4

An Infinitary Church-Rosser Property for Non-collapsing Orthogonal Term Rewriting Systems

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4.1 INTRODUCTION

There are at least two good reasons to study infinitary term rewriting. First, we believe that infinitary term rewriting is of interest for its own sake, as a natural extension of finitary term rewriting. Second, infinitary term rewriting provides a sound and thorough basis for term graph rewriting, a fruitful theoretical model for implementations of functional programming languages. Term graph rewriting has been defined by Barendregt and co-workers in [BvEG+87] and has been adopted as the central model by the ESPRIT BRA project *SemaGraph*.

Term rewriting is a general model of computation. Computations can be finite and infinite. The usual focus is on successful finite computations: finite derivations ending in finite normal form. However, infinite computations computing a possible infinite answer are of interest as well: recursive procedures enumerating some infinite set: e.g. the natural numbers or the Fibonacci numbers. Until recently, infinite computations have hardly seriously been considered in the theory of term rewriting.

In functional programming languages like Miranda or ML it is possible to manipulate with lazy expressions representing infinite objects, like lists. Graph rewriting has been introduced as a theoretical framework to show the soundness of such computing.

Term Graph Rewriting: Theory and Practice.
Eds. Ronan Sleep, Rinus Plasmeijer and Marko van Eekelen.

Infinitary term rewriting is a foundation for graph rewriting (cf. [KKSdV93] for an elaboration of this point): some instances of graph rewriting on shared graphs actually represent infinite computations on infinite terms.

At present the theory of infinitary rewriting for orthogonal Term Rewriting Systems is rapidly emerging in a series of papers. Dershowitz, Kaplan and Plaisted have opened the series with [DK89, DKP89, DKP91]. They take a rather topological approach and study Cauchy converging reduction sequences. A number of their results (Compression Lemma, Infinitary Projection Lemma, Infinitary Church-Rosser property) depend on the rather strong notion of a top-terminating orthogonal Term Rewriting System. Dropping the condition of top-termination introduces problems. Farmer and Watro [FW91] observed the necessity of strong convergence for some instances of compressing and pointed out the link between infinitary term rewriting and graph rewriting.

In [KKSdV90b] we developed the theory of infinite term rewriting based on strongly converging reductions after presenting counter-examples to the desired general results for Cauchy converging sequences. For the theory involving strong convergence the Infinitary Projection Lemma, the Compressing Lemma and the Unique Normal Form Property are provable, whereas counter-examples exist for these results in case of Cauchy convergence. We also showed that despite the nice theory one can develop for strongly converging reductions the infinitary Church-Rosser property does not hold. The presented counter-example shows that also for Cauchy-converging reductions there is no infinitary Church-Rosser property for arbitrary orthogonal infinitary Term Rewriting Systems.

In this chapter we will prove the infinitary Church-Rosser property for strongly converging reductions for orthogonal infinitary Term Rewriting Systems of which all rules are non-collapsing, except for at most one rule of the form $I(x) \to x$. We think that our proof is instructive and conceptually clear.

The present account improves our treatment in the early version [KKSdV90a].

4.1.1 Overview of this chapter

In section 4.2 we briefly introduce infinitary Term Rewriting Systems (TRS). Then, in section 4.3, we define depth-preserving orthogonal Term Rewriting Systems and prove the infinitary Church-Rosser property for strongly converging sequences in such systems. Using Park's idea of hiaton we show in section 4.4 that any orthogonal TRS can be transformed into a depth-preserving orthogonal TRS, via the so called ϵ-completion. This enables us to prove the infinitary Church-Rosser property for orthogonal TRS consisting of non-collapsing rules with at most one unary collapsing rule. Finally, we discuss our results and relate them with those of Dershowitz, Kaplan and Plaisted.

4.2 INFINITARY ORTHOGONAL TERM REWRITING SYSTEMS

We briefly recall the definition of a finitary Term Rewriting System, before we define infinitary orthogonal Term Rewriting Systems involving both finite and infinite terms. For more details the reader is referred to [DJ90] and [Klo92].

4.2.1 Finitary Term Rewriting Systems

A *finitary Term Rewriting System* over a signature Σ is a pair $(Ter(\Sigma), R)$ consisting of the set $Ter(\Sigma)$ of finite terms over the signature Σ and a set of rewrite rules $R \subseteq Ter(\Sigma) \times Ter(\Sigma)$.

The *signature* Σ consists of a countably infinite set Var of variables $(x, y, z, \ldots)$ and a non-empty set of function symbols $(A, B, C, \ldots, F, G, \ldots)$ of various finite arities ≥ 0. Constants are function symbols with arity 0. The set $Ter(\Sigma)$ of *finite terms* $(t, s, \ldots)$ over Σ is the smallest set containing the variables and closed under function application.

