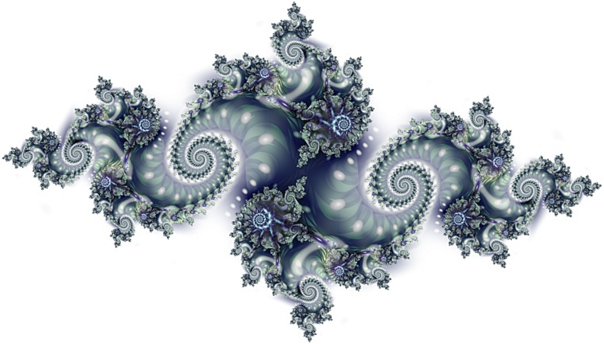
Fractals in Nature and Mathematics: From Simplicity to Complexity

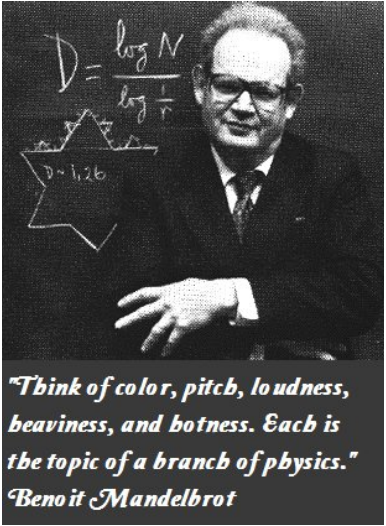
Dr. R. L. Herman, UNCW Mathematics & Physics



Outline

Benoît Mandelbrot (1924-2010)

- Grew up in France
- Paris and Caltech Education
- IBM Fellow
- Fractals
- Studied "roughness" in nature
- *Fractal Geometry*
- Mandelbrot Set
- TED Talk link



Clouds and Mountains

Figure: How would you measure roughness and complexity

Trees

Figure: Self-similarity

Lightning

Figure: Fractals - what do you see?

Capillaries

Figure: Leaf and rivers

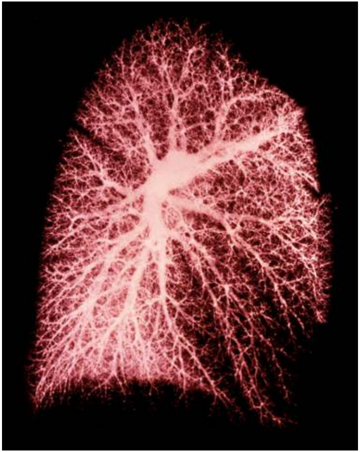


Figure: Lungs

Ferns

Figure: Do you see similarity at different scales?

Ferns - Example of Self-Similarity

Coastlines

Figure: What is the length of the coastline?

What are Fractals?

- From *fractus*, - “broken”
- Self-similarity
- Fractional Dimension

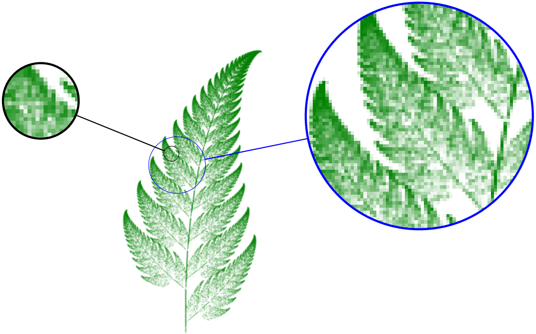


Figure: Fractal Fern

Fractal Trees

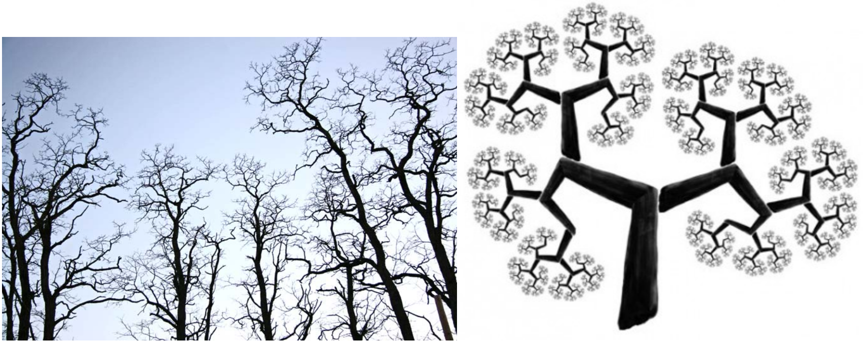


Figure: Fractals - self-similarity, roughness

Fractals in Nature

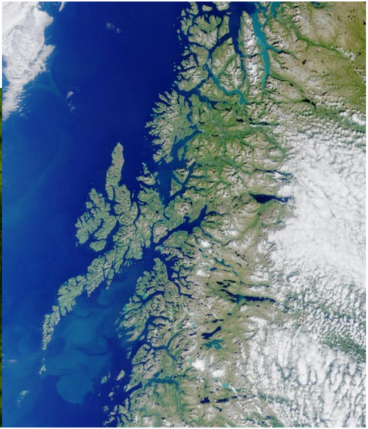



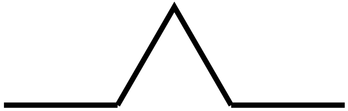
Figure: Fractals -what do you see?

