Educative Commentary on

JEE 2004 MATHEMATICS PAPERS

Note: The IIT's have continued the practice, begun in 2003, of collecting back the JEE question papers from the candidates. At the time of this writing, the JEE 2004 papers were not officially available to the general public, either on a website or on paper. So the text of the questions taken here was based on the memories of candidates and hence is prone to deviations from the original. In some cases, such deviations may result in a change of answers. However, from an educative point of view, they do not matter much. Subsequently, the text of the questions was made officially available. But, except for the order of the questions, there are no serious deviations and so, we have retained these memory-based versions. (Indeed, it was mind boggling to see how accurately even the numerical data was remembered in the case of most problems.) Unlike in 2003, the IITs have given no solutions for the 2004 papers. So there are no comments on them here.

For convenience, the questions in the Screening Paper are arranged topicwise. The actual order in the examination is quite different. Moreover, usually several versions of the question paper are prepared. They have the same questions but arranged in different orders. In the Main Paper, the first ten questions carry two points each and the remaining ones 4 points each. The total time allotted for the Main Paper was 2 hours. We reiterate that the solutions given here are far more detailed than what is expected in an examination.

SCREENING PAPER OF JEE 2004

Q. 1 If $f(x) = \sin x + \cos x$, and $g(x) = x^2 - 1$, then $g(f(x))$ is invertible in the interval

(A) $[0, \frac{\pi}{2}]$ (B) $[-\frac{\pi}{4}, \frac{\pi}{4}]$ (C) $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (D) $[0, \pi]$

Answer and Comments: (B). By a direct calculation, we have $g(f(x)) = 2\sin x \cos x = \sin 2x$ for all x. This is a continuous function and so, by the Intermediate Value Property (Comment No. 6, Chapter 16), it will be invertible on an interval if and only if it is either strictly increasing or strictly decreasing on that interval. Of the given four intervals, $[-\frac{\pi}{4}, \frac{\pi}{4}]$ is the only one on which $\sin 2x$ increases strictly. This can be seen from the fact that its derivative, viz., $2\cos 2x$ is positive for all x in the open interval $(-\frac{\pi}{4}, \frac{\pi}{4})$. Here we are implicitly using the Mean Value Theorem. Of course, the graphs of the sine and other trigonometric functions are so familiar, that one can also get the answer directly. The only catch is that we are dealing with the graph of $\sin 2x$ and not of $\sin x$. Effectively, this means we have to double the given intervals and see on which of them the sine function is increasing. Clearly, $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is such an interval (after doubling). Hence the original interval is $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

Normally, when we are given a function, say h, defined on a domain D and are asked to test whether it is invertible on a given subset, say S, of D, the question amounts to checking whether it is one-to-one on S, i.e. whether the equality $h(s_1) = h(s_2)$ for elements $s_1, s_2 \in S$ necessarily implies $s_1 = s_2$. In general, there is no golden way to do this and sometimes the answer has to be obtained by common sense. For example, suppose S is the set of all students in a class and h is the father function, i.e. $h(x)$ is the father of x. Then h is invertible on S if and only if the class contains no siblings. In the present problem, the function given is from $I\!R$ to $I\!R$ and hence methods based on calculus could be applied.

Note also that the function given was a composite function, viz. $g \circ f$. It was easy to calculate $g(f(x))$ explicitly and to answer the question from it. Sometimes this may not be so. Suppose for example, that $g(x)$ was given as $x - x^2 + x^3$. In this case, the formula for the composite $g(f(x))$ is horribly complicated. However, rewriting $g'(x)$ as $(1 - x)^2 + 2x^2$ we see that it is always positive. Hence, again by Lagrange's MVT, g is a strictly increasing function. So the composite $g \circ f$ will be strictly increasing or decreasing on an interval, depending upon whether f is so. This observation obviates the need to calculate $g(f(x))$ and reduces the problem to testing the behaviour of the function $f(x)$. (See Comment No. 12 of Chapter 13 for a problem based on similar considerations.)

Q. 2 If $f(x)$ is a strictly increasing and differentiable function with $f(0) = 0$, then $\lim\limits_{x \to 0} \dfrac{f(x^2) - f(x)}{f(x) - f(0)}$ is equal to

(A) 0 (B) 1 (C) -1 (D) 2

Answer and Comments: (C). Since $f(0) = 0$ the term $f(0)$ in the denominator is really redundant. But its presence suggests that derivatives may have to be taken for the solution. But, let us first try to tackle the problem without derivatives. Since $f(0) = 0$, the given ratio equals $\dfrac{f(x^2)}{f(x)} - 1$ and so its limit will depend only on the limit of the ratio $\dfrac{f(x^2)}{f(x)}$. First we let x tend to 0 through positive values. When x is small (and positive), x^2 is considerably smaller than x in the sense that the ratio x^2/x tends to 0 as $x \to 0$. As f is given to be strictly increasing, we expect intuitively that $f(x^2)$ should also be considerably smaller than $f(x)$, i.e. the ratio $f(x^2)/f(x)$ should approach 0 as $x \to 0^+$. A similar argument applies if x tends to 0 from the left. Thus we predict that $\lim\limits_{x \to 0} \dfrac{f(x^2)}{f(x)} = 0$.

As noted before, the given limit is obtained by subtracting 1 from this. Hence (C) is the correct answer.

In a multiple choice question where you do not have to show the work, an intuitive reasoning like this is quite valid. In fact, the ability to think like this shows a certain mathematical maturity. Still, one should be prepared to back intuition by a rigorous reasoning, should the need arise. In the present case, the way to do this is fairly obvious. Again, we work with the ratio $\dfrac{f(x^2)}{f(x)}$ instead of the given ratio and rewrite it as $\dfrac{f(x^2)-f(0)}{f(x)-f(0)}$. If we divide both the numerator and the denominator by x, the limit of the denominator as $x \to 0$ is simply $f'(0)$. The numerator becomes $\dfrac{f(x^2)-f(0)}{x}$ which can be rewritten as $\dfrac{f(x^2)-f(0)}{x^2} \times x$. As x tends to 0, so does x^2 and so the limit of the first factor is $f'(0)$ while that of the second factor is 0. Hence $\lim\limits_{x\to 0} \dfrac{f(x^2)}{f(x)} = \dfrac{f'(0)\times 0}{f'(0)} = 0$. This can also be seen from l'Hôpital's rule (a students' favourite!). The ratio $\dfrac{f(x^2)}{f(x)}$ is of the $\dfrac{0}{0}$ form as $x \to 0$. The derivative of the denominator at 0 equals $f'(0)$ while that of the numerator is $2 \times 0 \times f'(0)$, i.e. 0. Hence $\lim\limits_{x\to 0} \dfrac{f(x^2)}{f(x)} = 0$.

The trouble with this argument is that it is valid only if $f'(0) \neq 0$. Otherwise, l'Hôpital's rule is not applicable. Now, even though we are given that f is differentiable and strictly increasing, it does *not* follow that $f'(0) > 0$. All we can say is that $f'(x) \geq 0$ for all x. But as remarked in Exercise (13.8), f' may vanish at some points as we see from functions like $f(x) = x - \sin x$ which are strictly increasing. (A simpler example is the function $f(x) = x^3$.) So the given hypothesis does not necessarily imply that $f'(0) > 0$ as we need in the argument above.

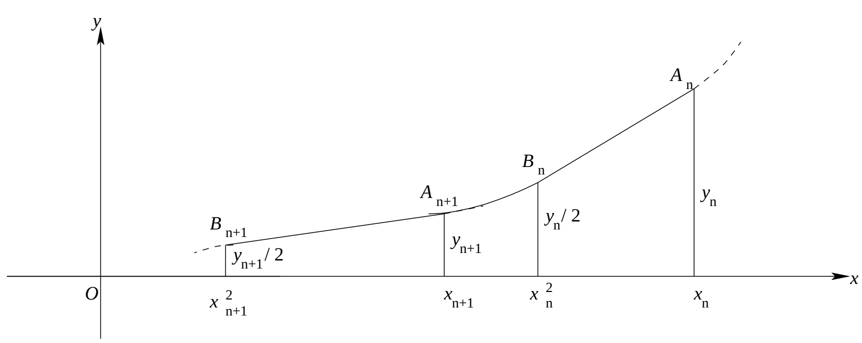
When $f'(0) = 0$, it is tempting to try to salvage the situation by applying the strong form of l'Hôpital's rule (Theorem 5 of Chapter 16). If we do so, then $\lim\limits_{x\to 0} \dfrac{f(x^2)}{f(x)}$ will equal $\lim\limits_{x\to 0} \dfrac{2xf'(x^2)}{f'(x)}$ provided the latter exists. Now, we cannot hastily say that the limit of the denominator is $f'(0)$ because f' is not given to be continuous at 0. But even if we assume this for a moment, the difficulty still remains if $f'(0) = 0$. In that case, we can differentiate the numerator and the denominator once more and write the limit of the ratio as $\lim\limits_{x\to} \dfrac{2f'(x^2) + 4x^2 f''(x^2)}{f''(x)}$. But there are still some problems. First, it is not given that f is twice differentiable. And even if we assume that it is so, what guarantee do we have that $f''(0)$ is non-zero? If $f''(0) = 0$, we can differentiate once more and pray that $f'''(0) \neq 0$. The trouble is that there do exist strictly increasing functions which have derivatives of all orders everywhere but all whose derivatives vanish at 0. One such example is the following function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -e^{-1/x^2} & \text{if } x < 0 \end{cases}$$

(Using the fact that the exponential function grows more rapidly than any polynomial function, it can be shown that $f^{(n)}(0) = 0$ for every positive integer n. Such functions are beyond the JEE level. For this function, one can show directly that $f(x^2)/f(x) \to 0$ as $x \to 0$. But this cannot be concluded from the knowledge of the successive derivatives of f at 0. The point is that even if we assume that f has derivatives of all orders, this line of attack, based on repeated applications of the l'Hôpital's rule is not always going to work in the present problem.)

These failures suggest that there is perhaps some serious lacuna in our original intuitive reasoning. From the fact that f is strictly increasing and x^2 is much smaller than x, we concluded that $f(x^2)$ must be considerably smaller than $f(x)$. But there is a catch here. In drawing a conclusion such as this, we are implicitly assuming that the rate of growth of f is more or less uniform. Without this assumption, the prediction may go wrong. As a social analogy, suppose we are given that the richer a person, the longer he tends to live. But this certainly does not mean that if A's income is considerably bigger (say 20 times bigger) than that of B, then A will live as much longer than B! Even though the income difference is very high, the corresponding difference in the life span may be only marginal, say, only 1.2 times.

Once this point is appreciated, it is easy to see that not only our reasoning is fallacious, but the conclusion itself, viz. that $\lim_{x \to 0} \frac{f(x^2)}{f(x)} = 0$ is probably wrong. Although it is rather difficult to actually give a counterexample by a succinct formula, the essential idea can be seen from the figure below in which we show a portion of the graph of $f(x)$ on the right of 0.



To construct this graph, we first fix some sequence $\{x_n\}_{n=1}^{\infty}$ of positive real numbers in which $x_1 < 1$ and

$$x_{n+1} < (x_n)^2 \tag{1}$$

for all $n \geq 1$. This ensures that the sequence is strictly monotonically decreasing and converges to 0. Similarly, let $\{y_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers with $y_1 = 1$ and

$$y_{n+1} < \frac{1}{2} y_n \tag{2}$$

for all $n \geq 1$, which ensures that the sequence $\{y_n\}$ is also monotonically decreasing and converges to 0. Now for each n, let A_n and B_n be the points (x_n, y_n) and $(x_n^2, y_n/2)$ respectively. Define f linearly on the interval $[x_n^2, x_n]$, i.e. let

$$f(x) = \frac{y_n}{2} + \frac{y_n}{2(x_n - x_n^2)}(x - x_n^2) \tag{3}$$

for $x_n^2 \leq x \leq x_n$. Then the graph of f is the line segment B_nA_n. Note further that $f'_+(x_n^2)$ and $f'_-(x_n)$ both equal the slope of this segment, viz. $\dfrac{y_n}{2(x_n - x_n^2)}$. By choosing the y_n's carefully, we can ensure that the slope of $B_{n+1}A_{n+1}$ is smaller than that of B_nA_n. It is now easy to join A_{n+1} and B_n by a strictly increasing smooth curve which touches the segment $A_{n+1}B_{n+1}$ at A_{n+1} and the segment B_nA_n at B_n. This way we get a function f which is strictly increasing and differentiable for $x > 0$. (Note that if we join A_{n+1} and B_n by a straight line segment, we would get a function which is strictly increasing and continuous but which is not differentiable at x_n and x_n^2.) Set $f(0) = 0$. While selecting the numbers $\{x_n\}$ and $\{y_n\}$ if we further ensure that $\dfrac{y_n}{x_{n+1}} \to 0$ as $n \to \infty$ then it can be shown that $f'_+(0)$ also exists and equals 0, because with this stipulation, we now have

$$0 \;\leq\; \frac{f(x) - 0}{x - 0} \;\leq\; \frac{y_n}{x_{n+1}} \tag{4}$$

for every $x \in [x_{n+1}, x_n]$. Finally, for $x < 0$, define $f(x) = -f(-x)$, i.e. make f an odd function. It is now easy to show that $f'(x) > 0$ for all $x \neq 0$ while $f'(0) = 0$. Hence f is strictly increasing.

But by very construction, this function has the property that for every n, $\dfrac{f(x_n^2)}{f(x_n)} = \dfrac{y_n/2}{y_n} = \dfrac{1}{2}$. Moreover, as $n \to \infty$, $x_n \to 0$. So, if at all $\displaystyle\lim_{x \to 0} \frac{f(x^2)}{f(x)}$ exists it will have to be $\frac{1}{2}$. Thus, what we thought intuitively clear is wrong. Consequently, for the limit in the given question, viz.

$\lim_{x \to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$, none of the given alternatives is correct. So the question is incorrect as it stands. (It is of course possible that in the original JEE question, the condition given was that $f'(x) > 0$ for all x but this was incorrectly paraphrased by the candidates to say that f is differentiable and strictly increasing. In that case the original JEE question is correct. It is impossible to decide the matter since the question papers are collected back from the candidates.)

In case of a wrongly set question like this, the only practical advice that can be given is that if the question becomes correct with a stronger or additional hypothesis, then by all means assume that this is what the papersetters intended. The logic is like this. Strictly speaking, none of the alternatives is correct. But it is *given* that one of them *is* correct. In that case, it has to be the one which holds true with a stronger hypothesis.

Q. 3 If y is a function of x and $\ln(x + y) - 2xy = 0$, then the value of $y'(0)$ is equal to

(A) -1 (B) 0 (C) 2 (D) 1

Answer and Comments: (D). An equation like this cannot be solved explicitly for y, even if we recast it as $x + y = e^{2xy}$. So, to find y' we have to resort to implicit differentiation. Differentiating both the sides of the given equation implicitly w.r.t. x, we get

$$\frac{1 + y'}{x + y} = 2xy' + 2y \tag{1}$$

This *can* be solved explicitly for y' as

$$y' = \frac{1 - 2xy - 2y^2}{2x^2 + 2xy - 1} \tag{2}$$

We want $y'(0)$. Even though the given equation cannot be solved for y, if we put $x = 0$ in it we get $\ln y = 0$ and hence $y = 1$. So putting $x = 0$ and $y = 1$ in (2), we get $y'(0) = \frac{1-2}{-1} = 1$. Note that we could also have gotten this directly from (1). In fact, this is a better idea, because although in the present problem, we can solve (1) explicitly for y' to get (2), this may not always be possible. And, even when possible, it does not really simplify the work, but in fact, increases the chances of numerical mistakes.

Q. 4 Let $f(x) = \begin{cases} x^\alpha \ln x, & x > 0 \\ 0, & x = 0 \end{cases}$. Then Rolle's theorem is applicable to f on $[0, 1]$ if α equals

(A) -2 (B) -1 (C) 0 (D) $\frac{1}{2}$

Answer and Comments: (D). We certainly have $f(0) = f(1) = 0$. In addition, for Rolle's theorem to apply f must be continuous on $[0, 1]$ and differentiable on $(0, 1)$. Both the requirements hold for $x > 0$. We only need to check what happens at 0. As $x \to 0^+$, $\ln x \to -\infty$. The other factor, viz. x^α tends to $0, 1$ or ∞ depending upon whether $\alpha > 0, \alpha = 0$ or $\alpha < 0$. Hence, the product $x^\alpha \ln x$ tends to $-\infty$ for $\alpha \leq 0$. For $\alpha > 0$, however, it is of the form $0 \times -\infty$ and can be shown to tend to 0, by applying l'Hôpital's rule to $\dfrac{\ln x}{x^{-\alpha}}$. Hence f is continuous at 0 if and only if $\alpha > 0$.

Q. 5 Let $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$. Then in $(-\infty, \infty)$, f

(A) is bounded (B) has a local maximum
(C) has a local minimum (D) is strictly increasing

Answer and Comments: (D). One can get this from the general properties of a cubic function. First of all such a function can never be bounded. Moreover, as the leading coefficient is positive, f will either be strictly increasing, or else it will have both a local maximum and a local minimum. Since only one of the alternatives is given to be correct, it has to be (D), by elimination.

The honest way, of course, is to consider $f'(x) = 3x^2 + 2bx + c$. This is a quadratic with discriminant $4(b^2 - 3c)$. Since $b^2 < c$ (which also implies $c > 0$), rewriting the discriminant as $4(b^2 - c) - 8c$ we see that it is always negative. Hence f' maintains its sign throughout. Moreover, this sign is positive. So f is strictly increasing all over $\mathbb{R}$.

Q. 6 Let $f(x)$ be a differentiable function and $\displaystyle\int_0^{t^2} xf(x)dx = \frac{2}{5}t^5$, $t > 0$. Then $f\left(\dfrac{4}{25}\right) =$

(A) $\dfrac{2}{5}$ (B) $\dfrac{5}{2}$ (C) $-\dfrac{2}{5}$ (D) 1

Answer and Comments: (A). Here the function $f(x)$ is not given directly. Instead, we are given the integral of its product with x. From this we can recover $f(x)$ using the second form of the Fundamental Theorem of Calculus. Differentiating both the sides of the given equality w.r.t. t, we get (using Equation (18) in Chapter 17, with a slight change of notation),

$$t^2 f(t^2) 2t = 2t^4$$

for $t > 0$. This gives $f(t^2) = t$. Putting $t = \dfrac{2}{5}$ gives the answer.

Q. 7 The value of the integral $\displaystyle\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$ is

(A) $\frac{\pi}{2}+1$ (B) $\frac{\pi}{2}-1$ (C) π (D) 1

Answer and Comments: (B). Rewrite the integrand as $\frac{\sqrt{1-x}}{\sqrt{1+x}}$. The substitutions $1-x=u^2$ or $1+x=v^2$ will get rid of one of the radical signs but not both. It would be nice if we could get rid of both. Luckily, putting $x=\cos 2\theta$ will do the job. This is the key idea. The rest is routine. (We could, of course, also put $x=\cos\theta$ instead of $x=\cos 2\theta$. Mathematically that hardly makes any difference. But then we would have to work in terms of the trigonometric functions of $\frac{\theta}{2}$ instead of θ and that may increase the chances of numerical errors as every time we would have to keep track of the fractional coefficient.) With $x=\cos 2\theta$, the integrand $\sqrt{\frac{1-x}{1+x}}$ becomes $\tan\theta$ and the given definite integral becomes $\int_0^{\pi/4}\sin^2\theta d\theta = 2\int_0^{\pi/4}(1-\cos 2\theta)d\theta$ which comes out as $\frac{\pi}{2}-1$.

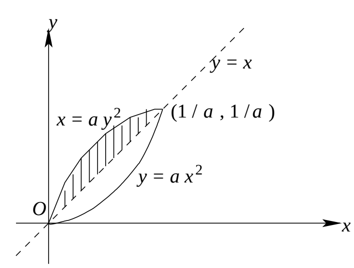
Note that the converted integrand was again in terms of $\cos 2\theta$ which is precisely our old variable x. So probably, the substitution was not needed after all! This comes as an afterthought. But to some persons who are more good at algebraic manipulations, this may strike as the very first idea. Indeed, if we rewrite $\frac{\sqrt{1-x}}{\sqrt{1+x}}$ as $\frac{1-x}{\sqrt{1-x^2}}$, then the indefinite integral $\sqrt{\frac{1-x}{1+x}}dx$ splits as $\int\frac{1}{\sqrt{1-x^2}}dx-\int\frac{x}{\sqrt{1-x^2}}dx$. For the first integral there is a standard formula. For the second one, the substitution $x^2=u$ suggests itself.

