# Geometric Methods in Engineering Applications

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***Abstract*— In this work, we introduce two set of algorithms inspired by the ideas from modern geometry. One is computational conformal geometry method, including harmonic maps, holomorphic 1-forms and Ricci flow. The other one is optimization method using affine normals.**

**In the first part, we focus on conformal geometry. Conformal structure is a natural structure of metric surfaces. The concepts and methods from conformal geometry play important roles for real applications in scientific computing, computer graphics, computer vision and medical imaging fields.**

**This work systematically introduces the concepts, methods for numerically computing conformal structures inspired by conformal geometry. The algorithms are theoretically rigorous and practically efficient.**

**We demonstrate the algorithms by real applications, such as surface matching, global conformal parameterization, conformal brain mapping etc.**

**In the second part, we consider minimization of a real-valued function $f$ over $\mathbb{R}^{n+1}$ and study the choice of the affine normal of the level set hypersurfaces of $f$ as a direction for minimization. The affine normal vector arises in affine differential geometry when answering the question of what hypersurfaces are invariant under unimodular affine transformations. It can be computed at points of a hypersurface from local geometry or, in an alternate description, centers of gravity of slices. In the case where $f$ is quadratic, the line passing through any chosen point parallel to its affine normal will pass through the critical point of $f$. We study numerical techniques for calculating affine normal directions of level set surfaces of convex $f$ for minimization algorithms.**

***Index Terms*— Conformal geometry, holomorphic 1-form, harmonic maps, Ricci flow, global conformal parametrization, Conformal brain mapping,**

## I. Introduction

Conformal structure is a natural geometric structure of a metric surface. It is more flexible than Riemannian metric structure and more rigid than topological structure, therefore it has advantages for many important engineering applications.

The first example is from computer graphics. Surface parameterization refers to the process to map a surface onto the planar domains, which plays a fundamental role in graphics and visualization for the purpose of texture mapping. Surface parameterization can be reformulated as finding a special Riemannian metric with zero Gaussian curvature everywhere, namely a flat metric. If the parameterization is known, then pull back metric induced by the map is the flat metric; conversely, if a flat metric of the surface is known, then the surface can be flattened onto the plane isometricly to induce the parameterization.

The second example is from geometric modeling. Constructing manifold splines on a surface is an important issue for modeling. In order to define parameters and the knots of the spline, special atlas of the surface is required such that all local coordinate transition maps are affine. One way to construct such an atlas is as follows, first a flat metric of the surface is found, then a collection of open sets are located to cover the whole surface, finally each open set is flattened using the flat metric to form the atlas.

The third example is from medical imaging. The human brain cortex surface is highly convolved. In order to compare and register brain cortex surfaces, it is highly desirable to canonically map them to the unit sphere. This is equivalent to find a Riemannian metric on the cortex surface, such that the Gaussian curvature induced by this metric equals to one everywhere. Once such a metric is obtained, the cortex surface can be coherently glued onto the sphere piece by piece isometricly.

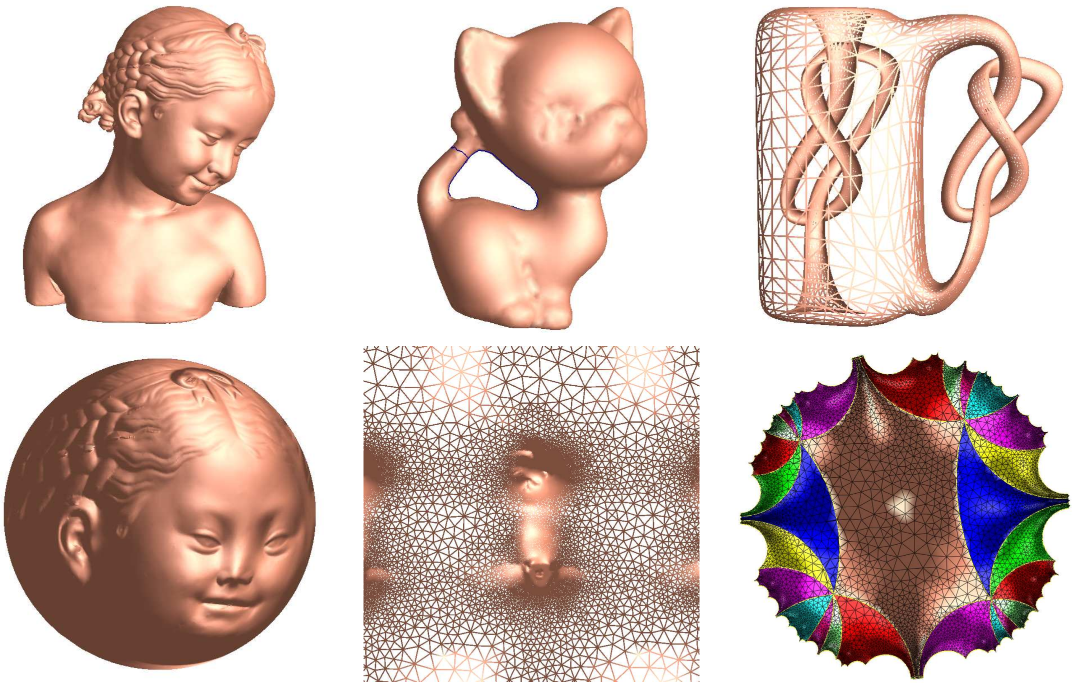
For most applications, the desired metrics should minimize the angle distortion and the area distortion. The angles measured by the new metric should be consistent with those measured by the original metric. The existence of such metrics can be summaried as Riemann uniformization theorem. Finding those metrics is equivalent to compute surface conformal structure. Therefore, it is of fundamental importance to compute conformal structures of general surfaces.

