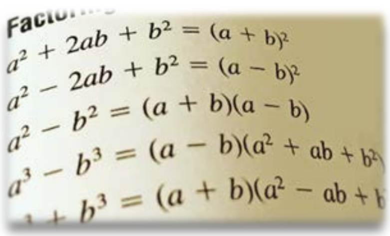
Factoring



Factoring is the reverse process of multiplication. Factoring polynomials in algebra has similar role as factoring numbers in arithmetic. Any number can be expressed as a product of prime numbers. For example, $6 = 2 \cdot 3$. Similarly, any polynomial can be expressed as a product of **prime** polynomials, which are polynomials that cannot be factored any further. For example, $x^2 + 5x + 6 = (x + 2)(x + 3)$. Just as factoring numbers helps in simplifying or adding fractions, factoring polynomials is very useful in simplifying or adding algebraic fractions. In addition, it helps identify zeros of polynomials, which in turn allows for solving higher degree polynomial equations.

In this chapter, we will examine the most commonly used factoring strategies with particular attention to special factoring. Then, we will apply these strategies in solving polynomial equations.

F.1 Greatest Common Factor and Factoring by Grouping

Prime Factors

When working with integers, we are often interested in their factors, particularly prime factors. Likewise, we might be interested in factors of polynomials.

Definition 1.1 ► To **factor** a polynomial means to write the polynomial as a **product** of 'simpler' polynomials. For example,

$$5x + 10 = 5(x + 2), \quad \text{or} \quad x^2 - 9 = (x + 3)(x - 3).$$

In the above definition, 'simpler' means polynomials of **lower degrees** or polynomials with coefficients that **do not contain common factors** other than 1 or -1. If possible, we would like to see the polynomial factors, other than monomials, having **integral coefficients** and a **positive leading term**.

When is a polynomial factorization complete?

In the case of natural numbers, the complete factorization means a factorization into prime numbers, which are numbers divisible only by their own selves and 1. We would expect that similar situation is possible for polynomials. So, which polynomials should we consider as prime?

Observe that a polynomial such as $-4x + 12$ can be written as a product in many different ways, for instance

$$-(4x + 12), \quad 2(-2x + 6), \quad 4(-x + 3), \quad -4(x - 3), \quad -12\left(\frac{1}{3}x + 1\right), \text{ etc.}$$

Since the terms of $4x + 12$ and $-2x + 6$ still contain common factors different than 1 or -1, these polynomials are not considered to be factored completely, which means that they should not be called prime. The next two factorizations, $4(-x + 3)$ and $-4(x - 3)$ are both complete, so both polynomials $-x + 3$ and $x - 3$ should be considered as prime. But what about the last factorization, $-12\left(\frac{1}{3}x + 1\right)$? Since the remaining binomial $\frac{1}{3}x + 1$ does not have integral coefficients, such a factorization is not always desirable.

Here are some examples of **prime polynomials**:

- any monomials such as $-2x^2$, πr^2, or $\frac{1}{3}xy$;
- any linear polynomials with integral coefficients that have no common factors other than 1 or -1, such as $x-1$ or $2x+5$;
- some quadratic polynomials with integral coefficients that cannot be factored into any lower degree polynomials with integral coefficients, such as x^2+1 or x^2+x+1.

For the purposes of this course, we will assume the following definition of a prime polynomial.

Definition 1.2 ► A polynomial with integral coefficients is called **prime** if one of the following conditions is true

- it is a **monomial**, or
- the only **common factors** of its terms are **1 or -1** and it **cannot be factored into any lower degree polynomials** with integral coefficients.

Definition 1.3 ► A **factorization** of a polynomial with integral coefficients is **complete** if all of its **factors** are **prime**.

Here is an example of a polynomial factored completely:

$$-6x^3-10x^2+4x=-2x(3x-1)(x+2)$$

In the next few sections, we will study several factoring strategies that will be helpful in finding complete factorizations of various polynomials.

Greatest Common Factor

The first strategy of factoring is to factor out the **greatest common factor** (**GCF**).

Definition 1.4 ► The **greatest common factor** (**GCF**) of two or more terms is the largest expression that is a factor of all these terms.

In the above definition, the "largest expression" refers to the expression with the most factors, disregarding their signs.

