11 - 19 P

Sample

Chapter 1 Discrete random variables

NEW FOR 2017

Edexcel AS and A level Further Mathematics

Further Statistics 1

FS1

Pearson

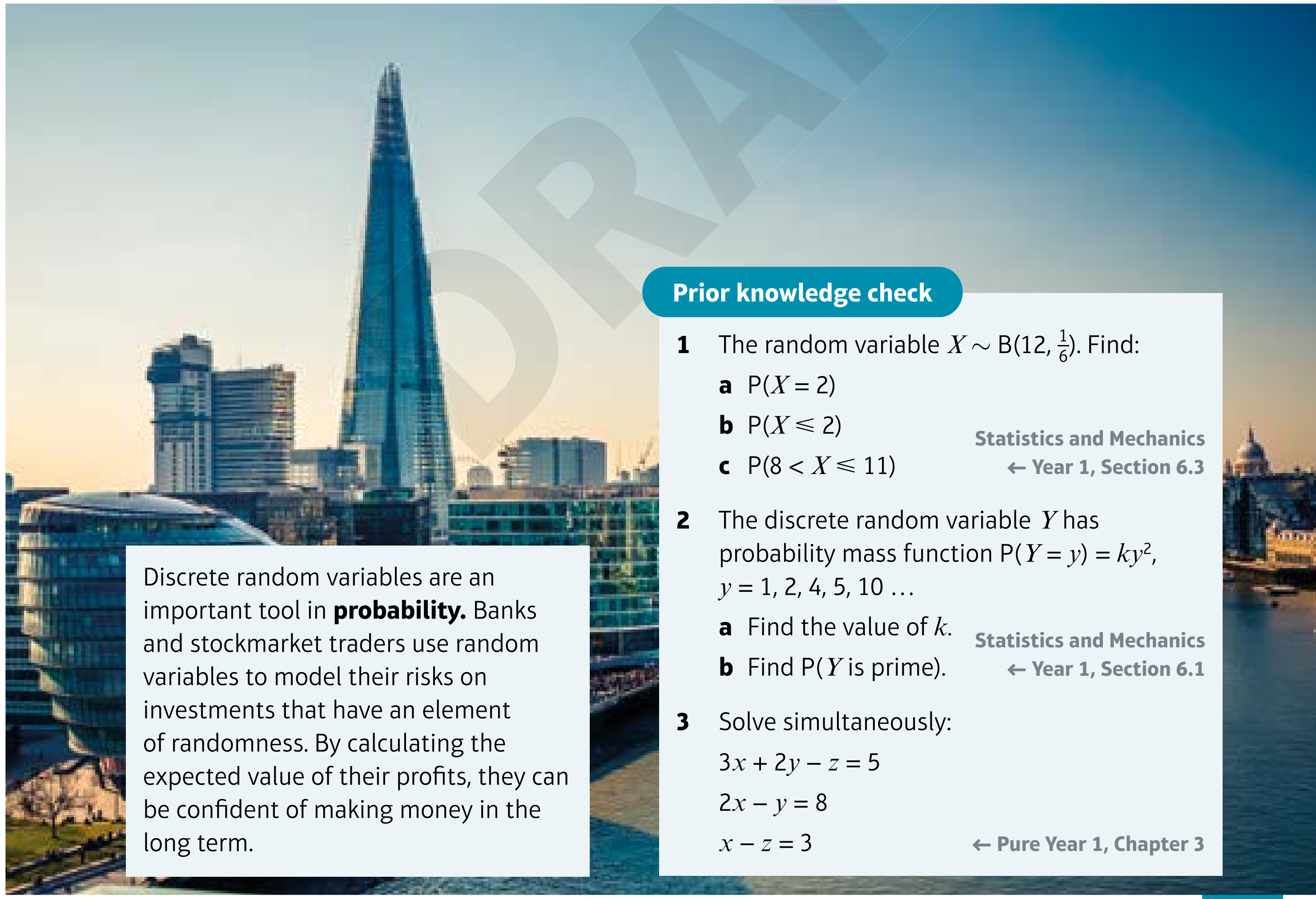
Discrete random variables

1

Objectives

After completing this chapter you should be able to:

- Find the expected value of a discrete random variable X → pages 2–5
- Find the expected value of X^2 → pages 3–5
- Find the variance of a random variable → pages 5–7
- Use the expected value and variance of a function of X → pages 7–11
- Solve problems involving random variables → pages 11–14



Discrete random variables are an important tool in **probability.** Banks and stockmarket traders use random variables to model their risks on investments that have an element of randomness. By calculating the expected value of their profits, they can be confident of making money in the long term.

Prior knowledge check

1 The random variable $X \sim B(12, \frac{1}{6})$. Find:

a $P(X = 2)$

b $P(X \leqslant 2)$

c $P(8 < X \leqslant 11)$

Statistics and Mechanics ← Year 1, Section 6.3

2 The discrete random variable Y has probability mass function $P(Y = y) = ky^2$, $y = 1, 2, 4, 5, 10 \ldots$

a Find the value of k.

b Find $P(Y \text{ is prime})$.