In modern geometry, conformal geometry of surfaces are studied in Riemann surface theory. Riemann surface theory is a rich and mature field, it is the intersection of many subjects, such as algebraic geometry, algebraic topology, differential geometry, complex geometry etc. This work focuses on converting theoretic results in Riemann surface theory to practical algorithms.

## II. Previous Works

Much research has been done on mesh parameterization due to its usefulness in computer graphics applications. The survey of [Floater and Hormann 2005] provides excellent reviews on various kinds of mesh parameterization techniques. Here, we briefly discuss the previous work on the conformal mesh parameterization.

Several researches on conformal mesh parameterization tried to discretize the nature of the conformality such that any intersection angle at any point on a given manifold is preserved on the parameterized one at the corresponding point. Floater [Floater 1997] introduced a mesh parameterization technique based on convex combinations. For each vertex, its 1-ring stencil is parameterized into a local parameterization space while preserving angles, and then the convex combination of the vertex is computed in the local parameterization spaces.



Spherical Euclidean Hyperbolic

The overall parameterization is obtained by solving a sparse linear system. [Sheffer and de Sturler 2001] presented a constrained minimization approach, so called angle-based flattening (ABF), such that the variation between the set of angles of an original mesh and one of 2D flatten version is minimized. In order to obtain a valid and flipping-free parameterization, several angular and geometric constraints are incorporated with the minimization processes. Lately, they improved the performance of ABF by using an advanced numerical approach and a hierarchical technique [Sheffer et al. 2005].

Recently, much research has been incorporated with the theories of differential geomety. [Levy et al. 2002] applied the Cauchy-Riemann equation for mesh parameterization and provided successful results on the constrained 2D parameterizations with free boundaries. [Desbrun et al. 2002] minimized the Dirichlet energy defined on triangle meshes for computing conformal parameterization. It has been noted that the approach of [Desbrun et al. 2002] has the same expressional power with [Levy et al. 2002]. Gu and Yau [Gu and Yau 2003] computed the conformal structure using the Hodge theory. A flat metric of the given surface is induced by computing the holomorphic 1-form with a genus-related number of singularities and used for obtaining a globally smooth parameterization. [Gortler et al. 2005] used discrete 1-forms for mesh parameterization. Their approach provided an interesting result in mesh parameterization with several holes, but they cannot control the curvatures on the boundaries. Ray et al. [Ray et al. 2005] used the holomorphic 1-form to follow up the principle curvatures on manifolds and computed a quad-dominated parameterization from arbitrary models. Kharevych et al. [34] applied the theory of circle patterns from [Bobenko and Springborn 2004] to globally conformal parameterizations. They obtain the uniform conformality by preserving intersection angles among the circum-circles each of which is defined from a triangle on the given mesh. In their approach, the set of angles is non-linear optimized first, and then the solution is refined with cooperating geometric constraints. They provide several parameterization results, such as 2D parameterization with predefined boundary curvatures, spherical parameterization, and globally smooth parameterization of a high genus model with introducing singularity points. [Gu et al. 2005] used the discrete Ricci flow [Chow and Luo 2003] for generating manifold splines with a single extraordinary point. The Ricci flow is utilized for obtaining 2D parameterization of high-genus models in their paper.

In theory, the Ricci flow [Chow and Luo 2003] and the variations with circle patterns [Bobenko and Springborn 2004] have the same mathematical power. However, because of the simplicity of the implementation, we adopt the Ricci flow as a mathematical tool for the parameterization process.

In contrast to all previous approaches, the parameterization spaces in our interests are not only the 2D space but also arbitrary hyperbolic spaces. As a result, we can provide novel classes of applications in this paper, such as parameterization with interior and exterior boundaries having prescribed curvatures, PolyCube-mapping, quasi-conformal cross-parameterization with high-genus surfaces, and geometry signatures.

## III. Theoretic Background

In this section, we introduce the theories of conformal geometry.

### A. Riemann Surface

Suppose $S$ is a two dimensional topological manifold covered by a collection of open sets $\{U_\alpha\}$, $S \subset \bigcup_\alpha U_\alpha$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{C}$ maps $U_\alpha$ to the complex plane. $(U_\alpha, \phi_\alpha)$ forms a local coordinate system. Suppose two open sets $U_\alpha$ and $U_\beta$ intersect, then each point $p \in U_\alpha \bigcap U_\beta$ has two local coordinates, the transformation between the local coordinates is defined as the *transition function*

$$\phi_{\alpha\beta} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\beta \cap U_\beta). \qquad (1)$$

Suppose a complex function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic*, if its derivative exists. If $f$ is invertible, and $f^{-1}$ is also holomorphic, then $f$ is called *bi-holomorphic*.

*Definition 1 (Conformal Structure):* A two dimensional topological manifold $S$ with an atlas $\{(U_\alpha, \phi_\alpha)\}$, if all transition functions $\phi_{\alpha\beta}$'s are bi-holomorphic, then the atlas is called a *conformal atlas*. The union of all conformal atlas is called the *conformal structure* of $S$.

