heuristic to produce a better initial feasible solution, for example, or improve the formulation with cutting planes, for example, to produce better dual bounds. To enable storing the *search\_progress\_log* set `store_search_progress_log` to True.

**Return type** `mip.ProgressLog`

property `seed`

Random seed. Small changes in the first decisions while solving the LP relaxation and the MIP can have a large impact in the performance, as discussed in [Fisch14]. This behaviour can be exploited with multiple independent runs with different random seeds.

**Return type** `int`

property `sense`

The optimization sense

**Return type** `str`

**Returns** the objective function sense, MINIMIZE (default) or (MAXIMIZE)

property `sol_pool_size`

Maximum number of solutions that will be stored during the search. To check how many solutions were found during the search use `num_solutions()`.

**Return type** `int`

property `start`

Initial feasible solution

Enters an initial feasible solution. Only the main binary/integer decision variables which appear with non-zero values in the initial feasible solution need to be informed. Auxiliary or continuous variables are automatically computed.

**Return type** Optional[List[Tuple[mip.Var, numbers.Real]]]

property `status`

optimization status, which can be OPTIMAL(0), ERROR(-1), INFEASIBLE(1), UNBOUNDED(2). When optimizing problems with integer variables some additional cases may happen, FEASIBLE(3) for the case when a feasible solution was found but optimality was not proved, INT\_INFEASIBLE(4) for the case when the lp relaxation is feasible but no feasible integer solution exists and NO\_SOLUTION\_FOUND(5) for the case when an integer solution was not found in the optimization.

**Return type** *mip.OptimizationStatus*

property `store_search_progress_log`

Whether *search\_progress\_log* will be stored or not when optimizing. Default False. Activate it if you want to analyze bound improvements over time.

**Return type** `bool`

property `threads`

number of threads to be used when solving the problem. 0 uses solver default configuration, -1 uses the number of available processing cores and  $\geq 1$  uses the specified number of threads. An increased number of threads may improve the solution time but also increases the memory consumption.

**Return type** `int`

`translate(ref)`

Translates references of variables/containers of variables from another model to this model. Can be used to translate references of variables in the original model to references of variables in the pre-processed model.

**Return type** `Union[List[Any], Dict[Any, Any], mip.Var]`

`validate_mip_start()`

Validates solution entered in MIPStart

If the solver engine printed messages indicating that the initial feasible solution that you entered in *start* is not valid then you can call this method to help discovering which set of variables is causing infeasibility. The current version is quite simple: the model is relaxed and one variable entered in mipstart is fixed per iteration, indicating if the model still feasible or not.

`var_by_name(name)`

Searchers a variable by its name

**Return type** `Optional[mip.Var]`

**Returns** Variable or None if not found

property `verbose`

0 to disable solver messages printed on the screen, 1 to enable

**Return type** `int`

`write(file_path)`

Saves a MIP model or an initial feasible solution.

One of the following file name extensions should be used to define the contents of what will be saved:

`.lp` mip model stored in the LP file format

`.mps` mip model stored in the MPS file format

`.sol` initial feasible solution

`.bas` optimal basis for the linear programming relaxation.

**Parameters** `file_path (str)` – file name



## 9.2 LinExpr

```
class LinExpr(variables=None, coeffs=None, const=0.0, sense="")
```

Linear expressions are used to enter the objective function and the model constraints. These expressions are created using operators and variables.

Consider a model object `m`, the objective function of `m` can be specified as:

```
m.objective = 10*x1 + 7*x4
```

In the example bellow, a constraint is added to the model

```
m += xsum(3*x[i] i in range(n)) - xsum(x[i] i in range(m))
```

A constraint is just a linear expression with the addition of a sense (`==`, `<=` or `>=`) and a right hand side, e.g.:

```
m += x1 + x2 + x3 == 1
```

If used in intermediate calculations, the solved value of the linear expression can be obtained with the `x` parameter, just as with a `Var`.

```
a = 10*x1 + 7*x4
print(a.x)
```

`add_const(val)`

adds a constant value to the linear expression, in the case of a constraint this correspond to the right-hand-side

**Parameters** `val` (*numbers.Real*) – a real number

`add_expr(expr, coeff=1)`

Extends a linear expression with the contents of another.

