# MATHEMATICS FOR ENGINEERING

## DIFFERENTIATION

## TUTORIAL 1 - BASIC DIFFERENTIATION

This tutorial is essential pre-requisite material for anyone studying mechanical engineering. This tutorial uses the principle of learning by example. The approach is practical rather than purely mathematical and may be too simple for those who prefer pure maths.

Calculus is usually divided up into two parts, integration and differentiation. Each is the reverse process of the other. Integration is covered in tutorial 1.

On completion of this tutorial you should be able to do the following.

- Explain differential coefficients.
- Apply Newton's rules of differentiation to basic functions.
- Solve basic engineering problems involving differentiation.
- Define higher differential coefficients.
- Evaluate higher order differential coefficient.

# DIFFERENTIATION

## 1. DIFFERENTIAL COEFFICIENTS

Differentiation is the reverse process of integration but we will start this section by first defining a differential coefficient.

Remember that the symbol $\Delta$ means a finite change in something. Here are some examples.

| | |
|---|---|
| Temperature change | $\Delta T = T_2 - T_1$ |
| Change in time | $\Delta t = t_2 - t_1$ |
| Change in Angle | $\Delta\theta = \theta_2 - \theta_1$ |
| Change in distance | $\Delta x = x_2 - x_1$ |
| Change in velocity | $\Delta v = v_2 - v_1$ |

The symbol $\delta$ means a small but finite change in something such as $\delta T$, $\delta t$, $\delta\theta$, $\delta x$, $\delta v$ and so on.

Consider the following. The distance moved by an object is directly proportional to time t as shown on the graph.

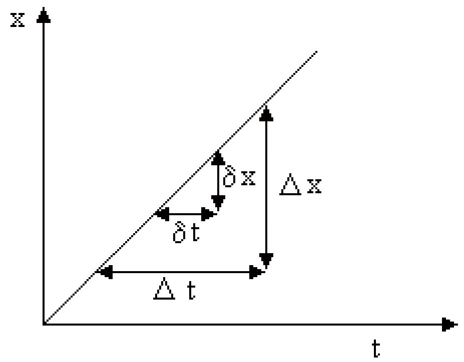



Figure 1

Velocity = Change in distance/change in time. $v = \Delta x/\Delta t$

This would be the same for a small change. $v = \delta x/\delta t = \Delta x/\Delta t$

The ratio $\Delta x/\Delta t$ is the same as the ratio $\delta x/\delta t$ and the ratio is the gradient of the straight line.

INFINITESIMALLY SMALL CHANGES'd'

The symbol d is used to denote a change that is infinitesimally small. On our graph the ratios are all the same and equal to the velocity. This value is the same at any point on a straight-line graph.

$$v = dx/dt = \delta x/\delta t = \Delta x/\Delta t.$$

This ratio holds true even when the changes approach zero.

Now consider the case when acceleration occurs and the velocity is changing. The graph of x against t is an upwards curve.

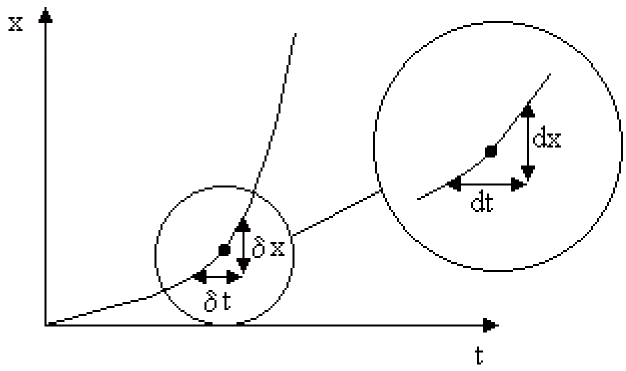



Figure 2

We can no longer say $v = \Delta x/\Delta t$ but $v = \delta x/\delta t$ might give a reasonable approximation if it is possible to measure the small changes. The result of evaluating $v = \delta x/\delta t$ would give the velocity at the time the measurements were made. At some other time the value would be different.

If we were able to take our measurements over smaller and smaller intervals, the velocity would become the instantaneous velocity. When the value of $\delta t$ tends to zero we would write $\delta t \rightarrow 0$. We would say that in the limit, as $\delta t \rightarrow 0$, the ratio $\delta x/\delta t$ becomes the differential coefficient (the true ratio) and we write it as dx/dt. It should be obvious now that the differential coefficient is the rate of change of one variable with another and is also the gradient of the graph at a given point.

$v = dx/dt$ gives the precise velocity at an instant in time but of course we could not find it by measuring dx and dt. Newton's calculus method allows us to find these differential coefficients if we have a mathematical equation linking the two variables.

When a body accelerates at 'a' $m/s^2$ the formula relating distance and time is $x = a\,t^2/2$.