Geometric Fractals - Koch Curve (1904)

Step 1 ($L = 1$)

Geometric Fractals - Koch Curve (1904)

Geometric Fractals - Koch Curve (1904)

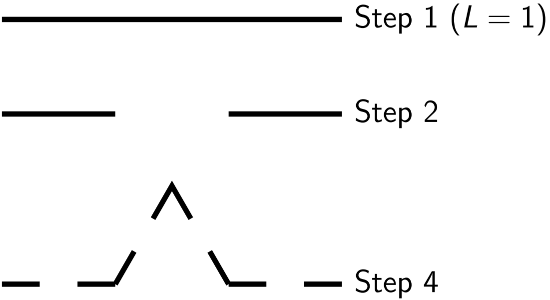


Step 1 ($L = 1$)

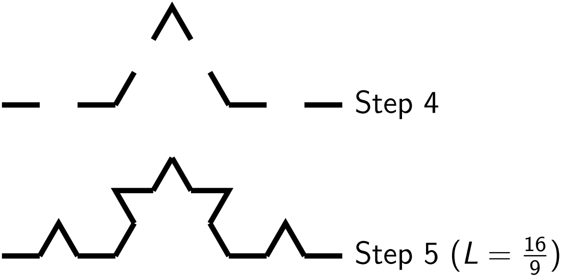
Step 2

Step 3 ($L = 4(\frac{1}{3}) = \frac{4}{3}$)

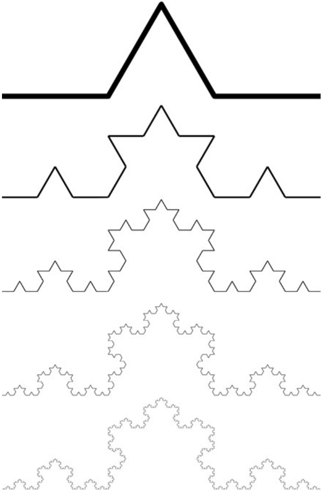
Geometric Fractals - Koch Curve (1904)



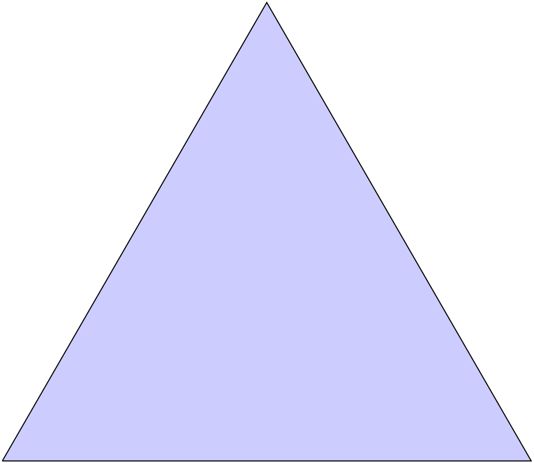
Geometric Fractals - Koch Curve (1904)



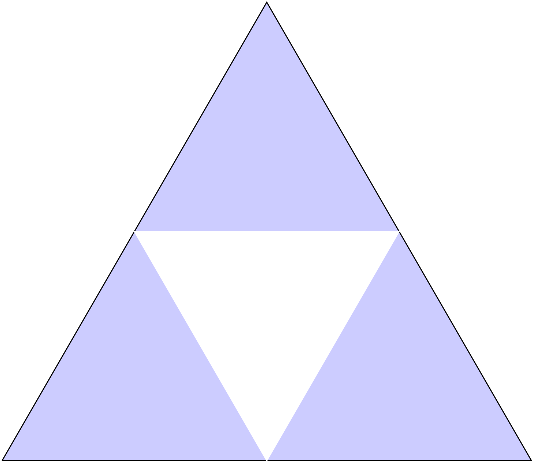
Koch Curve - Self Similarity ($L = \frac{4^n}{3^n} \to \infty$)



Simple Fractals - Sierpinski Triangle



Simple Fractals - Sierpinski Triangle



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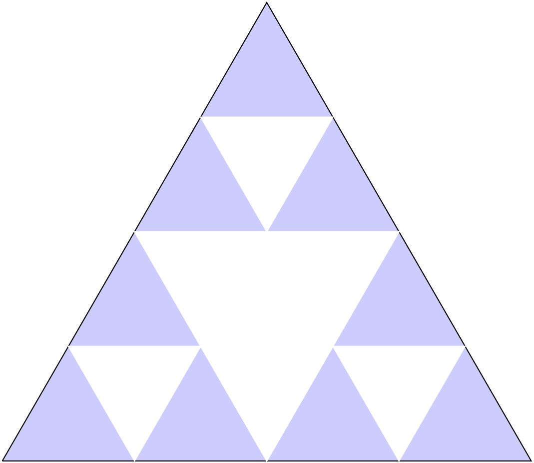