**Parameters**

- `expr` (*LinExpr*) – another linear expression
- `coeff` (*numbers.Real*) – coefficient which will multiply the linear expression added

`add_term(term, coeff=1)`

Adds a term to the linear expression.

**Parameters**

- `expr` (*Union [mip.Var, LinExpr, numbers.Real]*) – can be a variable, another linear expression or a real number.
- `coeff` (*numbers.Real*) – coefficient which will multiply the added term

`add_var(var, coeff=1)`

Adds a variable with a coefficient to the linear expression.

**Parameters**

- `var` (*mip.Var*) – a variable
- `coeff` (*numbers.Real*) – coefficient which the variable will be added

property `const`

constant part of the linear expression

**Return type** *Real*

`equals(other)`

returns true if a linear expression equals to another, false otherwise

**Return type** `bool`

property `expr`

the non-constant part of the linear expression

Dictionary with pairs: (variable, coefficient) where coefficient is a real number.

**Return type** `Dict[mip.Var, numbers.Real]`

property `model`

Model which this `LinExpr` refers to, `None` if no variables are involved.

**Return type** `Optional[mip.Model]`

property `sense`

sense of the linear expression

sense can be `EQUAL("=")`, `LESS_OR_EQUAL("<=")`, `GREATER_OR_EQUAL(">=")` or empty (`"`) if this is an affine expression, such as the objective function

**Return type** `str`

`set_expr(expr)`

Sets terms of the linear expression

**Parameters** `expr (Dict [mip.Var, numbers.Real])` – dictionary mapping variables to their coefficients in the linear expression.

property `violation`

Amount that current solution violates this constraint

If a solution is available, than this property indicates how much the current solution violates this constraint.

**Return type** `Optional[Real]`

property `x`

Value of this linear expression in the solution. `None` is returned if no solution is available.

**Return type** `Optional[Real]`

## 9.3 `LinExprTensor`

```
class LinExprTensor
```

## 9.4 `Var`

```
class Var(model, idx)
```

Decision variable of the `Model`. The creation of variables is performed calling the `add_var()`.

property `column`

Variable coefficients in constraints.

**Return type** `mip.Column`

property `lb`

Variable lower bound.

**Return type** `Real`

property `model`

Model which this variable refers to.

**Return type** `mip.Model`

- property name  
Variable name.  
**Return type** `str`
- property obj  
Coefficient of variable in the objective function.  
**Return type** `Real`
- property rc  
Reduced cost, only available after a linear programming model (only continuous variables) is optimized. Note that `None` is returned if no optimum solution is available  
**Return type** `Optional[Real]`
- property ub  
Variable upper bound.  
**Return type** `Real`
- property var\_type  
Variable type, ('B') BINARY, ('C') CONTINUOUS and ('I') INTEGER.  
**Return type** `str`
- property x  
Value of this variable in the solution. Note that `None` is returned if no solution is not available.  
**Return type** `Optional[Real]`
- $xi(i)$   
Value for this variable in the  $i$ -th solution from the solution pool. Note that `None` is returned if the solution is not available.  
**Return type** `Optional[Real]`

## 9.5 Constr

`class Constr(model, idx, priority=None)`

A row (constraint) in the constraint matrix.

A constraint is a specific *LinExpr* that includes a sense (<, > or == or less-or-equal, greater-or-equal and equal, respectively) and a right-hand-side constant value. Constraints can be added to the model using the overloaded operator += or using the method `add_constr()` of the *Model* class:

```
m += 3*x1 + 4*x2 <= 5
```

summation expressions are also supported:

```
m += xsum(x[i] for i in range(n)) == 1
```

property expr  
Linear expression that defines the constraint.

**Return type** `mip.LinExpr`

property name  
constraint name

**Return type** `str`

property pi  
Value for the dual variable of this constraint in the optimal solution of a linear programming *Model*. Only available if a pure linear programming problem was solved (only continuous variables).