The set $O(t)$ of *positions* of a term $t \in Ter(\Sigma)$ is defined by induction on the structure of t as follows: $O(t) = \{\lambda\}$, if t is a variable and $O(t) = \{\lambda\} \cup \{i \cdot u | 1 \leq i \leq n \text{ and } u \in O(t_i)\}$, if t is of the form $F(t_1, \ldots, t_n)$. If $u \in O(t)$ then the subterm t/u at position u is defined as follows: $t/\lambda = t$ and $F(t_1, \ldots, t_n)/i \cdot u = t_i/u$. The *depth* of a subterm of t at position u is the length of u.

Contexts are terms in $Ter(\Sigma \cup \{\Box\})$, in which the special constant $\Box$, denoting an empty place, occurs exactly once. Contexts are denoted by $C[\,]$ and the result of substituting a term t in place of $\Box$ is $C[t] \in Ter(\Sigma)$. A *proper* context is a context not equal to $\Box$.

Substitutions are maps $\sigma : Var \rightarrow Ter(\Sigma)$ satisfying the equation $\sigma(F(t_1, \ldots, t_n)) = F(\sigma(t_1), \ldots, \sigma(t_n))$.

The set R of *rewrite rules* contains pairs (l, r) of terms in $Ter(\Sigma)$, written as $l \rightarrow r$, such that the left-hand side l is not a variable and the variables of the right-hand side r are contained in l. The result l^σ of the application of the substitution σ to the term l is an *instance* of l. A *redex* (reducible expression) is an instance of a left-hand side of a rewrite rule. A reduction step $t \rightarrow s$ is a pair of terms of the form $C[l^\sigma] \rightarrow [r^\sigma]$, where $l \rightarrow r$ is a rewrite rule in R. Concatenating reduction steps we get a *finite reduction sequence* $t_0 \rightarrow t_1 \rightarrow \ldots \rightarrow t_n$ or an *infinite* reduction sequence $t_0 \rightarrow t_1 \rightarrow \ldots$.

4.2.2 Infinitary orthogonal Term Rewriting Systems

An *infinitary Term Rewriting System* (TRS, usually this abbreviation is reserved for the finitary Term Rewriting Systems only) over a signature Σ is a pair $(Ter^\infty(\Sigma), R)$ consisting of the set $Ter^\infty(\Sigma)$ of finite and infinite terms over the signature Σ and a set of rewrite rules $R \subseteq Ter(\Sigma) \times Ter(\Sigma)$. It takes some elaboration to define the set $Ter^\infty(\Sigma)$ of finite and infinite terms.

The set $Ter(\Sigma)$ of finite terms for a signature Σ can be provided with an metric $d : Ter(\Sigma) \times Ter(\Sigma) \rightarrow [0, 1]$. The *distance* $d(t, s)$ of two terms t and s is 0, if t and s are equal, and 2^{-k}, otherwise, where $k \in \omega$ is the largest natural number such that all nodes of s and t at depth less than or equal to k are equally labeled. The set of infinitary terms $Ter^\infty(\Sigma)$ is the metric completion of $Ter(\Sigma)$. (This is all well known, see for instance [AN80]). Substitutions, contexts and reduction steps generalize trivially to the set of infinitary terms $Ter^\infty(\Sigma)$.

To introduce the prefix ordering $\leq$ on terms we extend the signature Σ with a fresh symbol Ω. The prefix ordering $\leq$ on $Ter^\infty(\Sigma \cup \{\Omega\})$ is defined inductively: $x \leq x$ for any variable x, $\Omega \leq t$ for any term t and, if $t_1 \leq s_1, \ldots, t_n \leq s_n$, then $F(t_1, \ldots, t_n) \leq F(s_1, \ldots, s_n)$.

If all function symbols of Σ occur in R we will write just R for $(Ter^\infty(\Sigma), R)$. The usual properties for finitary Term Rewriting Systems extend verbatim to infinitary Term Rewriting Systems:

DEFINITION **4.2.1**

a. A rewrite rule $l \rightarrow r$ *is* left-linear *if no variable occurs more than once in the left-hand side* l.
b. R *is* non-overlapping *if for any two left-hand sides* s *and* t, *any position* u *in* t, *and any substitutions* σ *and* $\tau : Var \rightarrow Ter(\Sigma)$ *it holds that if* $(t/u)^\sigma = s^\tau$ *then either* t/u *is a variable or* t *and* s *are left-hand sides of the same rewrite rule and* $u = \lambda$ *(i.e. non-variable parts of different rewrite rules do not overlap and non-variable parts of the same rewrite rule overlap only entirely).*
c. A (in)finitary Term Rewriting System R *is* orthogonal *if its rules are left-linear and non-overlapping.*
d. A rewrite rule $l \rightarrow r$ *is* collapsing, *if* r *is a variable.*

It is well known (cf. [Ros73, Klo92]) that finitary orthogonal Term Rewriting Systems satisfy the *finitary Church-Rosser property*, i.e., ${}^*\!\!\leftarrow \circ \rightarrow^* \subseteq \rightarrow^* \circ {}^*\!\!\leftarrow$, where $\rightarrow^*$ is the transitive, reflexive closure of the relation $\rightarrow$. It is not difficult to see that infinitary orthogonal Term Rewriting Systems inherit this finitary Church-Rosser property. In this chapter we consider a generalization of the finite Church-Rosser property to infinite reductions. It is a rather subtle issue to decide on the appropriate class of infinite reductions. We will discuss this in the next section.

