Fisica della Materia allo Stato Fluido e di Plasma

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Prima di cominciare…

- Requisiti
  - Fisica I, Elettromagnetismo, termodinamica
  - Calcolo differenziale con funzioni di più variabili
  - Nozioni di: fluido dinamica, relatività (speciale)

- Approssimazioni

- Applicazioni:
  - Astrofisica (stelle, dischi di accrescimento, getti, ISM, …)
  - Fusione nucleare

- Modalità d’esame
  - Orale

Letture Consigliate

- Libro di Testo:
  - Chiuderi & Velli

    "Fisica del Plasma – Fondamenti e Applicazioni Astrofisiche", Springer.

  Complementi:
  - Diapositive (Power Point & PDF)
- Fisica dei Fluidi:
  - L.D. Landau, E.M.Lifshitz, *"Fluid Mechanics"* , Pergamon Press
- Fisica dei Plasmi
  - T. Boyd, J.Sanderson *"Plasma Dynamics"* Nelson & Sons
  - R Goldston, P Rutherford *"Plasma Physics"*,Taylor & Francis NY
  - Goedbloed, Keppens, Poedts, "Advanced Magnetohydrodynamics" Cambridge
  - Sturrock, "Plasma Physics: An Introduction" Cambridge

Operatori Differenziali

- Gradient:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- Divergence:

$$\operatorname{div}\ \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}.$$

- Curl:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}},$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix},$$

Alcune Identità Vettoriali

(1) $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{B}\times\mathbf{C}\cdot\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = \mathbf{C}\times\mathbf{A}\cdot\mathbf{B}$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{C}\times\mathbf{B})\times\mathbf{A} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

(3) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0$

(4) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$

(5) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D}$

(6) $\nabla(fg) = \nabla(gf) = f\nabla g + g\nabla f$

(7) $\nabla\cdot(f\mathbf{A}) = f\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla f$

(8) $\nabla\times(f\mathbf{A}) = f\nabla\times\mathbf{A} + \nabla f\times\mathbf{A}$

(9) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$

(10) $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

(11) $\mathbf{A}\times(\nabla\times\mathbf{B}) = (\nabla\mathbf{B})\cdot\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

(12) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(13) $\nabla^2 f = \nabla\cdot\nabla f$

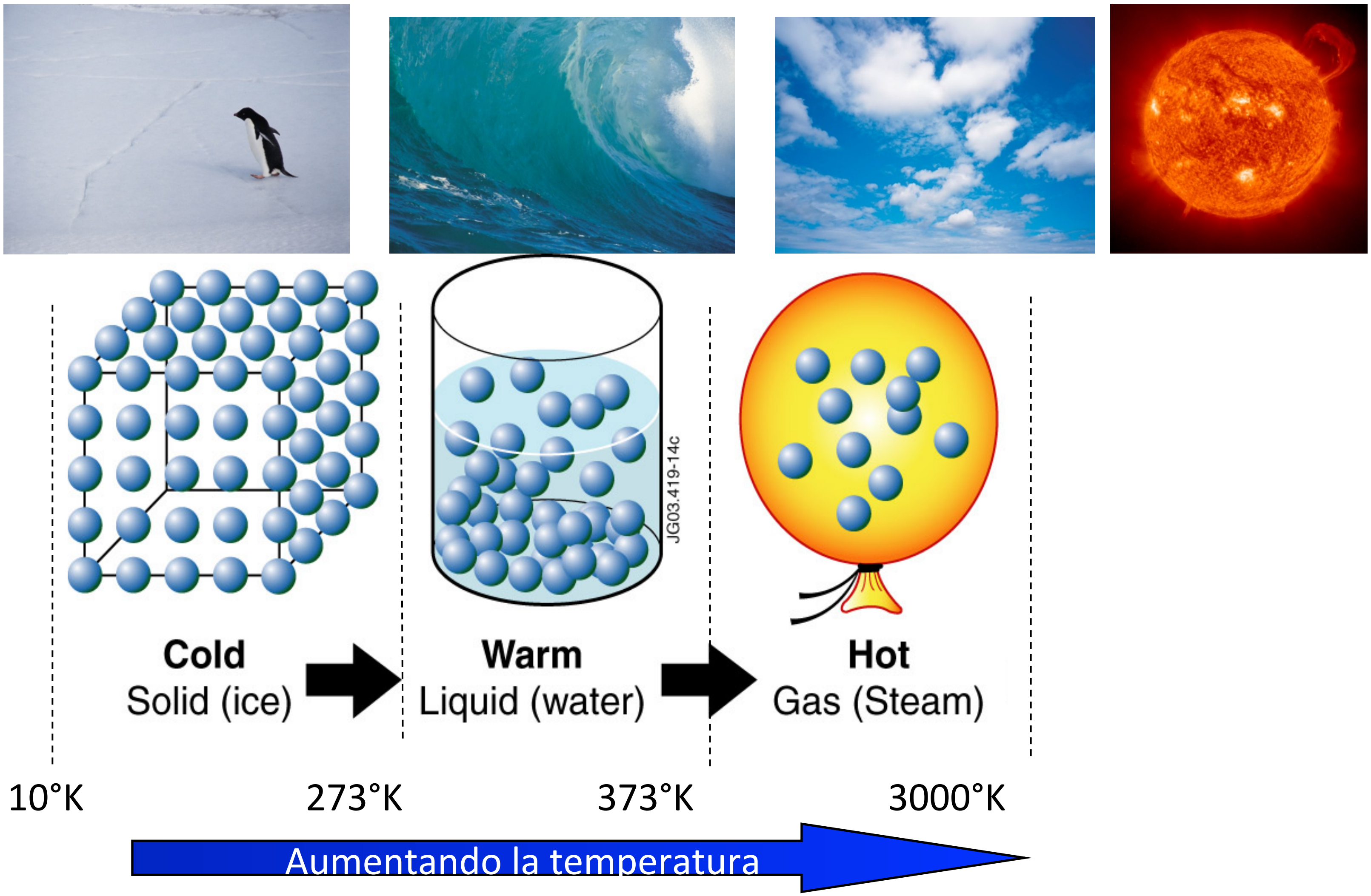
(14) $\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$