It is hard to say which method is better. The best thing is to have a mastery over both, choose the one that suits your inclination better and, in case you can afford the time, to verify the answer by the other method too.

Q. 8 If the area bounded by $y=ax^2$ and $x=ay^2$, $a>0$, is 1 then the value of a is

(A) 1 (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{3}$ (D) $-\frac{1}{\sqrt{3}}$

Answer and Comments: (B). The two parabolas intersect at the point $(1/a, 1/a)$ besides the origin. By symmetry, the area bounded splits into two equal parts, the one above the line $y=x$ and the other below it. Each has area $\frac{1}{2}$. Taking the upper one, this gives an equation



$$\int_0^{1/a}\sqrt{\frac{x}{a}}-x\,dx=\frac{1}{2}$$

Evaluating the integral, this becomes $\frac{2}{3\sqrt{a}}\frac{1}{a\sqrt{a}} - \frac{1}{2a^2} = \frac{1}{2}$. Upon simplification, this becomes $a^2 = \frac{1}{3}$. As $a > 0$, we must have $a = \frac{1}{\sqrt{3}}$.

Q. 9 If $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x,\ y(0) = 1$, then $y\left(\frac{\pi}{2}\right) =$

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

Answer and Comments: (C). This is a straightforward problem in which we have to find a particular solution of a first order differential equation. Fortunately, the d.e. can be cast easily into the separate variables form as

$$\frac{dy}{1+y} = -\frac{\cos x\ dx}{2+\sin x}$$

Integrating both the sides, we have

$$\ln(1+y) = -\ln(2+\sin x) + c$$

or equivalently

$$1+y = \frac{k}{2+\sin x}$$

where k is some constant. The initial condition $y = 1$ when $x = 0$, determines k as 4. Hence $y = \frac{4}{2+\sin x} - 1$. Putting $x = \frac{\pi}{2}$ gives the answer.

Q. 10 If θ and ϕ are acute angles satisfying $\sin\theta = \frac{1}{2}$ and $\cos\phi = \frac{1}{3}$, then $\theta + \phi$ belongs to

(A) $\left(\frac{\pi}{3}, \frac{\pi}{2}\right]$ (B) $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ (C) $\left(\frac{2\pi}{3}, \frac{5\pi}{6}\right)$ (D) $\left(\frac{5\pi}{6}, \pi\right)$

Answer and Comments: (B). The equation $\sin\theta = \frac{1}{2}$, along with the fact that θ is acute, determines θ as $\frac{\pi}{6}$. The equation $\cos\phi = \frac{1}{3}$ also determines ϕ uniquely. But there is no familiar angle whose cosine is $\frac{1}{3}$. So we do not know ϕ exactly. But we can estimate it. The cosine function decreases from 1 to 0. Moreover, $\cos^{-1}(\frac{1}{2})$ is a familiar angle, viz. $\frac{\pi}{3}$. So, ϕ must lie somewhere between $\frac{\pi}{3}$ and $\frac{\pi}{2}$. As $\theta = \frac{\pi}{6}$, the sum $\theta + \phi$ lies in the interval $\left(\frac{\pi}{6} + \frac{\pi}{3}, \frac{\pi}{6} + \frac{\pi}{2}\right)$. The key idea in this problem is that even though the exact value of $\cos^{-1}\frac{1}{3}$ is not a familiar figure, we can get easy lower and upper bounds on it from the properties of the cosine function.

Q. 11 The value of x for which $\sin\left(\cot^{-1}(1+x)\right) = \cos\left(\tan^{-1} x\right)$ is

(A) 1 (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) 0

Answer and Comments: (C). A straightforward way is to begin by expressing both the sides as algebraic functions of x (i.e. without involving any trigonometric functions). For example, using the identity $\sin\theta = \frac{1}{\text{cosec}\,\theta} = \pm\frac{1}{\sqrt{1+\cot^2\theta}}$, the L.H.S. equals $\pm\frac{1}{\sqrt{(x+1)^2+1}}$. Similarly, the R.H.S. equals $\pm\frac{1}{\sqrt{x^2+1}}$. Hence the given equation becomes

$$\frac{1}{\sqrt{x^2+2x+2}} = \pm\frac{1}{\sqrt{x^2+1}} \tag{1}$$

Because of the choice of the sign, this is equivalent to two separate equations. But no matter which sign holds, by squaring both the sides we get $x^2+2x+2 = x^2+1$ which gives $x = -\frac{1}{2}$. In a multiple choice question, there is no need to do anything further. But otherwise, one must show that with this value of x, the given equation indeed holds. This follows because $\tan^{-1}(-\frac{1}{2})$ lies in the fourth quadrant and so its cosine is positive while $\cot^{-1}(-\frac{1}{2}+1)$ lies in the first quadrant and hence its sine is positive.

An alternate approach is to solve the given equation as a trigonometric equation. Call $\cot^{-1}(x+1)$ as α and $\tan^{-1} x$ as β. Then both α, β lie in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. The given equation becomes simply $\sin\alpha = \cos\beta$. Writing $\sin\alpha$ as $\cos(\frac{\pi}{2}-\alpha)$, this further becomes

$$\cos(\frac{\pi}{2}-\alpha) = \cos\beta \tag{2}$$

The general solution of this trigonometric equation is

$$\frac{\pi}{2}-\alpha = 2n\pi \pm \beta \tag{3}$$

where n is an integer. Because α, β both lie in $(-\frac{\pi}{2}, \frac{\pi}{2})$, the only possible value of n is 0. So there are only two possibilities, viz.

$$\frac{\pi}{2}-\alpha = \beta \tag{4}$$

$$\text{or} \quad \frac{\pi}{2}-\alpha = -\beta \tag{5}$$

If (4) holds, then $\cot\alpha = \tan\beta$. But this means $x+1 = x$ which is impossible. Hence (5) must hold. In this case we have $\cot\alpha = -\tan\beta$, which gives $x+1 = -x$, i.e. $x = -\frac{1}{2}$. Note that now there is no need to check the original equation since all the conversions we made were reversible.

Q. 12 The sides a, b and c of a triangle are in the ratio $1 : \sqrt{3} : 2$. Then the angles A, B, and C of the triangle are in the ratio

(A) $3 : 5 : 2$ (B) $1 : \sqrt{3} : 2$ (C) $3 : 2 : 1$ (D) $1 : 2 : 3$

Answer and Comments: (D). The straightforward way is to determine the angles using the cosine formula. Since they depend only on the relative proportions of the sides, we may suppose without loss of generality that $a = 1$, $b = \sqrt{3}$ and $c = 2$. The cosine formula will give us the values of $\cos A, \cos B$ and $\cos C$, from which we shall get the values of $\angle A, \angle B$ and $\angle C$. This method would apply if the relative proportions of the sides were some other numbers too. But in the present problem, we get a much quicker solution if we observe that the sides can also be taken to be in the ratio $\frac{1}{2} : \frac{\sqrt{3}}{2} : 1$. By the sine rule, $\sin A, \sin B$ and $\sin C$ are also in the same ratio. But the numbers $\frac{1}{2}, \frac{\sqrt{3}}{2}$ and 1 *are* in fact the sines of $30°, 60°$ and $90°$. As these three figures add to 180 degrees, they must in fact be the angles of the triangle ABC. So they are in the ratio $1 : 2 : 3$.

The problem is a good example of how sometimes a particular numerical data can lead to a quicker solution than the general method. Dividing each of the three figures by 2 each did the trick.

Q. 13 The area of the triangle formed by the line $x + y = 3$ and the angle bisectors of the pair of straight lines $x^2 - y^2 + 2y = 1$ is

(A) 2 (B) 3 (C) 4 (D) 6

Answer and Comments: (A). The formula for the equation of the pair of angle bisectors of a general pair of straight lines is too complicated and too seldom used to deserve to be remembered. Nor will it be of much help here since in this problem we need to find the bisectors individually. The best bet is to take advantage of the special features of the data that permit an *ad-hoc* reasoning. The equation of the pair of straight lines can be rewritten as

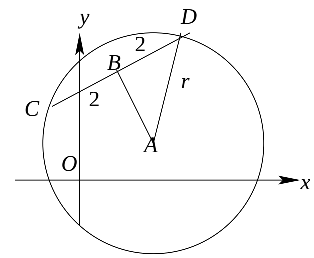
$$x^2 = (y - 1)^2 \qquad (1)$$

Hence the two lines are given by $y - 1 = \pm x$. Separately, the lines are $y = x + 1$ and $y = -x + 1$. They are inclined at $45°$ to the axes. So, even without drawing a diagram, their angle bisectors are parallel to the axes. The two lines intersect at the point $(0, 1)$. Call this point as A. Then the equations of the angle bisectors are $x = 0$ and $y = 1$. These meet the given line $x + y = 3$ at the points $B = (0, 3)$ and $C = (2, 1)$ (say) respectively. The area of the triangle ABC can be found from the determinant formula. Or, since the triangle is already known to be right-angled at A, we simply take half the product of the sides AB and AC each of which equals 2.

Q. 14 The radius of the circle, having centre at $(2, 1)$ whose one chord is a diameter of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is

(A) 1 (B) 2 (C) 3 (D) $\sqrt{3}$

Answer and Comments: (C). Let r be the desired radius. Let A denote $(2,1)$ and B denote the centre of the other circle, viz. $(1,3)$. Let CD be the diameter of the second circle which is also a chord of the first circle. Then B is the midpoint of CD and further $AB \perp CD$. The radius of the second circle is 2 units as we see from its equation by completing squares. So in the right-angled triangle ABD, r is the hypotenuse while $BD = 2$. From the coordinates of A and B, we get $AB = \sqrt{(2-1)^2 + (1-3)^2} = \sqrt{5}$. Hence $r = \sqrt{4+5} = 3$.



This solution is a good combination of coordinate geometry and pure geometry. Although the data is in terms of coordinates, it is utilised through pure geometry. Coordinate geometry was needed only to determine the radius and the centre of the second circle. The problem can also be done purely by coordinates (i.e. without using pure geometry). The equation of a circle with radius r and centre at $(2,1)$ is

$$x^2 + y^2 - 4x - 2y + 5 - r^2 = 0 \tag{1}$$

The equation of the other circle is given as

$$x^2 + y^2 - 2x - 6y + 6 = 0 \tag{2}$$

The equation of the common chord of these two circles is obtained by subtracting (1) from (2). It comes as

$$2x - 4y + 1 - r^2 = 0 \tag{3}$$

Now, this chord is given to be a diameter of the second circle. Hence its centre, viz. $(1,3)$, must satisfy (3). This gives $2 - 12 + 1 - r^2 = 0$, i.e. $r^2 = 9$. As r has to be positive we get $r = 3$.

In essence this solution amounts to considering the family of all circles centred at the point $(2,1)$. This is a 1-parameter family, the parameter being the radius r. We need one equation to determine the value of this parameter. Such an equation is provided by the piece of data which says that the common chord of this circle with the other given circle is a diameter of the latter.

The second solution is simpler and obviates the need to draw a diagram. Moreover, it also applies in more general situations. For example, instead of being given that the common chord is a diameter of the second circle, we could have been given that it has a certain length or that it subtends a certain angle at some given point. Any such piece of information would give us an equation in r, solving which we can get the answer.

The pure geometry solution can also be modified accordingly. But such a modification may not always be easy. But then, that's precisely why pure geometry solutions have their own charm.

Q. 15 The angle between the tangents drawn from the point $(1, 4)$ to the parabola $y^2 = 4x$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

Answer and Comments: (B). A very straightforward problem once the key idea is hit, viz., that to determine the desired angle, what we need is neither the points of contacts of the tangents, nor even their equations, but merely their slopes. So the right start is to express the equation of a typical tangent to the parabola $y^2 = 4x$ in terms of its slope, say m. This is a well-known equation given by

$$y = mx + \frac{1}{m} \tag{1}$$

Since this tangent passes through $(1, 4)$, we have $4 = m + \frac{1}{m}$, or equivalently,

$$m^2 - 4m + 1 = 0 \tag{2}$$

This is a quadratic in m consistent with the fact two tangents can be drawn from $(1, 4)$ to the given parabola. Their slopes, say m_1 and m_2 are precisely the roots of (2). We could identify them by solving (2). But that is hardly necessary. The (acute) angle, say θ, between the two tangents from $(1, 4)$ is given by

$$\tan\theta = \frac{|m_1 - m_2|}{1 + m_1 m_2} \tag{3}$$

From (2) we know $m_1 + m_2 = 4$ and $m_1 m_2 = 1$. To find θ from (3) we need to express $|m_1 - m_2|$ in terms of $m_1 + m_2$ and $m_1 m_2$. This can be done by writing $|m_1 - m_2|$ as $\sqrt{(m_1 - m_2)^2}$ and hence as

$$\begin{aligned} |m_1 - m_2| &= \sqrt{m_1^2 + m_2^2 - 2m_1 m_2} \\ &= \sqrt{(m_1 + m_2)^2 - 4m_1 m_2} \qquad (4) \end{aligned}$$

A straight substitution now gives $|m_1 - m_2| = \sqrt{12} = 2\sqrt{3}$. Hence from (3), $\tan\theta = \sqrt{3}$ which gives $\theta = \frac{\pi}{3}$.

Q. 16 The locus of the middle point of the intercept of the tangents drawn from an external point to the ellipse $x^2 + 2y^2 = 2$, between the coordinate axes is

(A) $\dfrac{1}{x^2} + \dfrac{1}{2y^2} = 1$ (B) $\dfrac{1}{4x^2} + \dfrac{1}{2y^2} = 1$

(C) $\dfrac{1}{2x^2} + \dfrac{1}{4y^2} = 1$ (D) $\dfrac{1}{2x^2} + \dfrac{1}{y^2} = 1$

Answer and Comments: (C). As in most locus problems, the correct start is to take the moving point as $P = (h, k)$. Then the extremities of the intercept of the tangent are $(2h, 0)$ and $(0, 2k)$ because P is given to be the midpoint of this intercept. So, in the intercepts form the equation of the tangent is

$$\frac{x}{2h} + \frac{y}{2k} = 1 \tag{1}$$

We can now apply the condition that this line be a tangent to the given ellipse, which when expressed in the standard form is

$$\frac{x^2}{2} + \frac{y^2}{1} = 2 \tag{2}$$

But instead of relying on such a readymade formula, we can, in effect, derive it. A typical point Q on (2) is of the form $(\sqrt{2}\cos\theta, \sin\theta)$, for some θ. The equation of the tangent at Q is

$$\frac{\sqrt{2}\cos\theta\ x}{2} + \frac{\sin\theta\ y}{1} = 1 \tag{3}$$

(This too, can be derived fresh, by first calculating the slope of the tangent at Q using differentiation. But that is probably going to the other extreme. The formula for the tangent at a point on a conic is both natural and of frequent use and hence worth remembering. The same cannot be said about the condition for tangency.)

Since (1) and (3) represent the same straight line, comparing the coefficients, we get

$$\frac{1}{2h} = \frac{\cos\theta}{\sqrt{2}} \tag{4}$$

$$\text{and} \quad \frac{1}{2k} = \sin\theta \tag{5}$$

Eliminating θ from (4) and (5) gives

$$\frac{1}{2h^2} + \frac{1}{4k^2} = 1 \tag{6}$$

As usual, to obtain the locus replace h and k by x and y respectively. (Note the advantage of the parametric equations of an ellipse here. Instead of taking Q as $(\sqrt{2}\cos\theta, \sin\theta)$, we could as well have taken it as (x_1, y_1). In that case instead of (3), we would have had

$$\frac{x_1 x}{2} + \frac{y_1 y}{1} = 1 \tag{7}$$

which looks simpler. Further, instead of (4) and (5), we would have gotten

$$\frac{1}{2h} = \frac{x_1}{2} \quad (8)$$

$$\text{and} \quad \frac{1}{2k} = y_1 \quad (9)$$

respectively. These too look better at least because of the absence of the ugly radical signs. But there is a price to pay. To get an equation in h and k by eliminating x_1 and y_1, we need one more equation besides (8) and (9). That comes from the fact that the point Q lies on the ellipse and hence satisfies

$$\frac{x_1^2}{2} + \frac{y_1^2}{1} = 1 \quad (10)$$

We can now get (6) by eliminating x_1 and y_1 from (8), (9) and (10). This is more time consuming than eliminating the single variable θ from (4) and (5). In effect, taking the point Q as $(\sqrt{2}\cos\theta, \sin\theta)$ instead of as (x_1, y_1) means we have already extracted whatever information is available from (10) and used it to reduce the number of parameters from 2 to 1.)

Q. 17 If the line $2x + \sqrt{6}y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is

(A) $(-2, \sqrt{6})$ (B) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (C) $(-5, 2\sqrt{6})$ (D) $(4, -\sqrt{6})$

Answer and Comments: (D). One method is to solve the equations of the line and the hyperbola simultaneously. Putting $x = 1 - \frac{\sqrt{3}}{\sqrt{2}}y$ in the equation of the hyperbola we get

$$1 - \sqrt{6}y + \frac{3}{2}y^2 - 2y^2 = 4 \quad (1)$$

which simplifies to $y^2 + 2\sqrt{6}y + 6 = 0$. This is a quadratic which has $-\sqrt{6}$ as a double root. This is consistent with the fact that the line touches the hyperbola, because a point of tangency can be thought of as a pair of coinciding points of intersection. In fact, if (1) had two distinct roots, that should alert you that something has gone wrong in your calculations. Knowing y as $-\sqrt{6}$, $x = 1 + \frac{\sqrt{3}}{\sqrt{2}}\sqrt{6} = 1 + 3 = 4$. Hence the point of contact is $(4, -\sqrt{6})$.

An alternative method is to take a general point on the hyperbola and equate the equation of the tangent at it with that of the given line. For this, first cast the equation of the hyperbola in the standard form as $\frac{x^2}{4} - \frac{y^2}{2} = 1$. Then a typical point, say P on it is of the form $(2\sec\theta, \sqrt{2}\tan\theta)$.

The equation of the tangent at P is

$$\frac{2\sec\theta\, x}{4} - \frac{\sqrt{2}\tan\theta\, y}{2} = 1 \tag{2}$$

Since (2) represents the same line as the given line, viz., $2x + \sqrt{6}y = 2$, by comparing the coefficients, we get

$$\begin{aligned} \sec\theta &= 2 & (3)\\ \text{and} \quad -\sqrt{2}\tan\theta &= \sqrt{6} & (4) \end{aligned}$$

which give the point of contact as $(4, -\sqrt{6})$.

Q. 18 If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{a} \cdot \vec{b} = 1$ and $\vec{a} \times \vec{b} = \hat{j} - \hat{k}$, then $\vec{b} =$

(A) $\hat{i}$ (B) $\hat{i} - \hat{j} + \hat{k}$ (C) $2\hat{j} - \hat{k}$ (D) $2\hat{i}$

Answer and Comments: (A). As the vectors are given in terms of their components, the most straightforward method to tackle the problem is to obtain $\vec{b}$ by finding its components. Let $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. We need three equations to determine the three unknowns b_1, b_2, b_3. Equality of the dot product $\vec{a} \cdot \vec{b}$ with 1 gives one such equation, viz.

$$b_1 + b_2 + b_3 = 1 \tag{1}$$

Also, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \end{vmatrix} = (b_3 - b_2)\hat{i} + (b_1 - b_3)\hat{j} + (b_2 - b_1)\hat{k}$. We are given that this vector equals $\hat{j} - \hat{k}$. Equating the components of both the sides, this gives a system of three equations :

$$\begin{aligned} b_3 - b_2 &= 0 & (2)\\ b_1 - b_3 &= 1 & (3)\\ \text{and} \quad b_2 - b_1 &= -1 & (4) \end{aligned}$$

Note that these three equations are not independent of each other. You can get any one of them from the other two. Consequently, they cannot determine the values of b_1, b_2, b_3 uniquely. This is to be expected because in general a vector $\vec{b}$ is not uniquely determined by its cross product with a fixed vector $\vec{a}$. Any multiple of $\vec{a}$ can be added to $\vec{b}$ without affecting the cross product $\hat{a} \times \vec{b}$. (Observations like this save you from being baffled. Otherwise you might think that the earlier equation (1) is superfluous and try to determine b_1, b_2, b_3 solely from (2), (3) and (4). This is destined to fail.)