4.2.3 Projecting infinitary reductions

In a complete metric space like $Ter^\infty(\Sigma)$, Cauchy sequences of any ordinal length have a limit. (Such transfinite Cauchy sequences are an instance of Moore-Smith convergence over a net indexed by the ordinal length of the sequence, see e.g. the text book [Kel55].) It is a natural idea to introduce *(transfinite) converging* reductions, as Dershowitz, Kaplan and Plaisted have done in [DJ90]. These are transfinite reduction sequences whose elements form a Cauchy sequence.

DEFINITION **4.2.2** *A reduction of ordinal length* α *is a set* $(t_\beta)_{\beta \leq \alpha}$ *of terms indexed by the ordinal* α *such that* $t_\beta \rightarrow t_{\beta+1}$ *for each* $\beta < \alpha$.

Note that when α is a limit ordinal, this definition does not stipulate any relationship between t_α and the earlier terms in the sequence. The obvious requirement to make is that the earlier terms should converge to t_α.

DEFINITION **4.2.3** *A reduction* $(t_\beta)_{\beta \leq \alpha}$ *is* Cauchy converging, *written* $t_0 \rightarrow^c_\alpha t_\alpha$), *in the following cases.*

a. $t_0 \rightarrow^c_0 t_0$,
b. $t_0 \rightarrow^c_{\beta+1} t_{\beta+1}$ *if* $t_0 \rightarrow^c_\beta t_\beta$,
c. $t_0 \rightarrow^c_\lambda t_\lambda$ *if* $t_0 \rightarrow^c_\beta t_\beta$ *for all* $\beta < \lambda$ *and* $\forall \epsilon < 0\, \exists \beta < \lambda\, \forall \gamma (\beta < \gamma < \lambda \rightarrow d(t_\gamma, t_\lambda) < \epsilon)$.

However, despite being apparently so natural, converging reductions are not well behaved even for orthogonal TRS.

- Converging reductions resist compression into converging reductions of length at most ω (cf. [FW91, KKSdV90b, DKP91]).
- Converging reductions do not project over finite reductions (cf. 4.2.4, [KKSdV90b, DKP91]).
- The infinitary Church-Rosser property does not hold (cf. 4.2.10, [KKSdV90b]).

The next example shows that the projection of a infinite converging reduction over a finite converging reduction need not be a converging reduction.

EXAMPLE **4.2.4** *[KKSdV90b, DKP91].*

$$\begin{array}{rl} \textit{Rules}: & A(x,y) \rightarrow A(y,x) \\ & C \rightarrow D \\ \textit{Sequences}: & A(C,C) \rightarrow A(C,C) \rightarrow A(C,C) \rightarrow A(C,C) \rightarrow \ldots \rightarrow_{\omega} A(C,C) \\ & A(C,D) \rightarrow A(D,C) \rightarrow A(C,D) \rightarrow A(D,C) \rightarrow \ldots \end{array}$$

Clearly $A(C,C) \rightarrow^{c}_{\omega} A(C,C)$. The second infinite reduction obtained by standard projection over the one step reduction $C \rightarrow D$ is not a converging reduction, and hence has no limit.

Strongly converging reductions, which generalize an idea in [FW91], have better properties. In [KKSdV90b] we have proved for orthogonal TRS that strongly converging reductions can be compressed and project over finite reductions. Informally, a strongly convergent reduction is such that for every depth d, there is some point in the reduction after which all contractions are performed at greater depth. By induction on α we define when a converging reduction $(t_\beta)_{\beta\leq\alpha}$ is strongly converging towards the limit t_α (notation $t_0 \rightarrow_\alpha t_\alpha$). By d_β we will denote the depth of the contracted redex in $t_\beta \rightarrow t_{\beta+1}$.

DEFINITION **4.2.5**

a. $t_0 \rightarrow_0 t_0$,
b. $t_0 \rightarrow_{\beta+1} t_{\beta+1}$ *if* $t_0 \rightarrow^{c}_{\beta+1} t_{\beta+1}$ *and* $t_0 \rightarrow_\beta t_\beta$,
c. $t_0 \rightarrow_\lambda t_\lambda$ *if* $t_0 \rightarrow^{c}_{\lambda} t_\lambda$ *and* $\forall\gamma<\lambda\ (t_0 \rightarrow_\gamma t_\gamma)$ *and* $\forall d>0\,\exists\beta<\lambda\,\forall\gamma\ (\beta<\gamma<\lambda \rightarrow d_\gamma > d)$.