The velocity is the ratio dx/dt and it may be found at any moment in time by applying Newton's rules for differentiation.

Remember **Differentiation gives the gradient of the function.**

## 2. NEWTON'S METHOD

### 2.1 DIFFERENTIATION OF AN ALGEBRAIC EXPRESSION

The equation x = a $t^2/2$ is an example of an algebraic equation. In general we use x and y and a general equation may be written as $y = Cx^n$ where 'C' is a constant and 'n' is a power or index. The rule for differentiating is :

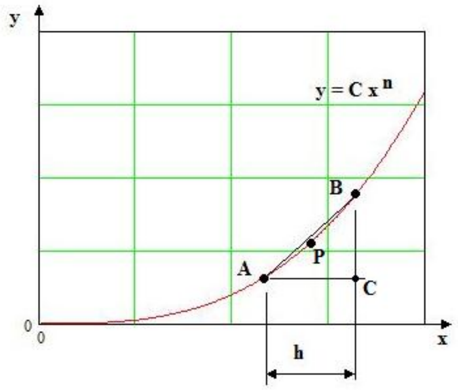
$$\mathbf{dy/dx = Cnx^{(n-1)} \text{ or } dy = Cnx^{(n-1)}\, dx}$$

Note that integrating returns the equation back to its original form.

**DERIVATION**

For those who want to know where this comes from here is the derivation.

Consider to points A and B on a curve of $y = Cx^n$
Join AB with a straight line and the gradient of the line is approximately the gradient at point P. The closer A and B are the more accurate this becomes.




Gradient at P ≈ BC/AC
h is the difference between the value of x at B and A.

$$Gradient \approx \frac{f(x)_B - f(x)_A}{h} = \lim_{h\to 0}\frac{f(x)_B - f(x)_A}{h}$$

$$\frac{\text{dy}}{\text{dx}} = \lim_{h\to 0}\frac{f(x_A + h) - f(x)_A}{h} = \lim_{h\to 0}\frac{C(x+h)^n - Cx^n}{h}$$

$$\frac{\text{dy}}{\text{dx}} = \lim_{h\to 0}\frac{Cx^n\left(1+\frac{h}{x}\right)^n - Cx^n}{h} = \lim_{h\to 0}\frac{Cx^n\left\{\left(1+\frac{h}{x}\right)^n - 1\right\}}{h}$$

If we expand $\left(1+\frac{h}{x}\right)^n$ we get a series $1 + n\frac{h}{x} + a\left(\frac{h}{x}\right)^2 + b\left(\frac{h}{x}\right)^3 + c\left(\frac{h}{x}\right)^4 \ldots\ldots$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{Cx^n\left\{1 + n\frac{h}{x} + a\left(\frac{h}{x}\right)^2 + b\left(\frac{h}{x}\right)^3 + c\left(\frac{h}{x}\right)^4 \ldots\ldots - 1\right\}}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{Cx^n\left\{n\frac{h}{x} + a\left(\frac{h}{x}\right)^2 + b\left(\frac{h}{x}\right)^3 + c\left(\frac{h}{x}\right)^4 \ldots\ldots\right\}}{h}$$

divide out the h

$$\frac{dy}{dx} = \lim_{h\to 0} Cx^n\left\{\frac{n}{x} + a\frac{h}{x^2} + b\frac{h^2}{x^3} + \cdots\right\}$$

Put h = 0 and

$$\frac{dy}{dx} = Cnx^{n-1}$$

## 2.2 DIFFERENTIATING A CONSTANT.

Consider the equation $y = a\,x^n$. When n = 0 this becomes $y = a\,x^0 = a$ (the constant). (Remember that anything to the power of zero is unity).

Using the rule for differentiation $dy/dx = anx^{0-1} = a\,(0)x^{-1} = 0$

The constant disappears when integrated. This explains why, when you do integration without limits, you must add on a constant that might or might not have been present before you differentiated. It is important to remember that:

**A constant disappears when differentiated.**

**WORKED EXAMPLE No.1**

Differentiate the function $x = 3\,t^2/2$ with respect to t and evaluate it when t = 2.

**SOLUTION**

$$x = \frac{3t^2}{2}$$

$$\frac{dx}{dt} = \frac{(2)(3)t^{2-1}}{2} = 3t$$

$$\text{Putting } t = 2 \text{ we find } \frac{dx}{dt} = 6$$

**WORKED EXAMPLE No.2**

Differentiate the function $y = 4 + x^2$ with respect to y and evaluate it when y = 5.

**SOLUTION**

$$y = 4 + x^2$$

$$\frac{dy}{dx} = 0 + 2x^{2-1} = 2x$$

$$\text{Putting } y = 5 \text{ we find } \frac{dy}{dx} = 10$$

**WORKED EXAMPLE No.3**

Differentiate the function $z = 2y^4$ with respect to y and evaluate it when y = 3.