Nevertheless, from (4) and (5) we get $b_2 = b_3$ and $b_1 = b_3 + 1$. Putting these into (1), we get $3b_3 + 1 = 1$ and hence $b_3 = 0$. So $b_1 = 1$ and $b_2 = 0$. Therefore $\vec{b} = \hat{i}$.

The solution above was purely algebraic in the sense that the problem about vectors was reduced to the problem of solving a system of three equations in three unknowns. This is analogous to solving a problem of geometry using coordinates. But just as there are pure geometry solutions, some problems on vectors are more conveniently amenable to vector methods, as elaborated in Comment No. 7 of Chapter 21. In the present problem, suppose, more generally, that we are given two vectors $\vec{a}$ and $\vec{v}$ and a scalar λ and the problem asks you to find a vector $\vec{b}$ which satisfies the conditions :

$$\vec{a} \cdot \vec{b} = \lambda \quad (5)$$

$$\text{and} \quad \vec{a} \times \vec{b} = \vec{v} \quad (6)$$

Obviously, the answer is to be expressed in terms of $\vec{a}, \vec{v}$ and λ. Here the algebraic method has little role. All we have is the equations (5) and (6). Taking cross product of the two sides of (6) with the vector $\vec{a}$ and using the standard expansion for the vector triple product, we get

$$\begin{aligned} \vec{a} \times \vec{v} &= \vec{a} \times (\vec{a} \times \vec{b}) \\ &= (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b} \\ &= \lambda\vec{a} - |\vec{a}|^2\vec{b} \quad (7) \end{aligned}$$

We assume $\vec{a} \neq \vec{0}$ (as otherwise the problem is degenerate). Then $|\vec{a}| \neq 0$ and so from (7), we get

$$\vec{b} = \frac{\lambda\vec{a} - \vec{a} \times \vec{v}}{|\vec{a}|^2} \quad (8)$$

This solves the more general problem. The solution to the given problem is obtained by putting $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ (so that $|\vec{a}|^2 = 3$), $\vec{v} = \hat{j} - \hat{k}$ and $\lambda = 1$. Then $\vec{a} \times \vec{v} = -2\hat{i} + \hat{j} + \hat{k}$ and we get $\vec{b} = \hat{i}$.

Q. 19 A unit vector in the plane of the vectors $2\hat{i} + \hat{j} + \hat{k}$ and $\hat{i} - \hat{j} + \hat{k}$ and orthogonal to the vector $5\hat{i} + 2\hat{j} + 6\hat{k}$ is

(A) $\dfrac{6\hat{i} - 5\hat{k}}{\sqrt{61}}$ (B) $\dfrac{3\hat{j} - \hat{k}}{\sqrt{10}}$ (C) $\dfrac{2\hat{i} - 5\hat{j}}{\sqrt{29}}$ (D) $\dfrac{2\hat{i} + \hat{j} - 2\hat{k}}{3}$

Answer and Comments: (B). Another straightforward problem about vectors, where the desired vector is found by solving a system of equations in its components. Let $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ be a desired vector. Since it is a unit vector, we have, first of all,

$$u_1^2 + u_2^2 + u_3^2 = 1 \quad (1)$$

Orthogonality of $\vec{u}$ with the vectors $5\hat{i} + 2\hat{j} + 6\hat{k}$ gives one more equation, viz.,

$$5u_1 + 2u_2 + 6u_3 = 0 \quad (2)$$

We need one more equation. This is provided by equating the box product of the vectors $\vec{u}, 2\hat{\imath}+\hat{\jmath}+\hat{k}$ and $\hat{\imath}-\hat{\jmath}+\hat{k}$ with 0, as these three vectors are given to lie in the same plane. This gives $\begin{vmatrix} u_1 & u_2 & u_3 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$, i.e.

$$2u_1 - u_2 - 3u_3 = 0 \quad (3)$$

(2) and (3) constitute a system of two homogeneous linear equations in three unknowns. By Theorem 7 in Chapter 3, the general solution of this system is

$$u_1 = 0r,\ u_2 = 27r \ \text{ and } \ u_3 = -9r \quad (4)$$

where r is some real number. To determine the value of r we use (1), i.e. the fact that $\vec{u}$ is a unit vector. Note that it is foolish to actually compute $(27)^2$ and 9^2 and add. All we want is a unit vector in the direction of the vector $27\hat{\jmath} - 9\hat{k}$. We might as well divide by the scalar 9 and get a unit vector in the direction of the vector $3\hat{\jmath} - \hat{k}$. There are two such vectors, viz. $\pm\frac{3\hat{\jmath} - \hat{k}}{\sqrt{61}}$. The question asks only for one of these two.

In the solution above, the coplanarity of the vectors $\vec{u}$, $2\hat{\imath}+\hat{\jmath}+\hat{k}$ and $\hat{\imath}-\hat{\jmath}+\hat{k}$ was used by setting their box product to 0 so as to get (3). We can use it in a more direct way and get an alternate solution as follows. Any vector, say $\vec{u}$ in the plane spanned by $2\hat{\imath}+\hat{\jmath}+\hat{k}$ and $\hat{\imath}-\hat{\jmath}+\hat{k}$ is a linear combination of them i.e. is of the form $\alpha(2\hat{\imath}+\hat{\jmath}+\hat{k})+\beta(\hat{\imath}-\hat{\jmath}+\hat{k})$ for some scalars α and β, or in terms of components,

$$\vec{u} = (2\alpha+\beta)\hat{\imath} + (\alpha-\beta)\hat{\jmath} + (\alpha+\beta)\hat{k} \quad (5)$$

We now have only two unknowns, viz., α and β and so we need only two equations to determine them. One such equation is provided by the fact that $\vec{u}$ is a unit vector while the other is provided by the fact that it is perpendicular to the vector $5\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$. These conditions give, respectively,

$$\begin{aligned} 6\alpha^2 + 4\alpha\beta + 3\beta^2 &= 1 \quad (6) \\ 18\alpha + 9\beta &= 0 \quad (7) \end{aligned}$$

From (7), $\beta = -2\alpha$. Putting this into (6) gives a quadratic in α, viz., $10\alpha^2 = 1$. This gives $\alpha = \pm\frac{1}{\sqrt{10}}$ and correspondingly $\beta = \mp\frac{1}{\sqrt{10}}$. Putting these into (5) gives $\vec{u} = \pm\frac{3\hat{\jmath}-\hat{k}}{\sqrt{10}}$ as before. This solution is quicker because the coplanarity of the vectors $\vec{u}$, $2\hat{\imath}+\hat{\jmath}+\hat{k}$ and $\hat{\imath}-\hat{\jmath}+\hat{k}$ has been used to reduce the number of variables by one. (In spirit this is analogous to taking a typical point on a conic in its parametric form.) Whenever a problem amounts to constructing a system of equations (based on the given data) in several variables and solving it, it is a good idea to see if you can use some piece of information cleverly to reduce the number of variables.

The problem can also be done by vector methods. Call the vectors $2\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\hat{j}+\hat{k}$ and $5\hat{i}+2\hat{j}+6\hat{k}$ as $\vec{a}, \vec{b}, \vec{c}$ respectively. We want to find a unit vector $\vec{u}$ in the plane spanned by $\vec{a}$ and $\vec{b}$ and which is perpendicular to $\vec{c}$. Letting $\vec{u} = \alpha\vec{a} + \beta\vec{b}$ we get a system of equations in α and β, viz.,

$$\alpha^2|\vec{a}|^2 + 2\alpha\beta(\vec{a}\cdot\vec{b}) + \beta^2|\vec{b}|^2 = 1 \quad (8)$$

$$\alpha(\vec{a}\cdot\vec{c}) + \beta(\vec{b}\cdot\vec{c}) = 0 \quad (9)$$

which is very analogous to (6) and (7) and is solved by exactly the same method to get two possible values for $\vec{u}$ in terms of the vectors $\vec{a}, \vec{b}, \vec{c}$, their lengths and their dot products with each other. Putting back the particular values of $\vec{a}, \vec{b}, \vec{c}$ will give the answer. So the second solution above was really a vector method in disguise.

Q. 20 If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then the value of k is

(A) $\frac{2}{9}$ (B) $\frac{9}{2}$ (C) 0 (D) -1

Answer and Comments: (B). Typical points on the two lines are of the form $(2r+1, 3r-1, 4r+1)$ and $(s+3, 2s+k, s)$ respectively where r and s are some real numbers. The lines will intersect if and only if the system of equations

$$\begin{aligned} 2r+1 &= s+3 && (1)\\ 3r-1 &= 2s+k && (2)\\ \text{and}\quad 4r+1 &= s && (3) \end{aligned}$$

has a solution. This can be done in a number of ways which differ more in the form than in their mathematical content. For example, we can solve any two of these, say (1) and (3) for r and s to get $r = -\frac{3}{2}$, $s = -5$. Putting these values into (2) we get

$$-\frac{9}{2} - 1 = -10 + k \quad (4)$$

which implies $k = \frac{9}{2}$. Or one can rewrite the equations in the form

$$\begin{aligned} 2r-s &= 2 && (5)\\ 3r-2s &= k+1 && (6)\\ \text{and}\quad 4r-s &= -1 && (7) \end{aligned}$$

which is a non-homogeneous system of three linear equations in the two unknowns r and s. By Theorem 6 in Chapter 3, this system is consistent (i.e. has a solution) if and only if the determinant $\begin{vmatrix} 2 & -1 & 2 \\ 3 & -2 & k+1 \\ 4 & -1 & -1 \end{vmatrix}$ vanishes. Upon expansion this is the same as (4).

Q. 21 The value of λ for which the system of equations $2x - y - z = 12$, $x - 2y + z = -4$, $x + y + \lambda z = 4$ has no solution is

(A) 3 (B) -3 (C) 2 (D) -2

Answer and Comments: (D). A straightforward problem about the existence of solution to a non-homogeneous system of linear equations in three unknowns. The determinant, say Δ, of the coefficients is

$$\Delta = \begin{vmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} \quad (1)$$

By direct expansion, $\Delta = -3\lambda - 6$. If $\Delta \neq 0$, by Cramer's rule the system has a unique solution. When $\Delta = 0$ either the system may have no solution or it may have infinitely many solutions. This gives $\lambda = -2$. In a multiple choice test this is good enough to tick the correct answer. But to complete the solution honestly, one must show that for this value of λ, the first possibility (viz. no solutions) indeed holds. This can be done by adding the last two equations, which gives $2x - y - z = 0$. But it is inconsistent with the first equation, viz., $2x - y - z = 12$.

Q. 22 If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then the value of α is

(A) ± 3 (B) ± 2 (C) ± 5 (D) 0

Answer and Comments: (A). It would be horrendous to first calculate the matrix A^3 and then take its determinant. The key idea is to use the product formula for the determinant which says that the determinant of the product of two square matrices (of the same order) is the product of their determinants. Applying this twice in succession, we have $|A^3| = |A|^3$. So in the present problem, $|A| = 5$. But, by a direct computation, $|A| = \alpha^2 - 4$. So we get $\alpha^2 - 4 = 5$ giving $\alpha = \pm 3$.

Q. 23 ω is an imaginary cube root of unity. If $(1 + \omega^2)^m = (1 + \omega^4)^m$, then the least positive integral value of m is

(A) 6 (B) 5 (C) 4 (D) 3

Answer and Comments: (D). It would be horrendous to apply the binomial theorem to expand the two sides. Instead, we note that ω satisfies $1 + \omega + \omega^2 = 0$. Further $1 + \omega^4 = 1 + \omega^3\omega = 1 + \omega = -\omega^2$. So the given equation is reduced to $(-\omega)^m = (-\omega^2)^m$, i.e. to $\omega^m = \omega^{2m}$. Cancelling ω^m from both the sides, we get $\omega^m = 1$. The least positive m for which this holds is 3.

Q. 24 If $x^2 + 2ax + 10 - 3a > 0$ for all $x \in I\!R$, then

(A) $-5 < a < 2$ (B) $a < -5$ (C) $a > 5$ (D) $2 < a < 5$

Answer and Comments: (A). The given expression is a quadratic in x with positive leading coefficient. So it will be positive for all x if and only if it never vanishes, i.e. if and only if it has no real roots. From the discriminant criterion, this is equivalent to saying that $a^2 < 10 - 3a$, i.e. $a^2 - 3a - 10 < 0$. The L.H.S. again is a quadratic expression in a with positive leading coefficient and roots -5 and 2. So the inequality $a^2 - 3a - 10 < 0$ will hold if and only if a lies between -5 and 2.

The problem requires two applications of the criterion for the sign of a quadratic expression. Because of their frequent occurrences in the JEE, such problems have become rather routine.

Q. 25 If one root of the equation $x^2 + px + q = 0$ is the square of the other, then

(A) $p^3 + q^2 - q(3p + 1) = 0$ (B) $p^3 + q^2 + q(1 + 3p) = 0$
(C) $p^3 + q^2 + q(3p - 1) = 0$ (D) $p^3 + q^2 + q(1 - 3p) = 0$

Answer and Comments: (D). Take the roots as α and α^2. Then

$$\alpha^2 + \alpha = -p \quad (1)$$

and

$$\alpha^3 = q \quad (2)$$

We need to eliminate α in these two equations. The most direct way to do so would be to write $\alpha = q^{1/3}$ from (2) and put this value in (1). However, instead of working with $q^{1/3}$, it is preferable to retain it as α and then whenever we encounter α^3, replace it with q. From (1),

$$\begin{aligned} -p^3 &= \alpha^3(\alpha + 1)^3 \\ &= q(\alpha^3 + 1 + 3\alpha^2 + 3\alpha) \\ &= q(q + 1 - 3p) \end{aligned}$$

which implies $p^3 + q^2 + q(1 - 3p) = 0$.

Q. 26 The first term of an infinite geometric progression is x and its sum is 5. Then

(A) $0 \leq x \leq 10$ (B) $0 < x < 10$ (C) $-10 < x < 0$ (D) $x > 10$

Answer and Comments: (B). Let r be the common ratio of the progression. The key idea is that for the infinite progression to have a sum it is necessary that $|r| < 1$, i.e.

$$-1 < r < 1 \tag{1}$$

The problem consists of translating this inequality to an inequality about x. From the formula of the sum of an infinite G.P., we have

$$\frac{x}{1-r} = 5 \tag{2}$$

or in other words,

$$r = 1 - \frac{x}{5} \tag{3}$$

Combining (3) and (1) we get,

$$-1 \ < \ 1 - \frac{x}{5} \ < \ 1 \tag{4}$$

Subtracting 1 from all the three terms and multiplying throughout by 5 we get

$$-10 \ < \ -x \ < \ 0 \tag{5}$$

which implies $0 < x < 10$.

Q. 27 ${}^{n-1}C_r = (k^2 - 3)\ {}^nC_{r+1}$, if $k \in$

(A) $[-\sqrt{3}, \sqrt{3}]$ (B) $(-\infty, -2)$ (C) $(2, \infty)$ (D) $(\sqrt{3}, 2]$

Answer and Comments: (D). The question is somewhat ambiguous because it is not specified for which values of n and r the given equality holds. Even though the binomial coefficient ${}^nC_r$ has a combinatorial significance only when $0 \leq r \leq n$, sometimes we have to consider it even when $n > r$ (in which case we set it equal to 0). In such a case the equality is of the form $0 = 0$ and hence is true for all k.

We take the question to mean that the given equality holds for some integers n, r with n positive and $0 \leq r \leq n - 1$. Using the identity

$${}^nC_{r+1} = \frac{n}{r+1} \times {}^{n-1}C_r \tag{1}$$

(which is proved by directly expanding both the sides), we get

$$k^2 = 3 + \frac{r+1}{n} \tag{2}$$

The problem amounts to asking which of the given lower and upper bounds on k will make (2) possible for suitable n and r. Because of our assumptions on n and r, we have $1 \leq r + 1 \leq n$. Hence the ratio in the R.H.S. of

(2) always lies between $\frac{1}{n}$ and 1. Hence by (2),

$$k^2 \in [3 + \frac{1}{n}, 4] \tag{3}$$

This closed interval is contained in the semi-open interval $(3, 4]$ for every $n \geq 1$. This implies that $k \in [-2, -\sqrt{3}) \cup (\sqrt{3}, 2]$. From this it *does not* follow that k is necessarily in the interval $(\sqrt{3}, 2]$. But the question only asks us to give a sufficient condition for the equality to hold. In that case (D) can be taken to be correct answer. (Strictly speaking this is also wrong, because it is not true that for *every* $k \in (\sqrt{3}, 2]$ the given equation will hold for some n and r. For example, if k^2 is irrational, then there cannot exist any integers n and r satisfying the given condition. This question is therefore poorly set and confusing. Again, it is hard to say whether the flaw lies in the original question or in its reported version.)

Q. 28 Three distinct numbers are selected from first 100 natural numbers. The probability that all three numbers are divisible by both 2 and 3 is

(A) $\frac{4}{35}$ (B) $\frac{4}{33}$ (C) $\frac{3}{55}$ (D) $\frac{4}{1155}$

Answer and Comments: (D). As 2 and 3 are relatively prime to each other, divisibility by both of them is equivalent to divisibility by 6. In the set $\{1, 2, \ldots, 99, 100\}$ there are exactly 16 numbers that are divisible by 6. So the total number of selections is $\binom{100}{3}$ while the number of favourable selections is $\binom{16}{3}$. Therefore the desired probability is

$$\frac{\binom{16}{3}}{\binom{100}{3}} = \frac{16 \times 15 \times 14}{100 \times 99 \times 98} = \frac{4}{1155}$$

MAIN PAPER OF JEE 2004

Problem No. 1: Find the centre and the radius of the circle given by $\frac{|z - \alpha|}{|z - \beta|} = k, k \neq 1$ where, $z = x + iy$ is a complex number and $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$ are fixed complex numbers.

Analysis and Solution: Let A, B be the points represented by the complex numbers α, β respectively. It is well-known (see the solution to Exercise (8.12)(b)) that the given locus is a circle (popularly called the Apollonius circle). Further, this circle has as a diameter the segment PQ where P and Q are the points which divide the segment AB in the ratio $k : 1$ internally and externally, respectively. The problem does not ask you to prove this. In fact the problem seems to be designed to test whether you already know this fact from geometry. For, those who do will have an easy time. All that they have to do is to identify P and Q. From the section formula, we have,

$$P = \frac{k\beta + \alpha}{k+1} \quad \text{and} \quad Q = \frac{k\beta - \alpha}{k-1} \tag{1}$$

Then the centre of the circle is simply the midpoint of PQ. From (1), this corresponds to the complex number

$$\frac{\frac{k\beta+\alpha}{k+1} + \frac{k\beta-\alpha}{k-1}}{2} \tag{2}$$

which comes out to be $\dfrac{k^2\beta - \alpha}{k^2 - 1}$ upon simplification. The radius is simply $\frac{1}{2}|PQ|$. Again, from (1) this equals $\dfrac{|\frac{k\beta+\alpha}{k+1} - \frac{k\beta-\alpha}{k-1}|}{2}$, which, upon simplification, becomes

$$\frac{k|\alpha - \beta|}{|k^2 - 1|} \tag{3}$$

(Note that if $k = 1$, then the locus is not a circle but a straight line, viz., the perpendicular bisector of the segment AB. This is consistent with (3) because a straight line can be considered as a limiting case of a circle as its radius tends to ∞.)