A surface with conformal structure is called a Riemann surface. All metric surfaces are Riemann surface.

### B. Uniformization Metric

Suppose $S$ is a $C^2$ smooth surface embedded in $\mathbb{R}^3$ with parameter $(u^1, v^2)$. The position vector is $\mathbf{r}(u^1, u^2)$, then tangent vector is $d\mathbf{r} = \mathbf{r}_1 du^1 + \mathbf{r}_2 du^2$, where $\mathbf{r}_1, \mathbf{r}_2$ are the partial derivatives of $\mathbf{r}$ with respect to $u^1, u^2$ respectively. The length of the tangent vector is represented as the *first fundamental form*

$$ds^2 = \sum g_{ij} du^1 du^2 \qquad (2)$$

where $g_{ij} =< \mathbf{r}_i, \mathbf{r}_j >$. The matrix $(g_{ij})$ is called the *Riemannian metric* matrix.

A special parameterization can be chosen to simplify the Riemannian metric, such that $g_{11} = g_{22} = e^{2\lambda}$ and $g_{12} = 0$, such parameter is called the *isothermal coordinates*. If all the local coordinates of an atlas are isothermal coordinates, then the atlas is the conformal atlas of the surface. For all orientable metric surfaces, such atlas exist, namely

*Theorem 2 (Riemann Surface):* All orientable metric surfaces are Riemann surfaces.

The *Gauss curvature* measures the deviation of a neighborhood of a point on the surface from a plane, using isothermal coordinates, the Gaussian curvature is calculated as

$$K = -\frac{2}{e^{2\lambda}} \Delta\lambda, \qquad (3)$$

where $\Delta$ is the Laplace operator on the parameter domain.

*Theorem 3 (Gauss-Bonnet):* Suppose a closed surface $S$, the Riemannian metric $\mathbf{g}$ induces the Gaussian curvature function $K$, then the total curvature is determined by

$$\int_S K dA = 2\pi\chi(S), \qquad (4)$$

where $\chi(S)$ is the Euler number of $S$.

Suppose $u : S \to \mathbf{R}$ is a function defined on the surface $S$, then $e^{2u}\mathbf{g}$ is another Riemannian metric on $S$. Given arbitrary two tangent vectors at one point, the angle between them can be measured by $\mathbf{g}$ or $e^{2u}\mathbf{g}$, the two measurements are equal. Therefore we say $e^{2u}\mathbf{g}$ is *conformal* (or angle preserving) to $\mathbf{g}$. $(S, \mathbf{g})$ and $(S, e^{2u}\mathbf{g})$ are endowed with different Riemannian metrics but the same conformal structure.

The following Poincaré uniformization theorem postulate the existence of the conformal metric which induces constant Gaussian curvature,

*Theorem 4 (Poincaré Uniformization):* Let $(S, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold, then there is a metric $\bar{\mathbf{g}}$ conformal to $\mathbf{g}$ which has constant Gauss curvature. Such a metric is called the *uniformization metric*. According to Gauss-Bonnet theorem 4, the sign of the constant Gauss curvature is determined by the Euler number of the surface. Therefore, all closed surfaces can be conformally mapped to three canonical surfaces, the sphere for genus zero surfaces $\chi > 0$, the plane for genus one surfaces $\chi = 0$, and the hyperbolic space for high genus surfaces $\chi < 0$.

### C. Holomorphic 1-forms

Holomorphic and meromorphic functions can be defined on the Riemann surface via conformal structure. Holomorphic differential forms can also be defined,

*Definition 5 (holomorphic 1-form):* Suppose $S$ is a Riemann surface with conformal atlas $\{(U_\alpha, z_\alpha\}$, where $z_\alpha$ is the local coordinates. Suppose a complex differential form $\omega$ is represented as

$$\omega = f_\alpha(z_\alpha) dz_\alpha,$$

where $f_\alpha$ is a holomorphic function, then $\omega$ is called a holomorphic 1-form.

Holomorphic 1-forms play important roles in computing conformal structures.

A holomorphic 1-form can be interpreted as a pair of vector fields, $\omega_1 + \sqrt{-1}\omega_2$, such that the curl and divergence of $\omega_1, \omega_2$ are zeros,

$$\nabla \times \omega_i = 0, \nabla \cdot \omega_i = 0, i = 1, 2,$$

and

$$\mathbf{n} \times \omega_1 = \omega_2,$$

everywhere on the surface. Both $\omega_i$ are *harmonic 1-forms*, the following Hodge theorem clarifies the existence and uniqueness of harmonic 1-forms,

*Theorem 6 (Hodge):* Each cohomologous class of 1-forms has a unique harmonic 1-form.

### D. Ricci Flow

In geometric analysis, *Ricci flow* is a powerful tool to compute Riemannian metric. Recently, Ricci flow is applied to prove the Poincaré conjecture. The Ricci flow is the process to deform the metric $\mathbf{g}(t)$ according to its induced Gauss curvature $K(t)$, where $t$ is the time parameter

$$\frac{dg_{ij}(t)}{dt} = -K(t) g_{ij}(t). \qquad (5)$$

It is proven that the curvature evolution induced by the Ricci flow is exactly like heat diffusion on the surface

$$\frac{K(t)}{dt} = -\Delta_{\mathbf{g}(t)} K(t), \qquad (6)$$

where $\Delta_{\mathbf{g}(t)}$ is the Laplace-Beltrami operator induced by the metric $\mathbf{g}(t)$. Ricci flow converges, the metric $\mathbf{g}(t)$ is conformal to the original metric at any time $t$. Eventually, the Gauss curvature will become to constant just like the heat diffusion $K(\infty) \equiv const$, the limit metric $\mathbf{g}(\infty)$ is the *uniformization metric*.

