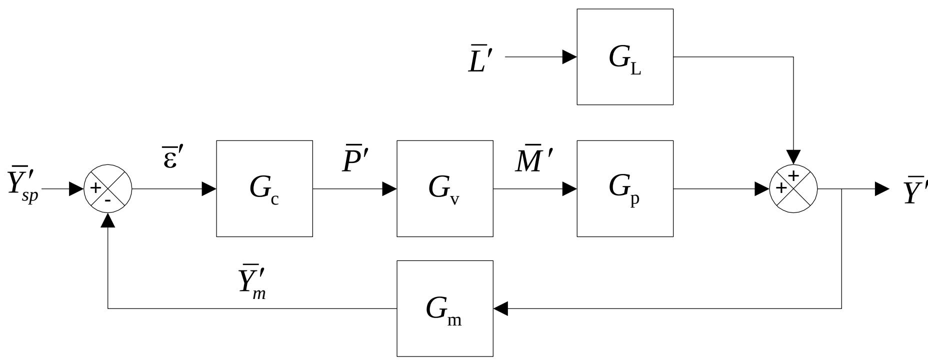
Direct Synthesis Controller Tuning

Direct synthesis methods are based upon prescribing a desired form for the system's response and then finding a controller strategy & parameters to give that response.



For the feedback control loop above the overall transfer functions between the output $\bar{Y}'$ and the set point & disturbance are:

$$\frac{\bar{Y}'}{\bar{Y}'_{sp}} = \frac{G_c G_v G_p}{1 + G_c G_v G_p G_m}$$

and:

$$\frac{\bar{Y}'}{\bar{L}'} = \frac{G_L}{1 + G_c G_v G_p G_m}.$$

Note that for the set point transfer function, we can manipulate it to give:

$$G_c = \frac{\left(\bar{Y}'/\bar{Y}'_{sp}\right)}{G_v G_p \left[1 - G_m \left(\bar{Y}'/\bar{Y}'_{sp}\right)\right]}.$$

One implication is that if we pick a desired form for the response to a set point change, $\bar{Y}'/\bar{Y}'_{sp}$, then we have set out the desired form for the controller. For example, we might think that it would be great to have the output to immediately track the set point change, i.e., $\bar{Y}' = \bar{Y}'_{sp}$. However, a more practical response would be a first order decay into the final value, or:

$$\frac{\bar{Y}'}{\bar{Y}'_{sp}} = \frac{1}{\tau_c s + 1}.$$

We can get this type of response with a controller strategy of the form:

$$G_c = \frac{\left(\dfrac{1}{\tau_c s + 1}\right)}{G_v G_p \left[1 - G_m \left(\dfrac{1}{\tau_c s + 1}\right)\right]} = \frac{1}{G_v G_p \left[\tau_c s + 1 - G_m\right]}.$$

The direct synthesis equations are shown for $\bar{Y}'/\bar{Y}'_{sp}$ but they really make more sense for looking at the dynamic response of the measured variable, $\bar{Y}'_m/\bar{Y}'_{sp}$. Then the transfer function with respect to the set point and the implied controller strategy is:

$$\frac{\bar{Y}'_m}{\bar{Y}'_{sp}} = \frac{G_m G_c G_v G_p}{1 + G_c G_v G_p G_m} \quad \Rightarrow \quad G_c = \frac{\left(\bar{Y}'_m/\bar{Y}'_{sp}\right)}{G_v G_p G_m \left[1 - \left(\bar{Y}'_m/\bar{Y}'_{sp}\right)\right]} = \frac{1}{\left(G_v G_p G_m\right)\left(\tau_c s\right)}.$$

Note that the product of the three transfer functions is simply the open-loop transfer function, represented usually as G_{OL} or some approximation of the actual transfer function, $\tilde{G}$:

$$\frac{\bar{Y}'_m}{\bar{Y}'_{sp}} = \frac{G_c \tilde{G}}{1 + G_c \tilde{G}} \quad \Rightarrow \quad G_c = \frac{\left(\bar{Y}'_m/\bar{Y}'_{sp}\right)}{\tilde{G}\left[1 - \left(\bar{Y}'_m/\bar{Y}'_{sp}\right)\right]} = \frac{1}{\tilde{G}} \cdot \frac{1}{\tau_c s}.$$

Notice that this shows there is an integral action to the Direct Synthesis controller strategy.