To find the greatest common factor, we take the product of the least powers of each type of common factor out of all the terms. For example, suppose we wish to find the GCF of the terms

$$6x^2y^3,\ -18x^5y, \text{ and } 24x^4y^2.$$

First, we look for the GCF of 6, 18, and 24, which is 6. Then, we take the lowest power out of x^2, x^5, and x^4, which is x^2. Finally, we take the lowest power out of y^3, y, and y^2, which is y. Therefore,

$$\mathrm{GCF}(6x^2y^3,\ -18x^5y,\ 24x^4y^2)=6x^2y$$

This GCF can be used to factor the polynomial $6x^2y^3-18x^5y+24x^4y^2$ by first seeing it as

$$6x^2y\cdot y^2-6x^2y\cdot 3x^3+6x^2y\cdot 4x^2y,$$

and then, using the **reverse distributing property**, 'pulling' the $6x^2y$ out of the bracket to obtain

$$6x^2y(y^2 - 3x^3 + 4x^2y).$$

Note 1: Notice that since 1 and -1 are factors of any expression, the GCF is defined up to the sign. Usually, we choose the positive GCF, but sometimes it may be convenient to choose the negative GCF. For example, we can claim that

$$\text{GCF}(-2x, -4y) = 2 \quad \text{or} \quad \text{GCF}(-2x, -4y) = -2,$$

depending on what expression we wish to leave after factoring the GCF out:

$$-2x - 4y = \underbrace{2}_{\substack{positive\\GCF}} \underbrace{(-x - 2y)}_{\substack{negative\\leading\\term}} \quad \text{or} \quad -2x - 4y = \underbrace{-2}_{\substack{negative\\GCF}} \underbrace{(x + 2y)}_{\substack{positive\\leading\\term}}$$

Note 2: If the GCF of the terms of a polynomial is equal to 1, we often say that these terms do not have any common factors. What we actually mean is that the terms do not have a common factor other than 1, as factoring 1 out does not help in breaking the original polynomial into a product of simpler polynomials. See *Definition 1.1*.

Example 1 ► **Finding the Greatest Common Factor**

Find the Greatest Common Factor for the given expressions.

a. $6x^4(x+1)^3,\ 3x^3(x+1),\ 9x(x+1)^2$ **b.** $4\pi(y-x),\ 8\pi(x-y)$
c. $ab^2,\ a^2b,\ b,\ a$ **d.** $3x^{-1}y^{-3},\ x^{-2}y^{-2}z$

Solution ►

a. Since $\text{GCF}(6,3,9) = 3$, the lowest power out of x^4, x^3, and x is x, and the lowest power out of $(x+1)^3$, $(x+1)$, and $(x+1)^2$ is $(x+1)$, then

$$\text{GCF}(6x^4(x+1)^3,\ 3x^3(x+1),\ 9x(x+1)^2) = \mathbf{3x(x+1)}$$

b. Since $y - x$ is opposite to $x - y$, then $y - x$ can be written as $-(x - y)$. So 4, π, and $(x - y)$ is common for both expressions. Thus,

$$\text{GCF}\big(4\pi(y-x),\ 8\pi(x-y)\big) = \boldsymbol{4\pi(x-y)}$$

Note: The Greatest Common Factor is unique up to the sign. Notice that in the above example, we could write $x - y$ as $-(y - x)$ and choose the GCF to be $4\pi(y - x)$.

c. The terms $ab^2,\ a^2b,\ b,$ and a have no common factor other than 1, so

$$\text{GCF}(ab^2,\ a^2b,\ b,\ a) = \mathbf{1}$$

d. The lowest power out of x^{-1} and x^{-2} is x^{-2}, and the lowest power out of y^{-3} and y^{-2} is y^{-3}. Therefore,

$$\mathrm{GCF}(3x^{-1}y^{-3},\ x^{-2}y^{-2}z) = \boldsymbol{x^{-2}y^{-3}}$$

Example 2 ► **Factoring out the Greatest Common Factor**

Factor each expression by taking the greatest common factor out. Simplify the factors, if possible.

a. $54x^2y^2 + 60xy^3$

b. $ab - a^2b(a-1)$

c. $-x(x-5) + x^2(5-x) - (x-5)^2$

d. $x^{-1} + 2x^{-2} - x^{-3}$

Solution ►

a. To find the greatest common factor of 54 and 60, we can use the method of dividing by any common factor, as presented below.

all common factors are listed in this column

$$\begin{array}{c|cc} 2 & 54, & 60 \\ 3 & 27, & 30 \\ \hline & 9, & 10 \end{array}$$

no more common factors for 9 and 10

So, $\mathrm{GCF}(54, 60) = 2 \cdot 3 = 6$.