(15) $\nabla\times\nabla f = 0$

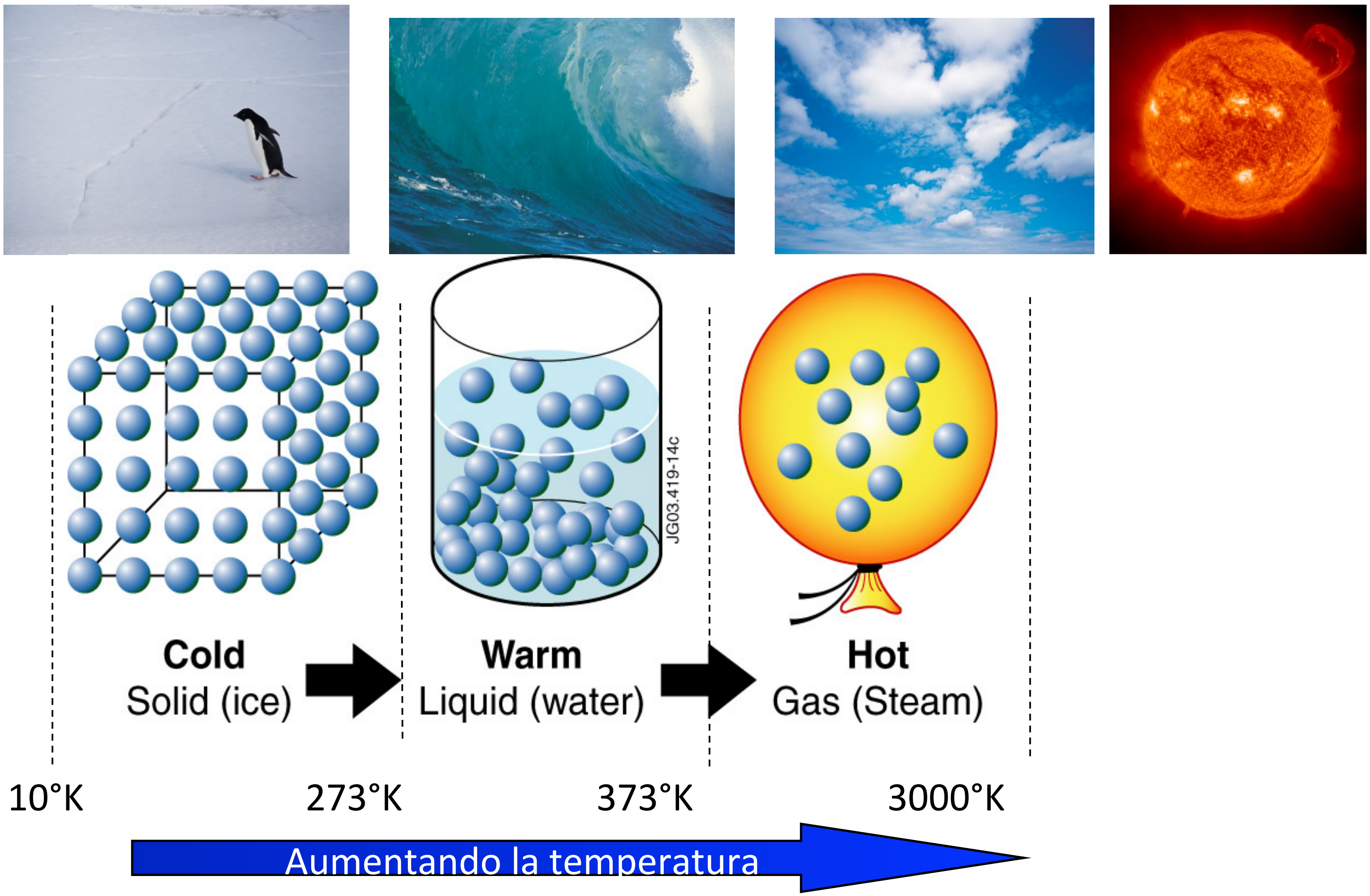
(16) $\nabla\cdot\nabla\times\mathbf{A} = 0$