First Order Process

Let's look at the Direct Synthesis controller strategy for a first order process:

$$\tilde{G} = \frac{K_p}{\tau_p s + 1}.$$

Then the controller strategy is:

$$G_c = \frac{1}{\frac{K_p}{\tau_p s + 1}} \cdot \frac{1}{\tau_c s} = \frac{\tau_p s + 1}{K_p \tau_c s} = \frac{\tau_p}{K_p \tau_c}\left(1 + \frac{1}{\tau_p s}\right).$$

Notice that this is simply PI control with the settings:

$$K_c = \frac{\tau_p}{K_p \tau_c} \text{ and } \tau_I = \tau_p .$$

Second Order Process

Let's create a 2nd order process by assuming the process & actuator (valve) are both first order. Then:

$$\tilde{G} = \frac{K_p}{\tau_p s + 1} \cdot \frac{K_v}{\tau_v s + 1}$$

$$G_c = \frac{\tau_p s + 1}{K_p} \cdot \frac{\tau_v s + 1}{K_v} \cdot \frac{1}{\tau_c s} = \frac{\tau_p \tau_v s^2 + \left(\tau_p + \tau_v\right) s + 1}{K_p K_v \tau_c s} = \frac{\tau_p + \tau_v}{K_p K_v \tau_c}\left[1 + \frac{1}{\tau_p + \tau_v} \cdot \frac{1}{s} + \frac{\tau_p \tau_v}{\tau_p + \tau_v} \cdot s\right].$$

Notice that this is PID control with the settings:

$$K_c = \frac{\tau_p + \tau_v}{K_p K_v \tau_c}, \quad \tau_I = \tau_p + \tau_v, \text{ and } \tau_D = \frac{\tau_p \tau_v}{\tau_p + \tau_v}.$$

Further note that integral & derivatives times are only dictated by the process & not the desired controller settling time.

Third Order & Higher Processes

Processes that are 3rd order and higher will lead to controller strategies that are different from PID controllers. However we can get PID controller settings by eliminating terms that correspond to high order derivatives. For example:

$$\tilde{G} = \frac{K}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

$$G_c = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}{K} \cdot \frac{1}{\tau_c s} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}{K \tau_c s}$$
$$= \frac{a_1}{K\tau_c}\left[1 + \frac{a_0}{a_1} \cdot \frac{1}{s} + \frac{a_2}{a_1} \cdot s + \frac{a_3}{a_1} \cdot s^2 + \cdots + \frac{a_n}{a_1} \cdot s^{n-1}\right]$$

So the appropriate PID control settings are:

$$K_c = \frac{a_1}{K\tau_c},\ \ \tau_I = \frac{a_1}{a_0},\ \text{and}\ \ \tau_D = \frac{a_2}{a_1}.$$

Again note that integral & derivatives times are only dictated by process parameters & not the desired controller settling time.

Processes with Dead Time

Let's consider a 1st order system with dead time (FOPDT):

$$\tilde{G} = \frac{K_p e^{-\theta_p s}}{\tau_p s + 1}.$$

If we keep the same desired form for the response to a set point change, then the controller strategy is:

$$G_c = \frac{\tau_p s + 1}{K_p e^{-\theta_p s}} \cdot \frac{1}{\tau_c s} = \frac{\tau_p}{K_p \tau_c}\left(1 + \frac{1}{\tau_p s}\right) e^{\theta_p s}.$$

This does not correspond to a standard controller strategy (because of the exponential term). In fact, this exponential term shows a requirement for prediction of what is going to happen, so this cannot be physically realized in a controller strategy.

