Mathematics: analysis and approaches
Standard level
Paper 1

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1 hour 30 minutes

Instructions to candidates

- You are not permitted access to any calculator for this paper.
- Section A: answer all questions.
- Section B: answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[80 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

1. [Maximum mark: 5]

Let A and B be events such that $\mathrm{P}(A) = 0.3$, $\mathrm{P}(B) = 0.75$ and $\mathrm{P}(A \cup B) = 0.9$.
Find $\mathrm{P}(B \mid A)$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \; *$$

$$0.9 = 0.3 + 0.75 - P(A \cap B)$$

$$P(A \cap B) = 0.3 + 0.75 - 0.9 = 0.15$$

Then $$P(B|A) = \frac{P(A \cap B)}{P(A)} \; *$$

$$= \frac{0.15}{0.3}$$

$$\boxed{P(B|A) = 0.5}$$

* These two formulae are in the exam Formula Booklet

2. [Maximum mark: 5]

Given that $\frac{dy}{dx} = 3x^2 \cos\left(3x^3 + \frac{\pi}{2}\right)$ and that the graph of y passes through the point $(0, -1)$, find an expression for y in terms of x.

Let $u = 3x^3 + \frac{\pi}{2}$.

Then $\frac{du}{dx} = 9x^2$, and $3x^2 = \frac{1}{3}\frac{du}{dx}$

Therefore

$$y = \int 3x^2 \cos\left(3x^3 + \frac{\pi}{2}\right) dx$$

$y = \int \frac{dy}{dx}\, dx$

$$= \int \frac{1}{3} \cos u \frac{du}{dx}\, dx$$

$$= \int \frac{1}{3} \cos u\, du$$

$$= \frac{1}{3} \sin u + C$$

C is the constant of integration

So

$$y = \frac{1}{3} \sin\left(3x^3 + \frac{\pi}{2}\right) + C$$

and $y = -1$ when $x = 0$, so

Because the graph goes through $(0, -1)$

$$-1 = \frac{1}{3} \sin\left(\frac{\pi}{2}\right) + C$$

$\sin\left(\frac{\pi}{2}\right) = 1$

$$-1 = \frac{1}{3} + C \implies C = -\frac{4}{3}$$

$$\boxed{y = \frac{1}{3} \sin\left(3x^3 + \frac{\pi}{2}\right) - \frac{4}{3}}$$

3. [Maximum mark: 5]

The functions f and g are defined such that $f(x) = 6x + 7$ and $g(x) = \frac{x-5}{3}$.

(a) Show that $(f \circ g)(x) = 2x - 3$. [2]

(b) Given that $(f \circ g)^{-1}(a) = 6$, find the value of a. [3]

a)
$$\boxed{\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= 6(g(x)) + 7 \\ &= 6\left(\frac{x-5}{3}\right) + 7 \\ &= 2(x-5) + 7 \\ &= 2x - 10 + 7 \\ &= 2x - 3\end{aligned}}$$

b)
$$\left.\begin{aligned} y &= 2x - 3 \\ x &= 2y - 3 \\ 2y &= x + 3 \\ y &= \frac{x+3}{2} \\ (f \circ g)^{-1} &= \frac{x+3}{2}\end{aligned}\right\} \text{Find inverse of } f \circ g$$

So $\frac{a+3}{2} = 6$ Because $(f \circ g)^{-1}(a) = 6$

$a + 3 = 12$

$\boxed{a = 9}$

Alternative method for part (b):

$(f \circ g)^{-1}(a) = 6 \implies a = (f \circ g)(6) \implies a = 2(6) - 3 = 12 - 3 = 9$

4. [Maximum mark: 5]

(a) (i) Expand $(2k-1)^3$.