Since $\mathrm{GCF}(54x^2y^2,\ 60xy^3) = 6xy^2$, we factor the $6xy^2$ out by dividing each term of the polynomial $54x^2y^2 + 60xy^3$ by $6xy^2$, as below.

$$\begin{aligned} &54x^2y^2 + 60xy^3 \\ &= \boldsymbol{6xy^2(9x + 10y)} \end{aligned}$$

$\frac{54x^2y^2}{6xy^2} = 9x$

$\frac{60xy^3}{6xy^2} = 10y$

Note: Since factoring is the reverse process of multiplication, it can be checked by finding the product of the factors. If the product gives us the original polynomial, the factorization is correct.

b. First, notice that the polynomial has two terms, ab and $-a^2b(a-1)$. The greatest common factor for these two terms is ab, so we have

$$\begin{aligned} ab - a^2b(a-1) &= ab(\mathbf{1} - a(a-1)) \\ &= ab(1 - a^2 + a) \\ &= ab(-a^2 + a + 1) \\ &= \boldsymbol{-ab(a^2 - a - 1)} \end{aligned}$$

remember to leave **1** for the first term

simplify and arrange in decreasing powers

take the "–" out

Note: Both factorizations, $ab(-a^2+a+1)$ and $-ab(a^2-a-1)$ are correct. However, we customarily leave the polynomial in the bracket with a positive leading coefficient.

c. Observe that if we write the middle term $x^2(5-x)$ as $-x^2(x-5)$ by factoring the negative out of the $(5-x)$, then $(5-x)$ is the common factor of all the terms of the equivalent polynomial

$$-x(x-5)-x^2(x-5)-(x-5)^2.$$

Then notice that if we take $-(x-5)$ as the GCF, then the leading term of the remaining polynomial will be positive. So, we factor

$$-x(x-5)+x^2(5-x)-(x-5)^2$$
$$=-x(x-5)-x^2(x-5)-(x-5)^2$$
$$=-(x-5)\left(x+x^2+(x-5)\right)$$

simplify and arrange in decreasing powers

$$=-(x-5)(\mathbf{x^2+2x-5})$$

d. The $\mathrm{GCF}(x^{-1},\ 2x^{-2},\ -x^{-3})=x^{-3}$, as -3 is the lowest exponent of the common factor x. So, we factor out x^{-3} as below.

$$x^{-1}+2x^{-2}-x^{-3}$$
$$=x^{-3}\,(\mathbf{x^2+2x-1})$$

the exponent 2 is found by subtracting -3 from -1

the exponent 1 is found by subtracting -3 from -2

To check if the factorization is correct, we multiply

$$x^{-3}\,(x^2+2x-1)$$

add exponents

$$=x^{-3}x^2+2x^{-3}x-1x^{-3}$$
$$=x^{-1}+2x^{-2}-x^{-3}$$

Since the product gives us the original polynomial, the factorization is correct.

Factoring by Grouping

When referring to a common factor, we have in mind a common factor other than 1.

Consider the polynomial $x^2+x+xy+y$. It consists of four terms that do not have any common factors. Yet, it can still be factored if we group the first two and the last two terms. The first group of two terms contains the common factor of x and the second group of two terms contains the common factor of y. Observe what happens when we factor each group.

$$\underbrace{x^2+x}+\underbrace{xy+y}$$
$$=x(x+1)+y(x+1)$$
$$=(x+1)(x+y)$$

now $(x+1)$ is the common factor of the entire polynomial

This method is called **factoring by grouping**, in particular, two-by-two grouping.

Warning: After factoring each group, make sure to write the "+" or "−" between the terms. Failing to write these signs leads to the false impression that the polynomial is already factored. For example, if in the second line of the above calculations we would fail to write the middle "+", the expression would look like a product $x(x+1)\,y(x+1)$, which is not the case. Also, since the expression $x(x+1)+y(x+1)$ is a sum, not a product, we should not stop at this step. We need to factor out the common bracket $(x+1)$ to leave it as a product.

A two-by-two grouping leads to a factorization only if **the binomials**, after factoring out the common factors in each group, **are the same**. Sometimes a rearrangement of terms is necessary to achieve this goal.

For example, the attempt to factor $x^3-15+5x^2-3x$ by grouping the first and the last two terms,

$$\underbrace{x^3-15}+\underbrace{5x^2-3x}$$

$$=(x^3-15)+x(5x-3)$$

does not lead us to a common binomial that could be factored out.