### E. Harmonic Maps

Suppose $S_1, S_2$ are metric surfaces embedded in $\mathbb{R}^3$. $\phi : S_1 \to S_2$ is a map from $S_1$ to $S_2$. The harmonic energy of the map is defined as

$$E(\phi) = \int_{S_1} < \nabla\phi, \nabla\phi > dA.$$

The critical point of the harmonic energy is called the *harmonic maps*.

The normal component of the Laplacian is

$$\Delta\phi^{\perp} =< \Delta\phi, \mathbf{n}\circ\phi > \mathbf{n},$$

If $\phi$ is a harmonic map, then the tangent component of Laplacian vanishes,

$$\Delta\phi = \Delta\phi^{\perp},$$

where $\Delta$ is the Laplace-Beltrami operator.

We can diffuse a map to a harmonic map by the heat flow method:

$$\frac{d\phi}{dt} = -(\Delta\phi - \Delta\phi^{\perp}).$$

## IV. Computational Algorithms

In practice, all surfaces are represented as simplicial complexes embedded in the Euclidean space, namely, triangular meshes. All the algorithms are discrete approximations of their continuous counter parts. We denote a mesh by $M$, and use $v_i$ to denote its ith vertex, edge $e_{ij}$ for the edge connecting $v_i$ and $v_j$, and $f_{ijk}$ for the triangle formed by $v_i, v_j$ and $v_k$, which are ordered counter-clock-wisely.

If a mesh $M$ is with boundaries, we fist convert it to a closed symmetric mesh $\bar{M}$ by the following *double covering* algorithm:

1) Make a copy mesh $M'$ of $M$.
2) Reverse the orientation of $M'$ by change the order of vertices of each face, $f_{ijk} \to f_{jik}$.
3) Glue $M$ and $M'$ along their boundaries to form a closed mesh $\bar{M}$.

In the following discussion, we always assume the surfaces are closed. We first introduce harmonic maps method for genus zero surfaces, then holomorphic 1-forms for genus one surfaces and finally Ricci flow method for high genus surfaces.

### A. Genus Zero Surfaces - Harmonic Maps

For genus zero surfaces, the major algorithm to compute their conformal mapping is *harmonic maps*, the basis procedure is to diffuse a degree one map until the map becomes harmonic.

1) Compute the normal of each face, then compute the normal of each vertex as the average of normals of neighboring faces.
2) Set the map $\phi$ equals to the Gauss map,
$$\phi(v_i) = \mathbf{n}_i,$$
3) Diffuse the map by Heat flow acting on the maps
$$\phi(v_i) -= (\Delta\phi(v_i) - \Delta\phi(v_i)^{\perp})\varepsilon$$
where $\Delta\phi(v_i))^{\perp}$ is defined as
$$< \Delta\phi(v_i), \phi(v_i) > \phi(v_i).$$
4) Normalize the map by set
$$\phi(v_i) = \frac{\phi(v_i) - \mathbf{c}}{|\phi(v_i) - \mathbf{c}|},$$
where $\mathbf{c}$ is the mass center defined as
$$\mathbf{c} = \sum_{v_i} \phi(v_i).$$
5) Repeat step 2 and 3, until $\Delta\phi(v_i))$ is very closed to $\Delta\phi(v_i))^{\perp}$.

where $\Delta$ is a discrete Laplace operator, defined as

$$\Delta\phi(v_i) = \sum_j w_{ij}(\phi(v_i) - \phi(v_i)),$$

where $v_j$ is a vertex adjacent to $v_i$, $w_{ij}$ is the edge weight

$$w_{ij} = \frac{\cot\alpha + \cot\beta)}{2},$$

$\alpha, \beta$ are the two angles against edge $e_{ij}$.

The harmonic maps $\phi : M \to \mathbb{S}^2$ is also conformal. The conformal maps are not unique, suppose $\phi_1, \phi_2 : M \to \mathbb{S}^2$ are two conformal maps, then $\phi_1 \circ \phi_2^{-1} : \mathbb{S}^2 \to \mathbb{S}^2$ is a conformal map from sphere to itself, it must be a so-called Möbius transformation. Suppose we map the sphere to the complex plane by a stereo-graphics projection

$$(x, y, z) \to \frac{2x + 2\sqrt{-1}y}{2 - z},$$

then the Möbius transformation has the form

$$w \to \frac{aw + b}{cw + d}, ad - bc = 1, a, b, c, d \in \mathbb{C}.$$

The purpose of normalization step is to remove Möbius ambiguity of the conformal map from $M$ to $\mathbb{S}^2$.

For genus zero open surfaces, the conformal mapping is straight forward

1) Double cover $M'$ to get $\bar{M}$.
2) Conformally map the doubled surface to the unit sphere
3) Use the sphere Möbius transformation to make the mapping symmetric.
4) Use stereographic projection to map each hemisphere to the unit disk.