**SOLUTION**

$$z = 2y^4$$

$$\frac{dz}{dy} = (2)(4)x^{4-1} = 8y^3$$

$$\text{Putting } y = 3 \text{ we find } \frac{dz}{dy} = 216$$

**WORKED EXAMPLE No.4**

Differentiate the function $p = 2q^3 + 3q^5 + 5$ with respect to q and evaluate it when q = 2.

**SOLUTION**

$$p = 2q^3 + 3q^5 + 5$$

$$\frac{dp}{dq} = (3)(2)q^{3-1} + (5)(3)q^{5-1} + 0 = 6q^2 + 15q^4$$

$$\text{putting } q = 2 \text{ we get } \frac{dp}{dq} = 24 + 240 = 264$$

**WORKED EXAMPLE No.5**

The equation linking distance and time is $x = 4t + ½ at^2$ where 'a' is the acceleration. Find the velocity at time t = 4 seconds given a = 1.5 m/s$^2$.

**SOLUTION**

$x = 4t + ½ at^2$ velocity = v = dx/dt = 4 + at = 4 +(1.5)(4) = 10 m/s

## SELF ASSESSMENT EXERCISE No.1

1. Find the gradient of the function $y = 2x - 5x^7$ when $x = 2$
(Answer -2238)

2. Find the gradient of the function $p = 2q + 2q^2 + 4q^3$ and evaluate when $q = 3$
(Answer 122)

3. Find the gradient of the function $u = 2v^2 + 4v^4$ and evaluate when $v = 5$
(Answer 2020)

## SELF ASSESSMENT EXERCISE No.2

1. The electric charge entering a capacitor is related to time by the equation $q = 3t^2$.

   Determine the current ($i = dq/dt$) after 5 seconds. (30 Amp)

2. The angle $\theta$ radians turned by a wheel after t seconds from the start of measurement is found to be related to time by the equation $\theta = \omega_1 t + ½ \alpha t^2$

   $\omega_1$ is the initial angular velocity (2 rad/s) and $\alpha$ is the angular acceleration (0.5 rad/s$^2$). Determine the angular velocity ($\omega = d\theta/dt$) 8 seconds from the start. (6 rad/s)

## 2.3 OTHER STANDARD FUNCTIONS

For other common functions the differential coefficients may be found from the look up table below.

$y = \sin(ax) \qquad \frac{dy}{dx} = a\cos(ax)$

$y = \cos(ax) \qquad \frac{dy}{dx} = -a\sin(ax)$

$y = \tan(ax) \qquad \frac{dy}{dx} = a + a\tan(ax)^2$

$y = \ln(ax) \qquad \frac{dy}{dx} = x^{-1} = \frac{1}{x}$

$y = ae^{kx} \qquad \frac{dy}{dx} = ake^{kx}$

**WORKED EXAMPLE No.6**

The distance moved by a mass oscillating on a spring is given by the equation:
$x = 5 \cos(8t)$ mm. Find the distance and velocity after 0.1 seconds.

**SOLUTION**

At 0.1 seconds $x = 5 \cos(0.8) = 3.48$ mm

$v = dx/dt = -40 \sin(8t) = -40 \sin(0.8) = -28.69$ mm/s

Note that your calculator must be in radian mode when looking up sine and cosine values.

**WORKED EXAMPLE No.7**

The distance moved by a mass is related to time by the equation :
$x = 20e^{0.5t}$ mm. Find the distance and velocity after 0.2 seconds.

**SOLUTION**

At 0.2 seconds $x = 20e^{0.5t} = 20e^{0.1} = 22.1$ mm

$v = dx/dt = (20)(0.5)e^{0.5t} = 10e^{0.1} = 11.05$ mm/s

**SELF ASSESSMENT EXERCISE No.3**

1. If the current flowing in a circuit is related to time by the formula $i = 4\sin(3t)$, find the rate of change of current after 0.2 seconds. (9.9 A/s)

2. The voltage across a capacitor C when it is being discharged through a resistance R is related to time by the equation $v = 4e^{-t/T}$ where T = is a time constant and T = RC.

   Find the voltage and rate of change of voltage after 0.2 seconds given R = 10 kΩ and C = 20 μF. (1.472 V and -7.36 V/s )

3. The voltage across a capacitor C when it is being charged through a resistance R is related to time by the equation $v = 4 - 4e^{-t/T}$ where T = is a time constant and T = RC.