Statistics and Mechanics ← Year 1, Section 6.1

3 Solve simultaneously:

$3x + 2y - z = 5$

$2x - y = 8$

$x - z = 3$

← Pure Year 1, Chapter 3

1.1 Expected value of a discrete random variable

Recall that a **random variable** is a variable whose value depends on a random event. The random variable is **discrete** if it can only take certain numerical values.

Links The probabilities of any discrete random variable add up to 1. For a discrete random variable, X, you write $\sum P(X = x) = 1$.
← Statistics and Mechanics Year 1, Chapter 6

If you take a set of observations from a discrete random variable, you can find the mean of those observations. As the number of observations increases, this value will get closer and closer to the **expected value** of the discrete random variable.

Watch out The expected value is a theoretical quantity, and gives information about the probability distribution of a random variable.

- **The expected value of the discrete random variable X is denoted E(X) and defined as E(X) = $\sum x P(X = x)$**

Notation The expected value is sometimes referred to as the **mean**, and is sometimes denoted by μ.

Example 1

A fair six-sided dice is rolled. The number on the uppermost face is modelled by the random variable X.

a Write down the probability distribution of X.

b Use the probability distribution of X to calculate E(X).

a

x	1	2	3	4	5	6
$P(x = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Since the dice is fair, each side is equally likely to end facing up, so the probability of any face ending up as the uppermost is $\frac{1}{6}$

b The expected value of X is:

$E(X) = \sum xP(X = x) = \frac{1}{6} + \frac{2}{6} + \ldots + \frac{6}{6}$

$= \frac{21}{6} = \frac{7}{2} = 3.5$

Substitute values from the probability distribution into the formula then simplify.

If you know the probability distribution of X then you can calculate the expected value. Notice that in Example 1 the expected value is 3.5, but $P(X = 3.5) = 0$. The expected value of a random variable does not have to be a value that the random variable can actually take. Instead this tells us that in the long run, we would expect the average of several rolls to get close to 3.5.

Example 2

The random variable X has a probability distribution as shown in the table.

x	1	2	3	4	5
p(x)	0.1	p	0.3	q	0.2

a Given that E(X) = 3, write down two equations involving p and q.

b Find the value of p and the value of q.

a $p + q + 0.1 + 0.3 + 0.2 = 1$

$p + q = 1 - 0.6$

$p + q = 0.4$ (1)

$(1 \times 0.1) + 2p + (3 \times 0.3) + 4q + (5 \times 0.2) = 3$

$2p + 4q = 3 - (0.1 + 0.9 + 1)$

$2p + 4q = 1$ (2)

b $2p + 4q = 1$

$2p + 2q = 0.8$

so $2q = 0.2$

$q = 0.1$

$p = 0.4 - q$

$= 0.4 - 0.1$

$= 0.3$

Problem-solving Remember that the probabilities must add up to 1. You will often have to use $\sum P(X = x) = 1$ when solving problems involving discrete random variables.

$E(X) = \sum xP(X = x)$

From (2).

Multiply (1) by 2.

Subtract bottom line from top line.

If X is a discrete random variable, then X^2 is also a discrete random variable. You can use this rule to determine the expected value of X^2.

- **E(X^2) = $\sum x^2 P(X = x)$**

Links Any function of a random variable is also a random variable. → Section 1.3

Example 3

A discrete random variable X has a probability distribution.

x	1	2	3	4
P($X = x$)	$\frac{12}{15}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

a Write down the probability distribution for X^2.

b Find E(X^2).

a The distribution for X^2 is

x	1	2	3	4
x^2	1	4	9	16
$P(X^2 = x^2)$	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

X can take values 1, 2, 3, 4, so X^2 can take values $1^2, 2^2, 3^2, 4^2$.

Note that because X takes only positive values, $P(X^2 = x^2) = P(X = x)$.

b $E(X^2) = \sum x^2 P(X = x^2)$

$= 1 \times \frac{12}{25} + 4 \times \frac{6}{25} + 9 \times \frac{4}{25} + 16 \times \frac{3}{25}$

$= \frac{120}{25}$

$= 4.8$

Watch out E(X^2) is, in general, not equal to $(E(X))^2$. In this example E(X) = 1.92 and $1.92^2 \neq 4.8$.

Exercise 1A

1 For each of the following probability distributions write out the distribution of X^2 and calculate both E(X) and E(X^2).

a

x	2	4	6	8
P($X = x$)	0.3	0.3	0.2	0.2

b

x	−2	−1	1	2
P($X = x$)	0.1	0.4	0.1	0.4

Watch out Note that, for example, $P(X^2 = 4) = P(X = 2) + P(X = -2)$.

2 The score on a biased dice is modelled by a random variable X with probability distribution

x	1	2	3	4	5	6
P($X = x$)	0.1	0.1	0.1	0.2	0.4	0.1

Find E(X) and E(X^2).

3 The random variable X has a probability function

$$P(X = x) = \frac{1}{x} \qquad x = 2, 3, 6$$

a Construct tables giving the probability distributions of X and X^2.

b Work out E(X) and E(X^2).

c State whether or not $(E(X))^2 = E(X^2)$.