The Möbius transformation on the disk is also a conformal map and with the form

$$w \to e^{i\theta} \frac{w - w_0}{1 - \bar{w}_0 w}, \tag{7}$$

where $w_0$ is arbitrary point inside the disk, *theta* is an angle. Figure IV illustrates two conformal maps from the David head surface to the unit disk, which differ by a Möbius transformation.

### *B. Genus One Surfaces - Holomorphic 1-forms*

For genus one closed surfaces, we compute the basis of holomorphic 1-form group, which induces the conformal parameterization directly. A holomorphic 1-form is formed by a pair of harmonic 1-forms $\omega_1, \omega_2$, such that $\omega_2$ is conjugate to $\omega_1$.

In order to compute harmonic 1-forms, we need to compute the homology basis for the surface. A homology base curve is a consecutive halfedges, which form a closed loop. First we compute a *cut graph* of the mesh, then extract a homology basis from the cut graph. Algorithm for cut graph:

1) Compute the dual mesh $\bar{M}$, each edge $e \in M$ has a unique dual edge $\bar{e} \in \bar{M}$.
2) Compute a spanning tree $\bar{T}$ of $\bar{M}$, which covers all the vertices of $\bar{M}$.
3) The cut graph is the union of all edges whose dual are not in $\bar{T}$,
$$G = \{e \in M | \bar{e} \notin \bar{T}\}.$$

Then, we can compute homology basis from $G$,

1) Compute a spanning tree $T$ of $G$.
2) $G$
$T = \{e_1, e_2, \cdots, e_n\}$.
3) $e_i \cup T$ has a unique loop, denoted as $\gamma_i$.
4) $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ form a homology basis of $M$.

A harmonic 1-form is represented as a linear map from the halfedge to the real number, $\omega : \{Half\ Edges\} \to \mathbb{R}$, such that

$$\left\{ \begin{array}{lll} \omega\partial f & \equiv & 0 \\ \Delta\omega & \equiv & 0 \\ \int_{\gamma_i} \omega & = & c_i \end{array} \right. \tag{8}$$

where $\partial$ represents boundary operator, $\partial f_{ijk} = e_{ij} + e_{jk} + e_{ki}$, therefore $\omega\partial f_{ijk} = \omega(e_{ij}) + \omega(e_{jk}) + \omega(e_{ki})$; $\Delta\omega$ represents the Laplacian of $\omega$,

$$\Delta\omega(v_i) = \sum_j w_{ij}\omega(h_{ij}),$$

$h_{ij}$ are the half edges from $v_i$ to $v_j$; $\{c_i\}$ are prescribed real numbers. It can be shown that the solution to the above equation group exists and unique.

On each face $f_{ijk}$ there exists a unique vector $\mathbf{t}$, such that on each edge, $\omega(h_{ij}) =< v_j - v_i, \mathbf{t} >, \omega(h_{jk}) =< v_k - v_j, \mathbf{t} >$ and $\omega(h_{ki}) =< v_i - v_k, \mathbf{t} >$. Let $\mathbf{t}' = \mathbf{n} \times \mathbf{t}$, then $\omega'(h_{ij}) =< v_j - v_i, \mathbf{t}' >$ defines another harmonic 1-form, which is conjugate to $\omega$, $(\omega, \omega')$ form a holomorphic 1-form.

We cut a surface $M$ along its cut graph to get an topological disk $D_M$, by gluing $D_M$ consistently, we can construct a finite portion of the universial covering space of $M$.

We then integrate a holomorphic 1-form to map $D_M$ to the plane conformally in the following way:

1) Fix one vertex $v_0 \in D_M$, and map it to the origin $\phi(v_0) = (0,0)$.
2) For any vertex $v \in D_M$, compute the shortest path $\gamma$ from $v_0$ to $v$ in $D_M$,
3) $\phi(v) = (\int_\gamma \omega, \int_\gamma \omega')$.

We then visualize the holomorphic 1-forms by texture mapping a checker board onto $D_M$ using texture coordinates $\phi$. Figure IV-B demonstrates the holomorphic 1-forms on three different surfaces.

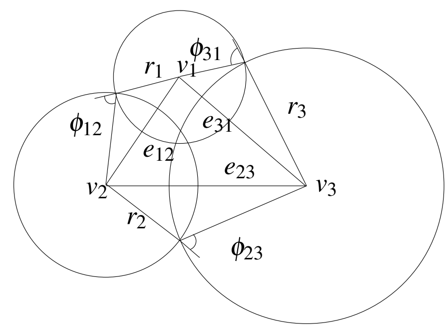
### *C. High Genus Surfaces - Discrete Ricci flow*

For high genus surfaces, we apply discrete Ricci flow method to compute their uniformization metric and then embed them in the hyperbolic space.

*1) Circle Packing Metric:* We associate each vertex $v_i$ with a circle with radius $\gamma_i$. On edge $e_{ij}$, the two circles intersect at the angle of $\Phi_{ij}$. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\Phi_{ij}$$

A circle packing metric is denoted by $\{\Sigma, \Phi, \Gamma\}$, where $\Sigma$ is the triangulation, $\Phi$ the edge angle, $\Gamma$ the vertex radii.




Two circle packing metrics $\{\Sigma, \Phi_1, \Gamma_1\}$ and $\{\Sigma, \Phi_2, \Gamma_2\}$ are conformal equivalent, if

- The radii of circles are different, $\Gamma_1 \neq \Gamma_2$.
- The intersection angles are same, $\Phi_1 \equiv \Phi_2$.