→ NRL Plasma Formulary

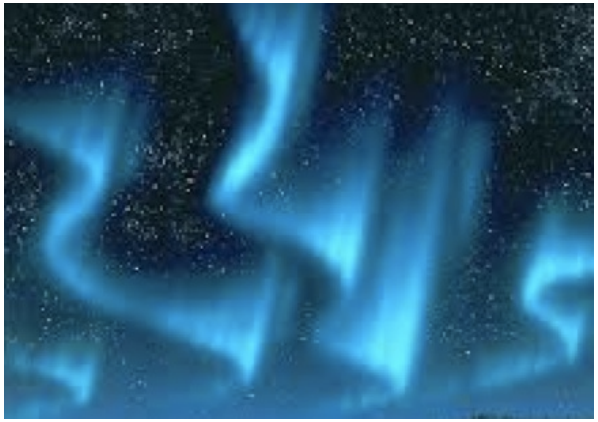
Stati di aggregazione della materia

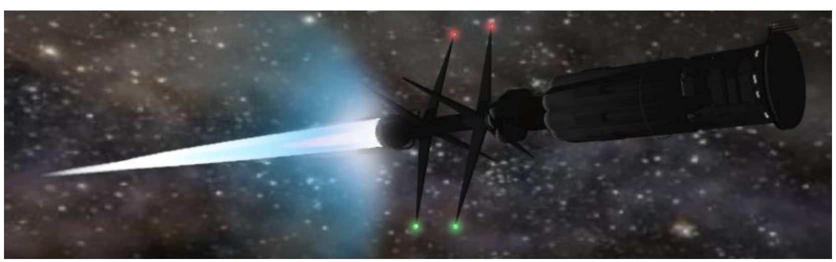


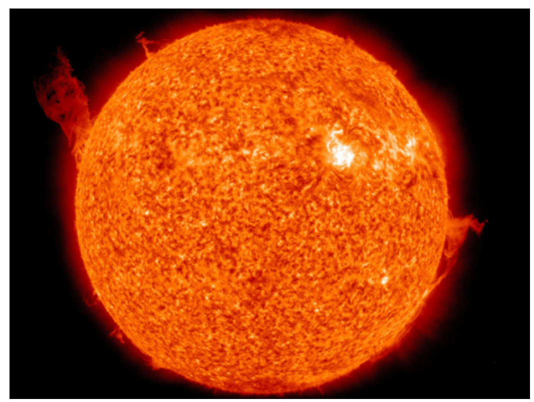
Stati di aggregazione della materia

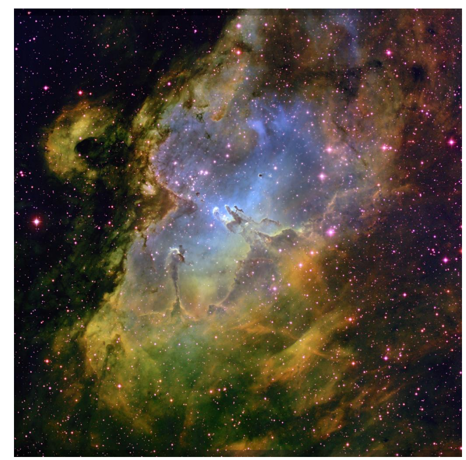


Examples of Plasmas









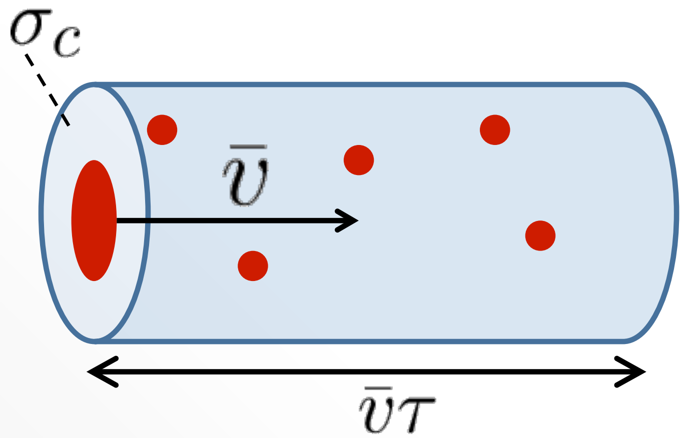
A note on the system of units

- The system adopted here is CGS-Gauss. CGS systems avoid introducing new base units and instead derives all electric and magnetic units directly from the centimeter, gram, and second based on the physical laws that relate electromagnetic phenomena to mechanics.
- The unit is therefore:
- Statcoulomb (Fr) = gr^(1/2) * cm^(3/2) * s^(-1)
- In the CGS system, the electron charge is
  e = 4.80320425e−10 statcoulombs (Fr)

Fisica della Materia allo Stato Fluido e di Plasma

1. Introduzione

Libero Cammino Medio



- Modello “cilindro”
- σ_c : sezione d’urto
- $\bar{v}$: velocità media
- $n = \dfrac{\Delta N}{\Delta \mathcal{V}}$: densità di particelle

- Volume del cilindro: $\sigma_c \bar{v} \tau = \Delta \mathcal{V} = \dfrac{\Delta N}{n}$

- Libero cammino medio: lunghezza del cilindro/numero di collisioni:

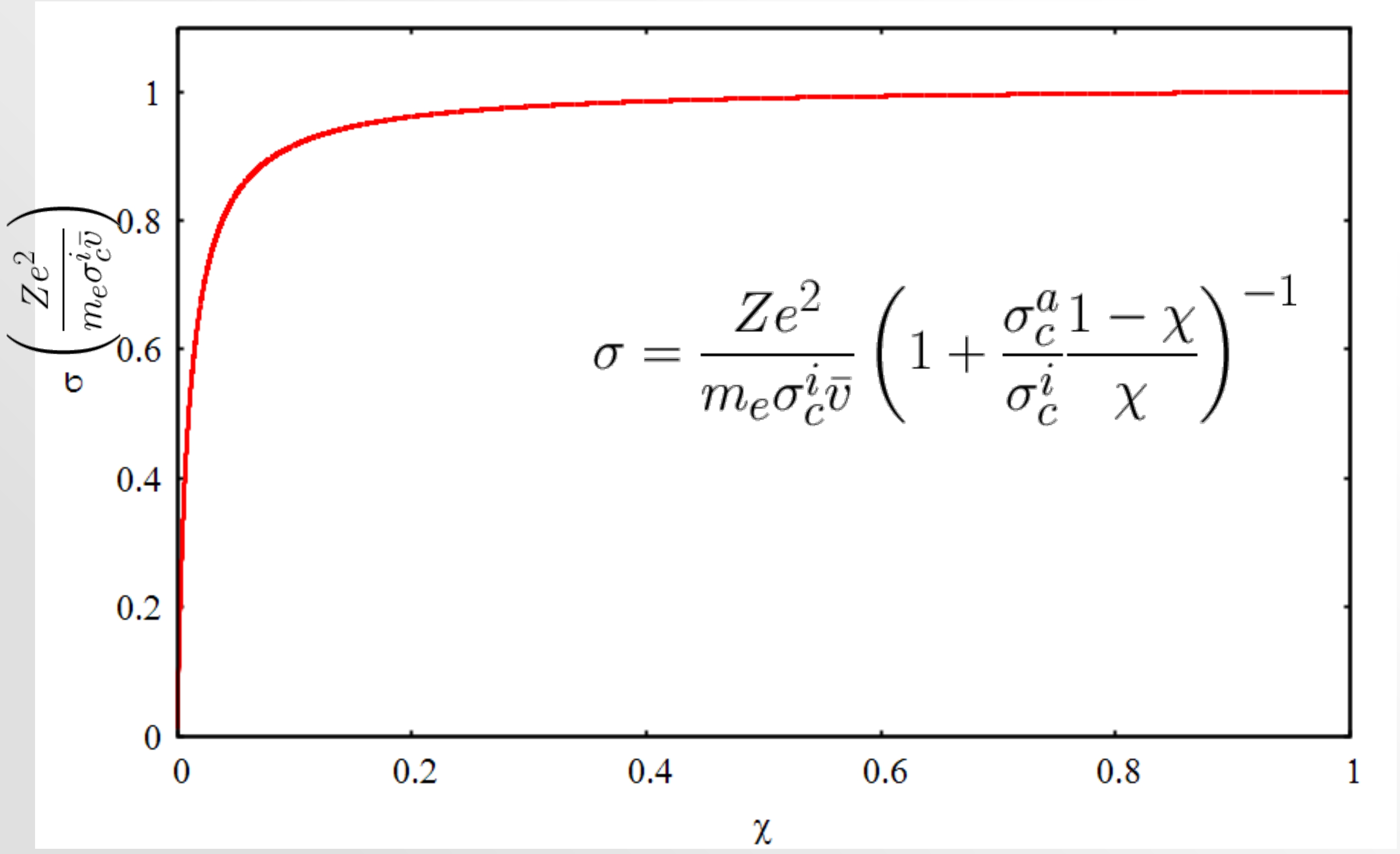
$$\lambda = \frac{\bar{v}\tau}{\Delta N} \qquad \Longrightarrow \qquad \lambda = \frac{1}{\sigma_c n}$$

- Frequenza:

$$\nu_c = \frac{\bar{v}}{\lambda} = n \sigma_c \bar{v}$$

Grado di ionizzazione

- E' sufficiente un piccolo grado di ionizzazione per avere alta conducibilità elettrica:



Saha Equation

1 Ionization Degree

The Saha equation describes the degree of ionization of a plasma as a function of the temperature, density, and ionization energies of the atoms:

$$\frac{n_{k+1} n_e}{n_k} = \frac{2}{\Lambda^3} \frac{g_{k+1}}{g_k} \exp\left[-\frac{E_{k+1} - E_k}{kT}\right] \tag{1}$$

where n_k is the density of atoms in the k-th state of ionization, g_k is the degeneracy of states for the $k-$th ions, E_k is the energy required to remove k electrons from a neutral atom, n_e is the electron density while

$$\Lambda \equiv \sqrt{\frac{h^2}{2\pi m_e kT}}$$

is the thermal de Broglie wavelenght of an electron. The difference $E_{k+1} - E_k$ is the energy required to remove the $(k+1)^{\text{th}}$ electron.

In the case of hydrogen, considering $k = 0$ as the ground state and $k = 1$ as the ionization level, Eq. (1) yields

$$\frac{n_i n_e}{n_0} = f(T) \approx 2.4 \times 10^{15}\, T^{3/2}\, e^{-1.58\times 10^5/T}$$

$(2g_1/g_0 = 1)$ where n_0 is the density of neutrals. But $n_i = n_e$ for a hydrogen gas and therefore using $n_i = \chi n_t$ where $n_t = n_i + n_0$ we obtain the following quadratic equation:

$$\chi^2 n_t + \chi f(T) - f(T) = 0 \qquad \Longrightarrow \qquad \chi = \frac{2f(T)}{f(T) + \sqrt{f(T)^2 + 4f(T) n_t}}$$

where the solution corresponds to the positive branch.