4 The random variable X has a probability function given by

$$P(X = x) = \begin{cases} 2^{-x} & x = 1, 2, 3, 4 \\ 2^{-4} & x = 5 \end{cases}$$

a Construct a table giving the probability distribution of X.

b Calculate E(X) and E(X^2).

c State whether or not $(E(X))^2 = E(X^2)$.

E/P **5** The random variable X has the following probability distribution.

x	1	2	3	4	5
P($X = x$)	0.1	a	b	0.2	0.1

Given that E(X) = 2.9 find the value of a and the value of b. **(5 marks)**

E/P **6** The random variable X has the following probability distribution.

x	−2	−1	1	2
P($X = x$)	0.1	a	b	c

Given that E(X) = 0.3 and E(X^2) = 1.9, find a, b and c. **(7 marks)**

Hint You can use the given information to write down simultaneous equations for a, b and c which can be solved using the matrix inverse operation on your calculator.
← Core Pure Book 1, Section 6.6

E/P **7** The discrete random variable X has probability function

$$P(X = x) = \begin{cases} a(1 - x) & x = -2, -1, 0 \\ b & x = 5 \end{cases}$$

Given that E(X) = 1.2, find the value of a and the value of b. **(6 marks)**

E/P **8** A biased six-sided dice has a $\frac{1}{8}$ chance of landing on any of the numbers 1, 2, 3 or 4. The probabilities of landing on 5 or 6 are unknown. The outcome is modelled as a random variable, X. Given that E(X) = 4.1,

a find the probability distribution of X. **(5 marks)**

The dice is rolled 10 times.

b Calculate the probability that the dice lands on 6 at least 3 times. **(3 marks)**

E **9** Jorge has designed a game for his school fete. Students can pay £1 to roll a fair six-sided dice. If they score a 6 they win a prize of £5. If they score a 4 or a 5 they win a smaller prize of £P.

By modelling the amount paid out in prize money as a discrete random variable, determine the maximum value of P in order for Jorge to make a profit on his game.

Hint The expected profit from the game is the cost of playing the game minus the expected value of the amount paid out in prize money.

Challenge

Three fair six-sided dice are rolled. The discrete random variable X is defined as the largest value of the three values shown. Find E(X).

1.2 Variance of a discrete random variable

If you take a set of observations from a discrete random variable, you can find the variance of those observations. As the number of observations increases, this value will get closer and closer to the **variance** of the discrete random variable.

- **The variance of X is usually written as Var(X) and is defined as $\text{Var}(X) = E((X - E(X))^2)$**
- **Sometimes it is easier to calculate the variance using the formula $\text{Var}(X) = E(X^2) - (E(X))^2$**

Notation The variance is sometimes denoted by σ^2, where σ is the standard deviation.

The random variable $(X - E(X))^2$ is the squared deviation from the expected value of X. It is large when X takes values that are very different to E(X).

From the definition you can see that $\text{Var}(X) \geqslant 0$ for any random variable X. The larger Var(X) the more variable X is. In other words, the more likely it is to take values very different to its expected value.

Example 4

A fair six-sided dice is rolled. The number on the uppermost face is modelled by the random variable X.

Calculate the variance using both formulae and check that you get the same answer.

We have that $E(X) = 3.5$

> This was calculated in the first example of the previous section.

The distributions of X, X^2 and $(X - E(X))^2$ are given by

x	1	2	3	4	5	6
x^2	1	4	9	16	26	36
$(x - E(X))^2$	6.25	2.25	0.25	0.25	2.25	6.25
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

So the variance is

$$\text{Var}(X) = \sum(x - E(X))^2 P(X = x)$$

$$= 6.25 \times \tfrac{2}{6} + 2.25 \times \tfrac{2}{6} + 0.25 \times \tfrac{2}{6}$$

> Substitute values into the formula for variance.

$$= (6.25 + 2.25 + 0.25) \times \tfrac{1}{3} = \tfrac{35}{12}$$

The expected value of X^2 is

$$E(X^2) = \sum x^2 P(X = x) = \tfrac{1}{6}(1 + 4 + \ldots + 36) = \tfrac{91}{6}$$

So using the alternative formula

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \tfrac{91}{6} - \tfrac{49}{4} = \tfrac{35}{12}$$

Exercise 1B

1 The random variable X has a probability distribution given by

x	−1	0	1	2	3
$P(X = x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

a Write down $E(X)$.

b Find $\text{Var}(X)$.

2 Find the expected value and variance of the random variable X with probability distribution given by

a

x	1	2	3
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

b

x	−1	0	1
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

c

x	−2	−1	1	2
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

3 Given that Y is the score when a single unbiased six-sided dice is rolled, find $E(Y)$ and $\text{Var}(Y)$.

(P) 4 Two fair cubical dice are rolled and S is the sum of their scores. Find:

a the distribution of S

b $E(S)$

c $\text{Var}(S)$

d the standard deviation.

5 Two fair tetrahedral (four-sided) dice are rolled and D is the difference between their scores. Find:

a the distribution of D and show that $P(D = 3) = \frac{1}{8}$

b $E(D)$

c $\text{Var}(D)$.