This solution depends on the knowledge of a fact from geometry which, although quite well-known in the past, is not so anymore because of the deemphasis on pure geometry in the school syllabi. So those who don't know it are at a slight disadvantage. The papersetters are apparently aware of this. For, the complex numbers z, α, β are also specified in terms of their real and imaginary parts, which are not at all needed in the solution above. The idea probably is that those who do not know the circle of Apollonius, can attempt a direct solution by converting the problem to a problem in coordinate geometry. If we do so, then the given equation, upon squaring, becomes

$$(x - \alpha_1)^2 + (y - \alpha_2)^2 = k^2[(x - \beta_1)^2 + (y - \beta_2)^2] \tag{4}$$

which can be rewritten as

$$x^2 + y^2 + \frac{2\alpha_1 - 2k^2\beta_1}{k^2 - 1}x + \frac{2\alpha_2 - 2k^2\beta_2}{k^2 - 1}y + \frac{k^2(\beta_1^2 + \beta_2^2) - \alpha_1^2 - \alpha_2^2}{k^2 - 1} = 0 \tag{5}$$

This is the equation of a circle with centre at $\left(\dfrac{k^2\beta_1 - \alpha_1}{k^2 - 1}, \dfrac{k^2\beta_2 - \alpha_2}{k^2 - 1}\right)$. This tallies with (2). The radius can be obtained by completing the squares. The calculation is not all that horrendous if we keep in mind that $\dfrac{1}{k^2 - 1}$ is a common factor for all the three terms involved. The radius then comes out to be

$$\frac{\sqrt{(\alpha_1 - k^2\beta_1)^2 + (\alpha_2 - k^2\beta_2)^2 - (k^2 - 1)[k^2(\beta_1^2 + \beta_2^2) - \alpha_1^2 - \alpha_2^2]}}{|k^2 - 1|} \tag{6}$$

Strictly speaking, this ought to be an acceptable answer, since it is in terms of the data of the problem. But, generally, obvious simplifications are mandatory. Carrying these out, the numerator of (6) becomes, after lots of cancellations, $k\sqrt{(\alpha_1-\beta_1)^2+(\alpha_2-\beta_2)^2}$. Going back to the complex numbers, this is the same as $k|\alpha-\beta|$ and so we see that (6) tallies with (3).

To conclude, two morals can be drawn. First, it pays to know pure geometry, even if it is not directly a part of the syllabus. Secondly, even if you don't, coordinates are a reliable device to lean onto, provided you are prepared to do the algebraic manipulations.

Problem No. 2: Suppose $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four distinct vectors satisfying the conditions $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$. Prove that $\vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{d}\neq\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{d}$.

Analysis and Solution: We are given

$$\begin{aligned} \vec{a}\times\vec{b} &= \vec{c}\times\vec{d} \qquad (1)\\ \text{and}\quad \vec{a}\times\vec{c} &= \vec{b}\times\vec{d} \qquad (2)\end{aligned}$$

Subtracting (2) from (1) we get $\vec{a}\times(\vec{b}-\vec{c})=(\vec{c}-\vec{b})\times\vec{d}$. The R.H.S. can be rewritten as $\vec{d}\times(\vec{b}-\vec{c})$. Taking all terms on one side this gives

$$(\vec{a}-\vec{d})\times(\vec{b}-\vec{c})=\vec{0} \qquad (3)$$

Now suppose, the assertion to be proved did not hold true, i.e. suppose we had,

$$\vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{d}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{d} \qquad (4)$$

Then we can rewrite this as

$$\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{d}-\vec{c}\cdot\vec{d} \qquad (5)$$

and further as

$$\vec{a}\cdot(\vec{b}-\vec{c})=\vec{d}\cdot(\vec{b}-\vec{c}) \qquad (6)$$

because the dot product is commutative. Bringing all terms on one side this becomes

$$(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c})=0 \qquad (7)$$

Now let $\vec{u}=\vec{a}-\vec{d}$ and $\vec{v}=\vec{b}-\vec{d}$. Then as the given vectors are all distinct, neither $\vec{u}$ nor $\vec{v}$ is a zero vector. By (3), $\vec{u}$ and $\vec{v}$ must be parallel to each other. But by (7), they must also be perpendicular to each other. This is a contradiction. So Equation (4) is false.

The problem is based on the basic properties of the cross and the dot products. Both are distributive over addition, which was used several times. Also needed was the commutativity of the dot product and the anti-commutativity of the cross product. The key idea was that the hypothesis implies Equation (3) because of the properties of the cross product, while the negation of the conclusion, (i.e., Equation (4)), implies (7) because of similar properties of the dot product. But (4) and (7) can never hold together. When you present the solution in an examination, it is not necessary to write so many intermediate steps. Nor is it necessary that you introduce separate symbols for the vectors $\vec{a} - \vec{d}$ and $\vec{b} - \vec{c}$.

Problem No. 3: Using permutations or otherwise, prove that $\dfrac{(n^2)!}{(n!)^n}$ is an integer, where n is a positive integer.

Analysis and Solution: The basic idea is that if we have n types of objects, with say k_i objects of type i for $i = 1, 2, \ldots, n$, then the number of permutations of these $k_1 + k_2 + \ldots + k_n$ objects equals

$$\frac{(k_1 + k_2 + \ldots + k_n)!}{k_1!k_2!\ldots k_n!} \tag{1}$$

As the number of permutations is always a whole number, i.e. an integer, (1) in particular implies that the numerator is divisible by the denominator. If we put each $k_i = n$, we get that $(n^2)!$ is divisible by $(n!)^n$.

The argument given here is combinatorial in that it consists of giving the ratio (1) a combinatorial interpretation as the cardinality of a suitable set S, in the present problem the set of all permutations of the given objects. An alternate proof can also be given which is purely number theoretic. It is based on the following simple result.

Lemma: The product of any n consecutive positive integers is divisible by $n!$.

Again, the best proof is combinatorial. Denote the consecutive integers by $k, k+1, k+2, \ldots, k+n-2$ and $k+n-1$. Then the ratio $\dfrac{k(k+1)\ldots(k+n-1)}{n!}$ is nothing but the binomial coefficient $\binom{k+n-1}{n}$. This, being the number of ways to choose n objects from $k+n-1$ ones, is an integer. A proof by induction is also possible. (See Comment No. 9 of Chapter 4.)

Now, assuming the Lemma, we can show that $\dfrac{(n^2)!}{(n!)^n}$ is an integer by directly expanding the numerator and grouping the factors suitably. The numerator is the product of the consecutive integers from 1 to n^2. Let us break these into n factors, each of which is itself the product of n consecutive integers. Thus, the first factor is the product $1 \times 2 \times 3 \times \ldots \times n$ (which is simply $n!$), the second factor is the product of the integers from $n+1$ to $2n$, and so on, the last factor being the product of the n consecutive integers from $n^2 - n + 1$ to n^2. By the Lemma, each product is divisible by $n!$. As there are n such factors, we see that the entire product $(n^2)!$ is divisible by the n-th power of $n!$, i.e. by $(n!)^n$, completing the solution.

Although the problem does not ask it, it is interesting to note that a stronger result is actually true. That is, $(n^2)!$ is divisible not only by the n-th power of $n!$, but by its $(n+1)$-th power, i.e. by $(n!)^{n+1}$. A combinatorial proof of this is given in the answer to Exercise (4.25)(b). A purely number-theoretic proof is also possible using the Lemma above, except that we apply it with n replaced by $n-1$, i.e. we shall use the fact that the product of any $n-1$ consecutive positive integers is divisible by $(n-1)!$.

As before, we write $(n^2)!$ out as the product of the integers from 1 to n^2. However, this time we mark with a hat every factor that is a multiple of n, i.e. the factors $n, 2n, 3n, \ldots, n^2-n$ and n^2. For example, for $n=5$, we have

$$25! = 1.2.3.4.\hat{5}.6.7.8.9.\hat{10}.11.12.13.14.\hat{15}.16.17.18.19.\hat{20}.21.22.23.24.\hat{25} \qquad (2)$$

Note that before the first hat and in between any two consecutive hats, we have a product of $n-1$ consecutive positive integers. So each such product is divisible by $(n-1)!$. As there are n such products, we see that the product of the non-hatted integers is divisible by $[(n-1)!]^n$. Now consider the hatted integers. Their product is simply $n^n \times n!$. Putting the two together, the product of all integers (hatted as well as non-hatted) from 1 to n^2 is divisible by $[(n-1)!]^n \times n^n \times n!$ which is nothing but $(n!)^{n+1}$.

Although it does not earn you any extra credit to prove a stronger result in an examination like the JEE, when you are using a problem as a motivation to learn, it is always a good idea to see if you can get a stronger conclusion with the same hypothesis or the same conclusion with a weaker hypothesis. This way you can sometimes anticipate a new question.

Continuing in the same vein, it is natural to see if we can strengthen the given problem even more. That is, can we prove that $(n^2)!$ is divisible by a still higher power of $(n!)$, i.e. by $(n!)^{n+2}$? But this is easily seen to be false for $n=2$ or 3. In fact, if you look keenly at the factorisation of $(n^2)!$ as given above, you will realise that if n is a prime, then $(n^2)!$ cannot be divisible by $(n!)^{n+2}$. For a composite n the situation is different.

Problem No. 4: If M is a 3×3 matrix with $M^T M = I$ (where I is the 3×3 identity matrix), and det $(M) = 1$, then prove that det $(M - I) = 0$.

Analysis and Solution: Matrices have been re-introduced into the JEE syllabus very recently. So it is not quite clear what degree of depth may be assumed of them. We first give a fairly sophisticated solution under the additional assumption that the entries of M are real and then follow it by an elementary one (which is valid even for complex matrices).

Let $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Let λ be a (possibly complex) variable and let

$$p(\lambda) = \det(M - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} \tag{1}$$

Then $p(\lambda)$ is a polynomial in λ of degree 3. The assertion det $(M - I) = 0$ is equivalent to saying that $\lambda = 1$ is a root of $p(\lambda)$, or using the terminology introduced in Exercise (2.37), that 1 is a characteristic root of the matrix M. So the problem amounts to showing that if M is a real 3×3 matrix with $M^T M = I$ and det $(M) = 1$ then M must have 1 as a characteristic root.

To see this, let λ_1, λ_2 and λ_3 be the characteristic roots of M, i.e. the roots of the polynomial $p(\lambda)$. As we are given that det $(M) = 1$, from Exercise (3.27), we have

$$\lambda_1 \lambda_2 \lambda_3 = 1 \tag{2}$$

The entries of M and hence the coefficients of $p(\lambda)$ are all real. Hence, even though some of the roots may be complex, they must occur in conjugate pairs. The product of each such pair is a positive real number. So from (2), the number of real characteristic roots is odd and further their product is positive. Hence there is at least one characteristic root of M which is positive. Without loss of generality, we suppose λ_1 is one such root. We claim $\lambda_1 = 1$. For this it suffices to show that $\lambda_1^2 = 1$. This can be done as follows.

We already know that det $(M - \lambda_1 I) = 0$. By Theorem 8 of Chapter 3, this is equivalent to saying that the system of real linear equations

$$(M - \lambda_1 I)\mathbf{x} = \mathbf{0} \tag{3}$$

has at least one non-trivial real solution. In slightly different terms this means that there exists a non-zero column vector $\mathbf{x}$ with real entries such that

$$M\mathbf{x} = \lambda_1 \mathbf{x} \tag{4}$$

Taking transpose of both the sides, we get

$$\mathbf{x}^T M^T = \lambda_1 \mathbf{x}^T \tag{5}$$

Multiplying (5) with (4),

$$\mathbf{x}^T M^T M \mathbf{x} = \lambda_1^2 \mathbf{x}^T \mathbf{x} \tag{6}$$

Now we use (for the first time) the hypothesis that $M^T M = I$. Because of this, the L.H.S. of (6) is simply $\mathbf{x}^T \mathbf{x}$ which is a positive real number as it is the sum of the squares of the entries of the real column vector $\mathbf{x}$ which is assumed to be non-zero. Hence from (6), we get $\lambda_1^2 = 1$. As proved before, this shows that $\lambda_1 = 1$ and completes the solution.

Note that to get the existence of a positive characteristic root of M from (2), we crucially needed that 3 is an odd integer. Indeed, the argument would go through if M were an $n \times n$ matrix for any odd positive integer n. The only change is that in deriving (3), we would need Theorem 4 of Chapter 3, which is the higher dimensional analogue of Theorem 8.

Although this solution is fairly lengthy (far more lengthy than justified by the number of marks the problem carries) and involved, it truly explains the significance of the condition det $(M - I) = 0$, viz. that the system $M\mathbf{x} = \mathbf{x}$ has at least one non-trivial solution. A slicker but less educative solution, based entirely on the properties of determinants, can be given as follows. Specifically, the properties that we shall need are :

(i) the determinant of the transpose of a (square) matrix is the same as the determinant of that matrix,

(ii) determinant of the product of two square matrices of the same order is the product of their determinants, and

(iii) if A is an $n \times n$ matrix and α is a real number then det $(\alpha A) = \alpha^n$det (A).

We are given a 3×3 matrix M with $M^T M = I$ and det $(M) = 1$. From (i), we have det $(M^T) = 1$. Further,

$$\begin{aligned}
\text{det }(M - I) &= \text{det }(M^T)\text{det }(M - I) \\
&= \text{det }(M^T(M - I)) \quad \text{by (ii)} \\
&= \text{det }(I - M^T) \\
&= \text{det }(I - M^T)^T \quad \text{by (i) again} \\
&= \text{det }(I - M) \\
&= (-1)^3\text{det }(M - I) \quad \text{by (iii)} \qquad (7)
\end{aligned}$$

Thus we have shown that det $(M - I) = -$ det $(M - I)$, which implies det $(M - I) = 0$ as was to be proved. Note again that this argument will remain valid if instead of 3 we have any odd positive integer. Note also that this solution does not require that the entries of M be real. So it is more general than the earlier solution. In this sense, it is the best solution.

Although neither solution made a crucial use of the fact that the size of the matrix M is 3, this case (along with the assumption that the entries of M are real) has an interesting application for vectors. Denote the row vectors of the matrix M by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ respectively. In other words, if we take M as above, then $\mathbf{u}_1 = a_{11}\mathbf{i} + a_{12}\mathbf{j} + a_{13}\mathbf{k}$ etc., where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is a fixed righthanded orthonormal basis. It is then easy to see (and fairly well-known) that the condition that $M^T M = I$ is equivalent to saying that $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is an orthonormal basis. (This follows because the (p, q)-th entry of the product matrix $M^T M$ is simply the dot product of the vectors $\mathbf{u}_p$ and $\mathbf{u}_q$, for $p = 1, 2, 3$ and $q = 1, 2, 3$.) The condition $M^T M = I$ implies det (M^T)det $(M) =$ det $(I) = 1$ and hence det $(M) = \pm 1$ since M^T and M have the same determinants. It is further true that the $+$ or $-$ sign holds depending upon whether the orthonormal basis $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is right-handed or left-handed. We are given that the first possibility holds. Thus the hypothesis of the problem can be paraphrased to say that M is a 3×3 matrix whose row vectors form a right-handed orthonormal basis.

The conclusion of the problem is that the determinant of the matrix $M - I$ is 0. If we write this matrix out, then we see that its row vectors are nothing but the vectors $\mathbf{u}_1 - \mathbf{i}, \mathbf{u}_2 - \mathbf{j}$ and $\mathbf{u}_3 - \mathbf{k}$. As the determinant equals the box product of its row vectors, its vanishing is equivalent to saying that the three vectors $\mathbf{u}_1 - \mathbf{i}, \mathbf{u}_2 - \mathbf{j}$ and $\mathbf{u}_3 - \mathbf{k}$ are linearly dependent. This can be further paraphrased to say that there exists a non-zero vector $\mathbf{v}$ which is perpendicular to all of them. Put differently this means that there exists a non-zero vector $\mathbf{v}$ such that

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{v} &= \mathbf{i} \cdot \mathbf{v} && (8)\\ \mathbf{u}_2 \cdot \mathbf{v} &= \mathbf{j} \cdot \mathbf{v} && (9)\\ \text{and} \quad \mathbf{u}_3 \cdot \mathbf{v} &= \mathbf{k} \cdot \mathbf{v} && (10) \end{aligned}$$

Here $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ was a fixed right-handed system we started with. We could as well replace it with some other right-handed system, say $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$. Then geometrically, this means that given any two right-handed systems $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, there exists a direction (viz., the direction of the vector $\mathbf{v}$), which makes the same angles with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ respectively as it does with the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ respectively. This is a purely geometric result. Thus we see that matrices have non-trivial applications to geometry. (A direct proof can be given using the concept of what are called direction cosines.) As already noted, the result can also be expressed in terms of the box product. Thus our little discovery can also be put as follows.

For any two right-handed systems $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, we have

$$[(\mathbf{u}_1 - \mathbf{w}_1)\ (\mathbf{u}_2 - \mathbf{w}_2)\ (\mathbf{u}_3 - \mathbf{w}_3)] = 0 \qquad (11)$$

whose direct proof is not easy.

Problem No. 5: A plane passes through the point $(1, 1, 1)$ and is parallel to the lines whose direction ratios are $(1, 0, -1)$ and $(-1, 1, 0)$. If it cuts the coordinate axes at A, B, C respectively, find the volume of the tetrahedron $OABC$.

Analysis and Solution: This is a very straightforward problem. Let the points A, B, C be respectively $(a, 0, 0), (0, b, 0)$ and $(0, 0, c)$. The volume of the tetrahedron $OABC$ equals $\frac{1}{6}|abc|$. So the real task is to determine a, b, c from the data. The equation of the plane passing through A, B, C is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (1)$$

A vector $\mathbf{n}$ perpendicular to this plane is given by

$$\mathbf{n} = \frac{1}{a}\mathbf{i} + \frac{1}{b}\mathbf{j} + \frac{1}{c}\mathbf{k} \qquad (2)$$

Direction ratios of a line are nothing but the components of any vector parallel to that line. Thus, we are given that the vectors $\mathbf{i} - \mathbf{k}$ and $-\mathbf{i} + \mathbf{j}$ are parallel to the plane (1). So each of them is perpendicular to the vector $\mathbf{n}$. This gives us two equations in a, b, c, viz.,

$$\frac{1}{a} - \frac{1}{c} = 0 \tag{3}$$

$$\text{and} \quad \frac{1}{a} - \frac{1}{b} = 0 \tag{4}$$

We need one more equation in a, b, c. This is provided by the hypothesis that the plane given by (1) passes through the point $(1, 1, 1)$. This means

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 \tag{5}$$

Solving (3) to (5) simultaneously is very easy, because (3) and (4) immediately give $a = b = c$. (5) then gives $a = b = c = 3$. The desired volume, therefore, is $\frac{1}{6} \times 3 \times 3 \times 3 = \frac{9}{2}$ cubic units. (One can also find the volume of the tetrahedron $OABC$ by taking the box product of the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ and writing this box product as a certain 3×3 determinant. But that is silly. When it is already known that the edges OA, OB, OC of the tetrahedron are at right angles to each other, its volume is simply one-sixth of the product of their lengths.)

The purpose of the problem is apparently to test whether a student knows how to apply vectors to the so-called solid coordinate geometry. This branch of mathematics is not elaborately studied in schools. But analogy with plane coordinate geometry and use of vectors can often pull you through at least as far as the lines and planes are concerned. For example, Equation (1) above is the direct analogue of the equation of a straight line in the two intercepts form. In going from (1) to (2) all you need is the dot product. More generally, the vector $p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$ is normal to any plane whose equation is of the form $px + qy + rz = s$ for any s. To see this, first fix any point $P_0 = (x_0, y_0, z_0)$ on the plane. Then $px_0 + qy_0 + rz_0 = s$. Moreover, a point $P = (x, y, z)$ will lie on the plane if and only if $p(x - x_0) + q(y - y_0) + r(z - z_0) = s - s = 0$. This amounts to saying that the vector $\overrightarrow{P_0 P}$ is perpendicular to the vector $p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$.

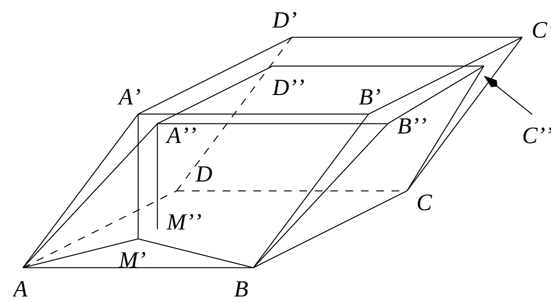
Problem No. 6: T is a parallelopiped in which A, B, C, D are vertices of one of the faces. The corresponding vertices of the opposite face are A', B', C', D'. T is compressed to form another parallelopiped S with the face $ABCD$ remaining the same and A', B', C', D' shifted to A'', B'', C'', D'' respectively. If the volume of S is 90% that of T, prove that the locus of A'' is a plane.