However, rearranging terms allows us to factor the original polynomial in the following ways:

$$x^3-15+5x^2-3x$$

$$=\underbrace{x^3+5x^2}+\underbrace{-3x-15}$$

$$=x^2(x+5)-3(x+5)$$

$$=(x+5)(x^2-3)$$

or

$$x^3-15+5x^2-3x$$

$$=\underbrace{x^3-3x}+\underbrace{5x^2-15}$$

$$=x(x^2-3)+5(x^2-3)$$

$$=(x^2-3)(x+5)$$

Factoring by grouping applies to polynomials with more than three terms. However, not all such polynomials can be factored by grouping. For example, if we attempt to factor x^3+x^2+2x-2 by grouping, we obtain

$$\underbrace{x^3+x^2}+\underbrace{2x-2}$$

$$=x^2(x+1)+2(x-1).$$

Unfortunately, the expressions $x+1$ and $x-1$ are not the same, so there is no common factor to factor out. One can also check that no other rearrangments of terms allows us for factoring out a common binomial. So, this polynomial cannot be factored by grouping.

Example 3 ► **Factoring by Grouping**

Factor each polynomial by grouping, if possible. Remember to check for the GCF first.

a. $2x^3 - 6x^2 + x - 3$ **b.** $5x - 5y - ax + ay$
c. $2x^2y - 8 - 2x^2 + 8y$ **d.** $x^2 - x + y + 1$

Solution ► **a.** Since there is no common factor for all four terms, we will attempt the two-by-two grouping method.

$$\underbrace{2x^3 - 6x^2} + \underbrace{x - 3}$$
$$= 2x^2(x-3) + 1(x-3)$$
$$= (\boldsymbol{x-3})(\boldsymbol{2x^2+1})$$

write the **1** for the second term

b. As before, there is no common factor for all four terms. The two-by-two grouping method works only if the remaining binomials after factoring each group are exactly the same. We can achieve this goal by factoring $-a$, rather than a, out of the last two terms. So,

$$\underbrace{5x - 5y}\ \underbrace{-ax + ay}$$
$$= 5(x-y) - a(x-y)$$
$$= (\boldsymbol{x-3})(\boldsymbol{2x^2+1})$$

reverse signs when 'pulling' a "–" out

c. Notice that 2 is the GCF of all terms, so we factor it out first.

$$2x^2y - 8 - 2x^2 + 8y$$
$$= 2(x^2y - 4 - x^2 + 4y)$$

Then, observe that grouping the first and last two terms of the remaining polynomial does not help, as the two groups do not have any common factors. However, exchanging for example the second with the fourth term will help, as shown below.

$$= 2(\underbrace{x^2y + 4y}\ \underbrace{-x^2 - 4})$$
$$= 2[y(x^2+4) - (x^2+4)]$$
$$= \boldsymbol{2(x^2+4)(y-1)}$$

the square bracket is essential here because of the factor of 2

reverse signs when 'pulling' a "–" out

now, there is no need for the square bracket as multiplication is associative

d. The polynomial $x^2 - x + y + 1$ does not have any common factors for all four terms. Also, only the first two terms have a common factor. Unfortunately, when attempting to factor using the two-by-two grouping method, we obtain

$$x^2 - x + y + 1$$
$$= x(x-1) + (y+1),$$

which cannot be factored, as the expressions $x - 1$ and $y + 1$ are different.

One can also check that no other arrangement of terms allows for factoring of this polynomial by grouping. So, this polynomial cannot be factored by grouping.

Example 4 ► **Factoring in Solving Formulas**

Solve $ab = 3a + 5$ for a.

Solution ► First, we move the terms containing the variable a to one side of the equation,

$$ab = 3a + 5$$
$$ab - 3a = 5,$$

and then factor a out

$$a(b - 3) = 5.$$

So, after dividing by $b - 3$, we obtain $\boldsymbol{a = \frac{5}{b-3}}$.

F.1 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***common factor, distributive, grouping, prime, product.***

1. To factor a polynomial means to write it as a ____________ of simpler polynomials.

2. The greatest ______________ _____________ of two or more terms is the product of the least powers of each type of common factor out of all the terms.

3. To factor out the GCF, we reverse the ______________ property of multiplication.

4. A polynomial with four terms having no common factors can be still factored by _____________ its terms.

5. A ___________ polynomial, other than a monomial, cannot be factored into two polynomials, both different that 1 or -1.