In practice, the circle radii and intersection angles are optimized to approximate the induced Euclidean metric of the mesh as close as possible.

*2) Poincaré Disk:* According to Riemann uniformization theorem, high genus surfaces can be conformally embed in hyperbolic space. Instead of treat each triangle as an Euclidean triangle, we can treat each triangle as a hyperbolic triangle. The hyperbolic space is represented using *Poincaré disk*, which is the unit disk on the complex plane, with Riemannian metric

$$ds^2 = \frac{4dwd\bar{w}}{(1 - \bar{w}w)^2}.$$

The rigid motion in Poincaré disk is Möbius transformation 7. The geodesics are circle arcs which are orthogonal to the unit circle. A hyperbolic circle in Poincaré disk with center $c$ and radius $r$ is also an Euclidean circle with center $C$ and radius $R$, such that $\mathbf{C} = \frac{2-2\mu^2}{1-\mu^2|\mathbf{c}|^2}$ and $R^2 = |\mathbf{C}|^2 - \frac{|\mathbf{c}|^2-\mu^2}{1-\mu^2|\mathbf{c}|^2}, \mu = \frac{e^r-1}{e^r+1}$.

*3) Hyperbolic Ricci Flow:* Let

$$u_i = \log tanh \frac{\gamma_i}{2}, \tag{9}$$

then discrete hyperbolic Ricci flow is defined as

$$\frac{du_i}{dt} = \bar{K}_i - K_i. \tag{10}$$

In fact, discrete Ricci flow is the gradient flow of the following *hyperbolic Ricci energy*

$$f(\mathbf{u}) = \int_{\mathbf{u}_0}^{\mathbf{u}} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i, \tag{11}$$

where $n$ is the number of edges, $\mathbf{u} = (u_1, u_2, \cdots, u_m)$, $m$ is the number of vertices. In practice, if we set $\bar{K}_i \equiv 0$ by minimizing the Ricci energy using Newton's method, the hyperbolic uniformization metric can be computed efficiently.

Once the hyperbolic metric for a mesh is calculated, the mesh can be flattened face by face in the Poincaré disk. Determining the position of a vertex in the Poincaré disk is equivalent to find the intersection between two hyperbolic circles, which can be converted as finding the intersection between two Euclidean circles.

## V. Applications

Conformal geometry has broad applications in medical imaging, computer graphics, geometric modeling and many other fields.

### A. Conformal Brain Mapping

Human cortex surfaces are highly convoluted, it is difficult to analyze and study them. By using conformal maps, we can map the brain surface to the canonical unit sphere and carry out all the geometric processing, analysis, measurement on the spherical domain. Because the conformal map preserves angle structure, local shapes are well preserved, it is valuable for visualization purpose. Different cortical surfaces can be automatically registered on the canonical parameter domain, it is more efficient to compare surfaces using conformal brain mapping.

Figure V-A illustrates an example of conformal brain mapping. The cortical surface is reconstructed from MRI images and converted as a triangular mesh.

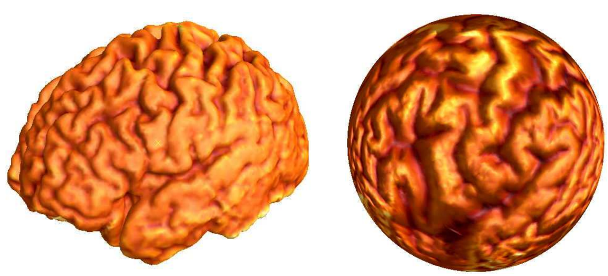


Fig. 3. Conformal Brain Mapping

### B. Global Conformal Parameterization

In computer graphics, surface parameterization plays an important role for various applications, such as texture mapping, texture synthesis.

Basically, a surface is mapped to the plane, the planar coordinates of each vertex are used as texture coordinates. It is highly desirable to reduce the distortion between the texture image and the geometric surface. Conformal mapping is useful because it is angle distortion free. Figure V-B illustrates an example for texture mapping using global conformal parameterization of a genus two surface.

Fig. 4. Texture mapping using conformal mapping.

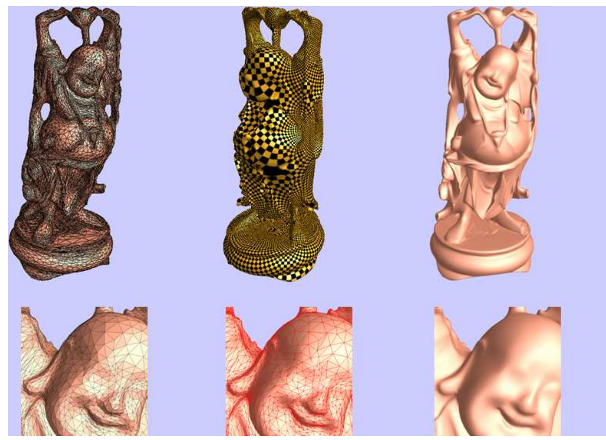


Fig. 5. Manifold Splines constructed from holomorphic 1-form.

### C. Manifold Splines

In geometric modeling, conventional splines are defined on the planar domains. It is highly desirable to define splines on surfaces with arbitrary topologies directly.

In order to define splines on manifolds, one needs to compute a special atlas of the manifold, such that all chart transition maps are affine. Such kind of atlas can be easily constructed by integrating a holomorphic 1-form.