By $t \rightarrow_{\leq\alpha} s$ we denote the existence of a strongly converging reduction from t with limit s of length less than or equal to α.

We end this section with some positive facts about strongly converging reductions that we will need in the sequel of this chapter.

Farmer and Watro have provided a necessary and sufficient condition when an infinite sequence of strongly converging reductions of length $\omega+1$ itself is strongly converging.

LEMMA **4.2.6** *[FW91]. Let $t_{n,0} \rightarrow_{\leq\omega} t_{n,\omega} = t_{n+1,0}$ be strongly converging for all $n \in \omega$. Let $d_{n,k}$ denote the depth of the contracted redex $R_{n,k}$ in $t_{n,k} \rightarrow t_{n,k+1}$. If for all n there is a d_n such that for all k it holds that $d_{n,k} > d_n$, and $\lim_{k\to\infty} d_k = \infty$, then there exists a term $t_{\omega,w}$ such that $t_{0,0} \rightarrow_{\leq\omega\times\omega} t_{\omega,\omega}$ via the strongly converging reduction $t_{0,0} \rightarrow_{\leq\omega} t_{0,\omega} = t_{1,0} \rightarrow_{\leq\omega} t_{1,\omega} = t_{2,0} \rightarrow_{\leq\omega} \ldots \rightarrow_{\leq\omega\times\omega} t_{\omega,\omega}$.* □

In order to state the Infinitary Projection Lemma for strongly convergent reductions we need the notion of descendant for transfinite reductions. We assume familiarity with the notion in finitary term rewriting of the descendants of of a position or set of positions by a finite reduction (cf. [HL91]). The existence of infinite terms does not complicate the notion, but for infinite sequences we must extend the definition to account for what happens at limit points.

DEFINITION **4.2.7** *Let R be a reduction sequence $t_0 \twoheadrightarrow_\alpha t_\alpha$ of length α. Denote the subsequence of R from t_{beta} to t_γ by $R_{\beta,\gamma}$ For a set of positions v of t_0 the set $v \setminus R$ of descendants of v by R in t_α is defined by induction on α. When α is finite, this is the standard notion. If α is a limit ordinal, then $v \setminus R$ is defined in terms of the sets $v \setminus R_{0,\beta}$ for all $\beta < \alpha$, as follows: $u \in v \setminus R$ if and only if $\exists \beta < \alpha\ \forall \gamma\ (\beta < \gamma < \alpha \rightarrow u \in v \setminus \gamma)$ If $\alpha = \lambda + n$ for a limit ordinal λ and a finite non-zero n, then $v \setminus R = v \setminus R_{0,\lambda} \setminus R_{\lambda,\lambda+n}$.*

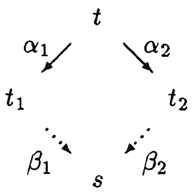
When contemplating this definition, note that the strong convergence of $t_0 \twoheadrightarrow_\alpha t_\alpha$ implies that for any position $u \in O(t_\alpha)$, either u is in every $v \setminus \gamma$ for sufficiently large γ, or u is in none of them. This is not the case for merely converging reductions, as Example 4.2.4 illustrates.

LEMMA **4.2.8** *Infinitary Projection lemma [KKSdV90b]. Let $(t_n)_{n \in \omega}$ be a strongly converging reduction of t_0 with limit t_ω and let $t_0 \rightarrow s_0$ be a reduction of a redex R of t_0. Then there is a strongly converging reduction $(s_n)_{n \in A}$ with limit s_ω, where for all $n \leq \omega$, s_n is obtained by contraction of all descendants of R in t_n.* □

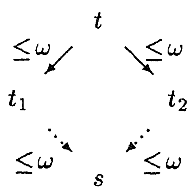
4.2.4 The infinitary Church-Rosser property

In the present infinitary context the natural generalization of the finite Church-Rosser property is to consider a peak of strongly converging reductions of arbitrary ordinal lengths.