(E) 6 A fair coin is tossed repeatedly until a head appears or three tosses have been made. The random variable T represents the number of tosses of the coin.

a Show that the distribution of T is

t	1	2	3
$P(T = t)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

(3 marks)

b Find the expected value and variance of T. **(6 marks)**

(E) 7 The random variable X has probability distribution given by

x	1	2	3
$P(X = x)$	a	b	a

where a and b are constants.

a Write down $E(X)$. **(2 marks)**

b Given that $\text{Var}(X) = 0.75$, find the values of a and b. **(5 marks)**

1.3 Expected value and variance of a function of X

If X is a discrete random variable, and g is a function, then $g(X)$ is also a discrete random variable. You can calculate the expected value of $g(X)$ using the formula:

- $\mathbf{E(g(X)) = \sum g(x)P(X = x)}$

This is a more general version of the formula for $E(X^2)$. For simple functions, such as addition and multiplication by a constant, you can learn the following rules:

- **If X is a random variable and a and b are constants, then $E(aX + b) = aE(X) + b$**
- **If X and Y are random variables, then $E(X + Y) = E(X) + E(Y)$.**

You can use a similar rule to simplify variance calculations for some functions of random variables:

- **If X is a random variable and a and b are constants then $\text{Var}(aX + b) = a^2\text{Var}(X)$.**

Example 5

A discrete random variable X has a probability distribution

x	1	2	3	4
$P(X = x)$	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

a Write down the probability distribution for Y where $Y = 2X + 1$.

b Find $E(Y)$.

c Compute $E(X)$ and verify that $E(Y) = 2E(X) + 1$.

a The distribution for Y is

x	1	2	3	4
y	3	5	7	9
$P(Y = y)$	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

When $x = 1$, $y = 2 \times 1 + 1 = 3$
$x = 2$, $y = 2 \times 2 + 1 = 5$
etc.

Notice how the probabilities relating to X are still being used, for example, $P(X = 3) = P(Y = 7)$.

b $E(Y) = \sum yP(Y = y)$

$= 3 \times \frac{12}{25} + 5 \times \frac{6}{25} + 7 \times \frac{4}{25} + 9 \times \frac{3}{25}$

$= \frac{121}{25}$

$= 4.84$

c $E(X) = \sum xP(X = x) = 1 \times \frac{12}{25} + 2 \times \frac{6}{25}$

$+ 3 \times \frac{4}{25} + 4 \times \frac{3}{25} = \frac{48}{25} = 1.92$

$2 \times 1.92 + 1 = 4.84$

If you know or are given $E(X)$ you can use the formula to find $E(Y)$ quickly.

Example 6

A random variable X has $E(X) = 4$ and $Var(X) = 3$.

Find:

a $E(3X)$ **b** $E(X - 2)$

c $Var(3X)$ **d** $Var(X - 2)$

e $E(X^2)$

a $E(3X) = 3E(X) = 3 \times 4 = 12$

b $E(X - 2) = E(X) - 2 = 4 - 2 = 2$

c $Var(3X) = 3^2\ Var(X) = 9 \times 3 = 27$

d $Var(X - 2) = Var(X) = 3$

e $E(X^2) = Var(X) + (E(X))^2 = 3 + 4^2 = 19$

Rearrange $Var(X) = E(X^2) - (E(X))^2$

Example 7

Two fair 10p coins are tossed. The random variable X represents the total value of the coins that land heads up.

a Find $E(X)$ and $Var(X)$.

The random variables S and T are defined as follows:

$S = X - 10$ and $T = \frac{1}{2}X - 5$

b Show that $E(S) = E(T)$.

c Find $Var(S)$ and $Var(T)$.

Susan and Thomas play a game using two 10p coins. The coins are tossed and Susan records her score using the random variable S and Thomas uses the random variable T. After a large number of tosses they compare their scores.

d Comment on any likely differences or similarities.

a The distribution of X is

x	0	10	20
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$E(X) = 10$ by symmetry.

The distribution of X is symmetric around 10. More precisely, X has the same distribution as $10 - (X - 10) = 20 - X$. Therefore $E(X) = E(20 - X) = 20 - E(X)$, so $E(X) = 10$.

$Var(X) = E(X^2) - (E(X))^2$

$Var(X) = 0^2 \times \frac{1}{4} + 10^2 \times \frac{1}{2} + 20^2 \times \frac{1}{4} - 10^2 = 50$

b $E(S) = E(X - 10) = E(X) - 10 = 10 - 10 = 0$

Use the formulae for the expected value of a sum.

$E(T) = E\left(\frac{1}{2}X - 5\right) = \frac{1}{2}E(X) - 5 = \frac{1}{2} \times 10 - 5 = 0$

c $Var(S) = Var(X) = 50$

Subtracting a constant doesn't change the variance, so $Var(S) = Var(X)$.

$Var(T) = \left(\frac{1}{2}\right)^2 Var(X) = \frac{50}{4} = 12.5$

d Their total scores should both be approximately zero, but Susan's scores should be more spread out than Thomas's.

Both random variables have expected value 0, so we would expect both Susan and Thomas to have a score of approximately 0. The random variable S which represents Susan's score has higher variance, meaning we should expect it to vary more.

Example 8

The random variable X has the following distribution

x	0°	30°	60°	90°
$P(X = x)$	0.4	0.2	0.1	0.3

Calculate $E(\sin X)$.