(ii) Hence, or otherwise, show that $(2k-1)^3-(2k-1)=8k^3-12k^2+4k$. [2]

(b) Thus prove, given $k>1,\ k\in\mathbb{N}$, that the difference between an odd natural number greater than 1 and its cube is always even. [3]

a) (i) $(2k-1)^2 = 4k^2-4k+1$

$(2k-1)^3 = (2k-1)(2k-1)^2$

$= (2k-1)(4k^2-4k+1)$

$= 8k^3-8k^2-4k^2+2k+4k-1$

You could also do this using the binomial theorem

$\boxed{(2k-1)^3 = 8k^3-12k^2+6k-1}$

(ii) $(2k-1)^3-(2k-1) = (8k^3-12k^2+6k-1)-(2k-1)$

$= 8k^3-12k^2+6k-1-2k+1$

$= 8k^3-12k^2+4k$

b) $k>1,\ k\in\mathbb{N}$ means $k\in\{2,3,4,5,\ldots\}$, so $2k-1\in\{3,5,7,9\ldots\}$

Therefore $2k-1$ represents any odd natural number greater than one.

The difference between $(2k-1)^3$ and $(2k-1)$ is

$$8k^3-12k^2+4k = 2(4k^3-6k^2+2k)$$

which is even because it is an integer multiplied by 2.

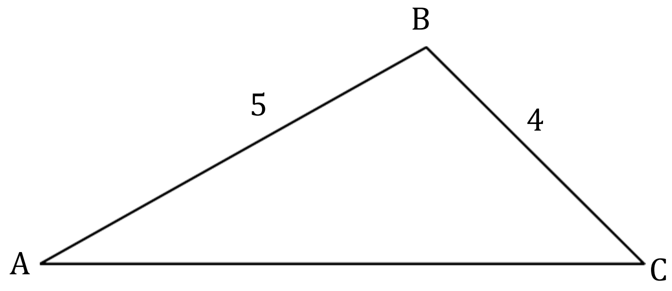
Therefore the difference between an odd natural number greater than one and its cube is always even.

$(4k^3-6k^2+2k)$ is an integer because k is an integer

5. [Maximum mark: 5]

The following diagram shows triangle ABC, with $\mathrm{AB} = 5$ and $\mathrm{BC} = 4$.

diagram not to scale



(a) (i) Given that $\sin \hat{\mathrm{B}} = \frac{3}{5}$, find the possible values of $\cos \hat{\mathrm{B}}$.

(ii) Given that $\hat{\mathrm{B}}$ is obtuse, find the precise value of $\cos \hat{\mathrm{B}}$. [3]

(b) Find the length of AC. [2]

a) (i) $\cos^2 \hat{B} + \sin^2 \hat{B} = 1$

Use Pythagorean identity from the exam Formula Booklet

$\cos^2 \hat{B} + \left(\frac{3}{5}\right)^2 = 1$

$\cos^2 \hat{B} + \frac{9}{25} = 1$

$\cos^2 \hat{B} = \frac{16}{25} \Rightarrow \cos \hat{B} = \pm\sqrt{\frac{16}{25}}$

$\boxed{\cos \hat{B} = \frac{4}{5} \text{ or } -\frac{4}{5}}$

(ii) Cosine is negative for obtuse angles, so $\boxed{\cos \hat{B} = -\frac{4}{5}}$

b) $AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos \hat{B}$

Use Cosine rule formula from the exam Formula Booklet

$AC^2 = 5^2 + 4^2 - 2(5)(4)(-\frac{4}{5})$

$AC^2 = 25 + 16 + 32 = 73$

$\boxed{AC = \sqrt{73} \text{ units}}$

6. [Maximum mark: 8]

(a) Show that $\log_4(\cos 2x + 13) = \log_2 \sqrt{\cos 2x + 13}$. [3]

(b) Hence or otherwise solve $\log_2(3\sqrt{2}\cos x) = \log_4(\cos 2x + 13)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. [5]

a)

$$\log_4(\cos 2x + 13) = \frac{\log_2(\cos 2x + 13)}{\log_2 4}$$

$$= \frac{\log_2(\cos 2x + 13)}{2}$$

$$= \frac{1}{2}\log_2(\cos 2x + 13)$$

$$= \log_2(\cos 2x + 13)^{\frac{1}{2}}$$

$$= \log_2 \sqrt{\cos 2x + 13}$$

$\log_a x = \dfrac{\log_b x}{\log_b a}$

$\log_a x^m = m \log_a x$

These two formulae are in the exam Formula Booklet

b)