DEFINITION **4.2.9** *An infinitary Term Rewriting System satisfies the infinitary Church-Rosser property for strongly converging reductions if for any peak $t \twoheadrightarrow_{\alpha_1} t_1$, $t \twoheadrightarrow_{\alpha_2} t_2$ there exists a joining valley $t_1 \twoheadrightarrow_{\beta_1} s$, $t_2 \twoheadrightarrow_{\beta_2} s$:*



Since strongly converging reductions can be compressed into reductions of length at most ω the infinitary Church-Rosser property follows if we can show that peaks of length ω can be joined:



Despite the Infinitary Projection Lemma for strongly converging reductions, the infinitary Church-Rosser property does not hold for strongly converging reductions (nor for converging reductions) in orthogonal TRS. The following TRS are counter-examples to the infinitary Church-Rosser property for both convergence and strong convergence:

EXAMPLE **4.2.10** *[KKSdV90b]*

a. *Rules* : $A(x) \to x$
$B(x) \to x$
$C \to A(B(C))$

Sequences : $C \to A(B(C)) \to A(C) \to_\omega A^\omega$
$C \to A(B(C)) \to B(C) \to_\omega B^\omega$

b. *Rules* : $D(x, y) \to x$
$C \to D(A, D(B, C))$

Sequences : $C \to D(A, D(B, C)) \to D(A, C) \to^* D(A, D(A, C)) \to^* \ldots$
$C \to D(A, D(B, C)) \to D(B, C) \to^* D(B, D(B, C)) \to^* \ldots$

Note that in these examples the rules involving C are not strictly necessary: e.g. for the first example one may consider then the infinite term $(AB)^\omega = A(B(A(B(\ldots))))$ instead.

4.3 DEPTH-PRESERVING ORTHOGONAL TERM REWRITING SYSTEMS

In this section and the next we consider two natural classes of orthogonal TRS in which the infinitary Church-Rosser property holds for strongly converging sequences. The counter-examples suggest that collapsing rules are destroying the Church-Rosser properties. In the next section we will prove the Church-Rosser property for strongly converging reductions in orthogonal TRS without collapsing rules.

In this section however we will consider the more restricted but easier to deal with orthogonal TRS whose rules are depth-preserving.

DEFINITION **4.3.1** *A* depth-preserving *TRS is a left-linear TRS such that for all rules the depth of any variable in a right-hand side is greater than or equal to the depth of the same variable in the corresponding left-hand side.*

THEOREM **4.3.2** *Any depth-preserving orthogonal TRS has the infinitary Church-Rosser property for strongly converging sequences.*

PROOF. Let $t_{0,0} \to t_{0,1} \to \ldots \to_{\leq\omega} t_{0,\omega}$ and $t_{0,0} \to t_{1,0} \to \ldots \to_{\leq\omega} t_{\omega,0}$ be strongly convergent.

a. Using the Infinitary Projection Lemma for strongly convergent reductions we construct the horizontal strongly converging sequences $t_{n,0} \to^* t_{n,1} \to^* \ldots \to_{\leq\omega} t_{n,\omega}$ for $0 < n < \omega$, as depicted in figure 4.1. The vertical reductions are constructed similarly.

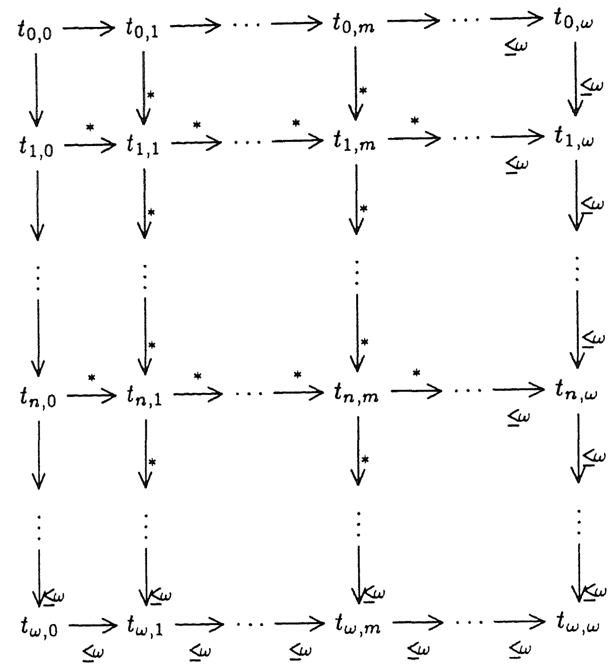


Figure 4.1

b. The construction of the Transfinite Projection Lemma also implies that the reduction $t_{n,\omega} \twoheadrightarrow_{\leq\omega} t_{n+1,\omega}$ is strongly converging.
By the depth-preserving property it holds for all $m, n \leq \omega$ that the depth of the reduced redexes in $t_{n,m} \twoheadrightarrow^* t_{n,m+1}$, which are all descendants of the redex $R_{0,m}$ in $t_{0,m} \rightarrow t_{0,m+1}$, is at least the depth of $R_{0,m}$ itself. Because $t_{0,0} \rightarrow t_{0,1} \rightarrow \ldots \twoheadrightarrow_{\leq\omega} t_{0,\omega}$ is strongly convergent we find by Lemma 4.2.6 that $t_{\omega,0} \twoheadrightarrow_{\leq\omega} t_{\omega,1} \twoheadrightarrow_{\leq\omega} t_{\omega,2} \ldots$ is strongly converging. Let us call its limit $t_{\omega,\omega}$.