Analysis and Solution: This is one of those problems which take a while to understand but which are very simple once you understand them correctly. Let M' be the foot of the perpendicular from the vertex A' to the plane of the face

$ABCD$. Then the volume, say V' of T is given by

$$V' = \text{area of } ABCD \times A'M' \tag{1}$$

When T is compressed to S, the vertices A, B, C, D remain fixed. But the vertices A', B', C', D' move to A'', B'', C'', D'' respectively. Let M'' be the foot of the perpendicular from A'' to the plane of the face $ABCD$. Since the volume of S is 90 % of V', we get



$$\frac{9}{10}V' = \text{area of } ABCD \times A''M'' \tag{2}$$

Dividing (2) by (1) and calling $A'M'$ as h, we get

$$A''M'' = \frac{9}{10}h \tag{3}$$

Here h is a constant. $A''M''$ is the (perpendicular) distance of the point A'' from the base plane $ABCD$. The points B'', C'', D'' are also at this height above the base plane. All these four points lie in a plane parallel to the base plane and at a height $\frac{9}{10}h$ above it. A'' is free to move anywhere on this plane. (Once the position of A'' is fixed, those of B'', C'' and D'' are automatically determined.) Hence the locus of A'' is a plane parallel to the plane of the face $ABCD$ and at a distance $\frac{9}{10}h$ from it, where h is the perpendicular distance between the faces $ABCD$ and $A'B'C'D'$.

In case you have difficulty in understanding a problem in solid geometry, an analogy with a problem in a plane often helps. In the present problem, for example, the plane analogue would be as follows. Suppose $ABB'A'$ is a parallelogram. Its vertices A', B' are moved to points A'', B'' respectively so that $ABB''A''$ is a parallelogram whose area is 90% of that of the parallelogram $ABCD$. Find the locus of A''. Since the area of a parallelogram with a fixed base is proportional to its height, we see that the vertex A'' has to move on a line parallel to AB. The key idea in the present problem is exactly same, except that instead of the area of a parallelogram, we are dealing with the volume of a parallelopiped with one face fixed.

(In the problem asked, the vertex A'' (and hence also the vertices B'', C'' and D'') could also lie on the other side of the face $ABCD$ at a distance $\frac{9}{10}h$ from it. In that case, the locus of A'' is not a single plane, but a union of two parallel planes lying on opposite sides of the plane of the face $ABCD$, each at a distance $\frac{9}{10}h$ from it. But in the statement of the problem, it is given that the original parallelopiped T is 'compressed' to the new parallelopiped S. Physically this means that the volume is gradually reduced from V' to $\frac{9}{10}V'$. In this process the moving vertices cannot possibly cross over to the other side of the plane $ABCD$. So this possibility is discarded.)

The solution given above was based on 'pure' solid geometry. The problem can also be done using coordinates or vectors (in which case the volume of the parallelopiped will be the scalar triple product of the vectors representing the three edges meeting at one of the vertices). But neither approach will simplify the problem. In fact, they may only serve to complicate it. The essential idea is very simple, viz., that the volume of a paralleopiped is the product of the area of any face of it and the perpendicular distance between that face and its opposite face. There is no need to complicate it with coordinates or vectors. Indeed, this problem is a good test of a student's ability to grasp the essence of a problem without cluttering it with unnecessary gadgets.

Problem No. 7: Given that $\left|\cos^{-1}\left(\frac{1}{n}\right)\right| < \frac{\pi}{2}$, evaluate

$$\lim_{n\to\infty} \quad \frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right) - n$$

[*Hint:* If f is differentiable at 0 and $f(0)=0$, then $\lim\limits_{n\to\infty} nf\left(\frac{1}{n}\right) = f'(0)$.]

Analysis and Solution: This is a problem about evaluating the limit of a sequence, specifically, the sequence $\{a_n\}$ where $a_n = \frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right) - n$ as n tends to ∞. The purpose of specifying that $\left|\cos^{-1}\left(\frac{1}{n}\right)\right| < \frac{\pi}{2}$ is not quite clear. Apparently, it comes from the fact that traditionally the inverse cosine function is not single-valued. Thus, depending on what value we assign to $\cos^{-1}\left(\frac{1}{n}\right)$, the value of a_n will change and so will the value of $\lim\limits_{n\to\infty} a_n$. In fact, with some interpretations of $\cos^{-1}\left(\frac{1}{n}\right)$, the given limit need not even exist. To avoid this confusion, it is stipulated that for a positive integer n, by $\cos^{-1}\left(\frac{1}{n}\right)$, we shall only mean that unique value of it which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. (However, nowadays, it is a standard practice that the values of $\cos^{-1}$ are always taken to lie in the interval $[0,\pi]$. With this convention, whenever n is an integer greater than 1, $\cos^{-1}\left(\frac{1}{n}\right)$, will automatically lie between 0 and $\frac{\pi}{2}$ and so there was really no need to specify this. In fact, today a stipulation like this is more likely to cause than to clear a confusion.)

Anyway, coming back to the limit $\lim\limits_{n\to\infty} a_n$, the hint given suggests that we should first cast a_n into the form $nf\left(\frac{1}{n}\right)$ for some suitable function $f(x)$ which is differentiable and vanishes at 0. We do not have much of a choice here. We want

$$nf\left(\frac{1}{n}\right) = \frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right) - n \tag{1}$$

to hold for every positive integer n. Dividing both the sides by n, and calling $\frac{1}{n}$ as x, this means we should take

$$f(x) = \frac{2}{\pi}(1+x)\cos^{-1}x - 1 \tag{2}$$

(Strictly speaking we need to have (2) only for x of the form $\frac{1}{n}$ where n is a positive integer. We are free to define it differently for other values of x. But nothing is to be gained by doing so. The definition above makes sense for all $x \in [-1, 1]$ and satisfies the conditions that $f(0) = 0$ and also that $f'(0)$ exists.)

To get the answer, all that is left is to find $f'(0)$ where f is as above. Direct differentiation gives

$$f'(x) = \frac{2}{\pi}(\cos^{-1}x - \frac{1+x}{\sqrt{1-x^2}}) \tag{3}$$

for all $x \in (-1, 1)$. In particular,

$$f'(0) = \frac{2}{\pi}(\frac{\pi}{2} - 1) = 1 - \frac{2}{\pi} \tag{4}$$

which is the answer to the problem.

The problem would have been more interesting had the hint not been given. With the hint thrown in, the only work needed is a manual one. But then again, without the hint, the problem would have been too complicated. The most straightforward approach in that case would be to replace the discrete variable n by a continuous variable x and consider

$$\lim_{x\to\infty} \quad \frac{2}{\pi}(x+1)\cos^{-1}\left(\frac{1}{x}\right) - x \tag{5}$$

Since $\cos^{-1}\left(\frac{1}{x}\right)$ tends to $\frac{\pi}{2}$ as $x \to \infty$ we see that (5) is an indeterminate form of the type $\infty - \infty$. There are several ways to convert this to some more manageable indeterminate form. For example, taking a factor x out, the limit above equals

$$\lim_{x\to\infty} \quad x \times \left[\frac{2}{\pi}\left(1 + \frac{1}{x}\right)\cos^{-1}\left(\frac{1}{x}\right) - 1\right] \tag{6}$$

which is an indeterminate form of the $\infty \times 0$ type. To cast it into the most familiar $\frac{0}{0}$ form, we make a change of variable. Put $u = \frac{1}{x}$. Then $u \to 0^+$ as

$x \to \infty$ and so the desired limit now becomes

$$\lim_{u \to 0^+} \frac{\frac{2}{\pi}(1+u)\cos^{-1} u - 1}{u} \tag{7}$$

We can now apply l'Hôpital's rule to evaluate this limit. Note that we now have to find the derivative of the numerator w.r.t. u. But the numerator is precisely $f(u)$ where f is as defined by (2) above. So ultimately this approach involves the same work at the end but we spend a lot more time in reaching that stage than we would with the hint. The result in the hint itself is hardly profound, as all it needs is the definition of a derivative. But had the hint not been given, only the highly experienced persons could have thought of applying the result contained in the hint to the present problem.

Problem No. 8: Let $p(x) = 51x^{101} - 2323x^{100} - 45x + 1035$. Using Rolle's theorem prove that $p(x)$ has at least one root in the interval $(45^{1/100}, 46)$.

Analysis and Solution: As in the last problem, in this problem, there is little for you to figure out. Everything is given to you. The conclusion of Rolle's theorem deals with a zero of the derivative of a function. So if we want to apply Rolle's theorem to get a root of $p(x)$ the first task is to get hold of a function, say $f(x)$, whose derivative is $p(x)$. Clearly, any such function is of the form

$$f(x) = \frac{1}{2}x^{102} - 23x^{101} - \frac{45}{2}x^2 + 1035x + c \tag{1}$$

where c is an arbitrary constant to be determined later. Evidently, $f(x)$ is differentiable on the entire real line and so there is no difficulty in applying Rolle's theorem to $f(x)$ from this angle. The only thing is that we have to ensure that

$$f(45^{1/100}) = 0 \tag{2}$$

$$\text{and} \qquad f(46) = 0 \tag{3}$$

by suitably choosing the constant c. (We are doing this because in the customary formulation of the Rolle's theorem, when it is to be applied to a function $f(x)$ over an interval $[a, b]$, it is required that $f(a) = f(b) = 0$. What is really vital is only that f takes equal values at a and b and not that this common value be 0. For, if $f(a) = f(b)$, one can apply Rolle's theorem to the new function $g(x) = f(x) - f(a)$ which indeed vanishes at both a and b. So, we could as well drop the constant c from (1) and instead of (2) and (3), verify merely that $f(45^{1/100}) = f(46)$. But we stick to the traditional formulation of Rolle's theorem.)

Instead of computing $f(46)$ and $f(45^{1/100})$ by directly evaluating every term of (1) for these values of x, a little regrouping will simplify the work considerably. (A hunch for this comes because 46 and 1035 are multiples of 23 and surely there must be some purpose behind choosing such clumsy figures in the problem.) For example, if we put $x = 46$ in (1), we get

$$\begin{aligned} f(46) &= \frac{1}{2} \times (46)^{102} - 23 \times (46)^{101} - \frac{45}{2} \times (46)^2 + 1035 \times 46 + c \\ &= 23 \times (46)^{101} - 23 \times (46)^{101} - 45 \times 23 \times 46 + 23 \times 45 \times 46 + c \\ &= c \qquad (4) \end{aligned}$$

So if we take $c = 0$, we have $f(46) = 0$. Putting $c = 0$ and $x = 45^{1/100}$ in (1), we now have

$$\begin{aligned} f(45^{1/100}) &= \frac{1}{2}(45)^{102/100} - 23 \times (45)^{101/100} - \frac{45}{2}(45)^{2/100} + 1035 \times (45)^{1/100} \\ &= \frac{45}{2} \times (45)^{2/100} - 23 \times 45 \times (45)^{1/100} - \frac{45}{2}(45)^{2/100} \\ &\quad +23 \times 45 \times (45)^{1/100} \\ &= 0 \qquad (5) \end{aligned}$$

Therefore, by Rolle's theorem, $f'(x)$, which is the same as $p(x)$, has at least one root in the interval $(45^{1/100}, 46)$.

The bulk of the computational work in the problem is in proving (4) and (5). This can be shortened a little by factorising the non-constant part of $f(x)$, i.e. by writing it as

$$\begin{aligned} f(x) &= \frac{1}{2}x^{101}(x - 46) - \frac{45}{2}x(x - 46) + c \\ &= \frac{x}{2}(x - 46)(x^{100} - 45) + c \qquad (6) \end{aligned}$$

which makes it easier to see that $f(45^{1/100})$ and $f(46)$ are equal, each being equal to c.

In fact, this alternate method would come to rescue even if the paper-setters had played a dirty trick in framing the problem. Suppose, for example, that the problem had asked you to show, using Rolle's theorem, that the given polynomial $p(x)$ had at least one root in the interval $(1, 50)$ (instead of the interval $(45^{1/100}, 46)$). Note that technically this *does not* mean that the papersetters have asked you to apply Rolle's theorem to the interval $[1, 50]$. Indeed, Rolle's theorem is not applicable to this interval because no matter what c is, $f(1)$ and $f(50)$ can never be equal. However, if we can find a suitable subinterval, say $[a, b]$ of $[1, 50]$, to which Rolle's theorem can be applied, then the interval (a, b), *and therefore the bigger interval* $(1, 50)$ would contain at least one zero of $p(x)$.

The crucial question now is how to identify these magic numbers a and b. Nobody can get the correct values $a = 45^{1/100}$ and $b = 46$ by trial and error or simply by guessing from (1). However, from the factorised form (6), it is not so difficult to get them. Fortunately, the papersetters have not played this dirty trick and so a solution based on the computations (4) and (5) is also possible.

Problem No. 9: If $y(x) = \displaystyle\int_{\pi^2/16}^{x^2} \frac{\cos x . \cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$, find $\dfrac{dy}{dx}$ at $x = \pi$.

Analysis and Solution: This is a straightforward application of the second form of the Fundamental Theorem of Calculus (or rather, its generalisation given in Equation (18) of Chapter 17). The only catch is that the variable x also occurs in the integrand besides appearing in the upper limit of integration. In such cases, we cannot apply the Fundamental Theorem blindly. (See Exercise (17.17)(b) for another problem where a similar difficulty arises.)

The correct procedure here is to recognise that as far as the given definite integral is concerned, x and hence any expression depending solely on x is a constant and is therefore to be pulled out of the integral sign. Thus, we have

$$y(x) = \cos x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta \qquad (1)$$

which is a product of two functions of x, the first being the function $\cos x$ and the second being $\int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$. So the derivative can be obtained by applying the product rule for derivatives. In doing this, we shall need the derivative of the second factor, for which we shall apply the Fundamental Theorem without any difficulty now because the integrand does not involve x at all.

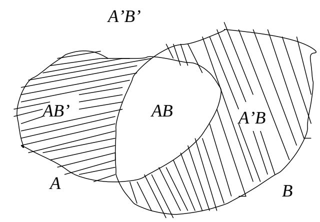
For the actual calculation,

$$\begin{aligned} y'(x) &= -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta + \cos x \frac{d}{dx} \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta \\ &= -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta + \cos x \frac{\cos\sqrt{x^2}}{1+\sin^2\sqrt{x^2}} \frac{d}{dx}(x^2) \\ &= -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta + \frac{2x\cos^2 x}{1+\sin^2 x} \qquad (2) \end{aligned}$$

We are interested in $y'(\pi)$. Since $\sin\pi = 0$, we are spared of the trouble of evaluating the horrible definite integral $\int_{\pi^2/16}^{\pi^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$. The second term has value $\frac{2\pi(-1)^2}{1+0}$ i.e., 2π at $x=\pi$. So $y'(\pi) = 2\pi$.

Problem No. 10: If A, B are two independent events, prove that $P(A\cup B).P(A'\cap B') \leq P(C)$, where C is the event that exactly one of A and B occurs.

Analysis and Solution: In symbols, the event C is the disjunction of the two mutually exclusive events $A \cap B'$ and $A' \cap B$. These events are shown by shaded regions in the accompanying Venn diagram. From the Venn diagram, we see at once that $P(C) = P(A \cup B) - P(A \cap B)$ and also that $P(A' \cap B') = 1 - P(A \cup B)$.



Substituting these values, the inequality we are asked to prove becomes

$$P(A \cup B) - [P(A \cup B)]^2 \leq P(A \cup B) - P(A \cap B) \qquad (1)$$

or, equivalently,

$$P(A \cap B) \leq [P(A \cup B)]^2 \qquad (2)$$

As the events A and B are given to be (mutually) independent, the L.H.S. equals $P(A)P(B)$ and so the problem is now reduced to proving that

$$P(A)P(B) \leq [P(A \cup B)]^2 \qquad (3)$$

But this is very easy to show. Since the event A is contained in the event $A \cup B$, (i.e., the occurrence of A always implies that of $A \cup B$), we have

$$P(A) \leq P(A \cup B) \qquad (4)$$

Similarly,

$$P(B) \leq P(A \cup B) \qquad (5)$$

If we multiply (4) and (5) and keep in mind that all the factors are non-negative, we get (3).

The problem can also be done by letting $x = P(A), y = P(B)$ and then expressing the various other probabilities in terms of x and y. But the solution inspired by the Venn diagram is natural and simple.

Problem No. 11: If a, b, c are positive real numbers, prove that

$$[(1+a)(1+b)(1+c)]^7 > 7^7 a^4 b^4 c^4$$

Analysis and Solution: The inequality to be proved can be rewritten as

$$\left[\frac{(1+a)(1+b)(1+c)}{7}\right]^7 > a^4 b^4 c^4 \qquad (1)$$

The very form of the L.H.S. suggests that the A.M.-G.M. inequality will be needed in the proof. Indeed, if the denominator in the brackets and the exponent of the L.H.S. were 8 instead of 7, then we could write the L.H.S. as the product of three factors, viz., as

$$\left(\frac{1+a}{2}\right)^8 \left(\frac{1+b}{2}\right)^8 \left(\frac{1+c}{2}\right)^8 \tag{2}$$

and applying the A.M.-G.M. inequality separately to each factor we would get

$$\begin{aligned}\left[\frac{(1+a)(1+b)(1+c)}{8}\right]^8 &= \left(\frac{1+a}{2}\right)^8 \left(\frac{1+b}{2}\right)^8 \left(\frac{1+c}{2}\right)^8 \\ &\geq (\sqrt{a})^8(\sqrt{b})^8(\sqrt{c})^8 \\ &= a^4b^4c^4 \end{aligned} \tag{3}$$

It is tempting to try to derive (1) from (3). This would be possible if we could show

$$\left[\frac{(1+a)(1+b)(1+c)}{7}\right]^7 > \left[\frac{(1+a)(1+b)(1+c)}{8}\right]^8 \tag{4}$$

Unfortunately (4) is not true. For, cancelling the common factors (which are all positive) (4) is equivalent to showing that

$$\frac{8^8}{7^7} > (1+a)(1+b)(1+c) \tag{5}$$

which is clearly false if a, b, c are sufficiently large as the L.H.S. is a fixed number.

Nevertheless, the failed attempt has a noteworthy feature. In the proof of (3), we expressed the L.H.S. as a certain product of three factors and showed separately that each factor was bigger than or equal to the corresponding factor of the R.H.S. In other words, the proof of (3) consisted of proving three independent but similar inequalities and multiplying them. It is worthwhile to paraphrase this proof slightly. Let

$$f(x) = \left(\frac{1+x}{2\sqrt{x}}\right)^8 \tag{6}$$

By the A.M.-G.M. inequality, $f(x) \geq 1$ for all $x > 0$. We applied this fact separately with $x = a$, $x = b$ and $x = c$ and multiplying the three inequalities got

$$f(a)f(b)f(c) \geq 1 \tag{7}$$

which is a mere recasting of (3). Let us try a similar approach for proving (1). We can paraphrase (1) to say that

$$g(a)g(b)g(c) > 1 \tag{8}$$

where the function $g(x)$ is defined by

$$g(x) = \frac{(1+x)^7}{7^{7/3}x^4} \tag{9}$$

for $x > 0$. Now comes the crucial observation. If at all a result like (8) is to hold for all positive a, b, c, then it is *mandatory* that $g(x) > 1$ for all $x > 0$. For suppose $g(u) \leq 1$ for some $u > 0$, then taking $a = b = c = u$ we would contradict (8). Thus we have reduced the inequality (1) to showing that $g(x) > 1$ for all $x > 0$. In effect, we have reduced a problem about a function of three variables to a similar problem about a function of just one variable. This simplification was possible because both the L.H.S. and the R.H.S. of (1) could be factorised as products of three highly similar factors, each involving only one of the three variables a, b, c. Had such a factorisation not been possible, this approach would not work. For example, suppose we want to prove the inequality

$$(a + b - c)(b + c - a)(c + a - b) \leq abc$$

for any three positive real numbers a, b, c. This can be done by other means (see Exercise (14.1)). But the method above would not apply.