**Return type** `Optional[Real]`

property `priority`  
priority value

**Return type** `ConstraintPriority`

property `rhs`  
The right-hand-side (constant value) of the linear constraint.

**Return type** `Real`

property `slack`  
Value of the slack in this constraint in the optimal solution. Available only if the formulation was solved.

**Return type** `Optional[Real]`

## 9.6 Column

```
class Column(constrs=None, coeffs=None)
```

A column contains all the non-zero entries of a variable in the constraint matrix. To create a variable see `add_var()`.

## 9.7 ConflictGraph

```
class ConflictGraph(model)
```

A conflict graph stores conflicts between incompatible assignments in binary variables.

For example, if there is a constraint  $x_1 + x_2 \leq 1$  then there is a conflict between  $x_1 = 1$  and  $x_2 = 1$ . We can state that  $x_1$  and  $x_2$  are conflicting. Conflicts can also involve the complement of a binary variable. For example, if there is a constraint  $x_1 \leq x_2$  then there is a conflict between  $x_1 = 1$  and  $x_2 = 0$ . We now can state that  $x_1$  and  $\neg x_2$  are conflicting.

```
conflicting(e1, e2)
```

Checks if two assignments of binary variables are in conflict.

**Parameters**

- `e1` (`Union [mip.LinExpr, mip.Var]`) – binary variable, if assignment to be tested is the assignment to one, or a linear expression like `x == 0` to indicate that conflict with the complement of the variable should be tested.
- `e2` (`Union [mip.LinExpr, mip.Var]`) – binary variable, if assignment to be tested is the assignment to one, or a linear expression like `x == 0` to indicate that conflict with the complement of the variable should be tested.

**Return type** `bool`

```
conflicting_assignments(v)
```

Returns from the conflict graph all assignments conflicting with one specific assignment.

**Parameters** `v` (`Union [mip.Var, mip.LinExpr]`) – binary variable, if assignment to be tested is the assignment to one or a linear expression like `x == 0` to indicate the complement.

**Return type** `Tuple[List[mip.Var], List[mip.Var]]`

**Returns** Returns a tuple with two lists. The first one indicates variables whose conflict occurs when setting them to one. The second list includes variable whose conflict occurs when setting them to zero.

## 9.8 VarList

```
class VarList(model)
```

List of model variables (*Var*).

The number of variables of a model *m* can be queried as `len(m.vars)` or as `m.num_cols`.

Specific variables can be retrieved by their indices or names. For example, to print the lower bounds of the first variable or of a variable named *z*, you can use, respectively:

```
print(m.vars[0].lb)
```

```
print(m.vars['z'].lb)
```

## 9.9 ConstrList

```
class ConstrList(model)
```

List of problem constraints

## 9.10 ConstrsGenerator

```
class ConstrsGenerator
```

Abstract class for implementing cuts and lazy constraints generators.

```
generate_constrs(model, depth=0, npass=0)
```

Method called by the solver engine to generate *cuts* or *lazy constraints*.

After analyzing the contents of the solution in model variables *vars*, whose solution values can be queried with the *x* attribute, one or more constraints may be generated and added to the solver with the `add_cut()` method for cuts. This method can be called by the solver engine in two situations, in the first one a fractional solution is found and one or more inequalities can be generated (cutting planes) to remove this fractional solution. In the second case an integer feasible solution is found and then a new constraint can be generated (lazy constraint) to report that this integer solution is not feasible. To control when the constraint generator will be called set your *ConstrsGenerator* object in the attributes `cuts_generator` or `lazy_constrs_generator` (adding to both is also possible).