Concept Check *True or false.*

6. The polynomial $6x + 8y$ is prime.

7. The factorization $\frac{1}{2}x - \frac{3}{4}y = \frac{1}{4}(2x - 3y)$ is essential to be complete.

8. The GCF of the terms of the polynomial $3(x - 2) + x(2 - x)$ is $(x - 2)(2 - x)$.

Concept Check *Find the GCF with a positive coefficient for the given expressions.*

9. $8xy,\ 10xz,\ -14xy$

10. $21a^3b^6,\ -35a^7b^5,\ 28a^5b^8$

11. $4x(x - 1),\ 3x^2(x - 1)$

12. $-x(x - 3)^2,\ x^2(x - 3)(x + 2)$

13. $9(a - 5),\ 12(5 - a)$

14. $(x - 2y)(x - 1),\ (2y - x)(x + 1)$

15. $-3x^{-2}y^{-3},\ 6x^{-3}y^{-5}$

16. $x^{-2}(x+2)^{-2},\ -x^{-4}(x+2)^{-1}$

Factor out the greatest common factor. Leave the remaining polynomial with a positive leading coeficient. Simplify the factors, if possible.

17. $9x^2 - 81x$

18. $8k^3 + 24k$

19. $6p^3 - 3p^2 - 9p^4$

20. $6a^3 - 36a^4 + 18a^2$

21. $-10r^2s^2 + 15r^4s^2$

22. $5x^2y^3 - 10x^3y^2$

23. $a(x-2) + b(x-2)$

24. $a(y^2-3) - 2(y^2-3)$

25. $(x-2)(x+3) + (x-2)(x+5)$

26. $(n-2)(n+3) + (n-2)(n-3)$

27. $y(x-1) + 5(1-x)$

28. $(4x-y) - 4x(y-4x)$

29. $4(3-x)^2 - (3-x)^3 + 3(3-x)$

30. $2(p-3) + 4(p-3)^2 - (p-3)^3$

Factor out the least power of each variable.

31. $3x^{-3} + x^{-2}$

32. $k^{-2} + 2k^{-4}$

33. $x^{-4} - 2x^{-3} + 7x^{-2}$

34. $3p^{-5} + p^{-3} - 2p^{-2}$

35. $3x^{-3}y - x^{-2}y^2$

36. $-5x^{-2}y^{-3} + 2x^{-1}y^{-2}$

Factor by grouping, if possible.

37. $20 + 5x + 12y + 3xy$

38. $2a^3 + a^2 - 14a - 7$

39. $ac - ad + bc - bd$

40. $2xy - x^2y + 6 - 3x$

41. $3x^2 + 4xy - 6xy - 8y^2$

42. $x^3 - xy + y^2 - x^2y$

43. $3p^2 + 9pq - pq - 3q^2$

44. $3x^2 - x^2y - yz^2 + 3z^2$

45. $2x^3 - x^2 + 4x - 2$

46. $x^2y^2 + ab - ay^2 - bx^2$

47. $xy + ab + by + ax$

48. $x^2y - xy + x + y$

49. $xy - 6y + 3x - 18$

50. $x^ny - 3x^n + y - 5$

51. $a^nx^n + 2a^n + x^n + 2$

Factor completely. Remember to check for the GCF first.

52. $5x - 5ax + 5abc - 5bc$

53. $6rs - 14s + 6r - 14$

54. $x^4(x-1) + x^3(x-1) - x^2 + x$

55. $x^3(x-2)^2 + 2x^2(x-2) - (x+2)(x-2)$

Discussion Point

56. One of possible factorizations of the polynomial $4x^2y^5 - 8xy^3$ is $2xy^3(2xy^2 - 4)$. Is this a complete factorization?

Use factoring the GCF strategy to ***solve*** *each formula* ***for the indicated variable****.*

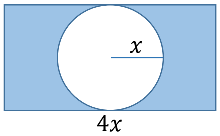
57. $A = \boldsymbol{P} + \boldsymbol{P}r$, for $\boldsymbol{P}$

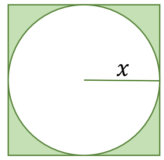
58. $M = \frac{1}{2}\boldsymbol{p}q + \frac{1}{2}\boldsymbol{p}r$, for $\boldsymbol{p}$

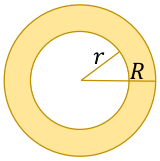
59. $2\boldsymbol{t} + c = k\boldsymbol{t}$, for $\boldsymbol{t}$

60. $w\boldsymbol{y} = 3\boldsymbol{y} - x$, for $\boldsymbol{y}$

Analytic Skills *Write the area of each shaded region in factored form.*

61. 

62. 

63. 

64.