> If we really wanted to do this though, we could approximate the exponential term with a truncated Taylor series expansion:
>
> $$e^{\theta_p s} \approx 1 + \theta_p s$$
>
> then:

$$G_c \approx \frac{\tau_p}{K_p \tau_c}\left(1+\frac{1}{\tau_p s}\right)\left(1+\theta_p s\right) = \frac{1}{K_p \tau_c}\left(\tau_p + \theta_p + \frac{1}{s} + \tau_p \theta_p s\right)$$
$$= \frac{\tau_p+\theta_p}{K_p \tau_c}\left(1+\frac{1}{\tau_p+\theta_p}\cdot\frac{1}{s}+\frac{\tau_p\theta_p}{\tau_p+\theta_p}\cdot s\right).$$

Now we have PID control with the settings:

$$K_c = \frac{\tau_p+\theta_p}{K_p\tau_c},\ \tau_I = \tau_p+\theta_p, \text{ and } \tau_D = \frac{\tau_p\theta_p}{\tau_p+\theta_p}.$$

However, we could change our desired form for the response to a set point change to something involving the same process dead time:

$$\frac{\bar{Y}'_m}{\bar{Y}'_{sp}} = \frac{e^{-\theta_p s}}{\lambda s+1}$$

then:

$$G_c = \frac{\left(\bar{Y}'_m/\bar{Y}'_{sp}\right)}{\tilde{G}\left[1-\left(\bar{Y}'_m/\bar{Y}'_{sp}\right)\right]} = \frac{1}{\tilde{G}}\cdot\frac{e^{-\theta_p s}}{\tau_c s+1-e^{-\theta_p s}}$$

and for this FOPDT process:

$$G_c = \frac{\tau_p s+1}{K_p e^{-\theta_p s}}\cdot\frac{e^{-\theta_p s}}{\tau_c s+1-e^{-\theta_p s}} = \frac{\tau_p s+1}{K_p\left(\tau_c s+1-e^{-\theta_p s}\right)}.$$

Now we can make a truncated Taylor series approximation:

$$e^{-\theta_p s} \approx 1-\theta_p s$$

to get:

$$G_c \approx \frac{\tau_p s+1}{K_p\left(\tau_c s+1-1+\theta_p s\right)} = \frac{\tau_p s+1}{K_p\left(\tau_c+\theta_p\right)s}$$
$$= \frac{\tau_p}{K_p\left(\tau_c+\theta_p\right)}\left(1+\frac{1}{\tau_p s}\right)$$

and we're back to having PI control with the settings:

$$K_c = \frac{\tau_p}{K_p\left(\tau_c + \theta_p\right)} \text{ and } \tau_I = \tau_p .$$

We used a simple truncated Taylor series expansion for the dead time term here. What happens if we use a Padé approximation instead? Using a 1st order Pade approximation the controller strategy for a FOPDT is:

$$G_c = \frac{\tau_p s + 1}{K_p\left(\tau_c s + 1 - \dfrac{1 - \frac{1}{2}\theta_p s}{1 + \frac{1}{2}\theta_p s}\right)}$$

$$G_c = \frac{\left(\tau_p s + 1\right)\left(1 + \frac{1}{2}\theta_p s\right)}{K_p\left[\left(\tau_c s + 1\right)\left(1 + \frac{1}{2}\theta_p s\right) - \left(1 - \frac{1}{2}\theta_p s\right)\right]}$$

$$G_c = \frac{\left(\tau_p + \frac{1}{2}\theta_p\right)s + 1 + \frac{1}{2}\tau_p\theta_p s^2}{K_p\left[\left(\left(\tau_c + \frac{1}{2}\theta_p\right)s + 1 + \frac{1}{2}\tau_c\theta_p s^2\right) - \left(1 - \frac{1}{2}\theta_p s\right)\right]}$$

$$G_c = \frac{\left(\tau_p + \frac{1}{2}\theta_p\right)s + 1 + \frac{1}{2}\tau_p\theta_p s^2}{K_p\left[\left(\tau_c + \theta_p\right)s + \frac{1}{2}\tau_c\theta_p s^2\right]}$$