### Parameters

- `model` (`mip.Model`) – model for which cuts may be generated. Please note that this model may have fewer variables than the original model due to pre-processing. If you want to generate cuts in terms of the original variables, one alternative is to query variables by their names, checking which ones remain in this pre-processed problem. In this procedure you can query model properties and add cuts (`add_cut()`) or lazy constraints (`add_lazy_constr()`), but you cannot perform other model modifications, such as add columns.
- `depth` (`int`) – depth of the search tree (0 is the root node)
- `npass` (`int`) – current number of cut passes in this node

## 9.11 IncumbentUpdater

`class IncumbentUpdater(model)`

To receive notifications whenever a new integer feasible solution is found. Optionally a new improved solution can be generated (using some local search heuristic) and returned to the MIP solver.

`update_incumbent(objective_value, best_bound, solution)`

Method that is called when a new integer feasible solution is found

### Parameters

- `objective_value` (*float*) – cost of the new solution found
- `best_bound` (*float*) – current lower bound for the optimal solution cost
- `solution` (*List [ Tuple [ mip.Var, float ] ]*) – non-zero variables in the solution

**Return type** `List[Tuple[mip.Var, float]]`

## 9.12 CutType

`class CutType(value)`

Types of cuts that can be generated. Each cut type is an implementation in the [COIN-OR Cut Generation Library](#). For some cut types multiple implementations are available. Sometimes these implementations were designed with different objectives: for the generation of Gomory cutting planes, for example, the GMI cuts are focused on numerical stability, while Forrest's implementation (GOMORY) is more integrated into the CBC code.

`CLIQUE = 12`

Clique cuts [Padb73].

`FLOW_COVER = 5`

Lifted Simple Generalized Flow Cover Cut Generator.

`GMI = 2`

Gomory Mixed Integer cuts [Gomo69], as implemented by Giacomo Nannicini, focusing on numerically safer cuts.

`GOMORY = 1`

Gomory Mixed Integer cuts [Gomo69], as implemented by John Forrest.

`KNAPSACK_COVER = 14`

Knapsack cover cuts [Bala75].

`LATWO_MIR = 8`

Lagrangean relaxation for two-phase Mixed-integer rounding cuts, as in LAGomory

`LIFT_AND_PROJECT = 9`

Lift-and-project cuts [BCC93], implemented by Pierre Bonami.

`MIR = 6`

Mixed-Integer Rounding cuts [Marc01].

`ODD_WHEEL = 13`

Lifted odd-hole inequalities.

`PROBING = 0`

Cuts generated evaluating the impact of fixing bounds for integer variables

`RED_SPLIT = 3`

Reduce and split cuts [AGY05], implemented by Francois Margot.

RED\_SPLIT\_G = 4  
Reduce and split cuts [AGY05], implemented by Giacomo Nannicini.

RESIDUAL\_CAPACITY = 10  
Residual capacity cuts [AtRa02], implemented by Francisco Barahona.

TWO\_MIR = 7  
Two-phase Mixed-integer rounding cuts.

ZERO\_HALF = 11  
Zero/Half cuts [Capr96].

## 9.13 CutPool

```
class CutPool
```

```
    add(cut)
        tries to add a cut to the pool, returns true if this is a new cut, false if it is a repeated one
```

**Parameters** *cut* (`mip.LinExpr`) – a constraint

**Return type** `bool`

## 9.14 OptimizationStatus

```
class OptimizationStatus(value)
```

Status of the optimization

```
CUTOFF = 7
    No feasible solution exists for the current cutoff
```

```
ERROR = -1
    Solver returned an error
```

```
FEASIBLE = 3
    An integer feasible solution was found during the search but the search was interrupted before
    concluding if this is the optimal solution or not.
```

```
INFEASIBLE = 1
    The model is proven infeasible
```

```
INT_INFEASIBLE = 4
    A feasible solution exist for the relaxed linear program but not for the problem with existing
    integer variables
```

```
LOADED = 6
    The problem was loaded but no optimization was performed
```

```
NO_SOLUTION_FOUND = 5
    A truncated search was executed and no integer feasible solution was found
```

```
OPTIMAL = 0
    Optimal solution was computed
```

```
UNBOUNDED = 2
    One or more variables that appear in the objective function are not included in binding
    constraints and the optimal objective value is infinity.
```