$\log_2(3\sqrt{2}\cos x) = \log_4(\cos 2x + 13)$

$\log_2(3\sqrt{2}\cos x) = \log_2\sqrt{\cos 2x + 13}$ using result from part (a)

$3\sqrt{2}\cos x = \sqrt{\cos 2x + 13}$

$(3\sqrt{2}\cos x)^2 = \cos 2x + 13$ square both sides

$18\cos^2 x = \cos 2x + 13$

$18\cos^2 x = (2\cos^2 x - 1) + 13$ Use double angle identity from the exam Formula Booklet

$16\cos^2 x = 12$

$\cos x = \pm\sqrt{\frac{12}{16}} = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$

But $\cos x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so $\cos x = \frac{\sqrt{3}}{2}$

$$\boxed{x = \frac{\pi}{6} \text{ or } -\frac{\pi}{6}}$$

Remember that, by symmetry of the cos function, $\cos(-x) = \cos x$

Section B

7. [Maximum mark: 16]

Let $f(x) = \frac{1}{3}x^3 - 2x^2 - 21x - 24$.

(a) Find $f'(x)$. [2]

The graph of f has horizontal tangents at the points where $x = a$ and $x = b$, $a < b$.

(b) Find the value of a and the value of b. [3]

(c) (i) Find $f''(x)$.

(ii) Hence show that the graph of f has a local maximum point at $x = a$. [2]

(d) (i) Sketch the graph of $y = f'(x)$.

(ii) Hence, use your answer to part (d)(i) to explain why the graph of f has a local minimum point at $x = b$. [4]

The tangent to the graph of f at $x = a$ and the normal to the graph of f at $x = b$ intersect At the point (p, q).

(e) Find the value of p and the value of q. [5]

a) $f'(x) = \frac{1}{3}(3x^2) - 2(2x) - 21(1)$

$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

$$\boxed{f'(x) = x^2 - 4x - 21}$$

b) A horizontal tangent has a gradient of zero, so $f'(x) = 0$ at the points where $x = a$ and $x = b$.

$x^2 - 4x - 21 = 0$

$(x - 7)(x + 3) = 0$

$x = 7$ or $x = -3$

You could also solve this using the quadratic formula

And $a < b$, so

$$\boxed{a = -3, \quad b = 7}$$

c) (i) $f''(x) = 2x - 4(1)$ Differentiate $f'(x)$ to find $f''(x)$

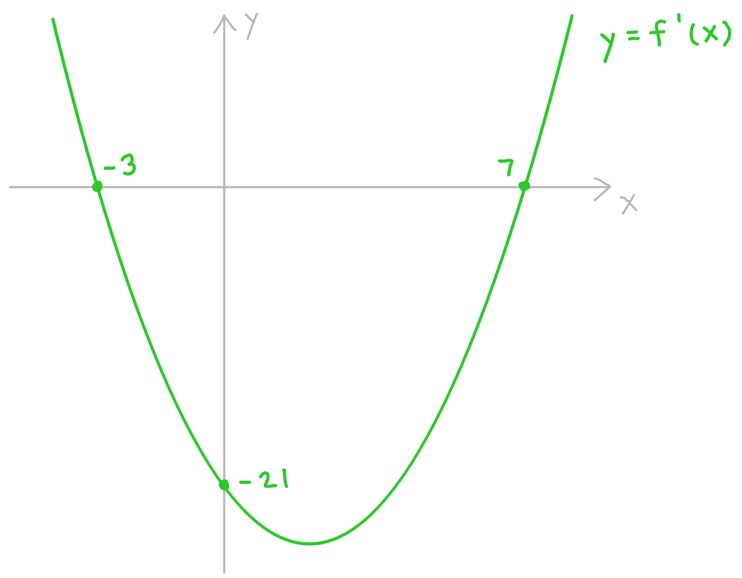
$$\boxed{f''(x) = 2x - 4}$$

(ii) $f''(a) = f''(-3) = 2(-3) - 4 = -6 - 4 = -10$

$f''(a) < 0$, therefore f is concave down at $x = a$.