The distribution of $\sin X$ is

$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$P(X = x)$	0.4	0.2	0.1	0.3

$E(\sin X) = \sum \sin x \, P(X = x) = 0 \times 0.4 + \frac{1}{2} \times 0.2 + \frac{\sqrt{3}}{2} \times 0.1 + 1 \times 0.3$

$= \frac{8 + \sqrt{3}}{20} \approx 0.487$

Using the general formula for $E(g(X))$.

Exercise 1C

1 The random variable X has distribution given by

x	1	2	3	4
$P(X = x)$	0.1	0.3	0.2	0.4

a Write down the probability distribution for Y where $Y = 2X - 3$.
b Find $E(Y)$.
c Calculate $E(X)$ and verify that $E(2X - 3) = 2E(X) - 3$.

2 The random variable X has distribution given by

x	−2	−1	0	1	2
$P(X = x)$	0.1	0.1	0.2	0.4	0.2

a Write down the probability distribution for Y where $Y = X^3$.
b Calculate $E(Y)$.

3 The random variable X has $E(X) = 1$ and $Var(X) = 2$. Find:
a $E(8X)$ **b** $E(X + 3)$ **c** $Var(X + 3)$
d $Var(3X)$ **e** $Var(1 - 2X)$ **f** $E(X^2)$

4 The random variable X has $E(X) = 3$ and $E(X^2) = 10$. Find:
a $E(2X)$ **b** $E(3 - 4X)$ **c** $E(X^2 - 4X)$
d $Var(X)$ **e** $Var(3X + 2)$

5 The random variable X has a mean μ and standard deviation σ.
Find, in terms of μ and σ:
a $E(4X)$ **b** $E(2X + 2)$ **c** $E(2X - 2)$
d $Var(2X + 2)$ **e** $Var(2X - 2)$

6 In a board game, players roll a fair six-sided dice each time they make it around the board. The score on the dice is modelled as a discrete random variable X.
a Write down $E(X)$.

They are paid £200 plus £100 times the score on the dice. The amount paid to each player is modelled as a discrete random variable Y.
b Write Y in terms of X.
c Find the expected pay out each time a player makes it around the board.

(P) 7 John runs a pizza parlour that sells pizza in three sizes: small (20 cm diameter), medium (30 cm diameter) and large (40 cm diameter). Each pizza base is 1 cm thick. John has worked out that on average, customers order a small, medium or large pizza with probabilities $\frac{3}{10}$, $\frac{9}{20}$ and $\frac{5}{20}$ respectively. Calculate the expected amount of pizza dough needed per customer.

(E/P) 8 Two tetrahedral dice are rolled. The random variable X represents the result of subtracting the smaller score from the larger.
a Find $E(X)$ and $Var(X)$. **(7 marks)**

The random variables Y and Z are defined as $Y = 2^X$ and $Z = \frac{4X + 1}{2}$.
b Show that $E(Y) = E(Z)$. **(3 marks)**
c Find $Var(Z)$. **(2 marks)**

Challenge
Show that $E((X - E(X))^2) = E(X^2) - (E(X))^2$.

Hint Remember that for two random variables X and Y we have $E(X + Y) = E(X) + E(Y)$

1.4 Solving problems involving random variables

Suppose we have two random variables X and $Y = g(X)$. If g is one-to-one, and we know the mean and variance of Y, then it is possible to deduce the mean and variance of X.

Example 9

X is a discrete random variable. The discrete random variable Y is defined as $Y = \frac{X - 150}{50}$
Given that $E(Y) = 5.1$ and $Var(Y) = 2.5$, find:
a $E(X)$
b $Var(X)$.

a $Y = \frac{X - 150}{50}$

$X = 50Y + 150$ — Rearrange to get an expression for X in terms of Y.

$E(X) = E(50Y + 150)$
$= 50E(Y) + 150$
$= 255 + 150$
$= 405$

b $\text{Var}(X) = \text{Var}(50Y + 150)$ — Use your expression for X in terms of Y. Remember that the '+150' does not affect the variance, and that you have to multiply Var(Y) by 50^2 to get Var(X).

$= 50^2\text{Var}(Y)$
$= 50^2 \times 2.5$
$= 6250$

Example 10

The discrete random variable X has probability distribution given by

x	−2	−1	0	1	2
$P(X = x)$	0.3	a	0.25	b	c

The discrete random variable Y is defined as $Y = 3X - 1$.
Given that $E(Y) = -2.5$ and $\text{Var}(Y) = 13.95$, find

a $E(X)$ and $E(X^2)$

b the values of a, b and c

c $P(X > Y)$.

a We have $X = \frac{Y+1}{3}$ — Rearrange the formula $Y = 3X - 1$ to get it in terms of X.

$E(X) = E\left(\frac{Y+1}{3}\right) = \frac{1}{3}(E(Y) + 1) = -0.5$

$\text{Var}(X) = \text{Var}\left(\frac{Y+1}{3}\right) = \frac{1}{9}\text{Var}(Y) = 1.55$ — Adding a constant does not change variance, so $\text{Var}(Y + 1) = \text{Var}(Y)$. $\text{Var}(aX + b) = a^2\text{Var}(X)$.