Saha Equation

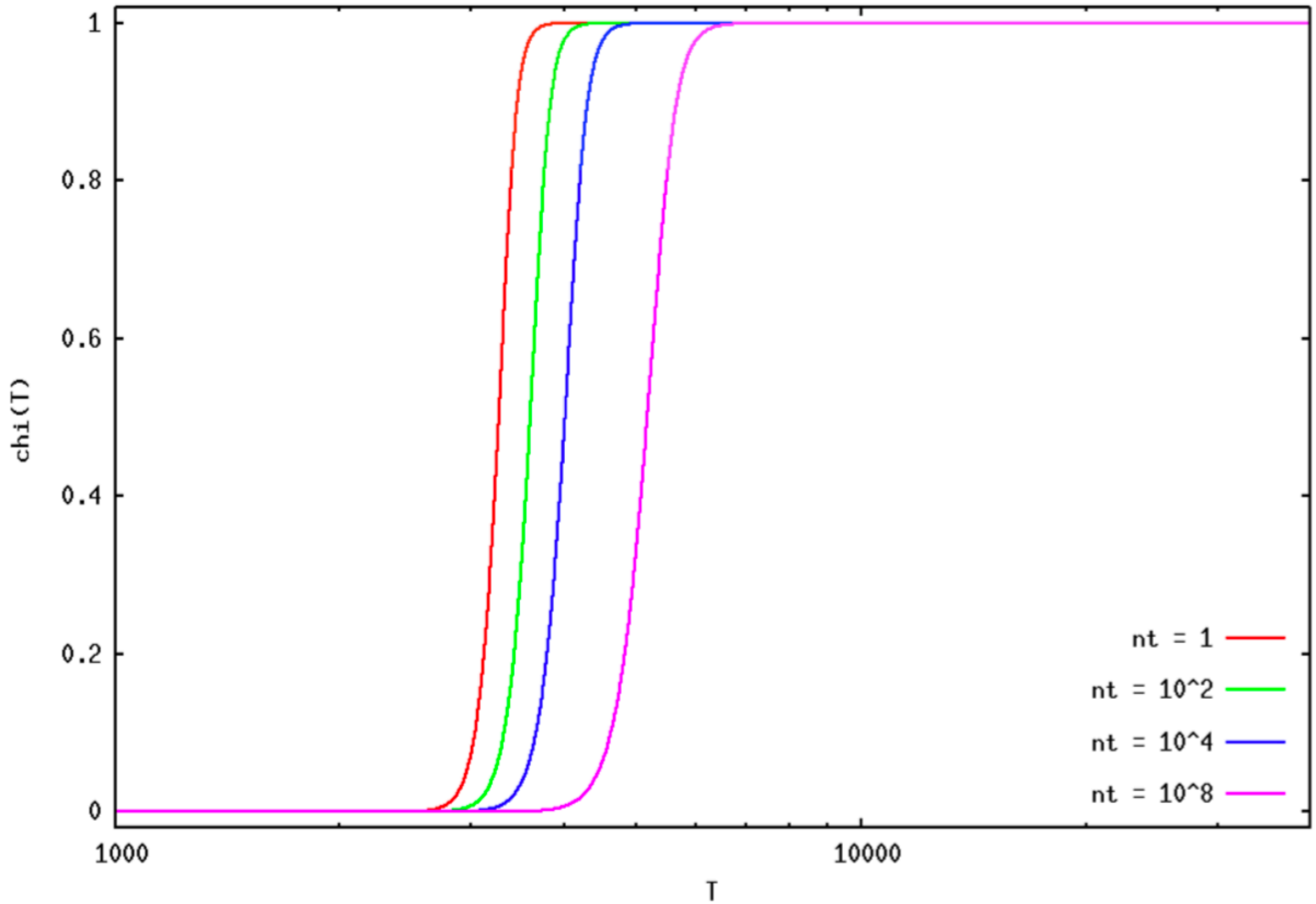


Figure 1: Ionization degree as a function of temperature for different values of the total number density n_t.

Debye Shielding [Bellan, Chap 1, page 8]

Let us now imagine slowly inserting a single additional particle (so-called "test" particle) with charge q_T into an initially unperturbed, spatially uniform neutral plasma. To keep the algebra simple, we define the origin of our coordinate system to be at the location of the test particle. Before insertion of the test particle, the plasma potential was $\phi = 0$ everywhere because the ion and electron densities were spatially uniform and equal, but now the ions and electrons will be perturbed because of their interaction with the test particle. Particles having the same polarity as q_T will be slightly repelled whereas particles of opposite polarity will be slightly attracted. The slight displacements resulting from these repulsions and attractions will result in a small, but finite potential in the plasma. This potential will be the superposition of the test particle's own potential and the potential of the plasma particles that have moved slightly in response to the test particle.

This slight displacement of plasma particles is called *shielding* or *screening* of the test particle because the displacement tends to reduce the effectiveness of the test particle field. To see this, suppose the test particle is a positively charged ion. When immersed in the plasma it will attract nearby electrons and repel nearby ions; the net result is an effectively negative charge cloud surrounding the test particle. An observer located far from the test particle and its surrounding cloud would see the combined potential of the test particle and its associated cloud. Because the cloud has the opposite polarity of the test particle, the cloud potential will partially cancel (i.e., *shield* or *screen*) the test particle potential.