Therefore, from now on we concentrate all our efforts on showing that $g(x) > 1$ for all $x > 0$. Taking the seventh root, this is equivalent to showing that

$$1 + x > 7^{1/3} x^{4/7} \tag{10}$$

for all $x > 0$. We shall give two proofs of (10). The first and the more straightforward proof uses calculus. To simplify the notations, we put $t = x^{1/7}$. Then (10) is equivalent to proving that

$$1 + t^7 > 7^{1/3} t^4 \tag{11}$$

for all $t > 0$. To do this we let $h(t) = 1 + t^7 - 7^{1/3} t^4$ and study the increasing/decreasing behaviour of this function for $0 < t < \infty$. Since $h'(t) = 7t^6 - 7^{1/3} 4t^3 = t^3(7t^3 - 7^{1/3} 4)$, the only critical point of $h(t)$ in $(0, \infty)$ is $t_0 = \dfrac{4^{1/3}}{7^{2/9}}$. Moreover $h'(t) < 0$ for $0 < t < t_0$ and $h'(t) > 0$ for $t_0 < t < \infty$. Hence the function $h(t)$ has its absolute minimum on $(0, \infty)$ at t_0. Therefore if we can show that $h(t_0) > 0$ then it would mean $h(t) > 0$ for all $t \in (0, \infty)$. And that will prove (10).

Thus we have to show that $h\left(\dfrac{4^{1/3}}{7^{2/9}}\right) > 0$. A direct computation gives

$$\begin{aligned} h\left(\frac{4^{1/3}}{7^{2/9}}\right) &= 1 + \frac{4^{7/3}}{7^{14/9}} - \frac{7^{1/3} 4^{4/3}}{7^{8/9}} \\ &= \frac{7^{14/9} + 4^{7/3} - 7 \times 4^{4/3}}{7^{14/9}} \\ &= \frac{7 \times 7^{5/9} + 16 \times 4^{1/3} - 7 \times 4 \times 4^{1/3}}{7^{14/9}} \\ &= \frac{7 \times 7^{5/9} - 12 \times 4^{1/3}}{7^{14/9}} \end{aligned} \tag{12}$$

It is not necessary to find the exact value of this quantity. All we want to show is that the numerator is positive i.e. to show that $7 \times 7^{5/9}$ is bigger than

$12 \times 4^{1/3}$. With a calculator, the approximate values of these two numbers are 20.6346426 and 19.048812624 respectively. However, calculators are not allowed at the JEE. Doing the comparison without calculators is a little messy, if attempted directly. But with a convenient middleman, it is not so bad. A good middleman is 20. By taking cubes, it is easy to show that $4^{1/3} < \frac{5}{3}$ and so we get $12 \times 4^{1/3} < 12 \times \frac{5}{3} = 20$. So we would be through if we show that $7 \times 7^{5/9} > 20$. This amounts to showing that $7^{14} > (20)^9 = 2^9 \times 10^9$. This can be simplified if we notice that $7^4 = 2401 > 2400 = 24 \times 10^2$. So $7^{14} = 49 \times 7^{12} > 49 \times (2400)^3 = 49 \times (24)^3 \times 10^6$. Therefore, it is enough to show that $49 \times (24)^3 > 2^9 \times 10^3$ or equivalently, to show that $49 \times 3^3 > 10^3$. This can be done by actually computing the L.H.S. by hand. It comes to 1323 which certainly exceeds 1000.

Admittedly, this argument is far too long to be expected for a 4 point question. But note that the gain is that we now got the actual minimum of the function $h(t)$ (viz., $h(t_0)$ where $t_0 = \dfrac{4^{1/3}}{7^{2/9}}$)) and hence also of the function $g(x)$ defined by (9), the minimum now being $g(x_0)$ with $x_0 = t_0^7$. This minimum is less than 1. So we not only have (8) and hence (1) but the best possible lower bound on $g(a)g(b)g(c)$, viz. $(g(x_0))^3$.

Our second proof of (10) is not so ambitious. But it is much shorter. It is based on the A.M.-G.M. inequality applied in a very clever way. Sometimes while applying the A.M.-G.M. inequality to a sum of two terms, it pays to split one or both the terms suitably. For example, showing directly that $2 \sec x + \cos^2 x \geq 3$ (where $0 < x < \frac{\pi}{2}$) is a little complicated. But if we write this expression as the sum of *three* terms, viz., as $\sec x + \sec x + \cos^2 x$ and *then* apply the A.M.-G.M. inequality, the result comes effortlessly.

More generally, suppose a, b are any two positive real numbers and m, n are positive integers. If we rewrite $ma + nb$ as the sum of $m + n$ terms, i.e. as $a + a + \ldots + a + b + b + \ldots + b$ and then apply the A.M.-G.M. inequality we get the following result

$$\left(\frac{ma + nb}{m + n}\right)^{m+n} \geq a^m b^n \tag{13}$$

with equality holding only if $a = b$. (Actually, this is true even when m, n are any two positive numbers, not just integers. But the proof is not so elementary. See Exercise (14.8).)

We can apply (13) to prove (9) as follows. In (13) take $m = 3, n = 4, a = \dfrac{1}{3}$ and $b = \dfrac{x}{4}$. Then we have

$$\left(\frac{1 + x}{7}\right)^7 \geq \frac{x^4}{3^3 4^4} \tag{14}$$

Taking the seventh roots of both the sides, we get

$$1 + x \geq 7 \frac{x^{4/7}}{3^{3/7} 4^{4/7}} \tag{15}$$

To derive (10) from (15), we must show that

$$\frac{7}{3^{3/7}4^{4/7}} > 7^{1/3} \tag{16}$$

which can be done by direct verification. Taking the 21-st powers of both the sides, (16) is equivalent to saying $7^{14} > 3^9 4^{12}$. Rewriting $3^9 4^{12}$ as $3^3 \times 3^6 \times (16)^6 = 27 \times (48)^6$ we indeed see that it is less than $49 \times (49)^6 = (49)^7 = 7^{14}$.

Thus we now have two proofs of the inequality asked, viz., (1). Both these proofs were based on the idea of reducing the inequality for a function of three variables (viz. a, b, c), to an inequality about a function of just one variable. We emphasize that such a reduction can work only when the three variables are completely independent of each other, i.e. we are free to assign any values to them within their permitted variations (e.g. such as that the values must be positive). Indeed we used this freedom in the important observation we made after (9). If there is some relationship (often called a constraint) which the variables have to satisfy together then this method no longer works. Suppose for example, that A, B, C are any three angles in the interval $[0, \pi]$. Then the maximum value of the function $\sin A \sin B \sin C$ is 1 obtained by taking the product of the maximum values of each factor. But if A, B, C are the angles of a triangle ABC then they are not independent of each other. They have to satisfy the constraint $A+B+C = \pi$. In this case, finding the maximum value of $\sin A \sin B \sin C$ is a problem in trigonometric optimisation. And the answer is no longer 1 but much less, viz., is $\frac{3\sqrt{3}}{8}$ attained when the triangle is equilateral.

Finally, we give a proof of (1) which is qualitatively different, that is, it does not go through the factorisation of the L.H.S. of (1). Instead, it begins by expanding the product $(1 + a)(1 + b)(1 + c)$ as the sum of 8 terms, viz.

$$(1 + a)(1 + b)(1 + c) = 1 + a + b + c + ab + bc + ca + abc \tag{17}$$

As already noted, the form of the L.H.S. of (1) suggests that the A.M.-G.M. inequality is to be applied. For this, the expression in the brackets of the L.H.S. ought to be the A.M. of seven terms. But by (17), the numerator is a sum of *eight* and not *seven* terms. So we simply drop one of these 8 terms! Which one should it be? In the interest of symmetry, it has to be either the first term 1 or the last term abc. We desire that (1) should hold for all (positive) values of a, b, c. If these values are large, then so will be abc and dropping such a large term may prove to be injudicious. So we drop the term 1. As the term dropped is positive, we now have

$$\left[\frac{(1 + a)(1 + b)(1 + c)}{7}\right]^7 > \left[\frac{a + b + c + ab + bc + ca + abc}{7}\right]^7 \tag{18}$$

The R.H.S. is indeed the seventh power of the A.M. of seven numbers and so we are now in a position to apply the A.M.-G.M. The product of the seven terms in the numerator of the bracketed expression of the R.H.S. is $a^4b^4c^4$. So their geometric mean is $a^{4/7}b^{4/7}c^{4/7}$. Hence the A.M.-G.M. inequality gives

$$\left[\frac{a+b+c+ab+bc+ca+abc}{7}\right]^7 \geq a^4b^4c^4 \tag{19}$$

Note that here we do not have strict inequality. In fact equality does hold when a, b, c all equal 1. However, in (18) we do have strict inequality. So combining (18) and (19) we get (1).

Admittedly, this is a very tricky solution. But questions about inequalities do sometimes require you to look at the problem with a keen eye and come up with the key idea. In the present problem dropping one of the terms from the R.H.S. of (17) was the key idea.

Problem No. 12: Find the equation of a circle which touches the line $2x + 3y + 1 = 0$ at the point $(1, -1)$ and cuts orthogonally the circle which has the line segment joining $(0, 3)$ and $(-2, -1)$ as a diameter.

Analysis and Solution: This is a straightforward problem about finding a member of a family of curves satisfying certain conditions. The circles touching a given line at a given point form a 1-parameter family of curves. The value of the parameter is then to be determined from the given additional piece of information. The choice of the parameter is left to us and depending upon which parameter we choose, the work will vary a little.

Call the given line L and the desired circle as C. Let C' be the circle having $(0, 3)$ and $(-2, -1)$ as the ends of a diameter. Then the equation of C' is $x(x + 2) + (y - 3)(y + 1) = 0$, i.e.

$$x^2 + y^2 + 2x - 2y - 3 = 0 \tag{1}$$

Using Equation (22) in Chapter 9, the equation of a typical circle which touches L at the point $(1, -1)$ is of the form

$$(x - 1)^2 + (y + 1)^2 + \lambda(2x + 3y + 1) = 0 \tag{2}$$

where λ is a parameter. If the circles represented by (1) and (2) cut each other orthogonally, then (applying the condition for orthogonality), we have

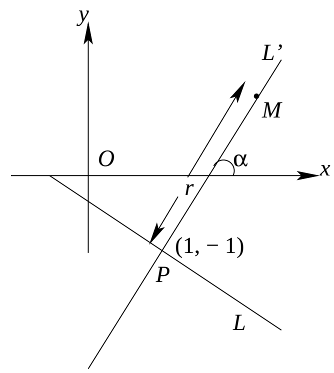
$$2(\lambda - 1) - 2\left(\frac{3\lambda}{2} + 1\right) = -3 + 2 + \lambda \tag{3}$$

This gives the value of λ as $-\frac{3}{2}$. Putting this into (2), the equation of the desired circle C is

$$x^2 + y^2 - 5x - \frac{5}{2}y + \frac{1}{2} = 0 \tag{4}$$

Although this solution is short and simple, it requires you to remember two things, first the parametrisation (given by (2)) of the family of circles touching a given line at a given point and, secondly, the condition for orthogonality of two circles in terms of their equations. Both these are fairly standard. But in case you are unfamiliar with or unsure of them, there is a way to get the answer with fresh thinking using only the very basic formulas and concepts. We give one such solution here.

Call the point $(1, -1)$ as P. Let L' be the line through P perpendicular to L. Then any circle which touches L at P must have its centre, say M, on L'. Moreover, the radius of such a circle must equal the distance between P and M (which is also the perpendicular distance of the centre M from the line L). Now let α be the angle L' makes with the positive x-axis. Then the point M can be written in the form $(1 + r\cos\alpha, -1+r\sin\alpha)$ for some real number r. (Here r may be taken as the algebraic distance of M from P. It is positive on one side of L and negative on the other. The geometric distance of M from P is $|r|$.) It is easy to calculate the angle α. the slope of L is $-\frac{2}{3}$. Hence that of the line L' is $\frac{3}{2}$, i.e., $\tan\alpha = \frac{3}{2}$. So $\sin\alpha = \frac{3}{\sqrt{13}}$ and $\cos\alpha = \frac{2}{\sqrt{13}}$.



With this background, the centre, say M, of a typical circle which touches the line L at P is of the form

$$M = (1 + \frac{2r}{\sqrt{13}}, -1 + \frac{3r}{\sqrt{13}}) \tag{5}$$

for some real number r and its radius is $|r|$. Our interest is in finding the value of r for which this circle cuts the circle C' orthogonally. Again, instead of writing the equation of C' in the form (1), we identify its centre, say M', and radius, say r', and then apply the condition of orthogonality in a geometric form. Since C' has $(0, 3)$ and $(-2, -1)$ as ends of a diameter, we get

$$M' = (\frac{-2+0}{2}, \frac{-1+3}{2}) = (-1, 1) \tag{6}$$

$$\text{and} \quad r' = \frac{1}{2}\sqrt{(0+2)^2 + (3+1)^2} = \frac{1}{2}\sqrt{20} = \sqrt{5} \tag{7}$$

The condition for orthogonality of the intersection of these two circles is

$$(MM')^2 = r^2 + r'^2 \tag{8}$$

From (5) to (8), we get

$$\left(2 + \frac{2r}{\sqrt{13}}\right)^2 + \left(-2 + \frac{3r}{\sqrt{13}}\right)^2 = r^2 + 5 \tag{9}$$

Upon simplification, this is a linear equation in r giving $r = \frac{3\sqrt{13}}{4}$. With this value of r, M becomes $(\frac{5}{2}, \frac{5}{4})$ and the equation of the desired circle C becomes

$$\left(x - \frac{5}{2}\right)^2 + \left(y - \frac{5}{4}\right)^2 = \frac{117}{16} \tag{10}$$

which tallies with (4).

Note that in this second solution, we found the equation of the circle C at the end because the question specifically asked for it. If the question had, instead, asked you to identify only the radius of C, then we already had the answer before getting (10). On the contrary, to get it from the earlier approach, we would have to complete the squares in (4). In that case, the second approach would be slightly better than the first one. But that is only a minor advantage. The real advantage, as pointed out earlier, is that it is based on the most basic and simple ideas and not on any readymade formulas. Such formulas, of course, save you precious time in an examination. But once in a while, you forget some formulas that are not so frequently used (e.g., the condition for orthogonality or for tangency) and in such a case it is important to be able to salvage the situation. (Note, incidentally, that from (5) we could get the equation of a typical circle touching the line L at the point $(1, -1)$ as

$$\left(x - 1 - \frac{2r}{\sqrt{13}}\right)^2 + \left(y + 1 - \frac{3r}{\sqrt{13}}\right)^2 = r^2 \tag{11}$$

or, in a simpler form,

$$(x-1)^2 + (y+1)^2 - \frac{2r}{\sqrt{13}}(2x + 3y + 1) = 0 \tag{12}$$

which is the same as (2) if we take $\lambda = -\frac{2r}{\sqrt{13}}$. This is not surprising. In effect, in the second approach we have *derived* the parametric equation of the given family of curves instead of using it readymade. That is why the second solution took longer than the first.)

Finally, there is also a brute force solution starting from taking the equation of the desired circle C in the general form, viz.,

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{13}$$

Here there are three unknowns, viz., g, f and c. To determine their values, we need three equations. These are given by the three conditions :

(i) that the point $(1, -1)$ lies on C,

(ii) that the line $2x + 3y + 1 = 0$ touches the circle C, and

(iii) that C and C' cut each other orthogonally.

These conditions imply, respectively, (using the condition for tangency, (1) and the condition for orthogonal intersection)

$$2 + 2g - 2f + c = 0 \quad (14)$$

$$(-2g - 3f + 1)^2 = 13(g^2 + f^2 - c) \quad (15)$$

$$\text{and} \quad 2g - 2f = c - 3 \quad (16)$$

Luckily, this system is not at all difficult to solve. (14) and (16) immediately give $c = \frac{1}{2}$. Also $g = f - \frac{5}{4}$. These substitutions reduce (15) to the quadratic

$$\left(-5f + \frac{7}{2}\right)^2 = 13\left(2f^2 - \frac{5}{2}f + \frac{17}{16}\right) \quad (17)$$

which simplifies to $f^2 + \frac{5}{2}f + \frac{25}{16} = 0$. This has $f = -\frac{5}{4}$ as a double root. Hence finally, $g = -\frac{5}{2}$. The equation of the desired circle now comes exactly as (4).

The brute force method, as the name indicates, is often chided by authors and is rarely recommended. The criticism is indeed valid in general. For example, in the present problem, in the brute force method we had three unknowns while in the earlier two solutions we had just one each (viz., λ and r respectively). This happened because we carefully utilised the conditions in the problem to get rid of two unknowns. And we also got the reward because surely it is easier to solve a single equation in one unknown than to solve a system (especially a non-linear one such as containing Equation (15) above) of three equations in three unknowns.

So generally, brute force methods are not the right ones. But once in a while, there are exceptions. The present problem is one such. As a slight variation, instead of starting with (13) one can take the equation of the circle C in the form

$$(x - a)^2 + (y - b)^2 = r^2 \quad (18)$$

Now that the centre is at (a, b) and the radius is r, the conditions (i), (ii) and (iii) can be applied in a more direct and geometric manner (using (6) and (7) for the last condition). We then get the following three non-linear equations in the three unknowns a, b, c.

$$(a - 1)^2 + (b + 1)^2 = r^2 \quad (19)$$

$$(2a + 3b + 1)^2 = 13r^2 \quad (20)$$

$$\text{and} \quad (a + 1)^2 + (b - 1)^2 = r^2 + 5 \quad (21)$$

which, despite being a non-linear system, is easy to solve.

Problem No. 13: At a point P on the parabola $y^2 - 2y - 4x + 5 = 0$, a tangent is drawn which meets the directrix at Q. Find the locus of the point R which divides QP externally in the ratio $\dfrac{1}{2} : 1$.

Analysis and Solution: Locus problems are generally straightforward and the present one is no exception. But there is a slight hitch right at the start. The equation of the parabola is not in the standard from. Fortunately, it is possible to convert the given equation to the standard form merely by shifting the origin suitably. (The problem would have been considerably more complicated if we also had to rotate the axes to do so.)

Completing squares, the equation of the parabola can be written as

$$(y-1)^2 = 4(x-1) \tag{1}$$

We now have a choice. We can shift the origin to the point $(1,1)$ and introduce new coordinates (X,Y) by $X = x-1, Y = y-1$. That would translate (1) to the equation

$$Y^2 = 4X \tag{2}$$

which is in the standard form, We can then do the problem entirely in these new coordinates. The desired locus will then be an equation in X and Y. At the end we put back $X = x-1$ and $Y = y-1$ to get the equation of the locus in the given coordinates x and y.

However, we can follow essentially the same procedure implicitly without introducing X and Y. For example, the equation of the directrix of (2) is $X = -1$. So the directrix of (1) is given by $x-1 = -1$ i.e. $x = 0$. This could as well be done by inspection directly from (1). We shall follow this direct approach.

First, note that a typical point P on (1) can be represented parametrically as $(t^2+1, 2t+1)$. The equation of the tangent at P can be obtained by remembering the equation of the tangent to a parabola in the standard form. But we can as well get it directly by first finding its slope $\frac{dy}{dx}$ from the parametric equations

$$x = t^2+1 \tag{3}$$

$$y = 2t+1 \tag{4}$$

Differentiating, $\frac{dy}{dt} = 2$ and $\frac{dx}{dt} = 2t$. So $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{t}$. Therefore the equation of the tangent to the parabola at the point $P = (t^2+1, 2t+1)$ is

$$y - 2t - 1 = \frac{1}{t}(x - t^2 - 1) \tag{5}$$

This meets the directrix $x = 0$ at $Q = (0, t+1-\frac{1}{t})$. The point R divides QP externally in the ratio $\frac{1}{2} : 1$ which is the same as the ratio $1:2$. Letting (h,k) be the current coordinates of R, by the section formula we get

$$h = \frac{2 \times 0 - 1 \times (t^2+1)}{-1+2} = -t^2 - 1 \tag{6}$$

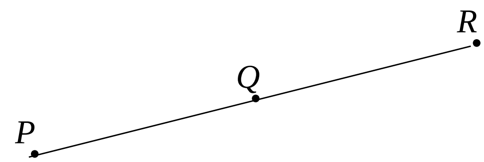
$$\text{and} \quad k = \frac{2 \times (t+1-\frac{1}{t}) - 1 \times (2t+1)}{-1+2} = 1 - \frac{2}{t} \tag{7}$$

We need to eliminate t from these two equations. (7) gives $t = \frac{2}{1-k}$. Putting this in (6) and simplifying, we get

$$(h+1)(k-1)^2 + 4 = 0 \tag{8}$$

Hence the desired locus is $(x+1)(y-1)^2 + 4 = 0$.