So $E(X^2) = \text{Var}(X) + (E(X))^2 = 1.55 + 0.25 = 1.8$.

b We have

$a + b + c = 1 - 0.3 - 0.25 = 0.45$ — The probabilities must sum to 1.

$E(X) = -2 \times 0.3 - 1 \times a + 0 \times 0.25 + 1 \times b + 2 \times c$
$= -0.5$ — We know that $E(X) = -0.5$ from part **a**.

So

$-a + b + 2c = -0.5 + 0.6 = 0.1$

$E(X^2) = 4 \times 0.3 + 1 \times a + 0 \times 0.25 + 1 \times b + 4 \times c$
$= 1.8$

So

$a + b + 4c = 1.8 - 1.2 = 0.6$ — We know that $E(X^2) = 1.8$ from part **a**.

In matrix form this is

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.45 \\ 0.1 \\ 0.6 \end{pmatrix}$$

So, by inverting the matrix we find — By inverting the matrix on a calculator (or by hand) we find values for a, b and c.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 2 & -3 & 1 \\ 6 & 3 & -3 \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0.45 \\ 0.1 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.05 \end{pmatrix}$$

So $a = 0.2$, $b = 0.2$ and $c = 0.05$.

c $P(X > Y) = P(X > 3X - 1) = P(1 - 2X > 0)$ — Use the expression for Y to write everything in terms of X only.

So $P(X > Y) = 0.3 + 0.2 + 0.25 = 0.75$. — $1 - 2X > 0$ when $X = -2, -1, 0$. So $P(X > Y) = P(X = -2) + P(X = -1) + P(X = 0)$.

Exercise 1D

1 X is a discrete random variable. The random variable Y is defined by $Y = 4X - 6$. Given that $E(Y) = 2$ and $\text{Var}(Y) = 32$, find:

a $E(X)$

b $\text{Var}(X)$

c the standard deviation.

2 X is a discrete random variable. The random variable Y is defined by $Y = \frac{4 - 3X}{2}$

Given that $E(Y) = -1$ and $\text{Var}(Y) = 9$, find:

a $E(X)$

b $\text{Var}(X)$

c $E(X^2)$

(P) **3** The discrete random variable X has probability distribution given by

x	1	2	3	4
$P(X = x)$	0.3	a	b	0.2

The random variable Y is defined by $Y = 2X + 3$. Given that $E(Y) = 8$, find the values of a and b.

(E/P) **4** The discrete random variable X has probability distribution given by

x	90°	180°	270°
$P(X = x)$	a	b	0.3

The random variable Y is defined as $Y = \sin X°$.

a Find the range of possible values of $E(Y)$. **(5 marks)**

b Given that $E(Y) = 0.2$, write down the values of a and b. **(2 marks)**

(P) **5** The discrete random variable X has probability distribution given by

x	−2	−1	0	1	2
$P(X = x)$	a	b	c	b	a

The random variable Y is defined $Y = (X + 1)^2$. Given that $E(Y) = 2.4$ and $P(Y > 2) = 0.4$,

a show that:

$2a + 2b + c = 1$

$10a + 4b + c = 2.4$

$a + b = 0.4$

b Hence find the values of a, b, and c.

c Find $P(2X + 3 \leqslant Y)$.

(E/P) **6** The discrete random variable X has probability distribution is given by

$$P(X = x) = \begin{cases} a & x = 1, 2, 3 \\ b & x = 4, 5 \\ c & x = 6 \end{cases}$$

Suppose that Y is defined by $Y = 1 - 2X$. Given that $E(X) = -5.6$ and $P(Y \leqslant -5) = 0.6$,

a write down the value of $E(X)$ **(1 mark)**

b show that:

$3a + 2b + c = 1$

$2a + 3b + 2c = 1.1$

$a + 2b + c = 0.6$ **(4 marks)**

c Solve the system to find values for a, b, c. **(2 marks)**

d Find $P(X > 5 + Y)$. **(2 marks)**

Mixed exercise 1

1 The random variable X has probability function

$$P(X = x) = \frac{x}{21} \qquad x = 1, 2, 3, 4, 5, 6.$$

a Construct a table giving the probability distribution of X.

Find:

b $P(2 < X \leqslant 5)$ **c** $E(X)$ **d** $Var(X)$

e $Var(3 - 2X)$ **f** $E(X^3)$

2 The discrete random variable X has the probability distribution in the table below.

x	−2	−1	0	1	2	3
$P(X = x)$	0.1	0.2	0.3	t	0.1	0.1

Find:

a r **b** $P(-1 \leqslant X < 2)$ **c** $E(2X + 3)$ **d** $Var(2X + 3)$

3 A discrete random variable X has the probability distribution shown in the table below.

x	0	1	2
$P(X = x)$	$\frac{1}{5}$	b	$\frac{1}{5} + b$

a Find the value of b. **b** Show that $E(X) = 1.3$.

c Find the exact value of $Var(X)$. **d** Find the exact value of $P(X \leqslant 1.5)$.

4 The discrete random variable X has a probability function

$$P(X = x) = \begin{cases} k(1 - x) & x = 0, 1 \\ k(x - 1) & x = 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

a Show that $k = \frac{1}{4}$ **b** Find $E(X)$ and show that $E(X^2) = 5.5$.

c Find $Var(2X - 2)$.