Screening is calculated using Poisson's equation with the source terms being the test particle and its associated cloud. The cloud contribution is determined using the Boltzmann relation for the particles that participate in the screening. This is a 'self-consistent' calculation for the potential because the shielding cloud is affected by its self-potential.

Debye Shielding *[Bellan, Chap 1, assignment 4, page 24]*

- Start from Poisson equation:

$$\nabla \cdot \vec{E} = -\nabla^2 \Phi = 4\pi e(n_i - n_e) + 4\pi q \delta(\vec{r})$$

where $\quad n_i = n_0 e^{-e\Phi/kT}, \quad n_e = n_0 e^{e\Phi/kT}$

- Use Taylor expansion to get

$$\frac{1}{r^2}\partial_r(r^2 \partial_r \Phi) = \frac{8\pi n_0 e^2}{kT}\Phi - 4\pi q \delta(\vec{r}) = \frac{\Phi}{\lambda_D^2} - 4\pi q \delta(\vec{r})$$

- Where the Debye length is defined as

$$\lambda_D = \sqrt{\frac{kT}{8\pi e^2 n_0}}$$

Debye Shielding [Bellan, Chap 1, assignment 4, page 24]

- Solve for a point charge, $$\frac{1}{r^2}\partial_r(r^2\partial_r\Phi_q) = -4\pi q\delta(\vec{r}) \implies \Phi_q = \frac{q}{r}$$

  where we have used the fact that: $$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{1}{4\pi r^2}\delta(r-r_0)$$
- We seek for a solution in the form $\Phi(r) = \Phi_q(r)\, f(r)$ in the original equation:

$$\frac{1}{r^2}\partial_r\left[r^2\partial_r(\Phi_q f)\right] - \frac{\Phi_q f}{\lambda_D^2} = -4\pi q\delta(\vec{r})$$

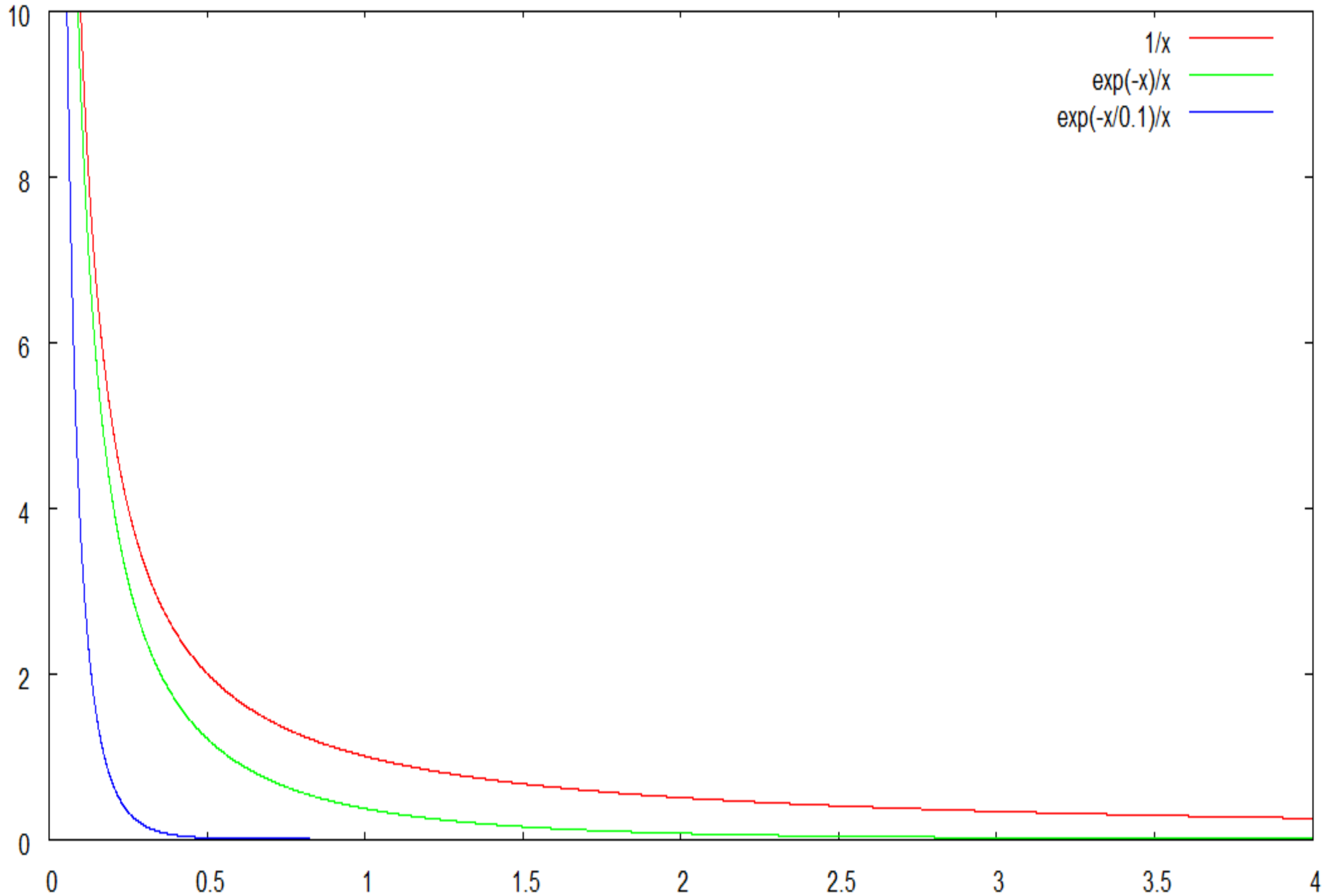
- Subtracting the two equations gives $$\frac{1}{r^2}\partial_r\left[r^2\partial_r(\Phi_q f - \Phi_q)\right] - \frac{\Phi_q f}{\lambda_D^2} = 0$$