The equations (6) and (7) could have been obtained slightly more easily if we had observed that to say that R divides QP externally in the ratio 1 : 2 means that R lies on the (extended) ray PQ and $PR = 2QR$. But this is the same thing as saying that Q is the midpoint of PR. Hence we get $t^2+1+h = 0$ and $2(t+1-\frac{1}{t}) = 2t+1+k$.



Problem No. 14: Suppose $|c| \leq \dfrac{1}{2}$ and let

$$f(x) = \begin{cases} b\sin^{-1}\left(\dfrac{x+c}{2}\right), & -\dfrac{1}{2} < x < 0 \\ \dfrac{1}{2}, & x = 0 \\ \dfrac{e^{\frac{a}{2}x} - 1}{x}, & 0 < x < \dfrac{1}{2} \end{cases}$$

If $f(x)$ is differentiable at $x = 0$, then find the value of a and prove that $64b^2 = 4 - c^2$.

Analysis and Solution: Problems of this type are very common at the JEE. The stipulation $|c| < \frac{1}{2}$ ensures that $|c+x| \leq 1$ for all $x \in (-\frac{1}{2}, 0)$ and hence that $\sin^{-1}\left(\frac{x+c}{2}\right)$ is defined for these values of x. (Actually for this purpose it would suffice if $-\frac{3}{2} \leq c \leq 2$. An unnecessarily strong stipulation can sometimes confuse a student by leading him to wrongly believe that the full strength of the stipulation is crucially needed in the solution.)

We are given that f is differentiable at 0. This also implies that it is continuous at 0. So let us first see what inference we can draw from the left and right continuity of f at 0. The left handed limit of f at 0 is easy to compute since the $\sin^{-1}$ function is continuous everywhere in its domain. This gives

$$b\sin^{-1}\left(\frac{c}{2}\right) = \lim_{x\to 0^-} b\sin^{-1}\left(\frac{x+c}{2}\right) = f(0) = \frac{1}{2} \tag{1}$$

As for the right-handed limit $f(0^+)$, we see that it is nothing but the right handed derivative of the function $e^{\frac{a}{2}x}$ at the point $x = 0$. As this function is differentiable everywhere with derivative $\dfrac{a}{2}e^{\frac{a}{2}x}$ we get

$$\frac{a}{2} = f(0^+) = f(0) = \frac{1}{2} \tag{2}$$

from which we at once get $a = 1$.

Equation (1) gives us a relation in b and c. But the relationship which is asked, viz., $64b^2 = 4 - c^2$ does not follow from this. This is not surprising, because if it did then the hypothesis that f is differentiable at 0 would be superfluous as both the conclusions would follow simply from the continuity of f at 0. Once in a while, the papersetters do include some redundant hypothesis. For example, in the Main Paper of JEE 2003, Problem 8 never required that the function $p(x)$ was a polynomial. It could just as well have been any continuously differentiable function. But this is an exception. Normally, every bit of hypothesis is needed.

The trouble is that in the present problem, not only is (1) inadequate to imply the desired relationship, viz. $64b^2 = 4 - c^2$, but in fact the two relationships can never hold together. Even if we ignore the given restrictions on the values of c, we know that $\sin^{-1}\left(\frac{c}{2}\right)$ is at most $\frac{\pi}{2}$ in magnitude. So, from (1) we get

$$|b| \geq \frac{1}{2} \times \frac{2}{\pi} = \frac{1}{\pi} \tag{3}$$

On the other hand, if $64b^2 = 4 - c^2$ is to hold true, then $64b^2$ is at most 4, which gives

$$|b| \leq \frac{1}{4} \tag{4}$$

Since $\pi < 4$, $\frac{1}{4} < \frac{1}{\pi}$. So we see that (3) and (4) are inconsistent with each other.

A candidate who sees such inconsistency is bound to be confused. The only practical advice that can be given is that after pointing out such inconsistency, simply ignore it and try to derive the relationship asked by some other means. In the present problem, so far we have used only continuity of the given function. Let us now see what we can get from the differentiability of f at $\frac{1}{2}$. For this we first need to find $f'_-(0)$ and $f'_+(0)$. From the very definition,

$$\begin{aligned} f'_-(0) &= \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0^-} \frac{b\sin^{-1}(\frac{h+c}{2}) - b\sin^{-1}(\frac{c}{2})}{h} \quad \text{by (1)} \end{aligned} \tag{5}$$

But this limit is nothing but the left handed derivative of the function $b\sin^{-1}\left(\frac{x+c}{2}\right)$ at the point $x = 0$. As the $\sin^{-1}(u)$ function is differentiable at all $u \in (-1, 1)$, by the chain rule, the derivative of $b\sin^{-1}\left(\frac{x+c}{2}\right)$ at the point $x = 0$ is simply the value of $\frac{b}{2}\frac{1}{\sqrt{1-(\frac{x+c}{2})^2}}$ at $x = 0$. This value is $\frac{b}{\sqrt{4-c^2}}$. Hence from (5) we get

$$f'_-(0) = \frac{b}{\sqrt{4-c^2}} \tag{6}$$

Let us now compute $f'_+(0)$. As we Aleda know that $a = 1$ and $f(0) = \frac{1}{2}$, analogously to (5) we have

$$\begin{aligned} f'_-(0) &= \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0^+} \frac{\frac{e^{h/2}-1}{h} - \frac{1}{2}}{h} \end{aligned} \tag{7}$$

The catch now is that unlike in the approach we followed for (5), we cannot quite write this limit as the (right handed) derivative of the function $\dfrac{e^{x/2}-1}{x}$ at the point $x = 0$ because this function is not even defined at 0. It is tempting to simply *define* it to be $\frac{1}{2}$ at 0. That would make it continuous at 0. But to find its derivative will take us back to square one. There are three ways to overcome this difficulty. The first, and the most sophisticated, method is to apply what is called the generalised Mean Value Theorem (which is an extension of the Lagrange Mean Value Theorem and a special case of the Taylor's theorem) to the function $e^{x/2}$ which has derivatives of all orders everywhere. Its value at $x = 0$ is 1 while that of its derivative is $\frac{1}{2}$. Its second derivative is $\frac{1}{4}e^{x/2}$. So applying this theorem for this function to the interval $[0, h]$, we get

$$e^{h/2} = 1 + \frac{1}{2}h + \frac{h^2}{2!}\frac{1}{4}e^{u/2} \tag{8}$$

for some $u \in (0, h)$. (This u could depend on h. So, if one prefers, one can write $u(h)$ instead of mere u. Note, however, that $u(h)$ need not be single valued.)

Combining (7) and (8) together, we get

$$\begin{aligned} f'_+(0) &= \lim_{h \to 0^+} \frac{\frac{h^2}{8}e^{u/2}}{h} \\ &= \lim_{h \to 0^+} \frac{e^{u/2}}{8} \end{aligned} \tag{9}$$

Now note that although we do not know the exact value of u we do know that it lies in the interval $(0, h)$. As the function $e^{x/2}$ is monotonically increasing, we have $1 < e^{u/2} < e^{h/2}$ for $h > 0$. As $h \to 0^+$, both 1 (which is a constant) and $e^{h/2}$ tend to 1. So by the Sandwich Theorem for limits, we get that $\lim\limits_{h \to 0^+} e^{u/2} = 1$. Therefore we now have

$$f'_+(0) = \frac{1}{8} \tag{10}$$

We are given that f is differentiable at 0. This means that $f'_-(0) = f'_+(0)$. So from (4) and (8) we get $\dfrac{b}{\sqrt{4-c^2}} = \dfrac{1}{8}$. Squaring both the sides we get $64b^2 = 4 - c^2$ as was to be proved. (As remarked before, (1) gives us another equation in b and c which is inconsistent with the present one. But we are ignoring this inconsistency.)

A somewhat simpler way to evaluate the limit in (7) is also based on the MVT. As mentioned before, the limit we are after is simply the right-handed derivative $g'_+(0)$ of the function $g(x)$ defined by

$$g(x) = \begin{cases} \frac{e^{x/2}-1}{x}, & x > 0 \\ \frac{1}{2}, & x = 0 \end{cases} \tag{11}$$

We already saw that g is right continuous at 0. Also it is differentiable at every $x > 0$. Under these conditions, it can be shown as a standard application of the Lagrange's MVT, that *in case the right handed limit of the derivative of g exists* at 0, then it is also the right handed derivative of g at 0. In symbols, if $g'(0^+)$ exists then $g'_+(0)$ also exists and the two are equal. (Note, however, that $g'(0^+)$ need not always exist as we see from a function like $g(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $g(0) = 0$. Here $g'_+(0)$ does exist and equals 0. But it cannot be obtained from $\lim\limits_{x \to 0^+} g'(x) = \lim\limits_{x \to 0^+} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}$ which fails to exist.)

From (11) we have that

$$g'(x) = \frac{\frac{x}{2}e^{x/2} - e^{x/2} + 1}{x^2} \tag{12}$$

for $x > 0$. To evaluate $\lim\limits_{x \to 0^+} g'(x)$ we apply l'Hôpital's rule, giving

$$\begin{aligned} g'_+(0) = g'(0^+) &= \lim_{x \to 0^+} \frac{\frac{x}{2}e^{x/2} - e^{x/2} + 1}{x^2} \quad (\frac{0}{0} \text{ form}) \\ &= \lim_{x \to 0^+} \frac{\frac{1}{2}e^{x/2} + \frac{x}{4}e^{x/2} - \frac{1}{2}e^{x/2}}{2x} \\ &= \lim_{x \to 0^+} \frac{1}{8}e^{x/2} = \frac{1}{8} \end{aligned} \tag{13}$$

As noted before, this proves (10).

The third way to go from (7) to (10) is to write $\dfrac{\frac{e^{h/2}-1}{h} - \frac{1}{2}}{h}$ as $\dfrac{e^{h/2} - 1 - \frac{h}{2}}{h^2}$ and find its limit by applying the l'Hôpital's rule twice. This would give

$$\begin{aligned} f'_+(0) &= \lim_{h \to 0^+} \frac{e^{h/2} - 1 - \frac{h}{2}}{h^2} \quad (\frac{0}{0} \text{ form}) \\ &= \lim_{h \to 0^+} \frac{\frac{1}{2}e^{h/2} - \frac{1}{2}}{2h} \quad (\frac{0}{0} \text{ form again}) \end{aligned}$$

$$= \lim_{h \to 0^+} \frac{\frac{1}{4}e^{h/2}}{2} = \frac{1}{8} \tag{14}$$

Thus we see that all the three methods give the same value of $f'_+(0)$, as they ought to, of course. In the first method, we did not apply l'Hôpital's rule at all. In the second one we applied it once, while in the third one we applied it twice. So it might appear that the l'Hôpital's rule simplifies the work and further, the more you apply it the more simple the calculations become. For the lovers of l'Hôpital's rule (and this includes the vast majority of the JEE candidates!) this is surely a feather in the cap of that rule. These lovers should bear in mind that what is needed in the second and the third method is the strong form of the l'Hôpital's rule whose proof requires the Mean Value Theorem in one form or the other. Because of this, the three derivations are not radically different. The second and the third one appear short because you are borrowing readymade theorems based on the Mean Value theorem. In the first method you are using the generalised MVT directly.

In passing, we mention one more method of evaluating the limit in (7). It is based on the power series expansion of the exponential function, viz.,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \tag{15}$$

If we apply this with $x = \frac{h}{2}$ we get

$$\begin{aligned} \frac{\frac{e^{h/2}-1}{h} - \frac{1}{2}}{h} &= \frac{1}{h}\left(\frac{\frac{h}{2} + \frac{h^2}{8} + \frac{h^3}{48} + \ldots + \frac{h^n}{2^n n!} + \ldots}{h} - \frac{1}{2}\right) \\ &= \frac{1}{h^2}\left(\frac{h^2}{8} + \frac{h^3}{48} + \ldots + \frac{h^n}{2^n n!} + \ldots\right) \\ &= \frac{1}{8} + \frac{h}{48} + \ldots + \frac{h^{n-2}}{2^n n!} + \ldots \end{aligned} \tag{16}$$

for every $h \neq 0$. On the R.H.S. we have a sum of an infinite number of terms, each being a power of h. As $h \to 0^+$, the first term, being a constant, stays as it is. But each of the remaining terms tends to 0. Hence the limit of the expression is $\frac{1}{8}$.

This argument is certainly attractive and often finds quick acceptance by those who are not worried about rigour. But those who are, will object, first of all, that the expansion of e^x (or of any other function) as a power series is beyond the JEE level. Serious questions of convergence have to be answered to make it valid. Moreover, in the argument above, the limit of an *infinite* number of terms was found by adding the limit of each one of them. This is not valid in general and to make it so in the present case again requires a lot of work well beyond the JEE level.

So, the acceptability of this method at the JEE is questionable. Still, the method is not wrong in itself if handled with care. Unfortunately, many old texts manipulate infinite series with little regard to rigour. For example, a quick proof of the fact that $\frac{d}{dx}e^x = e^x$ is obtained if we differentiate the R.H.S. of (15) term-by-term. As a result, many students are tempted to use them. It is best to avoid power series in the JEE. But there is nothing wrong in using them for a quick verification of the answer arrived at by other methods, or to predict the answer before applying other methods. And, in situations where you don't have to show your reasoning, who is going to prevent you from using them?

Problem No. 15: Prove that for $x \in \left[0, \frac{\pi}{2}\right]$, $\sin x + 2x \geq \frac{3x(x+1)}{\pi}$.

Analysis and Solution: If we let

$$f(x) = \sin x + 2x - \frac{3x(x+1)}{\pi} \tag{1}$$

then the problem is equivalent to showing that $f(x) \geq 0$ for all $x \in [0, \frac{\pi}{2}]$. This elementary simplification enables us to do the problem by studying the properties of a single function rather than those of two separate functions represented by the two sides of the inequality to be proved. Note that $f(0) = 0$. So if we could show that f is increasing on $[0, \frac{\pi}{2}]$ then the assertion would follow. For this we consider

$$f'(x) = \cos x + 2 - \frac{6x+3}{\pi} \tag{2}$$

We would be through if $f'(x) \geq 0$ for all $x \in [0, \frac{\pi}{2}]$. Unfortunately, this is not the case. Even though $f'(0) = 3 - \frac{3}{\pi} > 0$, at the other end we have $f'(\frac{\pi}{2}) = 2 - \frac{3\pi+3}{\pi} = -1 - \frac{3}{\pi} < 0$. So this simple-minded approach will not work. The function $f(x)$ is not increasing throughout. It is increasing near 0 but decreasing near the other end $\frac{\pi}{2}$.

But something can be salvaged. Note that $f(\frac{\pi}{2}) = 1 + \pi - \frac{3\pi}{4} - \frac{3}{2} = \frac{\pi}{4} - \frac{1}{2} > 0$ since $\pi > 3$. In other words, the graph of $y = f(x)$, although sloping downward near $\frac{\pi}{2}$, is nevertheless above the x-axis. Of course, it could have gone below the x-axis at some intermediate points. We have to rule out this possibility.

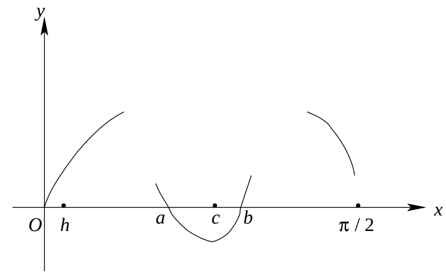
This can be done in several ways. Each one of them looks intuitively obvious but requires non-trivial results from theoretical calculus for a rigorous justification. Suppose $c \in (0, \frac{\pi}{2})$ is a point such that

$$f(c) < 0 \tag{3}$$

From the fact that $f(0) = 0$ while $f'(0) > 0$ it follows that if h is sufficiently small and positive then

$$f(h) > 0 \tag{4}$$

Clearly we may suppose $h < c$. Then from (3), (4) and the Intermediate Value Property (IVP) for continuous functions, there exists some $a \in (h, c)$ such that $f(a) = 0$. Similarly, from $f(c) < 0$ and $f(\frac{\pi}{2}) > 0$, we get that there exists some $b \in (c, \frac{\pi}{2})$ such that $f(b) = 0$. Thus we have



$$0 < a < b < \frac{\pi}{2} \tag{5}$$

$$\text{and} \quad f(0) = f(a) = f(b) = 0 \tag{6}$$

We now apply Rolle's theorem to the intervals $[0, a]$ and $[a, b]$ to get points $c_1 \in (0, a)$ and $c_2 \in (a, b)$ such that

$$f'(c_1) = f'(c_2) = 0 \tag{7}$$

Clearly, $c_1 < c_2$ since $c_1 < a < c_2$. We now apply Rolle's theorem to the function f' over the interval $[c_1, c_2]$ to get yet another point $c_3 \in (c_1, c_2)$ such that

$$f''(c_3) = 0 \tag{8}$$

But if we differentiate (2), we get $f''(x) = -\sin x - \frac{6}{\pi}$ which is negative throughout the interval $(0, \frac{\pi}{2})$. This contradicts (3). So $f(x) \geq 0$ for all $x \in [0, \frac{\pi}{2}]$.

Our second argument is based on the concept of a local minimum. Suppose, the graph of $f(x)$ does go below the x-axis for some values of x in the interval $[0, \frac{\pi}{2}]$. Then consider the lowest point, say $(c, f(c))$ on the graph. That such a point exists follows from the fact that every continuous function on a closed, bounded interval has an absolute minimum. Moreover c must be an interior point since at both the end points, f is already shown to be non-negative. Therefore the point c is also a point of local minimum of $f(x)$. Therefore $f''(c)$, in case it exists must be non-negative. But as we already saw, $f''(x)$ is negative everywhere. So again we get a contradiction.

The second argument appears shorter. But the proof of the property of points of local minima used in it is again based on the Mean Value Theorems. So the two arguments are not radically different.

Note that we made little use of the particular function f. In essence the theorem we proved is that if a function is non-negative at both the end points of an interval and its second derivative is negative throughout that interval, then the function is non-negative throughout that interval.

Problem No. 16: Evaluate $\displaystyle\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$.

Analysis and Solution: From the very expression of the integrand, we see that finding an andiderivative for it is not going to be easy. So this is one of those definite integrals which have to be evaluated by some other means. Also we note that the interval of integration, viz., $[-\pi/3, \pi/3]$ is symmetric about the origin. Chances are that this fact will be crucially needed in the solution, especially if even or odd functions are involved. (Of course, sometimes such predictions may turn out to be false. But it is a good habit to make a note of the special features of a given problem before attacking it.)

We also note that the denominator of the integrand is an even function of x. The numerator is neither even nor odd. In fact, it is a sum of two terms of which the first, being a constant, is an even function of x, while the second, viz., $4x^3$ is an odd function of x. This suggests that if we split the numerator, then the integrand will be a sum of two functions, one even and the other odd. In symbols, let I denote the given integral and let

$$I_1 = \int_{-\pi/3}^{\pi/3} \frac{\pi}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx \tag{1}$$

$$\text{and} \quad I_2 = \int_{-\pi/3}^{\pi/3} \frac{4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx \tag{2}$$

Then clearly

$$I = I_1 + I_2 \tag{3}$$

Let us tackle I_1 and I_2 separately. I_2 is easier, because since the integrand is odd and the interval of integration is symmetric about the origin, without doing any further work we get that

$$I_2 = 0 \tag{4}$$

For I_1, too, the symmetry of the interval and the evenness of the integrand imply

$$I_1 = 2\int_0^{\pi/3} \frac{\pi}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx \tag{5}$$

This may not appear as much of a simplification. But the gain is that now we can get rid of the nagging absolute value sign because throughout the interval of integration $|x| = x$. Hence

$$I = I_1 = 2\int_0^{\pi/3} \frac{\pi}{2 - \cos\left(x + \frac{\pi}{3}\right)} dx \tag{6}$$

We can find an antiderivative for the integrand by expanding $\cos(x + \frac{\pi}{3})$. But that would involve both $\sin x$ and $\cos x$ and hence will be rather complicated. Instead, let us make a change of variable by setting $u = x + \frac{\pi}{3}$. Then we get

$$I = I_1 = 2\pi \int_{\pi/3}^{2\pi/3} \frac{du}{2 - \cos u} \tag{7}$$

We have just about exhausted whatever simplification was possible. We now have to get to the task of finding an antiderivative. The standard substitution $t = \tan\frac{u}{2}$ converts the integrand to a rational function in t and we get

$$I = I_1 = 2\pi \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2\,dt}{3t^2+1} \tag{8}$$

which can be further evaluated by the standard techniques as

$$\begin{aligned} I = I_1 &= \frac{4\pi}{3} \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dt}{t^2 + (1/\sqrt{3})^2} \\ &= \frac{4\pi}{3}\sqrt{3}\left[\tan^{-1}\sqrt{3}t\right]_{1/\sqrt{3}}^{\sqrt{3}} \\ &= \frac{4\pi}{\sqrt{3}}\left(\tan^{-1} 3 - \tan^{-1} 1\right) \qquad (9) \\ &= \frac{4\pi}{\sqrt{3}}(\tan^{-1}\left(\frac{1}{2}\right) \qquad (10) \end{aligned}$$

The conversion from (9) to (10) is based on the trigonometric identity

$$\tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\frac{a-b}{1+ab}\right)$$

As the focus of the problem is on integration and not trigonometry, such a conversion is not very vital. But if you leave the answer at (9) you better at least replace $\tan^{-1} 1$ by $\frac{\pi}{4}$.