5 A discrete random variable X has the probability distribution

x	0	1	2	3
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$

Find:

a $P(1 < X \leqslant 2)$ **b** $E(X)$ **c** $E(3X - 1)$

d $Var(X)$ **e** $E(\log(X + 1))$

6 A discrete random variable X has the probability distribution

x	1	2	3	4
$P(X = x)$	0.4	0.2	0.1	0.3

Find:

a $P(3 < X^2 < 10)$ **b** $E(X)$ **c** $Var(X)$

d $E\left(\frac{3 - X}{2}\right)$ **e** $E(\sqrt{X})$ **f** $E(2^{-x})$

7 A discrete random variable is such that each of its values is assumed to be equally likely.

a Write the name of the distribution. **b** Give an example of such a distribution.

A discrete random variable X as defined above can take values 0, 1, 2, 3, and 4.

Find:

c $E(X)$ **d** $Var(X)$ **e** the standard deviation.

(P) **8** The random variable X has a probability distribution.

x	1	2	3	4	5
$P(X = x)$	0.1	p	q	0.3	0.1

a Given that $E(X) = 3.1$, write down two equations involving p and q. Find:

b the value of p and the value of q

c $Var(X)$

d $Var(2X - 3)$.

E **9** The random variable X has probability function

$$P(X = x) = \begin{cases} kx & x = 1, 2 \\ k(x-2) & x = 3, 4, 5 \end{cases}$$

where k is a constant.

a Find the value of k. **(2 marks)**

b Find the exact value of $E(X)$. **(2 marks))**

c Show that, to three significant figures, $Var(X) = 2.02$. **(2 marks)**

d Find, to one decimal place, $Var(3 - 2X)$. **(2 marks)**

E **10** The random variable X has the discrete uniform distribution

$$P(X = x) = \tfrac{1}{6} \qquad x = 1, 2, 3, 4, 5, 6$$

a Write down $E(X)$ and show that $Var(X) = \frac{35}{12}$. **(4 marks)**

b Find $E(2X - 1)$. **(2 marks)**

c Find $Var(3 - 2X)$. **(2 marks)**

d Find $E(2^x)$. **(3 marks)**

11 The random variable X has probability function

$$p(x) = \frac{3x - 1}{26} \qquad x = 1, 2, 3, 4.$$

a Construct a table giving the probability distribution of X.

Find:

b $P(2 < X \leqslant 4)$

c the exact value of $E(X)$.

d Show that $Var(X) = 0.92$ to two significant figures.

e Find $Var(1 - 3X)$.

12 The random variable Y has mean 2 and variance 9.

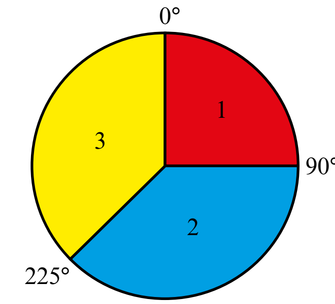
Find:

a $E(3Y + 1)$ b $E(2 - 3Y)$ c $Var(3Y + 1)$

d $Var(2 - 3Y)$ e $E(Y^2)$ f $E((Y - 1)(Y + 1))$.

13 The random variable T has a mean of 20 and a standard deviation of 5.
It is required to scale T by using the transformation $S = 3T + 4$.
Find $E(S)$ and $Var(S)$.

14 A fair spinner is made from the disc in the diagram and the random variable X represents the number it lands on after being spun.



a Write down the distribution of X. b Work out $E(X)$.

c Find $Var(X)$. d Find $E(2X + 1)$.

e Find $Var(3X - 1)$.

15 The discrete variable X has the probability distribution

x	−1	0	1	2
$P(X = x)$	0.2	0.5	0.2	0.1

Find:

a $E(X)$ b $Var(X)$ c $E(\frac{1}{3}X + 1)$ d $Var(\frac{1}{3}X + 1)$

E/P **16** The discrete random variable X has probability distribution given by

x	−1	0	1	2
$P(X = x)$	0.1	0.3	a	b

The random variable Y is defined $Y = 1 - 3X$. Given that $E(Y) = 1.1$,

a find the values of a and b. **(5 marks)**

b Calculate $E(X^2)$ and $Var(X)$ using the values of a and b that you found in part **a**. **(3 marks)**

c Write down the value of $Var(Y)$. **(1 mark)**

d Find $P(Y + 2 > X)$. **(2 marks)**

E/P **17** The discrete random variable X has probability distribution given by

x	−2	0	2	3	4
$P(X = x)$	a	b	a	b	c

The random variable Y is defined as $Y = \dfrac{2 - 3X}{5}$

You are given that $E(Y) = -0.98$ and $P(Y \geqslant -1) = 0.4$

a Write down three simultaneous equations in a, b and c. **(4 marks)**

b Solve this system to find the values of a, b and c. **(3 marks)**

c Find $P(-2X > 10Y)$. **(2 marks)**

Challenge

Let n be a positive integer and suppose that X is a discrete random variable with $P(X = i) = \frac{1}{n}$ for $i = 1, \ldots, n$.