Figure V-C demonstrates one example of genus 6 surface. The holomorphic 1-form induces an affine atlas with singularities, the planar powell-sabin splines are defined on the atlas directly.

## VI. Affine Normal

Many problems in engineering field can be formulated as optimization problem. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, finding its critical points is the basic task.

Situated at any point, a natural direction to choose for minimization is the steepest descent direction. This is the direction along which the function is locally diminishing the most rapidly. The steepest descent direction can be computed from derivative information of $f$ through the form $-\nabla f$. Unfortunately, while this direction is intuitively sound, it shows slow convergence.

Newton's method use quadratic approximation at the origin,

$$f(\mathbf{x}) \approx \mathbf{x}^T \nabla^2 f(0)\mathbf{x} + \nabla f^T(0)\mathbf{x} + c.$$

When the quadratic approximation is taken at a point $\mathbf{y}$, the critical point is at $\mathbf{x} = \mathbf{y} - (\nabla^2 f(\mathbf{y})^{-1} \nabla f(\mathbf{y})$, therefore, one use $\mathbf{x}$ to replace $\mathbf{y}$ as the next guess. By iteration, the critical point can be reached. Newton's method quadratically converges. But it is expensive to compute the inverse of the second derivative matrices, the Hessian matrices.

### A. Affine Normal

Let $M$ be a hypersurface in $\mathbb{R}^{n+1}$, $N$ is the normal vector field on $M$. Now if $X$ and $Y$ are vector fields on $M$ and $D_X Y$ is the flat connection on $\mathbb{R}^{n+1}$, then we decompose

$$D_X Y = \nabla_X Y + h(X,Y)N,$$

where $\nabla_X Y$ is the tangential part of $D_X Y$ and $h(X,Y)$ the normal part, also known as the second fundamental form. Furthermore, in this case, $\nabla_X Y$ is the Levi-Civita connection of the Riemannian metric induced by $\mathbb{R}^{n+1}$ on $M$.

We choose an arbitrary local frame field $e_1, e_2, \cdots, e_n$ tangent to $M$ and $det(e_1, e_2, \cdots, e_{n+1}) = 1$, we may define $h$ as $D_X Y = \nabla_X Y + h(X,Y)e_{n+1}$, to arrive at the affine metric

$$II_{ik} = H^{-\frac{1}{n+2}} h_{ik},$$

where $H$ is the determinant $det\{h_{ik}\}$. In this case, the affine normal field is given by $\Delta M$, where the Laplacian is with respect to the affine metric $II$ and $M$ is the position vector of $M$M.

Suppose $M$ is a level set surface of the function $f$, we can derive the affine normal field as

$$H^{\frac{1}{n+2}} \begin{pmatrix} f^{ij}(-\frac{n}{n+2} f^{pq} f_{pqi} + n \frac{f_{n+1,i}}{|\nabla f|}) \\ -\frac{n}{|\nabla f|} \end{pmatrix},$$

where the coordinates $x_i$ used are rotated so that $x_{n+1}$ is in the normal direction. It can be shown that when the hypersurface is an ellipsoid, all affine normals point towards its center. in fact, the affine normals of the level sets of a quadratic polynomial will point toward the unique critical point, even if that critical point is unstable.

### B. Affine Normal Descent Algorithm

Using this affine normal field, we can summarize our algorithm in the following steps, iterated to convergence:

1) Compute the affine normal direction to the level set of the function at the current approximation location.
2) Use a line search to find the minimum of the function along that direction. This location serves as the new approximation.

We call this the affine normal descent algorithm. For the quadratic minimization problem, due to the nature of the affine normal and the ellipsoidal level sets of $f$, the approximations of this algorithm will take on the value of the exact minimum after one iteration. Thus, the affine normal and the vector $-(\nabla^2 f)^{-1} \nabla f$ used in Newton's method are parallel to each other in this case. This means we may view this algorithm as an extension of the steepest descent method, using the affine normal direction, which points at the center of ellipsoids, instead of the steepest descent direction, which points at the center of spheres. On the other hand, we may view it as a relative of Newton's method, both exact for the quadratic minimization problem but with different higher order terms.

### C. Efficiency

In terms of computational costs, we note that the previously derived formula for the affine normal direction requires first, second, and third derivatives of $f$, as well as inversion of an $n \times n$n matrix of second derivatives. While it may be possible to generate other forms or approximations of the affine normal direction that simplify the inversion or diminish the need of derivative information. Instead, we consider a different viewpoint of the affine normal to bypass the need for such information. Consider a convex hypersurface, a point on that surface, and the tangent plane located there. Furthermore, consider the class of planes intersecting with the surface and parallel to the tangent plane. On each of these planes, we look at the center of gravity of the region enclosed by the intersection of the plane with the surface. The union of these centers of gravity forms a curve. It turns out that the one-sided tangent direction of this curve at the point of interest is the affine normal vector. Thus, an alternate approach for calculating the affine normal vector involves calculating centers of gravity, completely bypassing the need for derivative information higher than that of the first derivative which is required for tangent planes.