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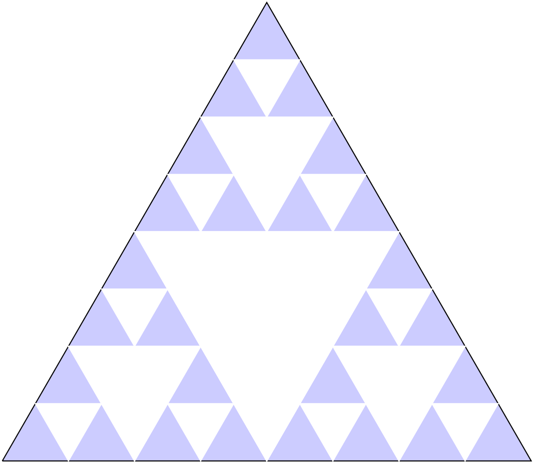
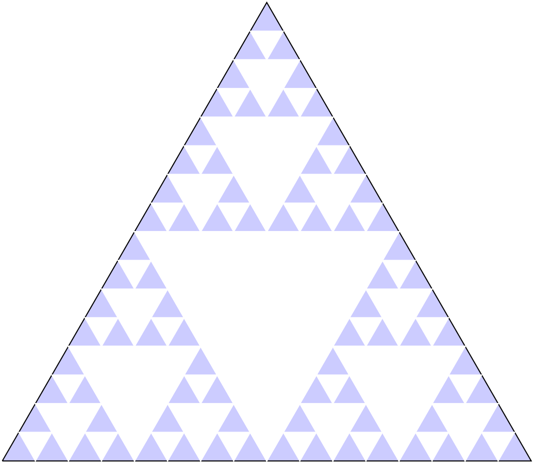
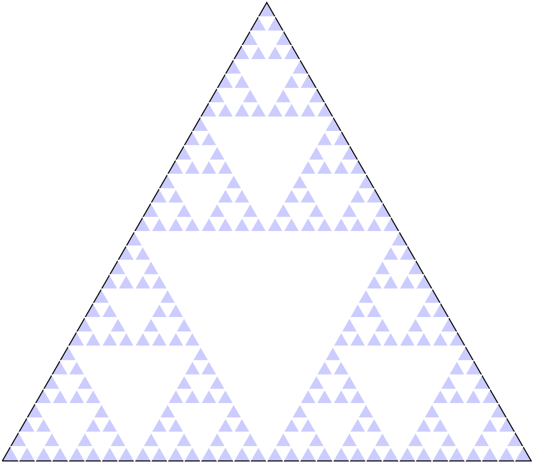


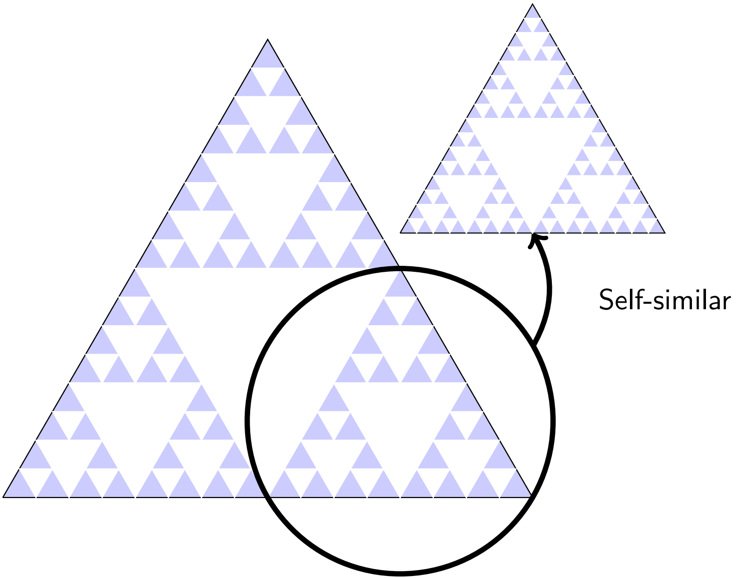
Simple Fractals - Sierpinski Triangle



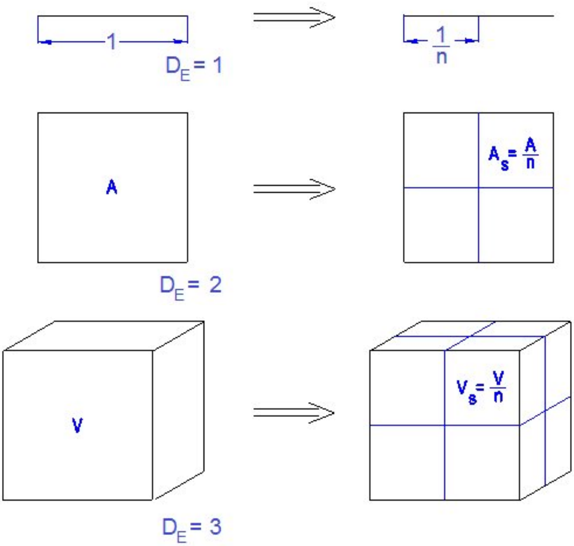
Simple Fractals - Sierpinski Triangle



Simple Fractals - Sierpinski Triangle



Dimensions - $r =$ magnification, $n =$ Number of shapes



$n = 2, \quad r = 2$

$\Rightarrow \ln 2 = \ln 2.$

$n = 4, \quad r = 2$

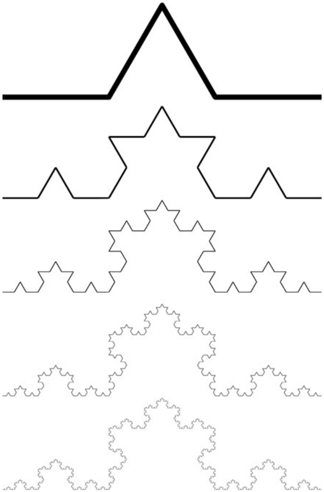
$\Rightarrow \ln 4 = 2 \ln 2.$

$n = 8, \quad r = 2$

$\Rightarrow \ln 8 = 3 \ln 2.$

$$D = \frac{\ln n}{\ln r}.$$

Fractal Dimensions - Koch Curve



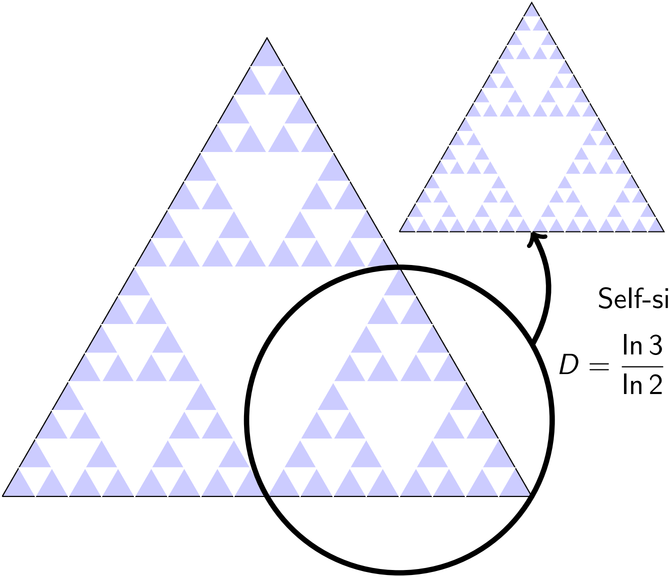
For each step length of line segment is reduced by $r = 3$.