Show that $E(X) = \dfrac{n+1}{2}$ and $Var(X) = \dfrac{(n+1)(n-1)}{12}$

Hint You can make use of the following results:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

← Core Pure Book 1, Chapter 3

Summary of key points

1 The expected value of the discrete random variable X is denoted E(X) and defined as
$\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$.

2 The expected value of X^2 is **$\mathrm{E}(X^2) = \sum x^2\mathrm{P}(X = x)$**.

3 The variance of X is usually written as Var(X) and is defined as
$\mathrm{Var}(X) = \mathrm{E}((X - \mathrm{E}(X))^2)$

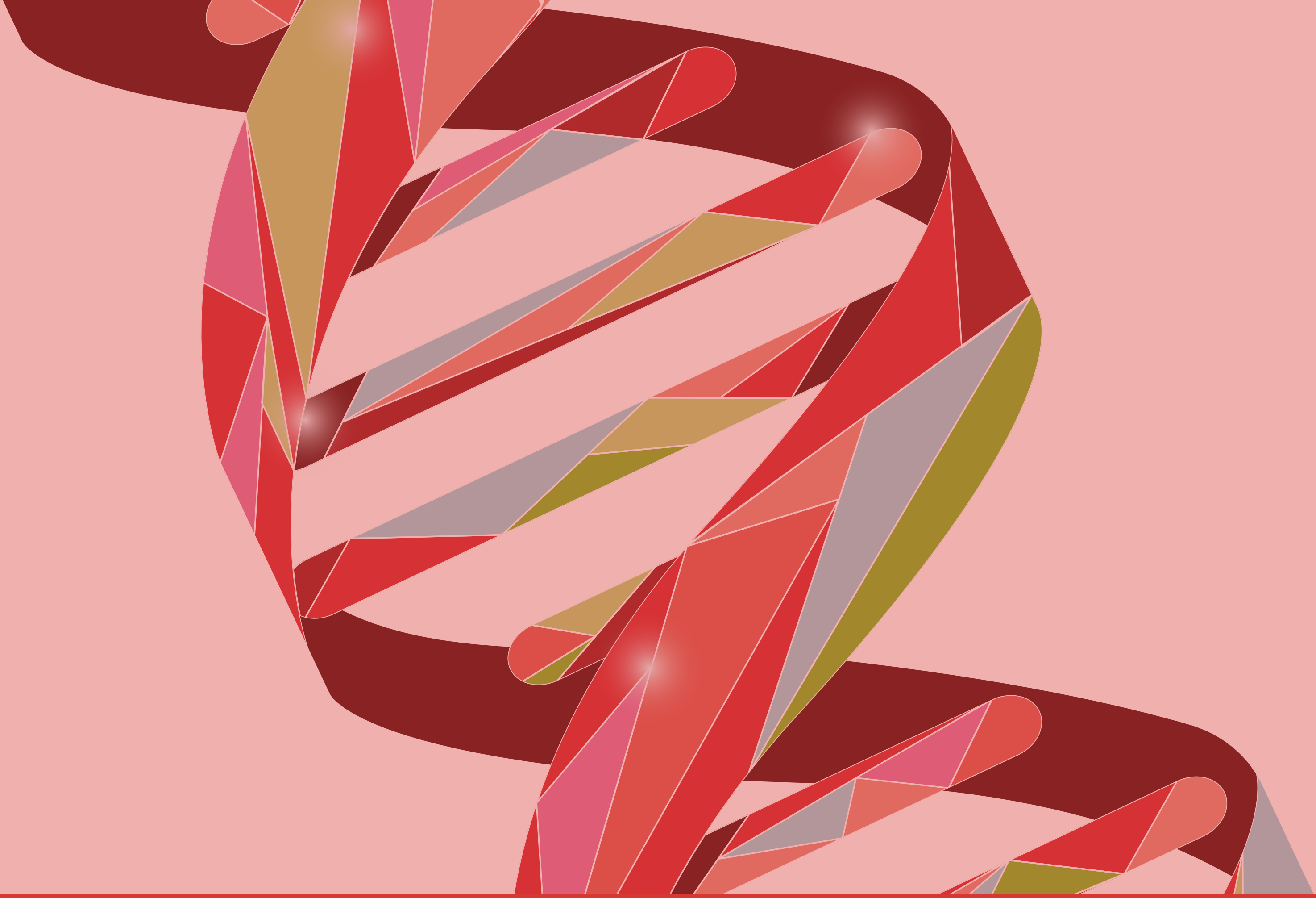
4 Sometimes, it is easier to calculate the variance using the formula
$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$

5 **$\mathrm{E}(\mathrm{g}(X)) = \sum \mathrm{g}(x)\,\mathrm{P}(X = x)$**.

6 If X is a random variable and a and b are constants, then **$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$**.

7 If X and Y are random variables, then **$\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$**.

8 If X is a random variable and a and b are constants then **$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$**.



Order now!

Visit our webpage to see all products in this series, and order now:
www.pearsonschools.co.uk/ALM1

Speak to us

For help and advice you can request a call back or appointment with one of our maths experts:
www.pearsonschools.co.uk/speakalmaths

PEUK W876e