- Substitute the solution for $\Phi_q(r)$:

$$\frac{1}{r^2}\partial_r\left[r^2\partial_r\left(\frac{f-1}{r}\right)\right] = \frac{f}{r\lambda_D^2} \implies \partial_{rr}^2 f = \frac{f}{\lambda_D^2}$$

- The final solution is $$\boxed{\Phi(r) = \frac{q}{r}e^{-r/\lambda_D}}$$

Debye Shielding *[Bellan, Chap 1, page 9]*



Debye Shielding [Bellan, Chap 1, page 9]

For $r << \lambda_D$ the potential $\phi(\mathbf{r})$ is identical to the potential of a test particle in vacuum whereas for $r >> \lambda_D$ the test charge is completely screened by its surrounding shielding cloud. The nominal radius of the shielding cloud is λ_D. Because the test particle is completely screened for $r >> \lambda_D$, the total shielding cloud charge is equal in magnitude to the charge on the test particle and opposite in sign. This test-particle/shielding-cloud analysis makes sense only if there is a macroscopically large number of plasma particles in the shielding cloud; i.e., the analysis makes sense only if $4\pi n_0 \lambda_D^3/3 >> 1$. This will be seen later to be the condition for the plasma to be nearly collisionless and so validate assumption #1 in Sec.1.6.

In order for shielding to be a relevant issue, the Debye length must be small compared to the overall dimensions of the plasma, because otherwise no point in the plasma could be outside the shielding cloud. Finally, it should be realized that *any* particle could have been construed as being 'the' test particle and so we conclude that the time-averaged effective potential of any selected particle in the plasma is given by Eq. (1.8) (from a statistical point of view, selecting a particle means that it no longer is assumed to have a random thermal velocity and its effective potential is due to its own charge and to the time average of the random motions of the other particles).

Debye Shielding [Chen end of Sec. 1.4]

We are now in a position to define "quasineutrality." If the dimensions L of a system are much larger than λ_D, then whenever local concentrations of charge arise or external potentials are introduced into the system, these are shielded out in a distance short compared with L, leaving the bulk of the plasma free of large electric potentials or fields. Outside of the sheath on the wall or on an obstacle, $\nabla^2\phi$ is very small, and n_i is equal to n_e, typically, to better than one part in 10^6. It takes only a small charge imbalance to give rise to potentials of the order of KT/e. The plasma is "quasineutral"; that is, neutral enough so that one can take $n_i \simeq n_e \simeq n$, where n is a common density called the *plasma density*, but not so neutral that all the interesting electromagnetic forces vanish.

A criterion for an ionized gas to be a plasma is that it be dense enough that λ_D is much smaller than L.

Confronto di parametri per plasmi

	n	T	λ_D	$n\lambda_D{}^3$	$\ln\Lambda$	ν_{ee}	l_{mfp}	L
units	m^{-3}	eV	m			s^{-1}	m	m
Solar corona (loops)	10^{15}	100	10^{-3}	10^{7}	19	10^{2}	10^{5}	10^{8}
Solar wind (near earth)	10^{7}	10	10	10^{9}	25	10^{-5}	10^{11}	10^{11}
Magnetosphere (tail lobe)	10^{4}	10	10^{2}	10^{11}	28	10^{-8}	10^{14}	10^{8}
Ionosphere	10^{11}	0.1	10^{-2}	10^{4}	14	10^{2}	10^{3}	10^{5}
Mag. fusion (tokamak)	10^{20}	10^{4}	10^{-4}	10^{7}	20	10^{4}	10^{4}	10
Inertial fusion (imploded)	10^{31}	10^{4}	10^{-10}	10^{2}	8	10^{14}	10^{-7}	10^{-5}
Lab plasma (dense)	10^{20}	5	10^{-6}	10^{3}	9	10^{8}	10^{-2}	10^{-1}
Lab plasma (diffuse)	10^{16}	5	10^{-4}	10^{5}	14	10^{4}	10^{1}	10^{-1}

Plasma Frequency *[Goedbloed & Poedts, section 2.3.2]*

- In plasma oscillations (also called Langmuir waves 1929) heavy ions may be considered as a fixed ($u_i = 0$) background in which only the light electrons move ($u_e \neq 0$).
- Perturb a small region by displacing the electrons $n_e = n_0 + n'_e$ ($n'_e << n_0$).
- An electric field E proportional to n'_e is created.
- This small electric field creates a small electron flow velocity v , which is also proportional to n'_e.
- The equation of motion and electric field will be

$$m_e \partial_t \vec{v} = -e\vec{E}$$

$$\nabla \cdot \vec{E} = -4\pi e n'_e$$

- Particle conservation (continuity) demands

$$\partial_t (n_0 + n'_e) + \nabla \cdot [(n_0 + n'_e)\vec{v}] = 0$$

- Linearization is appropriate: terms involving products of perturbations are neglected since they are small compared to linear terms:

$$\partial_t n'_e + n_0 \nabla \cdot \vec{v} = 0$$

Plasma Frequency *[Goedbloed & Poedts, section 2.3.2]*

- Taking the divergence of the equation of motion

$$m_e \partial_t \nabla \cdot \vec{v} = -e \nabla \cdot \vec{E} = 4\pi e^2 n_e'$$