If we assume that the dead time is much less than the controller settling time ($\theta_p << \tau_c$) then we can neglect the quadratic term in the denominator.

$$G_c = \frac{\left(\tau_p + \frac{1}{2}\theta_p\right)s + 1 + \frac{1}{2}\tau_p\theta_p s^2}{K_p\left(\tau_c + \theta_p\right)s}$$

$$G_c = \frac{\tau_p + \frac{1}{2}\theta_p}{K_p\left(\tau_c + \theta_p\right)}\left[1 + \frac{1}{\left(\tau_p + \frac{1}{2}\theta_p\right)s} + \frac{\tau_p\theta_p}{2\tau_p + \theta_p}s\right]$$

This is in the form of a PID controller where:

$$K_c = \frac{\tau_p + \frac{1}{2}\theta_p}{K_p\left(\tau_c + \theta_p\right)},\ \tau_I = \tau_p + \frac{1}{2}\theta_p, \text{ and } \tau_D = \frac{\tau_p\theta_p}{2\tau_p + \theta_p}.$$

To get the form reported by Smith & Corropio we have to make the further assumption that the dead time is also much less than the process time constant ($\theta_p << \tau_p$):

$$G_c = \frac{\tau_p}{K_p\left(\tau_c + \theta_p\right)}\left[1 + \frac{1}{\tau_p s} + \frac{\theta_p}{2} s\right]$$

and this is still in the form of a PID controller but now the settings are:

$$K_c = \frac{\tau_p}{K_p\left(\tau_c + \theta_p\right)}, \ \tau_I = \tau_p, \text{ and } \tau_D = \tfrac{1}{2}\theta_p.$$

FOPDT Example

Let's look at the response to a unit step change in the set point for a FOPDT process where $K_p = 0.25$, $K_L = 0.75$, $\tau_p = \tau_L = 10$, and $\theta_p = \theta_L = 1$.

Though it's not needed for the Direct Synthesis method, let's calculate the stability limit for P-control. Under P-control the characteristic equation is:

$$1 + G_c G_v G_p G_m = 0 \ \Rightarrow \ 1 + K_c \frac{K_p e^{-\theta_p s}}{\tau_p s + 1} = 0 \ \Rightarrow \ \tau_p s + 1 + K_c K_p e^{-\theta_p s} = 0.$$

The stability limit can be determined by the direct substitution method:

$$\tau_p \omega_u j + 1 + K_{cu} K_p e^{-\theta_p \omega_u j} = 0 \ \Rightarrow \ \tau_p \omega_u j + 1 + K_{cu} K_p \left[\cos\left(\theta_p \omega_u\right) - j \sin\left(\theta_p \omega_u\right)\right] = 0$$

which leads to the two simultaneous equations:

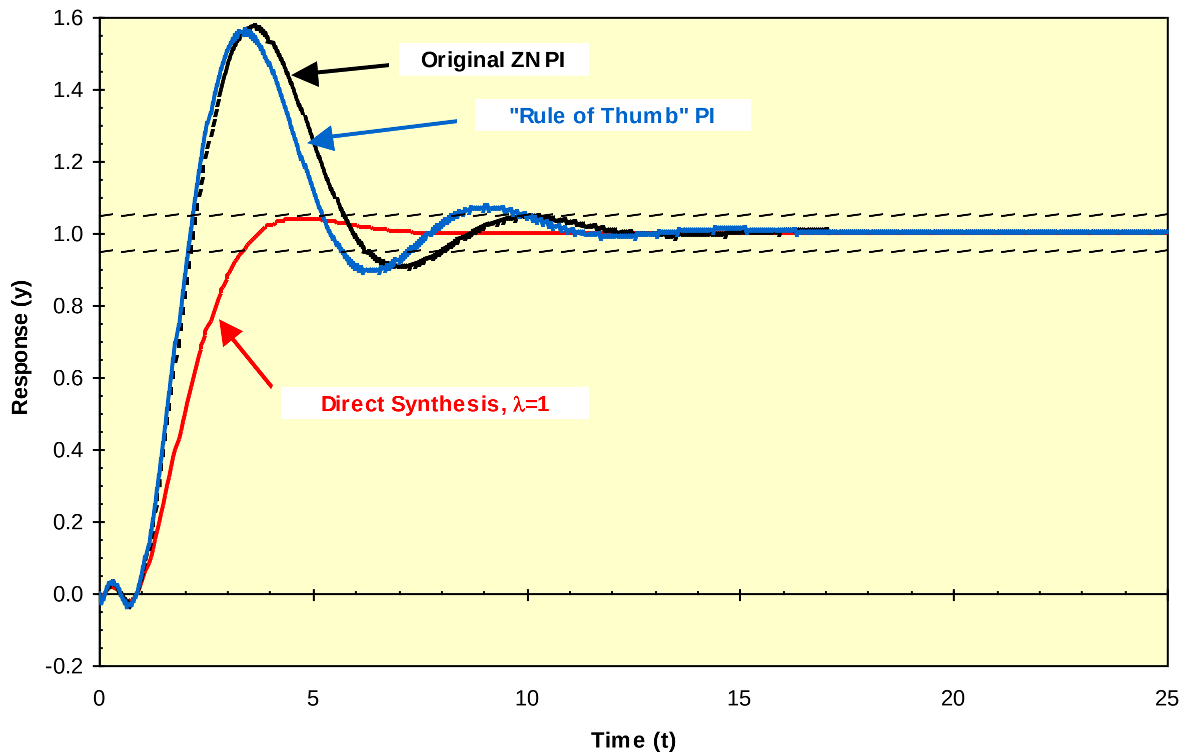
$$\left.\begin{array}{r} 1 + K_{cu} K_p \cos\left(\theta_p \omega_u\right) = 0 \\ \tau_p \omega_u - \sin\left(\theta_p \omega_u\right) K_{cu} K_p = 0 \end{array}\right\} \ \Rightarrow \ \begin{cases} K_{cu} = -\dfrac{1}{K_p \cos\left(\theta_p \omega_u\right)} \\ \tau_p \omega_u + \tan\left(\theta_p \omega_u\right) = 0 \end{cases}$$

The 2nd equation must be solved numerically. For this particular problem the equation & solution are:

$$10\omega_u + \tan\left(\omega_u\right) = 0 \ \Rightarrow \ \omega_u = 1.632 \ \Rightarrow \ P_u = \frac{2\pi}{\omega_u} = 3.85 \ \text{ and } \ K_{cu} = 65.40.$$

The following table gives controller settings from various methods. The response time is considered that time at which the final response stays within ±5% of the final value (0.95 to 1.05). Three of the responses are depicted in the figure following this table.

Method	K_c	τ_I	τ_D	Comments
Direct Synthesis - $\tau_c = 1$	20	10	—	Slight overshoot – max value 1.04. Response time is actually before the first peak, about 3.4 min.
Direct Synthesis - $\tau_c = 0.5$	26.7	10	—	Even larger overshoot – max value 1.2. Response time is after the first peak, about 5 min.
Direct Synthesis - $\tau_c = 2$	13.3	10	—	No apparent overshoot – Response time about 6.4 min.
Original Zeigler-Nichols	29.7	3.2	—	Large overshoot – max value about 1.6. Response time after 1st trough, about 8 min.
Original Zeigler-Nichols	38.5	1.9	0.5	Multiple harmonics at early times. Again large overshoot – max value about 1.5. 1st trough just outside range – response time about 8 min.
ZN Some Overshoot	21.6	1.9	1.3	Multiple harmonics over an extended period of time. Response time about 15 min.
ZN No Overshoot	14.4	1.9	1.3	Long, broad cycling. Response time about 19 min.
Rule of Thumb	32.7	3.9		$\tau_{Iu} = 1.32$. Large overshoot – max value 1.6. Response time 9 min.



Original ZN PI
"Rule of Thumb" PI
Direct Synthesis, λ=1
Response (y)
Time (t)
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
5
10
15
20
25