c. In the same way the terms $t_{n,\omega}$ are part of a strongly converging sequence. The limit of this sequence is also equal to $t_{\omega,\omega}$, as can be seen with the following argument.
Let $\epsilon > 0$. Because $(t_{\omega,n})_{n\leq\omega}$ is a Cauchy sequence, there is an N_1 such that for all $m \geq N_1$ we have $d(t_{\omega,m}, t_{\omega,\omega}) < \frac{1}{3}\epsilon$.
Because $t_{0,0} \rightarrow t_{1,0} \rightarrow \ldots \twoheadrightarrow_{\leq\omega} t_{\omega,0}$ is strongly converging, there is an N_2 such that for $n \geq N_2$ we have that $2^{-d_n} < \frac{1}{3}\epsilon$ where d_n is the depth of the redex R_n reduced at step $t_{n,0} \rightarrow t_{n+1,0}$. Since the descendants of this redex R_n occur at least at the same depth, and since the TRS R is depth-preserving, we get $d(t_{n,m}, t_{\omega,m}) < \frac{1}{3}\epsilon$ for all $m \leq \omega$ and all $n \geq N_2$.
For similar reasons there is an N_3 such that for all $n \leq \omega$ and all $m \geq N_3$ we have that $d(t_{n,\omega}, t_{n,m}) < \frac{1}{3}\epsilon$.
Concluding: Let N be the maximum of N_1, N_2 and N_3. Then for $n \geq N$ we find using the triangle inequality for metrics that

$$\begin{aligned} d(t_{n,\omega}, t_{\omega,\omega}) &\leq d(t_{n,\omega}, t_{n,n}) + d(t_{n,n}, t_{\omega,\omega}) \\ &\leq d(t_{n,\omega}, t_{n,n}) + d(t_{n,n}, t_{\omega,n}) + d(t_{\omega,n}, t_{\omega,\omega}) \\ &\leq \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon \\ &\leq \epsilon. \end{aligned}$$

□

Observe that in this proof there are two places where it is essential that the reductions are strongly convergent. The first is the appeal to the Infinitary Projection Lemma. The second is in the argument that the sequences $(t_{\omega,n})_{n\in\omega}$ and $(t_{n,\omega})_{n\in\omega}$ have the same limit.

4.4 NON-COLLAPSING ORTHOGONAL TERM REWRITING SYSTEMS

DEFINITION **4.4.1** *A TRS R is non-collapsing if all its rewrite rules are non-collapsing, i.e. there is no rewrite rule in R whose right-hand side is a single variable.*

We will show that any non-collapsing orthogonal TRS satisfies the infinitary Church-Rosser property for strongly converging reductions. The proofs will use a variant of Park's notion of hiaton (cf. [Par83]). The idea is to replace a depth losing rule like $A(x, B(y)) \to B(y)$ by a depth-preserving variant $A(x, B(y)) \to B(\epsilon(y))$. In order to keep the rewrite rules applicable to terms involving hiatons, we also have to add more variants like $A(x, \epsilon^m(B(y))) \to B(\epsilon^{m+k+1}(y))$ for $k, m > 0$. By adding to a TRS all depth-preserving variants of its rewrite rules, we transform it into a depth-preserving TRS.

DEFINITION **4.4.2** *Let R be a TRS based on the alphabet Σ. Let Σ_ϵ be the extension of Σ with a fresh unary symbol ϵ.*

a. *Let the ϵ-hiding function $\rho : Ter^\infty(\Sigma_\epsilon) \to Ter^\infty(\Sigma)$ be partially defined by induction as follows:*

   (1) $\rho(x) = x$,
   (2) $\rho(f(t_1, \ldots, t_n)) = f(\rho(t_1), \ldots, \rho(t_n))$ *for f in Σ and $t_i \in Ter^\infty(\Sigma_\epsilon)$ for $0 \leq i \leq n$,*
   (3) $\rho(\epsilon(t)) = \rho(t)$ *for $t \in Ter^\infty(\Sigma_\epsilon)$.*

   Hence ρ is well-defined on terms in $Ter^\infty(\Sigma_\epsilon)$ containing no infinite string of ϵs.

b. *A term $t \in Ter^\infty(\Sigma_\epsilon)$ is an* ϵ-variant *of a term $s \in Ter^\infty(\Sigma)$ if $\rho(t) = s$, that is, if hiding the ϵs in t results in s.*

c. *An ϵ-variant of a rule $l \to r$ is a pair of terms (l_ϵ, r_ϵ) such that*

   (1) $\rho(l_\epsilon) = l$.
   (2) $\rho(r_\epsilon) = r$.
   (3) the root symbol of l_ϵ is not ϵ.