The number of lines increases by factor $n = 4$.

Therefore

$$
\begin{aligned}
D &= \frac{\ln n}{\ln r} \\
&= \frac{\ln 4}{\ln 3} \\
&= 1.26.
\end{aligned}
$$

Sierpinski Triangle - Dimension



Self-similar

$$D = \frac{\ln 3}{\ln 2} = 1.585$$

Coastlines - Great Britain, $D = 1.25$

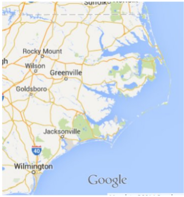
Unit = 200 km,
Length = 2400 km (approx.)

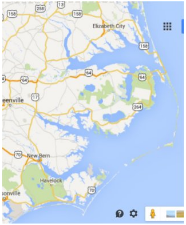
Unit = 100 km,
Length = 2800 km (approx.)

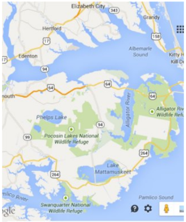
Unit = 50 km,
Length = 3400 km (approx.)

Coastlines -North Carolina

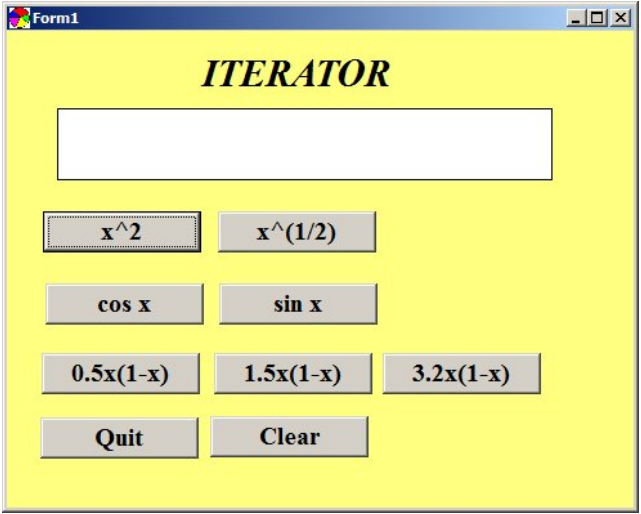
What is the fractal dimension of the NC coast?