### D. Experimental Results

We tested our method for several cases and measure the accuracy and efficiency. For a five-dimensional convex function, let

$$f(\mathbf{x}) = \sum_{i=1}^{5} (x_i^2 + \sin x_i).$$

Let $(-0.2, 0, 0.4, 1, -0.3)$ be the starting point. The iterations of our algorithm are shown in Table VI-D, the algorithm converge to the point

$$\begin{pmatrix} -0.45018354967147 \\ -0.45018354967140 \\ -0.45018354967139 \\ -0.45018354967125 \\ -0.45018354967129 \end{pmatrix}$$

From the statistics, we can see that the affine normal method is efficient and practical.

| j | $f(p_j)$ | $\lvert p_j - p_{j-1} \rvert$ |
|---|---|---|
| 0 | 2.02669978966015 | |
| 1 | −0.87543513107826 | 2.17147295853185 |
| 2 | −1.16211480429203 | 0.13960261665982 |
| 3 | −1.16232787552543 | 0.00003483822789 |
| 4 | −1.16232787579106 | 0.00001466875411 |
| 5 | −1.16232787579106 | 0.00000000001789 |

TABLE I

Five-Dimensional Result: In this five-dimensional example, convergence is achieved after 5 iterations.

## VII. Conclusion

This paper introduces some algorithms inspired by geometric insights.

We first introduce a series of computational algorithms to compute conformal Riemannian metrics on surfaces, especially the uniformization metrics. The algorithms include harmonic maps, holomorphic 1-forms and surface Ricci flow on discrete meshes. The methods are applied for various applications in computer graphics, medical imaging and geometric modeling.

In the future, we will generalize these algorithms for discrete 3-manifolds represented as tetrahedral meshes.

Second, we introduce an efficient optimization algorithm based on affine differential geometry, which reaches cirtical point for quadratic functions in one step. The method is practical and efficient. In the future, we will improve the method for computing affine normals.

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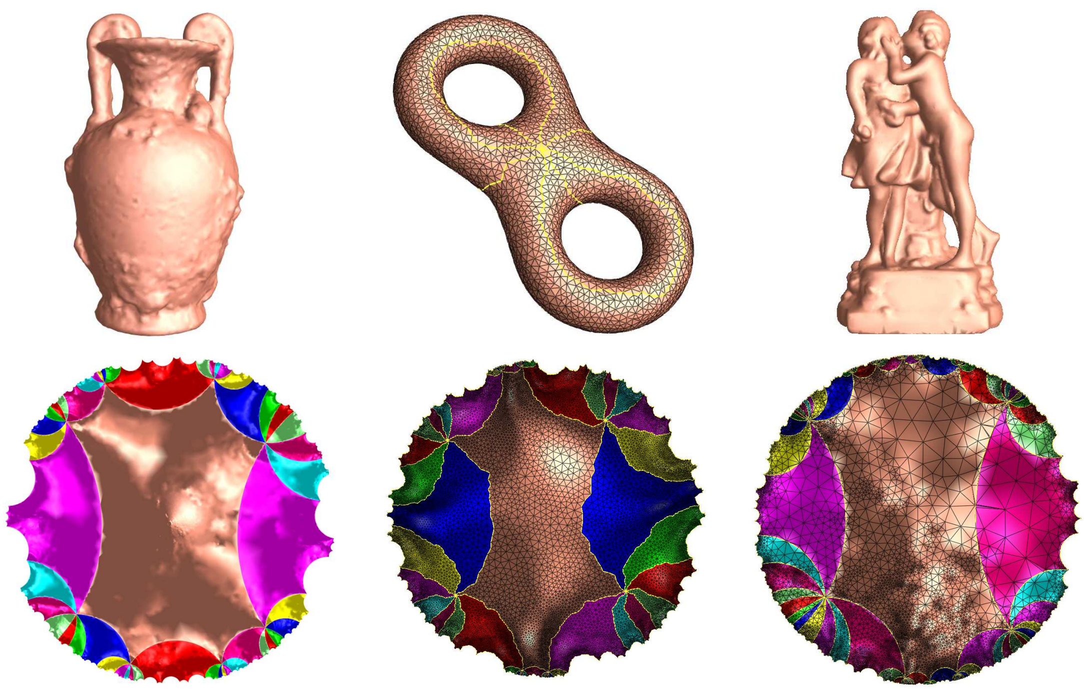
Fig. 1. According to Riemann Mapping theorem, a topological disk can be conformally mapped to the unit disk. Two such conformal maps differ by a Möbius transformation of the unit disk



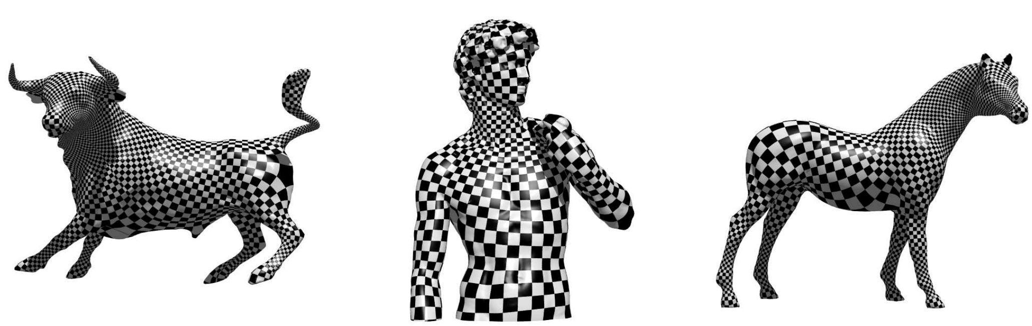


Fig. 2. Holomorphic 1-forms on different surfaces.