(4) l_ϵ *does not contain a subterm of the form* $\epsilon(x)$ *for any variable* x.
(5) *the root symbol of* r_ϵ *is not* ϵ *unless* r *is a variable,*

d. *The* ϵ*-completion* R^ϵ *of* R *has alphabet* Σ_ϵ. *Its rules are the* depth-preserving ϵ-*variants of rules of* R. *We denote reduction in* R^ϵ *by* $\rightarrow^\epsilon$.

The proof of the following lemma is straightforward and omitted.

LEMMA **4.4.3** *The* ϵ*-completion of an orthogonal TRS is depth-preserving and orthogonal.* □

LEMMA **4.4.4** *Let* R *be a non-collapsing orthogonal TRS.*

a. *Let* t_ϵ *be an* ϵ*-variant of a term* t *of* R. *If* t_ϵ *strongly* ϵ*-converges in* ω *steps to some term* s *in* R^ϵ, *then* s *does not contain a branch ending in an infinite string of* ϵ*s.*
b. *Let* t_0 *be the* ϵ*-variant of some term* s_0. *If* $t_0 \rightarrow^\epsilon_\omega t_\omega$ *is a strongly converging reduction in* R^ϵ, *then so is* $s_0 \rightarrow_\omega s_\omega$ *in* R, *where* $s_i = \rho(t_i)$ *for* $0 \leq i \leq \omega$.
c. *Let* $t_0 \rightarrow_\omega t_\omega$ *be a strongly converging reduction in* R. *Let* s_0 *be an* ϵ*-variant of* t_0. *Then there exists a strongly converging reduction* $s_0 \rightarrow^\epsilon_\omega s_\omega$ *in* R^ϵ *such that each* s_i *is an* ϵ*-variant of the corresponding* t_i *and similar for the reduction rules used.*

PROOF.

a. Since there are no collapsing rules, a string of ϵs can only be made longer by a reduction occurring at its top. Strong convergence implies that only finitely many such reductions can be made, and therefore that an infinite string of ϵs cannot be created.
b. Since t_0 is an ϵ-variant it does not contain an infinite string of ϵs. Neither does any of the t_i for $i \in \omega$, nor t_ω itself by the previous item 4.4.4(a). Hence, $\rho(t_n)$ is a well-defined term for all $0 \leq n \leq \omega$.
Because there are no infinite strings of ϵs in t_ω, every infinite path from the root of t_ω must contain infinitely many occurrences of members of Σ. Note also that t_ω is necessarily an infinite term.
Since by the previous item 4.4.4(a) t_ω contains no infinite string of ϵs, it must contain occurrences of members of Σ at arbitrarily great depth.
Given any finite number k, consider those occurrences v of t_ω, such that the path from the root to v contains at least k occurrences of symbols in Σ. By the preceding remarks, there must be at least one such occurrence. Let N_k be the minimum length of all such v. Because there are no infinite strings of ϵs, N_k must tend to infinity with k. Since $t_0 \rightarrow_\omega t_\omega$ is strongly converging there exists for any $k > 0$ an N such that for $n > N$, the depth of the redex reduced in $t_{n-1} \rightarrow t_n$ is at least N_k. This implies that the corresponding redex in $s_{n-1} \rightarrow s_n$ is at depth at least k, and hence $s_0 \rightarrow_\omega s_\omega$ is strongly convergent.
c. Trivial. The ϵ-variant s_0 of t_0 contains the corresponding ϵ-variant of the redex reduced in t_0. Apply an ϵ-variant of the corresponding rule. The resulting reduction satisfies the required properties. □

THEOREM **4.4.5** *Any non-collapsing orthogonal TRS satisfies the infinitary Church-Rosser property for strongly converging reductions.*

PROOF. Let R be an orthogonal TRS. Construct its ϵ-completion R^ϵ. By Theorem 4.3.2 the depth-preserving orthogonal TRS R^ϵ satisfies the infinitary Church-Rosser property. So if we start with two strongly converging reductions $t \rightarrow_{\leq\omega} s_1$ and $t \rightarrow_{\leq\omega} s_2$, then by Lemma 4.4.4(c) these reductions lift to two strongly converging reductions in R^ϵ, let us say $t \rightarrow^{\epsilon}_{\leq\omega} r_1$ and $t \rightarrow^{\epsilon}_{\leq\omega} r_2$. By Theorem 4.3.2 there exists a join u for the two lifted reductions such that $r_1 \rightarrow^{\epsilon}_{\leq\omega} u$ as well as $r_2 \rightarrow^{\epsilon}_{\leq\omega} u$. Erasing all ϵs using Lemma 4.4.4(b) we see that the term $\rho(u)$ is the join in R of $t \rightarrow_{\leq\omega} s_1$ and $t \rightarrow_{\leq\omega} s_2$. □

THEOREM **4.4.6** *An orthogonal TRS, each of whose rules is non-collapsing except for at most one rule of the form $I(x) \rightarrow x$, satisfies the infinitary Church-Rosser property for strongly converging reductions.*