Iterations



Form1
ITERATOR
x^2
x^(1/2)
cos x
sin x
0.5x(1-x)
1.5x(1-x)
3.2x(1-x)
Quit
Clear

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

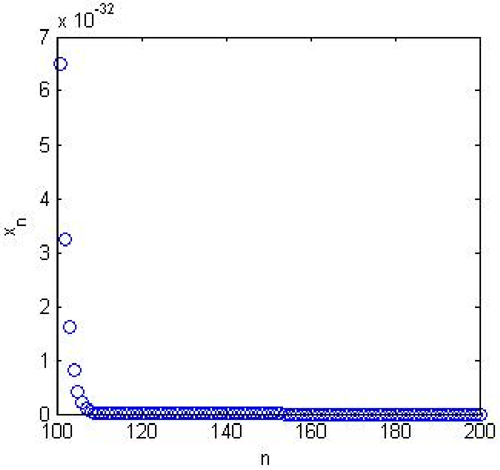


Figure: $r = 0.05$

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

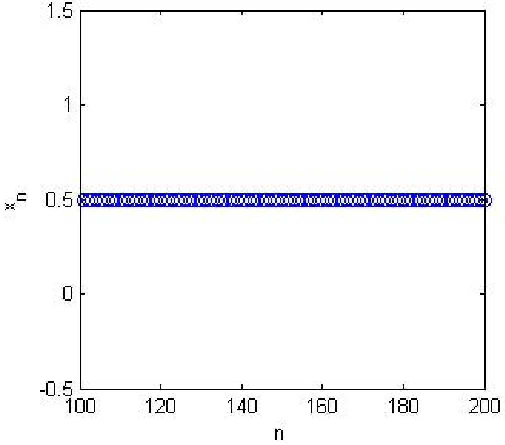


Figure: $r = 2.0$

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

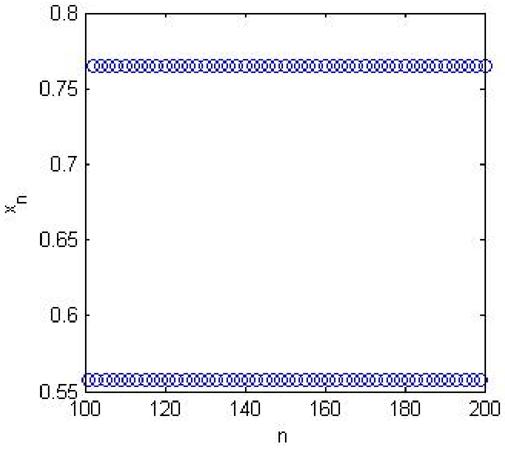


Figure: $r = 3.1$

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

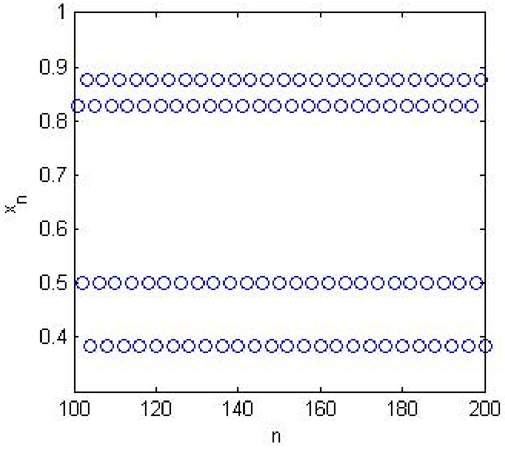


Figure: $r = 3.5$

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

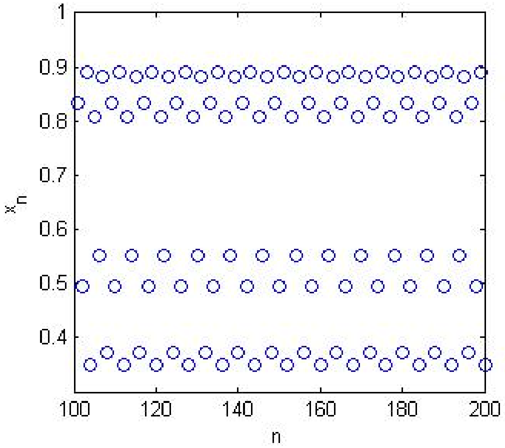


Figure: $r = 3.56$

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

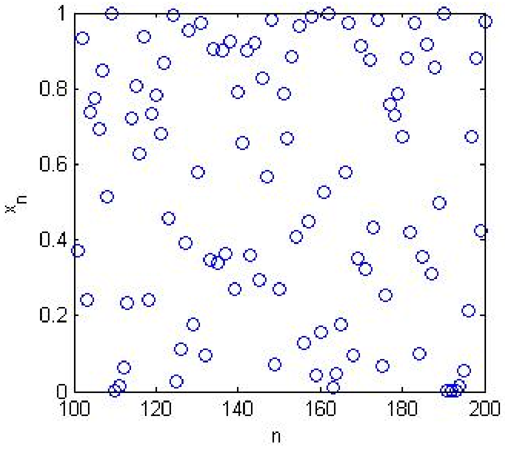
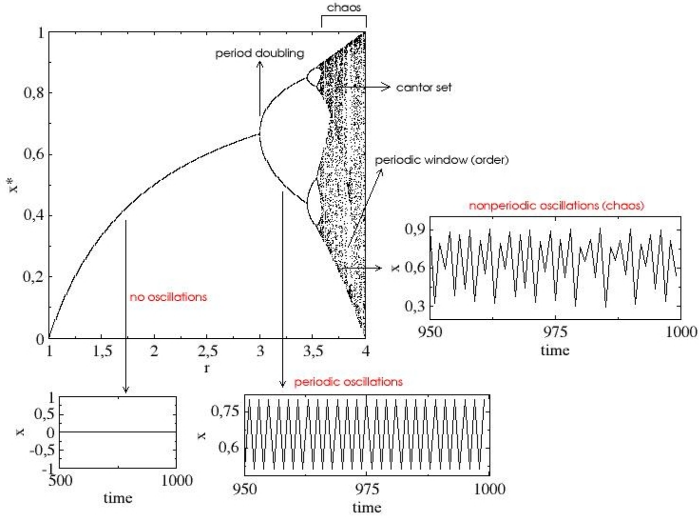


Figure: $r = 4.0$

Bifurcations of Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0



Mandelbrot Set

Iterate complex numbers
$z = a + bi$, $i = \sqrt{-1}$.

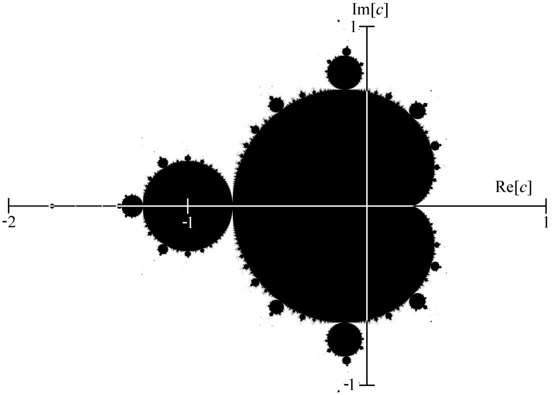
$$z_{n+1} = z_n^2 + c, \quad z_0 = 0.$$

Example: $c = 1$:

$$0, 1, 2, 5, 26 \ldots.$$

Example: $c = -1$:

$$0, -1, 0, -1, 0, \ldots.$$



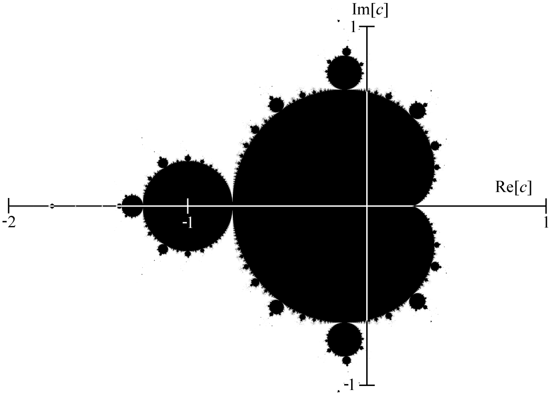
Mandelbrot Set - $z_{n+1} = z_n^2 + c, \quad z_0 = 0.$

Example: $c = i$:

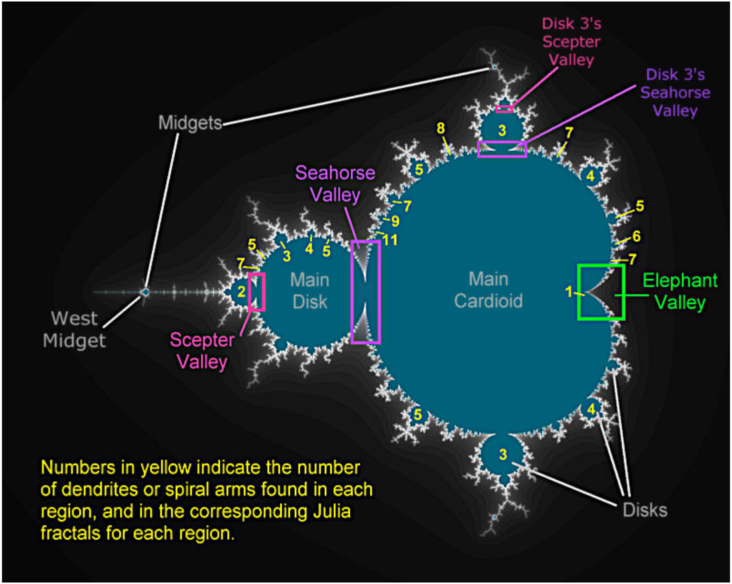
$$\begin{aligned} x_0 &= 0 \\ x_1 &= 0^2 + i = i \\ x_2 &= i^2 + i = -1 + i \\ x_3 &= (-1+i)^2 + i = -i \\ x_4 &= (-i)^2 + i = -1 + i \\ x_4 &= (-i)^2 + i = -i \end{aligned}$$

Gives period 2 orbit =
$\{-1+i, -i, -1+i, -i, \ldots\}$.

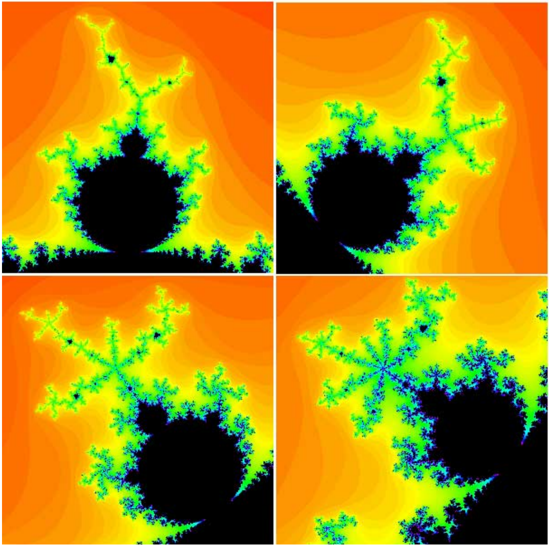
- *Show more points at Fractals site.*
- *Mandelbrot Set Zoom online.*



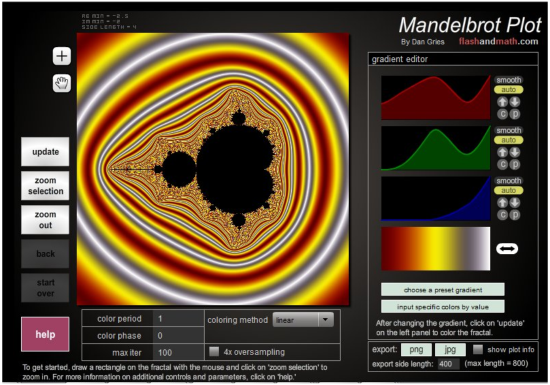
Mandelbrot Set - Bulbs



Mandelbrot Set - Bulbs

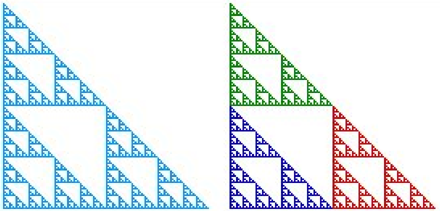


Mandelbrot Set - Plot and Zoom

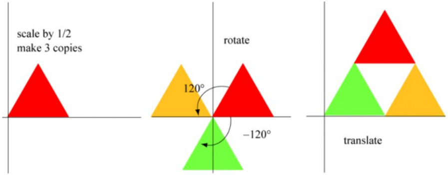


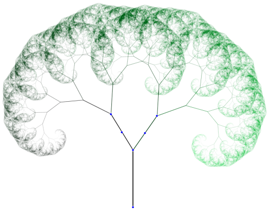
http://www.flashandmath.com/advanced/mandelbrot/MandelbrotPlot.html