Since we also have $f'(a) = 0$, this shows that f has a local maximum at $x = a$.

d) (i)



(ii) $f'(b) = 0$, and the graph shows that $f'(x)$ changes from negative to positive at $x = b$.

Therefore f has a local minimum at $x = b$.

e) $f(a) = f(-3) = \frac{1}{3}(-3)^3 - 2(-3)^2 - 21(-3) - 24 = -9 - 18 + 63 - 24 = 12$

So the tangent at $x = a$ is a horizontal line through $(-3, 12)$ with equation $y = 12$.

The normal at $x = b$ is a vertical line through $(7, f(7))$ with equation $x = 7$.

Those lines intersect at the point $(7, 12)$.

$$\boxed{p = 7 \qquad q = 12}$$

8. [Maximum mark: 16]

Let $f(x) = \dfrac{\ln px}{qx}$ where $x > 0,\ \ p, q \in \mathbb{R}^+$.

(a) Show that $f'(x) = \dfrac{1 - \ln px}{qx^2}$. [3]

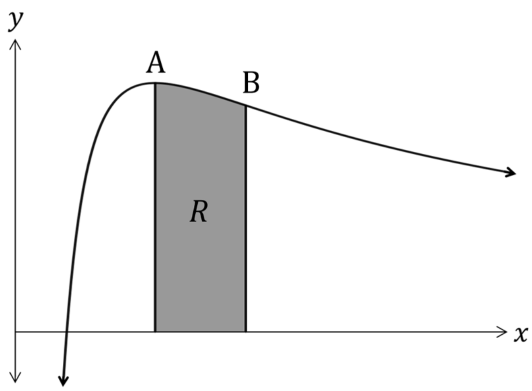
The graph of f has exactly one maximum point A.

(b) Find the x-coordinate of A. [3]

The second derivative of f is given by $f''(x) = \dfrac{2\ln px - 3}{qx^3}$. The graph of f has exactly one point of inflexion B.

(c) Show that the x-coordinate of B is $\dfrac{e^{\frac{3}{2}}}{p}$. [3]

The region R is enclosed by the graph of f, the x-axis, and the vertical lines through the maximum point A and the point of inflexion B.



(d) Calculate the area of R in terms of q and show that the value of the area is independent of p. [7]

a) First let $y = \ln(px)$ and let $u = px$.

Then $y = \ln(u)$, $\frac{dy}{du} = \frac{1}{u}$, and $\frac{du}{dx} = p$.

So the derivative of $\ln(px)$ is

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left(\frac{1}{u}\right)(p) = \left(\frac{1}{px}\right)(p) = \frac{1}{x}$$

Use Chain rule formula from the exam Formula Booklet

Now let $u = \ln(px)$ and let $v = qx$.

Then $f(x) = \frac{u}{v}$, $\frac{du}{dx} = \frac{1}{x}$, and $\frac{dv}{dx} = q$.

$$\text{So } f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{qx\left(\frac{1}{x}\right) - \ln(px)(q)}{(qx)^2}$$

$$= \frac{q - q\ln(px)}{q^2x^2} = \frac{q(1 - \ln(px))}{q^2x^2}$$

$$= \frac{1 - \ln(px)}{qx^2}$$

Use Quotient rule formula from the exam Formula Booklet

b) A maximum point occurs where $f'(x) = 0$, so

$$\frac{1 - \ln(px)}{qx^2} = 0 \implies 1 - \ln(px) = 0 \implies \ln(px) = 1$$

$$\implies px = e^1 = e \implies \boxed{x = \frac{e}{p}}$$

$\ln(a) = b \iff a = e^b$

c) A point of inflexion occurs where $f''(x) = 0$, so

$$\frac{2\ln(px) - 3}{qx^3} = 0 \implies 2\ln(px) - 3 = 0 \implies \ln(px) = \frac{3}{2}$$

$$\implies px = e^{3/2} \implies x = \frac{e^{3/2}}{p}$$

$\ln(a) = b \iff a = e^b$

d) The area can be found by integrating:

$$\text{Area} = \int_{\frac{e}{p}}^{\frac{e^{3/2}}{p}} \frac{\ln(px)}{qx}\, dx$$

see part (a)