PROOF. First, note that the proof of the previous theorem cannot be directly applied in the presence of the rule $I(x) \rightarrow x$. Consider the rules $A(x) \rightarrow I(x)$, $B(x) \rightarrow I(x)$, $I(x) \rightarrow x$. There are obvious reductions of the term $A(B(A(B(\cdots))))$ to both A^ω and B^ω. These lift to reductions ending with $A(\epsilon(A(\epsilon(\cdots))))$ and $\epsilon(B(\epsilon(B(\cdots))))$ respectively. If we now apply the Church-Rosser property of the depth-balanced system, we obtain reductions of these terms to $\epsilon(\epsilon(\epsilon(\epsilon(\cdots))))$, which cannot be lifted to strongly convergent reductions in the original system.

A simple modification of the previous proof establishes the present theorem. We modify the depth-preserving transformation by introducing two versions of ϵ: ϵ itself, and ϵ'. The rule $I(x) \rightarrow x$ is replaced by the depth-preserving version $I(x) \rightarrow \epsilon'(x)$. The other rules are transformed as before, except that wherever ϵ would appear on the left-hand side in the original transformation, either ϵ or ϵ' is used, in all possible combinations. On the right-hand sides, only ϵ is used. It is easy to see that the resulting system is depth-preserving and orthogonal, and hence that the infinite Church-Rosser property holds.

The distinction between ϵ and ϵ' can be thought of as labeling those occurrences of ϵ which arise from reductions of the I-rule.

Now consider two strongly converging reductions $t \rightarrow_{\leq\omega} s_1$ and $t \rightarrow_{\leq\omega} s_2$. As in the proof of the previous theorem, we obtain in R^ϵ a term u and two strongly converging reductions $r_1 \rightarrow^{\epsilon}_{\leq\omega} u$ and $r_2 \rightarrow^{\epsilon}_{\leq\omega} u$, where r_1 and r_2 are ϵ-variants of s_1 and s_2.

We cannot in general erase all the ϵs and ϵ's from these sequences to obtain a join for s_1 and s_2, since u may contain infinite branches of ϵs and ϵ's (which we shall call ϵ-branches for short). But we will show that we can transform these sequences in such a way as to eliminate such branches, after which the erasing process can be performed safely.

In every ϵ-branch in u, there must be infinitely many ϵ's. This follows for the same reason that in the non-collapsing case, no infinite branch of ϵs can arise.

Now consider an occurrence of ϵ' in an ϵ-branch of u. This must arise from a reduction by the rule $I(x) \rightarrow \epsilon'(x)$ at some point in each of the sequences $r_1 \rightarrow^{\epsilon}_{\leq\omega} u$ and $r_2 \rightarrow^{\epsilon}_{\leq\omega} u$. This reduction is performed on a subterm of the form $I(T)$, where T reduces to a ϵ-branch. By orthogonality, it is impossible for the reduction of the I-redex to be necessary for any later step of the sequence to be possible. If we omit it, the only effect is that certain occurrences of ϵ' later in the sequence are replaced by I.

We therefore omit from both $r_1 \twoheadrightarrow^{\epsilon}_{\leq\omega} u$ and $r_2 \twoheadrightarrow^{\epsilon}_{\leq\omega} u$ every I-reduction which gives rise to an occurrence of ϵ' in any ϵ-branch of u. This gives a term u' containing no such occurrences of ϵ', and reduction sequences $r_1 \twoheadrightarrow^{\epsilon}_{\leq\omega} u'$ and $r_2 \twoheadrightarrow^{\epsilon}_{\leq\omega} u'$. These sequences have the property that they contain no ϵ-branch anywhere. They may therefore be lifted to strongly convergent reductions in the original system, providing a strongly convergent joining of the original reduction sequences. □

4.5 CONCLUSION

The results of Dershowitz, Kaplan and Plaisted in [DKP91] imply that top-terminating orthogonal TRS satisfy the infinitary Church-Rosser property for Cauchy converging reductions which start from a finite term. (Cf. [DKP91]: combine their Theorem 3.3, Proposition 5.1, Theorem 6.4, and Theorem 6.3.) The property *top-termination*, that is, there are no derivations of infinite length starting from a finite term with infinitely many rewrites at topmost position, is rather strong and not very syntactic.

Our Theorems 4.4.5 and 4.4.6 show that for strongly converging reductions, orthogonal systems with no collapsing rules, other than possibly one of the form $I(x) \to x$, have the infinitary Church-Rosser property without conditions on the finiteness of the initial term. Theorem 4.4.6 is the best possible result for orthogonal TRS, since the counter-examples in 4.2.10 make it clear that no larger class of orthogonal TRS is Church-Rosser.

We do not know what the situation is for Cauchy converging reductions. For example, do non-collapsing orthogonal TRS have the infinitary Church-Rosser property for Cauchy converging reductions?

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