- Using the continuity equation gives:

$$\partial_{tt}^2 n_e' = -\frac{4\pi e^2 n_0}{m_e} n_e'$$

- Harmonic oscillator with frequency

$$\omega_{pe} = \sqrt{\frac{4\pi e^2 n_0}{m_e}}$$

- ➔ Plasma Frequency 🡐

Summary

- Electric conducibility: $$\sigma = \frac{e^2 n_e}{m_e \nu_c}$$

- Debye length $$\lambda_D = \sqrt{\frac{kT}{4\pi n e^2}} = 7.43 \cdot 10^2 \sqrt{\frac{T}{n}}\ \mathrm{cm}$$

- Plasma frequency $\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$ →el: $\omega_{pe} = 5.64 \cdot 10^4 \sqrt{n_e} \quad \mathrm{rad/s}$

Collisions [Bellan, Sect. 1.9, pag 14]

1.9 Electron and ion collision frequencies

One of the fundamental physical constants influencing plasma behavior is the ion to electron mass ratio. The large value of this ratio often causes electrons and ions to experience qualitatively distinct dynamics. In some situations, one species may determine the essential character of a particular plasma behavior while the other species has little or no effect. Let us now examine how mass ratio affects:

1. Momentum change (scattering) of a given incident particle due to collision between
   (a) like particles (i.e., electron-electron or ion-ion collisions, denoted ee or ii),
   (b) unlike particles (i.e., electrons scattering from ions denoted ei or ions scattering from electrons denoted ie),
2. Kinetic energy change (scattering) of a given incident particle due to collisions between like or unlike particles.

Momentum scattering is characterized by the time required for collisions to deflect the incident particle by an angle $\pi/2$ from its initial direction, or more commonly, by the inverse of this time, called the collision frequency. The momentum scattering collision frequencies are denoted as $\nu_{ee}, \nu_{ii}, \nu_{ei}, \nu_{ie}$ for the various possible interactions between species and the corresponding times as τ_{ee}, etc. Energy scattering is characterized by the time required for an incident particle to transfer all its kinetic energy to the target particle. Energy transfer collision frequencies are denoted respectively by $\nu_{Eee}, \nu_{Eii}, \nu_{Eei}, \nu_{Eie}$.

We now show that these frequencies separate into categories having *three distinct orders of magnitude* having relative scalings $1 : (m_i/m_e)^{1/2} : m_i/m_e$. In order to estimate the

Collisions [Bellan, pag 15]

Momentum Exchange

We normalize all collision frequencies to ν_{ee}, and for further simplification assume that the ion and electron temperatures are of the same order of magnitude. First consider ν_{ei}: the reduced mass for ei collisions is the same as for ee collisions (except for a factor of 2 which we neglect), the relative velocity is the same — hence, we conclude that $\nu_{ei} \sim \nu_{ee}$. Now consider ν_{ii}: because the temperatures were assumed equal, $\sigma^*_{ii} \approx \sigma^*_{ee}$ and so the collision frequencies will differ only because of the different velocities in the expression $\nu = n\sigma v$. The ion thermal velocity is lower by an amount $(m_e/m_i)^{1/2}$ giving $\nu_{ii} \approx (m_e/m_i)^{1/2}\nu_{ee}$.

Energy Exchange

Now consider energy changes in collisions. If a moving electron makes a head-on collision with an electron at rest, then the incident electron stops (loses all its momentum and energy) while the originally stationary electron flies off with the same momentum and energy that the incident electron had. A similar picture holds for an ion hitting an ion. Thus, like-particle collisions transfer energy at the same rate as momentum so $\nu_{Eee} \sim \nu_{ee}$ and $\nu_{Eii} \sim \nu_{ii}$.

Collisions *[Bellan, pag 15]*

Inter-species collisions are more complicated. Consider an electron hitting a stationary ion head-on. Because the ion is massive, it barely recoils and the electron reflects with a velocity nearly equal in magnitude to its incident velocity. Thus, the change in electron momentum is $-2m_e\mathbf{v}_e$. From conservation of momentum, the momentum of the recoiling ion must be $m_i\mathbf{v}_i = 2m_e\mathbf{v}_e$. The energy transferred to the ion in this collision is $m_iv_i^2/2 = 4(m_e/m_i)m_ev_e^2/2$. Thus, an electron has to make $\sim m_i/m_e$ such collisions in order to transfer all its energy to ions. Hence, $\nu_{Eei} = (m_e/m_i)\nu_{ee}$.

Similarly, if an incident ion hits an electron at rest the electron will fly off with twice the incident ion velocity (in the center of mass frame, the electron is reflecting from the ion). The electron gains energy $m_ev_i^2/2$ so that again $\sim m_i/m_e$ collisions are required for the ion to transfer all its energy to electrons.

We now summarize the orders of magnitudes of collision frequencies in the table below.

~ 1	$\sim (m_e/m_i)^{1/2}$	$\sim m_e/m_i$
ν_{ee}	ν_{ii}	ν_{ie}
ν_{ei}	ν_{Eii}	ν_{Eei}
ν_{Eee}		ν_{Eie}