Let $u = \ln(px)$. Then $\frac{du}{dx} = \frac{1}{x}$, and

$$\int \frac{\ln(px)}{qx}\, dx = \frac{1}{q}\int \ln(px)\left(\frac{1}{x}\right) dx$$

$$= \frac{1}{q}\int u \frac{du}{dx}\, dx = \frac{1}{q}\int u\, du$$

Also $x = \frac{e}{p} \Rightarrow u = \ln\left(p\left(\frac{e}{p}\right)\right) = \ln(e) = 1$

and $x = \frac{e^{3/2}}{p} \Rightarrow u = \ln\left(p\left(\frac{e^{3/2}}{p}\right)\right) = \ln\left(e^{3/2}\right) = \frac{3}{2}$

$\ln\left(e^{a}\right) = e^{\ln a} = a$

Therefore

$$\text{Area} = \frac{1}{q}\int_{1}^{3/2} u\, du = \frac{1}{q}\left[\frac{1}{2}u^2\right]_{1}^{3/2}$$

$$= \frac{1}{q}\left(\frac{1}{2}\left(\frac{3}{2}\right)^2 - \frac{1}{2}(1)^2\right)$$

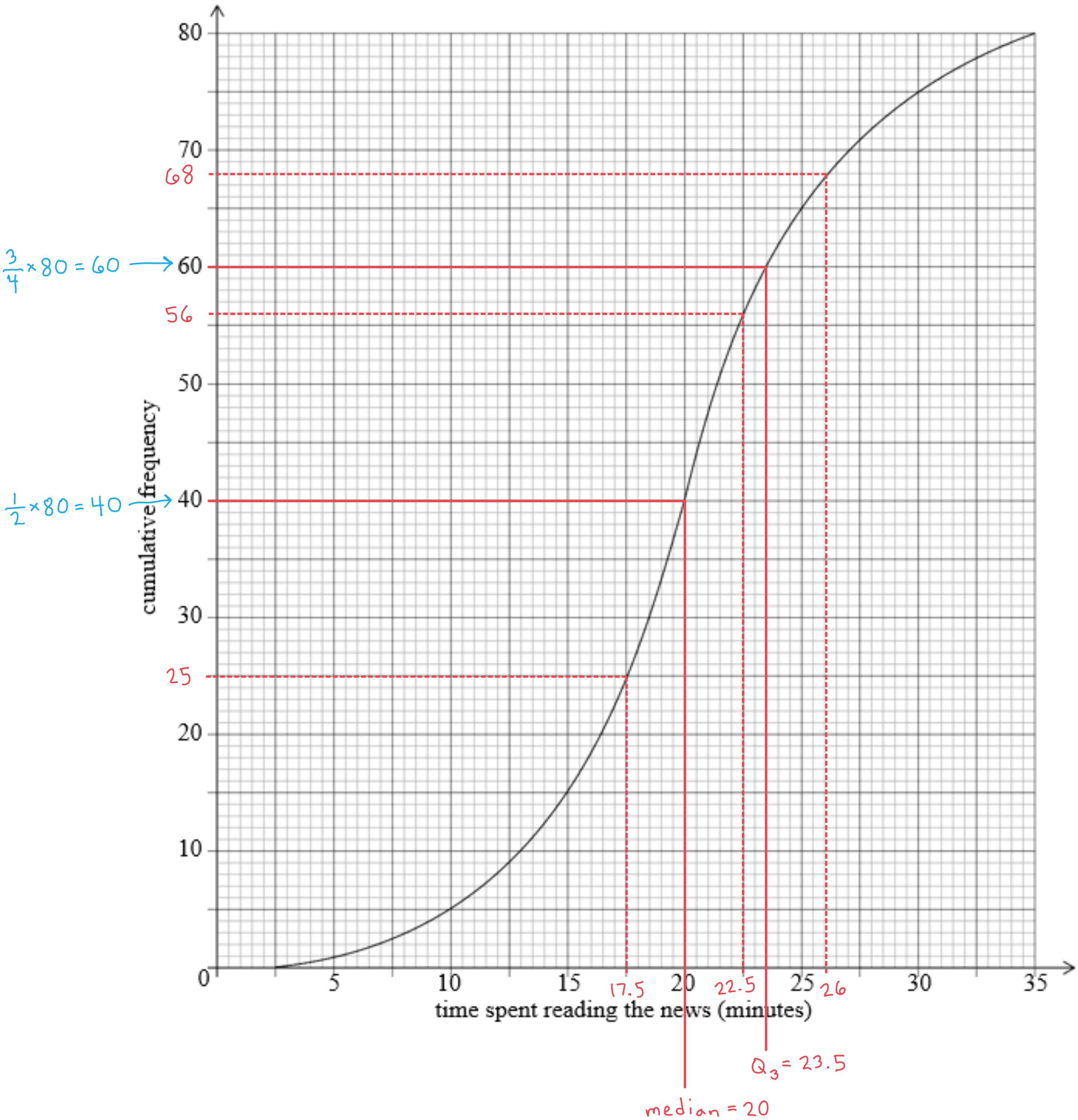
$$= \frac{1}{q}\left(\frac{9}{8} - \frac{1}{2}\right) = \frac{1}{q}\left(\frac{5}{8}\right)$$