- Iterated Function Systems
- Scalings, Rotations, and Translations

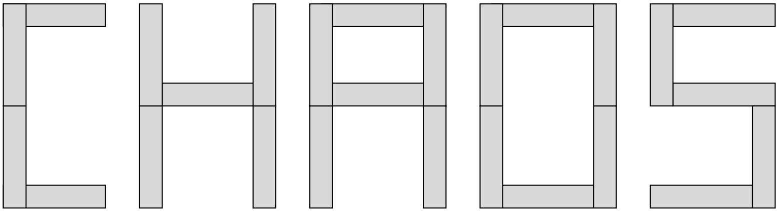


r	s	θ	φ	e	f
0.5	0.5	0	0	0.0	0.0
0.5	0.5	0	0	0.5	0.0
0.5	0.5	0	0	0.0	0.5

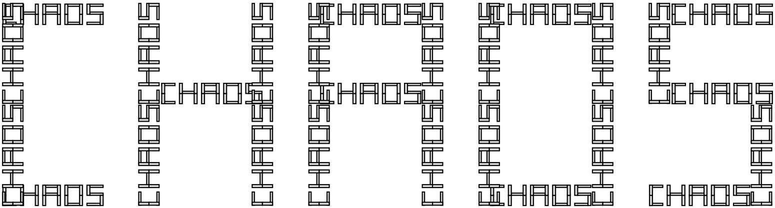




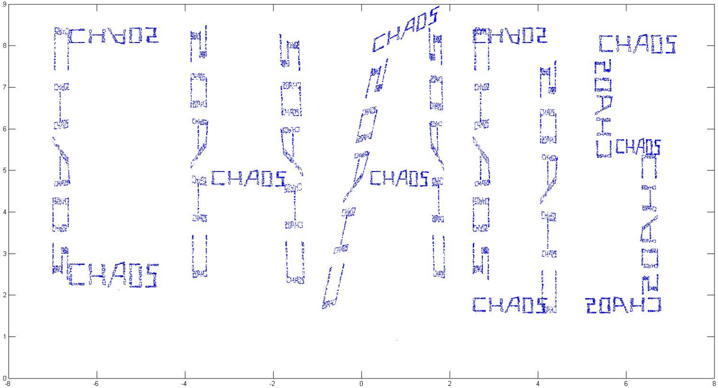
IFS - Turning CHAOS into a Fractal



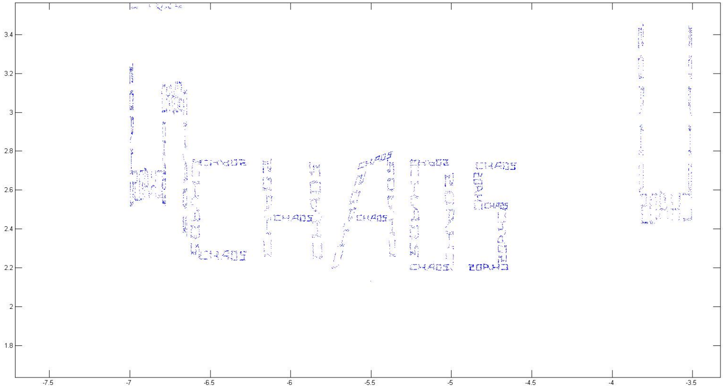
IFS - CHAOS



IFS Chaos

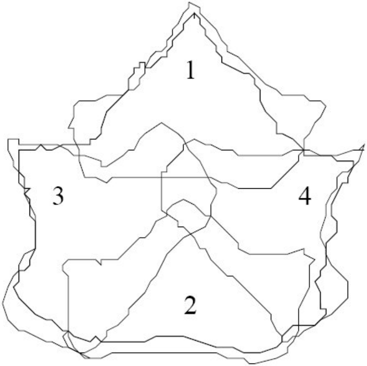


IFS Chaos



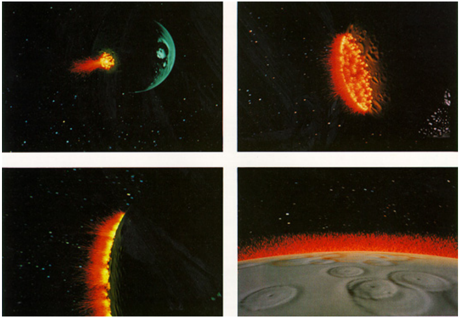
Maple Leaf - Barnsley *Fractals Everywhere*

The Collage Theorem and Fractal Image Compression.



The Genesis Effect - *Star Trek II: The Wrath of Khan* (1982)

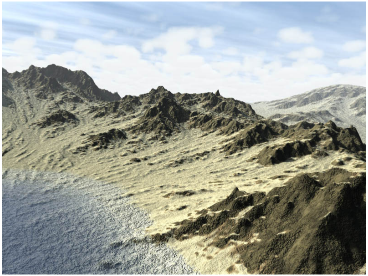
- Fractal Landscapes
- First completely computer-generated sequence in a film `http://design.osu.edu/carlson/history/tree/images/pages/genesis1_jpeg.htm`

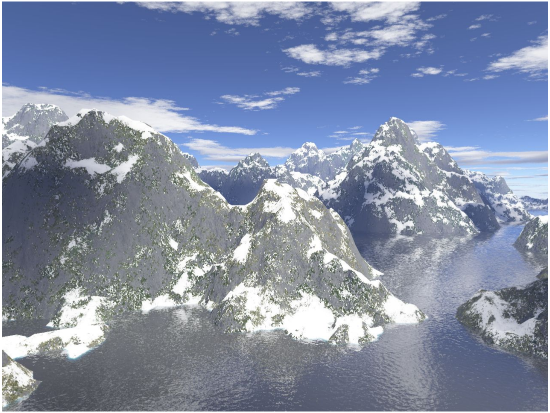


`https://www.youtube.com/watch?v=QXbWCrzWJo4`

Fractal Landscapes - Roughness in Nature

- Mountains
- Clouds





1D Midpoint Displacement Algorithm

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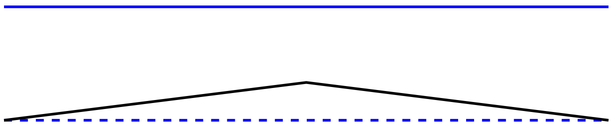
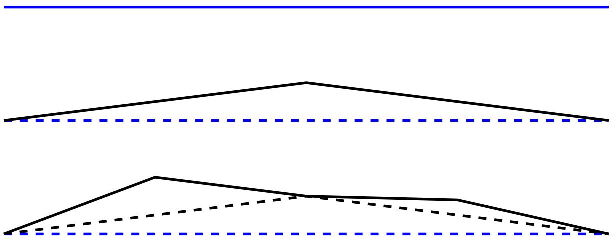
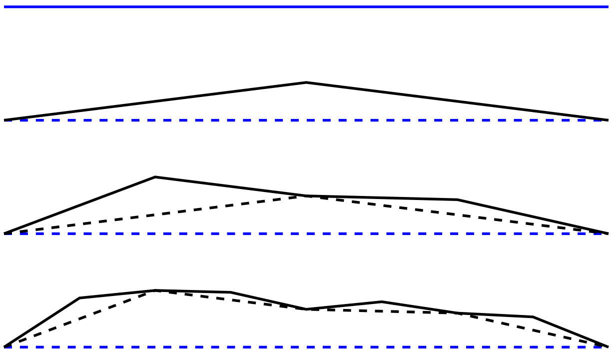