## 9.15 SearchEmphasis

```
class SearchEmphasis(value)
    An enumeration.

    DEFAULT = 0
        Default search emphasis, try to balance between improving the dual bound and producing
        integer feasible solutions.

    FEASIBILITY = 1
        More aggressive search for feasible solutions.

    OPTIMALITY = 2
        Focuses more on producing improved dual bounds even if the production of integer feasible
        solutions is delayed.
```

## 9.16 LP\_Method

```
class LP_Method(value)
    Different methods to solve the linear programming problem.

    AUTO = 0
        Let the solver decide which is the best method

    BARRIER = 3
        The barrier algorithm

    DUAL = 1
        The dual simplex algorithm

    PRIMAL = 2
        The primal simplex algorithm
```

## 9.17 ProgressLog

```
class ProgressLog
    Class to store the improvement of lower and upper bounds over time during the search. Results
    stored here are useful to analyze the performance of a given formulation/parameter setting for
    solving a instance. To be able to automatically generate summarized experimental results, fill the
    instance and settings of this object with the instance name and formulation/parameter setting
    details, respectively.

    log
        List of tuples in the format (time, (lb, ub)), where time is the processing time in seconds and
        lb and ub are the lower and upper bounds, respectively

        Type List[Tuple[float, Tuple[float, float]]]

    instance
        instance name

        Type str

    settings
        identification of the formulation/parameter settings used in the optimization (whatever is
        relevant to identify a given computational experiment)

        Type str

    read(file_name)
        Reads a progress log stored in a file
```



```
write(file_name='')
```

Saves the progress log. If no extension is informed, the `.plog` extension will be used. If only a directory is informed then the name will be built considering the `instance` and `settings` attributes

## 9.18 Exceptions

```
class MipBaseException
```

Base class for all exceptions specific to Python MIP. Only sub-classes of this exception are raised. Inherits from the Python builtin `Exception`.

```
class ProgrammingError
```

Exception that is raised when the calling program performs an invalid or nonsensical operation. Inherits from `mip.MipBaseException`.

```
class InterfacingError
```

Exception that is raised when an unknown error occurs while interfacing with a solver. Inherits from `mip.MipBaseException`.

```
class InvalidLinExpr
```

Exception that is raised when an invalid linear expression is created. Inherits from `mip.MipBaseException`.

```
class InvalidParameter
```

Exception that is raised when an invalid/non-existent parameter is used or set. Inherits from `mip.MipBaseException`.

```
class ParameterNotAvailable
```

Exception that is raised when some parameter is not available or can not be set. Inherits from `mip.MipBaseException`.

```
class InfeasibleSolution
```

Exception that is raised the produced solution is unfeasible. Inherits from `mip.MipBaseException`.

```
class SolutionNotAvailable
```

Exception that is raised when a method that requires a solution is queried but the solution is not available. Inherits from `mip.MipBaseException`.

## 9.19 Useful functions

```
minimize(objective)
```

Function that should be used to set the objective function to MINIMIZE a given linear expression (passed as argument).

**Parameters** `objective (Union [mip.LinExpr, Var])` – linear expression

**Return type** `mip.LinExpr`

```
maximize(objective)
```

Function that should be used to set the objective function to MAXIMIZE a given linear expression (passed as argument).

**Parameters** `objective (Union [mip.LinExpr, Var])` – linear expression

**Return type** `mip.LinExpr`

```
xsum(terms)
```

Function that should be used to create a linear expression from a summation. While the python function `sum()` can also be used, this function is optimized version for quickly generating the linear expression.

**Parameters** `terms` – set (ideally a list) of terms to be summed

**Return type** *mip.LinExpr*

`compute_features(model)`

This function computes instance features for a MIP. Features are instance characteristics, such as number of columns, rows, matrix density, etc. These features can be used in machine learning algorithms to recommend parameter settings. To check names of features that are computed in this vector use `features()`

**Parameters** `model` (`Model`) – the MIP model where features will be extracted

**Return type** `List[float]`

`features()`

This function returns the list of problem feature names that can be computed `compute_features()`

**Return type** `List[str]`

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