Problem No. 17: A curve passes through $(2, 0)$ and the slope of the tangent to it at a point (x, y) on it is $\frac{(x+1)^2 + (y-3)}{x+1}$. Find the equation of the curve and the area bounded by it and the x-axis in the fourth quadrant.

Analysis and Solution: This is a combination of two problems. The first part consists of obtaining a particular solution of a differential equation, while the second one is a problem of finding the area of a plane region.

For the first part, the differential equation

$$\frac{dy}{dx} = \frac{(x+1)^2 + (y-3)}{x+1} \tag{1}$$

is to be solved subject to the initial condition that

$$y = 0 \text{ when } x = 2 \tag{2}$$

The very form of the R.H.S. of (1) suggests that a change of variables will simplify the equation considerably. Let $u = x + 1$ and $v = y - 3$. Then

$\frac{dv}{du} = \frac{dv}{dy}\frac{dy}{dx}\frac{dx}{du} = 1 \times \frac{dy}{dx} \times 1 = \frac{dy}{dx}$ and (1) changes to

$$\frac{dv}{du} = \frac{u^2 + v}{u} \tag{3}$$

which can be rewritten as

$$\frac{dv}{du} - \frac{v}{u} = u \tag{4}$$

This is a linear equation with integrating factor $e^{-\int \frac{du}{u}} = e^{-\ln u} = \frac{1}{u}$. Multiplying (4) by the I.F. we get

$$\frac{1}{u}\frac{dv}{du} - \frac{v}{u^2} = 1, \quad \text{i.e.,} \quad \frac{d}{du}(\frac{v}{u}) = 1 \tag{5}$$

from which the general solution of (4) is $\frac{v}{u} = u + c$. (Of course this could also have been written down from the readymade formula for the solution of a linear differential equation. But, once again, we are illustrating how the burden on the memory can often be lessened.)

Translated in terms of the original variables, the general solution of (1) is

$$\frac{y-3}{x+1} = x + 1 + c \tag{6}$$

The initial condition (2) gives $0 - 3 = (2+1)(3+c)$ i.e., $c = -4$. Hence the equation of the given curve is

$$y = 3 + (x+1)(x-3) = x^2 - 2x \tag{7}$$

The curve therefore is a vertically upward parabola which cuts the x-axis at the points $(0, 0)$ and $(2, 0)$. Being such a familiar curve, the desired region can be identified mentally even without a diagram. Its area is

$$\int_0^2 -(x^2 - 2x)dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = \frac{4}{3} \tag{8}$$

square units.

The two parts of the problem are completely independent of each other. the second part is too trivial to be asked separately. That is why it is probably clubbed with the first part. The problem would have been more interesting if in finding the area, the original differential equation could have been used. For example, the region could have involved the normal to the curve at some point on it. In that case, instead of finding the slope of the normal from the solution (7) one could save time by getting it directly from (1).

Problem No. 18: A box contains 12 red and 6 white balls. Balls are drawn from it one at a time without replacement. If in 6 draws there are at least 4 white balls, find the probability that exactly one white ball is drawn in the next two draws. (The answer may be left in terms of the binomial coefficients.)

Analysis and Solution: The wording of the question can be misleading to someone not having a good command over English. He is likely to think that 6 balls are drawn in all and at least the first 4 are white. The correct interpretation is, of course, that 8 balls are drawn in all and it is given further that at least 4 of the first 6 balls are white. It would have been better if this was spelt out in the statement of the problem.

Now coming to the solution, let A be the event that at least 4 of the first 6 balls are white and B be the event that at least one ball in the 7th and the 8th draws is white. We have to find the conditional probability of B given A, or in symbols, $P(B|A)$. By the law of conditional probability, this equals

$$\frac{P(A \cap B)}{P(A)} \tag{1}$$

We proceed to compute the numerator and the denominator separately. First consider $P(A)$. Here A is the event that at least 4 balls in the first 6 draws are white. It is convenient to apply 'divide and rule' here, That is we break up A into three mutually exclusive events depending upon the number of white balls in the first 6 draws. Specifically, let A_1 be the event that exactly 4 of the first 6 balls are white, A_2 be the event that exactly 5 are white white, and A_3 be the event that exactly 6 are white. It is easy to find the probabilities of these events. For simplicity we distinguish balls of the same colour. (The problem can also be done assuming that they are indistinguishable. But the solution will be a bit complicated since we shall have to regard various outcomes as identical. The answer will not change.)

In all 6 balls can be chosen from the balls in the bag in $\binom{18}{6}$ ways. The event A_1 occurs in exactly $\binom{12}{2} \times \binom{6}{4}$ ways. Hence

$$P(A_1) = \frac{\binom{12}{2}\binom{6}{4}}{\binom{18}{6}} \tag{2}$$

Similarly,

$$P(A_2) = \frac{\binom{12}{1}\binom{6}{5}}{\binom{18}{6}} \quad \text{and} \quad P(A_3) = \frac{\binom{12}{0}\binom{6}{6}}{\binom{18}{6}} \tag{3}$$

(Note that we are not distinguishing between two sequences of draws as long as the balls drawn in them are the same but possibly in a different order. If we wanted to make a distinction of this kind, then to begin with, the number of ways to draw 6 balls from 18 would no longer be $\binom{18}{6}$ but ${}^{18}P_6$, i.e. the number of permutations of 6 symbols out of 18. Similar changes will be needed to find the numbers of favourable cases. The probabilities will, however, remain the same since the extra factor we have to put in, viz. 6!, will cancel out anyway.)

We are specifically permitted to leave the answer in terms of the binomial coefficients. This is a welcome declaration on the part of the papersetters. In absence of such a declaration, many students spend their precious time in doing the purely arithmetical task of evaluating the coefficients. Worse still, if they make a numerical mistake, they stand to lose even though the conceptual part of their solutions is quite correct.

Of course, just because we have a certain freedom does not mean we *have to* use it every time. (Or else it is not a freedom in the first place!). The binomial coefficients in the numerators of (1) and (2) are very easy to compute. And since we want to add the three probabilities and their denominators are common, it will be better to simplify the numerators and add to make the answer a little more appealing. So

$$P(A) = P(A_1) + P(A_2) + P(A_3) = \frac{66 \times 15 + 12 \times 6 + 1}{\binom{18}{6}} = \frac{1063}{\binom{18}{6}} \qquad (4)$$

We now turn to the numerator of (1), viz., $P(A \cap B)$. Since $A = A_1 \cup A_2 \cup A_3$, we have

$$A \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \qquad (5)$$

Further as A_1, A_2, A_3 are mutually exclusive, so are $A_1 \cap B, A_2 \cap B$ and $A_3 \cap B$. Hence from (5),

$$P(A \cap B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \qquad (6)$$

Let us compute $P(A_1 \cap B)$. This is, the probability that exactly 4 white balls came in the first 6 draws and exactly one white ball came in the next two draws. Now, after A_1 takes place, the bag contains 10 red and 2 white balls. So 2 balls can be drawn out of these 12 balls in $\binom{12}{2}$ ways and out of these, exactly one white ball will occur in $\binom{2}{1}\binom{10}{1}$ i.e. in 20 ways. (Note again that we are not distinguishing whether the white ball is the seventh or the eighth.)

So, given A_1, B can occur in 20 out of the $\binom{12}{2}$ ways. As $P(A_1)$ is already known from (2), we get

$$P(A_1 \cap B) = P(A_1) \times P(B|A_1) = P(A_1) \times \frac{20}{\binom{12}{2}} = \frac{990}{\binom{18}{6}} \times \frac{20}{66} = \frac{300}{\binom{18}{6}} \qquad (7)$$

Similarly,

$$P(A_2 \cap B) = P(A_2) \times \frac{11}{\binom{12}{2}} = \frac{72}{\binom{18}{6}} \times \frac{11}{66} = \frac{12}{\binom{18}{6}} \qquad (8)$$

while $P(A_3 \cap B) = 0$ because after A_3 no white balls are left. Hence from (6), (7) and (8),

$$P(A \cap B) = \frac{312}{\binom{18}{6}} \qquad (9)$$

Hence ultimately, from (1), (4) and (9), the desired probability is

$$\frac{312}{\binom{18}{6}} \times \frac{\binom{18}{6}}{1063} = \frac{312}{1063} \qquad (10)$$

Although the answer could have been left in terms of the binomial coefficients, we have calculated it fully. This was not so difficult because in most of the binomial coefficients we encountered, the lower index was small. The only exception was $\binom{18}{6}$. But it got cancelled out in (10) anyway. So the papersetters could have as well insisted upon the answer in a numerical form as a ratio of two integers. The right strategy in such cases is to evaluate the easy binomial coefficients as they come along and leave the tough ones hoping that they may get cancelled later.

In fact, insistence on a numerical answer in a problem like this makes life easier for the examiner. For, nobody can arrive at such a crazy answer by bluffing, or by clever guesswork or by sheer luck. If a student comes up with the answer $\frac{312}{1063}$ it is safe to assume that his reasoning as well as calculations are correct. This spares the examiner the need to go through each and every line. All he needs to do is to ensure that the answer is not a borrowed one. It is only when the answer is not correct that he need check where the mistake has occurred and how serious it is.

Problem No. 19: Let

$$A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}, B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}, U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}, V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

If $AX = U$ has infinitely many solutions, prove that $BX = V$ has no unique solution. If further, $afd \neq 0$, then show that $BX = V$ has no solution.

Analysis and Solution: Although matrices are involved in the statement of the problem, the problem itself is not so much about matrices as about the existence and uniqueness of solutions of systems of linear equations in which the number of variables equals the number of unknowns. Matrices are only a convenient tool. The basic result that is needed is that a system such as

$$AX = U$$

has a unique solution if $\Delta \neq 0$, where Δ is the determinant of the matrix A. When $\Delta = 0$, then the system has,

(i) no solution if at least one of Δ_1, Δ_2 and Δ_3 is non-zero

(ii) either no solution or else infinitely many solutions if $\Delta_1 = \Delta_2 = \Delta_3 = 0$,

where, for $i = 1, 2, 3$, Δ_i is the determinant of the matrix obtained by replacing the i-th column of A by the column vector U.

First we apply this result to the system

$$AX = U \tag{1}$$

which is given to have infinitely many solutions. As a result, $\Delta, \Delta_1, \Delta_2, \Delta_3$ all vanish, where

$$\Delta = \begin{vmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} f & 1 & 0 \\ g & b & d \\ h & b & c \end{vmatrix}, \tag{2}$$

$$\Delta_2 = \begin{vmatrix} a & f & 0 \\ 1 & g & d \\ 1 & h & c \end{vmatrix} \quad \text{and} \quad \Delta_3 = \begin{vmatrix} a & 1 & f \\ 1 & b & g \\ 1 & b & h \end{vmatrix} \tag{3}$$

We could expand these determinants right now. But we are not sure exactly which ones among them will be needed. So let us postpone this and consider the system

$$BX = V \tag{4}$$

about whose solutions we have to prove something. We apply the same criterion as above. However, this time let us denote the corresponding determinants by $\Delta', \Delta'_1, \Delta'_2, \Delta'_3$. That is,

$$\Delta' = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix}, \quad \Delta'_1 = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix}, \tag{5}$$

$$\Delta'_2 = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} \quad \text{and} \quad \Delta'_3 = \begin{vmatrix} a & 1 & a^2 \\ 0 & d & 0 \\ f & g & 0 \end{vmatrix} \tag{6}$$

The first part of the problem amounts to showing that if $\Delta, \Delta_1, \Delta_2, \Delta_3$ are all 0, then so is Δ'. For this it is hardly necessary to expand Δ'. From (3) and (5), we see that if we take the transpose of Δ_2 and interchange the second and the third rows, we get Δ'. Hence $\Delta' = -\Delta_2$. Since Δ_2 vanishes so does Δ'. Hence the system (4) can have no unique solution.

For the second part, we have to show that under the additional hypothesis $afd \neq 0$, (4) has no solution. From the criterion above, this amounts to showing that at least one of Δ'_1, Δ'_2 and Δ'_3 is non-zero. By direct expansions, these determinants equal, respectively, $a^2(dh - gc), a^2cf$ and $-a^2df$. We are given that $afd \neq 0$. This means a, f, d are all non-zero. Hence $-a^2df$ is also non-zero. In other words, $\Delta'_3 \neq 0$. As $\Delta' = 0$, this means (4) has no solution.

Problems based on the criterion for the existence and uniqueness of solutions of systems of three linear equations in three unknowns are quite common at the JEE. But in most of these problems, the coefficients of the system are in terms of some single parameter (often denoted by λ) and you are asked to identify the values of this parameter for which the system has a unique solution, no solution etc. (Q. 21 in the Screening Paper is a typical illustration.) The present problem is a welcome departure from these stereotyped problems. Another nice feature of the problem is the use of the properties of determinants made in its solution (for example, that the determinant of the transpose of a matrix is the same as that of the original matrix). This saved a lot of computations. For example, we could show $\Delta' = 0$ without calculating it because we could relate it to Δ_2 which was given to vanish. Many students believe that the use of matrices in systems of equations is only to provide a compact notation. Such students feel more comfortable in solving systems of linear equations (especially when the system has only 2 or 3 equations) by writing them out fully and manipulating them. In the present problem, such an approach would be very complicated to say the least.

Problem No. 20: P_1 and P_2 are two planes passing through the origin. L_1 and L_2 are two lines passing through the origin such that L_1 is in P_1 but not in P_2 while L_2 is in P_2 but not in P_1. Show that there exist points A, B, C (none of them being the origin) and a suitable permutation A', B', C' of these points such that

(i) A is on L_1, B is on P_1 but not on L_1 and C is not in P_1, and

(ii) A' is on L_2, B' is on P_2 but not on L_2 and C' is not in P_2.

Analysis and Solution: This problem is more like a puzzle. Very little mathematics is needed in it except the knowledge of some very elementary facts from solid geometry. For example, we need to know that $P_1 \cap P_2$ is a straight line, say L. Further the lines L_1, L_2 and L are concurrent at O. Let O be the origin and $\mathbb{R}^3$ the euclidean (3-dimensional) space. Note that we have two chains of subsets here, each inclusion being strict.

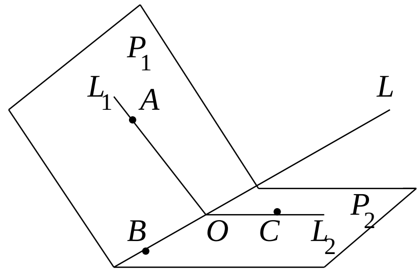
$$\{O\} \subset L_1 \subset P_1 \subset \mathbb{R}^3 \tag{1}$$
$$\{O\} \subset L_2 \subset P_2 \subset \mathbb{R}^3 \tag{2}$$

Conditions (i) and (ii) can be stated as

$$A \in L_1 - \{O\}, B \in P_1 - L_1 \text{ and } C \in \mathbb{R}^3 - P_1 \tag{3}$$
$$A' \in L_2 - \{O\}, B' \in P_2 - L_2 \text{ and } C' \in \mathbb{R}^3 - P_2 \tag{4}$$

respectively. Further we want A', B', C' to be the same points as A, B, C but possibly in a different order. Now A is already in L_1 and hence in P_1 too. We claim that $A \notin P_2$. For, if A were in P_2, then it would lie on $P_1 \cap P_2$ which is the line L. As $A \in L_1$ already, this would mean that $A \in L_1 \cap L = \{O\}$, i.e. A would equal the origin O, a contradiction.

So we must have $A \notin P_2$. Therefore A will have to equal C' since A', B' lie in P_2 by (2) and (4). By a similar reasoning, we must have $A' = C$. Therefore, by elimination, $B' = B$. This determines the permutation completely. With this knowledge, it is also easy to obtain the desired points. Since $B = B'$, this common point must lie on $P_1 \cap P_2$, i.e. on L. Then take A to be any point on L_1 other than O and C to be any point on L_2 other than O. This ensures $C \notin P_1$ and $C'(= A) \notin P_2$ as required.



CONCLUDING REMARKS

Although there were a couple of mistakes (Q. 2 and Q. 27) in the Screening Paper, the Main Paper has only one mistake, viz., the mistake in Problem 14. While a mistake in a question is always bad, it hurts more in a multiple choice paper. In a conventional paper, a student who feels that there is a mistake in some problem can elaborate his contention. And if found correct, some remedial measures can be taken. In a multiple choice question paper, the answers are to be shown only on a special card by blackening appropriate boxes and so there is simply no provision to write anything else.

Slicker solutions were possible for several questions in the screening paper. In Q. 12 it was probably intended. But in Q. 5, 11 and 21 it was possible to get the correct answer without doing the full honest work.

On the background of the total absence of Number Theory in 2003, this year Q. 28 and Problem 3, make at least a passing reference to divisibility. This year the axe has fallen on the binomial and trigonometric identities.

There is some duplication of ideas within the same paper. For example, Problem 1 and Problem 13 in the Main Paper both involve the section formula. Q. 15, 16 and 17 in the Screening Paper all involve tangents to a conic. So there is not only a duplication, but a triplication of ideas!

There is also some duplication of ideas between the Screening and the Main Papers taken together. For example Q. 16 and Problem 13 both ask for loci of a point which divide a certain portion of the tangent to a conic in a certain ratio. Similarly, Q. 14 in the Screening Paper is not qualitatively different from Problem 12 in the Main Paper.

Such duplication is probably unavoidable as the two papers are set by different teams working independently of each other. Over the coming years, a general consensus will hopefully emerge as to which problems are suitable for which paper. In that case, such a duplication will probably be minimised. At present there seems to be little awareness of the expected difference of the purpose and the standards of the two papers. The Screening Paper is mostly to weed out those candidates who need not have even appeared for the JEE. So, simple, strightforward questions spread over the entire syllabus make sense in the Screening Paper. In the Main Paper, on the other hand, the very presumption is that the candidates have already proved their basic credentials. So the testing has to be more delicate and imaginative, designed more to test the innate qualities than familiarity with certain topics.

With this yardstick, problems like Problems 5, 12, 13, 16, 17 and 18 have no place in the Main Paper. Problems 3, 7 and 8 are good problems basically, but have been diluted by the hints thrown in. The really good problems are Problems 2, 4, 6, 10, 11, 19 and 20. For the reasons elaborated earlier, Problem 19 is innovative and probably the best problem in the whole paper. Also, instead of asking a stereotype problem on finding maxima/minima, it is commendable that the inequalities in Problems 11 and 15 test this indirectly. (It is doubtful, of course, if anybody would try Problem 11 by minimising the function $g(x)$ we considered in its solution.)

The tragedy is that when such good problems are clubbed together with the mediocre ones, the selection is dominated by the latter. For example, it takes some time and intelligence even to understand Problem 6 and 20. But a mediocre student can simply leave them and comfortably bag more marks by doing Problems 12 and 18. It can be argued that an intelligent student has an equal opportunity to do the mediocre problems. But when the time is so severely limited, this is a lame argument.

The present Main Paper would have been a more valid test of a student's qualities if the mediocre problems listed above had been dropped, the three problems 3, 7 and 8 made more challenging (in the manner indicated in the respective comments about them) and one problem each on binomial identities and trigonometry thrown in (with no reduction in the total time allowed). The mediocre problems could also be included by allowing three hours instead of two. That would indeed give the intelligent student the chance to finish the mediocre problems first and then to devote the rest of the time to the challenging problems.