   Find the voltage and rate of change of voltage after 0.2 seconds given R = 10 kΩ and C = 20 μF. (2.528 V and 7.36 V/s )

4. The distance moved by a mass is related to time by the equation :

   $x = 17e^{0.3t}$ mm. Find the distance and velocity after 0.4 seconds.
   (19.17 mm and 5.75 mm/s)

5. The angle turned by a simple pendulum is given by the equation:
   $\theta = 0.05 \sin(6t)$ mm. Find the angle and angular velocity ($d\theta/dt$) after 0.2 seconds.
   (0.0466 radian and 0.1087 rad/s)

## 3. HIGHER ORDER DIFFERENTIALS

Consider the function $y = x^3$. The graph looks like this.

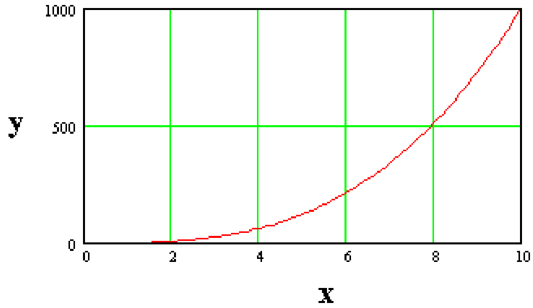



Figure 3

The gradient of the graph at any point is $dy/dx = 3x^2$. This may be evaluated for any value of x. If we plot dy/dx against x we get the following graph.

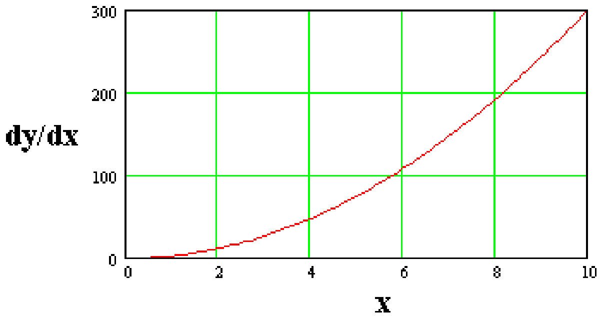



Figure 4

This graph is also a curve. We may differentiate again to find the gradient at any point. This is the gradient of the gradient. We write it as follows.

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2} = 6x$$

The graph is a straight line as shown with a gradient of 6 at all points.

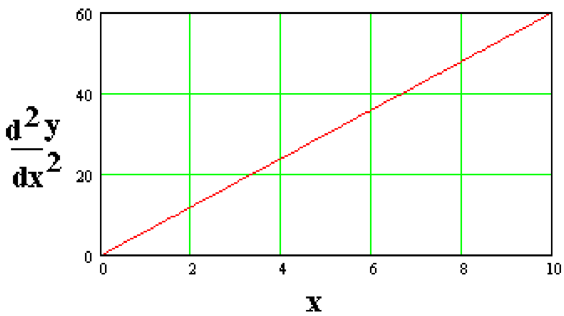



Figure 5

If we differentiate again we get

$$\frac{d\left(\frac{d^2y}{dx^2}\right)}{dx} = \frac{d^3y}{dx^3} = 6$$

**WORKED EXAMPLE No.8**

The distance moved by a body (in metres) with uniform acceleration is given by s = $5t^2$. Find the distance moved, velocity and acceleration after 12 seconds.

**SOLUTION**

$$\text{distance} = s = 5t^2 = 720 \text{ m}$$

$$\text{velocity} = v = \frac{ds}{dt} = 10t = 120 \text{ m/s}$$

$$\text{acceleration} = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 10 \text{ m/s}^2$$

**WORKED EXAMPLE No.9**

The distance moved by an oscillating body is related to time by the function:

x = 1.2 sin(2t) mm.

Find the distance moved, velocity and acceleration after 0.3 seconds.

**SOLUTION**

$$\text{distance} = x = 1.2\sin(2t) = 1.2\sin(0.6) = (1.2)(.5646) = 0.678 \text{ mm}$$

$$\text{velocity} = v = \frac{dx}{dt} = (2)(1.2)\cos(2t) = (2.4)\cos(0.6) = 1.981 \text{ mm/s}$$

$$\text{acceleration} = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -(2)(2)(1.2)\sin(2t) = -4.8\sin(0.6) = -2.71 \text{ mm/s}^2$$

**SELF ASSESSMENT EXERCISE No.4**

1. Evaluate the first and second derivative of the function $p = 8e^{-0.2t}$ when t = 2.
   (Answers -1.073 and 0.215)

2. The motion of a mechanism is described by the equation x = 50 Cos(0.5t) mm. Calculate the distance, velocity and acceleration after 0.3 seconds.
   (Answers 49.44 mm, -3.74 mm/s and -12.36 mm/s$^2$)

3. Evaluate the first and second derivatives of the function $z = 2x^4 + 3x^3 + 2x - 5$ when x = 4
   (Answers 658 and 456)

4. The motion of a body is described the equation x = Asin(ωt) where x is the distance moved and t is the variable time. Show by successive differentiation and a substitution that the acceleration is given by a = -ωx.