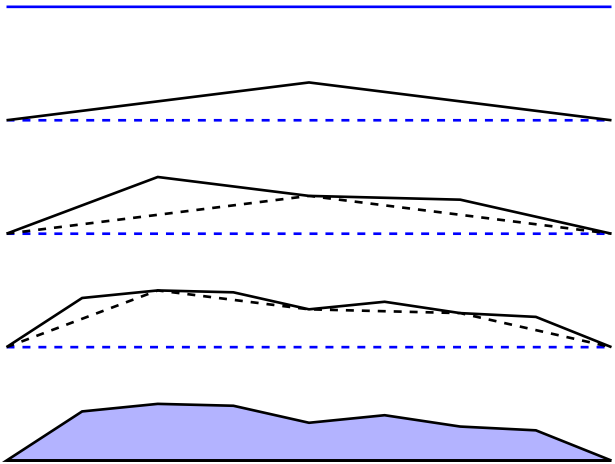
1D Midpoint Displacement Algorithm



1D Midpoint Displacement Algorithm

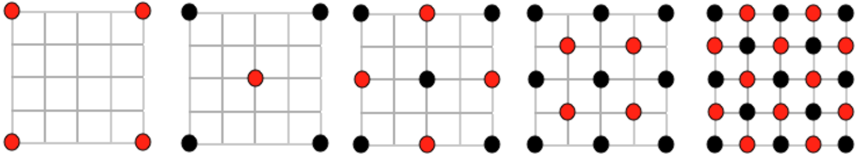


1D Midpoint Displacement Algorithm



Diamond - Square Algorithm

- Square of size $2^n + 1$.
- Find Midpoint, adding random small hieghts.
- Create Diamond.
- Edge midpoints, ...



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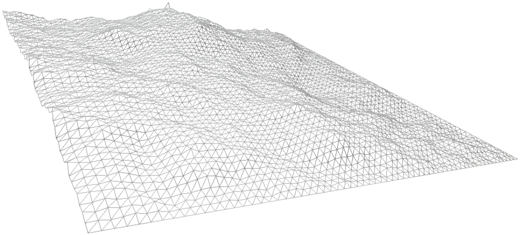
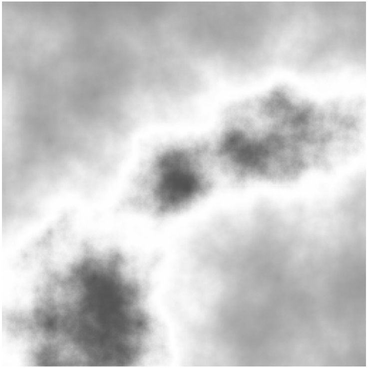
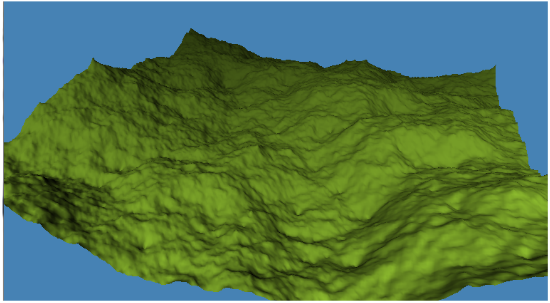


Height Maps: Clouds and Coloring





Other Applications

- Video Games
- Fracture - link
- Ceramic Material - link
- Biology - wrinkles, lungs, brain, ...
- Astrophysics

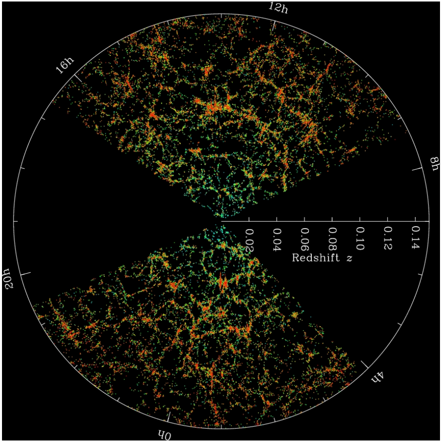
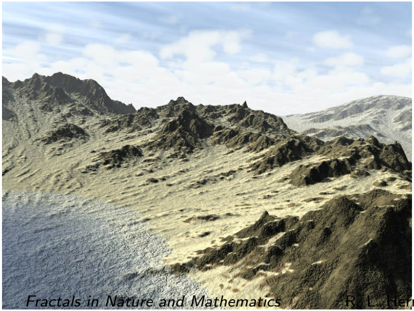


Figure: The Sloan Digital Sky Survey

Conclusion

- Fractals - Measure of Roughness
- Fractal Dimension
- Function Iteration - Mandelbrot Set
- Applications



- J. Gleick, *Chaos, Making a New Science*, 1987/2008
- E. Lorenz, *The Essence of Chaos, Making a New Science*, 1995
- B. Mandelbrot, *The Fractal Geometry of Nature*, 1982
- Barnsley, *Fractals Everywhere*, 1988/2012