$$\text{Area} = \frac{5}{8q}$$

The value of the area depends on q, but because p doesn't appear in that expression it is independent of p.

9. [Maximum mark: 15]

A school surveyed 80 of its final year students to find out how much time they spent reading the news on a given day. The results of the survey are shown in the following cumulative frequency diagram.



(This question continues on the following page)

(Question 9 continued)

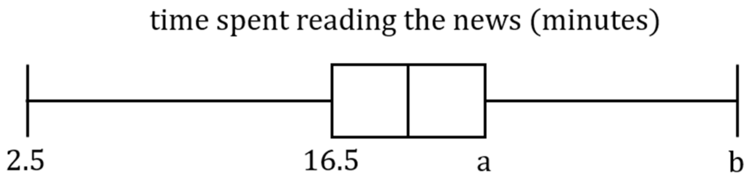
(a) Find the median number of minutes spent reading the news. [2]

(b) Find the number of students whose reading time is within 2.5 minutes of the median. [3]

Only 15% of students spent more than k minutes reading.

(c) Find the value of k. [3]

The results of the survey can also be displayed on the following box-and-whisker diagram.



(d) Write down the value of b. [1]

(e) (i) Find the value of a.

(ii) Hence, find the interquartile range. [4]

(f) Determine whether someone who spends 30 minutes reading the news would be an outlier. [2]

a) $\boxed{\text{median} = 20 \text{ minutes}}$ working is on the graph

b) $56 - 25 = 31$ $\boxed{31 \text{ students}}$ working is on the graph

c) 15% of $80 = 12$ $80 - 12 = 68$

$\boxed{k = 26}$ working is on the graph

d) $\boxed{b = 35}$ this is the maximum value in the data set

e) (i) $\boxed{a = 23.5}$ this is the third quartile (Q_3); working is on the graph

(ii) $\underbrace{IQR = Q_3 - Q_1}_{\text{This formula is in the exam Formula Booklet}} = 23.5 - 16.5 = 7$ $\boxed{IQR = 7}$

f) $1.5 \times IQR = 1.5 \times 7 = 10.5$

$Q_3 + 10.5 = 23.5 + 10.5 = 34$

Someone who spends 30 minutes reading the news would not be an outlier.

An outlier is any value at least $1.5 \times IQR$ above Q_3 or below Q_1.