DIFFERENTIAL GEOMETRY NOTES

HAO (BILLY) LEE

Abstract. These are notes I took in class, taught by Professor Andre Neves. I claim no credit to the originality of the contents of these notes. Nor do I claim that they are without errors, nor readable.

Reference: Do Carmo Riemannian Geometry

1. Review

Example 1.1. When $M = \left\{(x, |x|) \in \mathbb{R}^2 : x \in \mathbb{R}\right\}$, we have one chart $\phi : \mathbb{R} \to M$ by $\phi(x) = (x, |x|)$. This is bad, because there's no tangent space at $(0, 0)$. Therefore, we require that our collection of charts is maximal.

Really though, when we extend this to a maximal completion, it has to be compatible with ϕ, but the map $F : M \to \mathbb{R}^2$ is not smooth.

Definition 1.2. $F : M \to N$ smooth manifolds is differentiable at $p \in M$, if given a chart ϕ of p and ψ of $f(p)$, then $\psi^{-1} \circ F \circ \phi$ is differentiable.

Given $p \in M$, let D_p be the set of functions $f : M \to \mathbb{R}$ that are differentiable at p. Given $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$ and α differentiable at 0, we set $\alpha'(0) : D_p \to \mathbb{R}$ by

$$\alpha'(0)(f) = \frac{d}{dt}\left(f \circ \alpha\right)(0).$$

Then

$$T_pM = \left\{\alpha'(0) : D_p \to \mathbb{R} : \alpha \text{ is a curve with } \alpha(0) = p \text{ and diff at } 0\right\}.$$

This is a finite dimensional vector space. Let ϕ be a chart, and $\alpha_i(t) = \phi(te_i)$, then the derivative at 0 is just $\frac{d}{dx_i}$, and claim that this forms a basis.

For $F : M \to N$ differentiable, define $dF_p : T_pM \to T_{F(p)}N$ by

$$\left(dF_p\right)(\alpha'(0))(f) = \left(f \circ F \circ \alpha\right)'(0).$$

Need to check that this is well-defined.

Example 1.3. If $F : \mathbb{R}^n \to \mathbb{R}^k$ with $k \leq n$, and dF_α is surjective for all $x \in F^{-1}(0)$, then $F^{-1}(0)$ is a smooth manifold.

$TM = \{(x, V) : X \in M, v \in T_pM\}$.

M smooth manifold, G a group acting on M properly discontinuously, that is, $F : G \times M \to M$ so that for all $g \in G$, $F_g : M \to M$ by $x \mapsto F(g, x)$ is a diffeomorphism. $F_{gh} = F_gF_h$ and $F_e = 1$. For all $x \in M$, there exists U neighbourhood of x such that $F_g(U) \cap U = \emptyset$ for all $g \in G$, $g \neq e$. Set $N = M^n / \{x \sim F_g(x) : g \in G\}$ is a manifold.

Recall: $F : M \to M$ is a diffeo if F bijective, differentiable, AND F^{-1} is diff

Theorem 1.4. *Given M smooth manifold, there is $\tilde{M}$ simply connected manifold such that $\pi_1(M)$ acts properly discontinuously on $\tilde{M}$ and $M^n = \tilde{M}/\pi_1(M)$.*

Example 1.5. $\mathbb{RP}^n = S^n / \{x \sim -x\}$. $\mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\} / \{x \sim \lambda x : \lambda \neq 0\}$.

Definition 1.6. A vector field X is a map $M \to TM$. Let $\mathfrak{X}(M)$ be the set of all vector fields. Given $X, Y \in \mathfrak{X}(M)$, define the Lie bracket to act by

$$[X, Y](f) = X\left(Y(f)\right) - Y\left(X(f)\right).$$

Satisfying

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0.$$

2. Riemannian Metric

Definition 2.1. Let M be an n-dimensional manifold. A metric g on M^n is a symmetric bilinear map $g_p : T_pM\times T_pM \to \mathbb{R}$ for all $p\in M$ so that

(1) $g_p(aX+Y,Z)=ag_p(X,Z)+g_p(Y,Z)$ all $a\in\mathbb{R}$, $X,Y,Z\in T_pM$
(2) $g_p(X,Y)=g_p(Y,X)$
(3) $g_p(X,X)>0$ for all $X\in T_pM-0$
(4) If $X,Y\in\mathfrak{X}(M)$, then $p\mapsto g_p(X(p),Y(p))$ is a smooth function

Theorem 2.2. *Every manifold has a metric (partition of unity)*

Let (U,φ) be a chart for M^n. Then $g_{ij}(x)=g_{\varphi(x)}(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})$ for all $x\in U$ and $i,j=1,...,n$. Then the matrix (g_{ij}) is positive definite and symmetric. Additionally, $g_{ij}(x)$ are smooth functions of x.

If we write $X,Y\in T_{\varphi(x)}M$ as $X=\sum a_i\frac{\partial}{\partial x_i}$ and $Y=\sum b_j\frac{\partial}{\partial x_j}$ then

$$g_{\varphi(x)}(X,Y)=\sum_{i,j}a_ib_jg_\varphi\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=\sum_{i,j}a_ib_jg_{ij}.$$

(V,φ) is another chart, g_{ij}^{φ} from first chart, g_{ij}^{ψ} from another chart. Let $h=\psi^{-1}\circ\phi$, then

Claim 2.3. $\frac{\partial}{\partial x_i}=\sum_k\frac{\partial h_k}{\partial x_i}\frac{\partial}{\partial y_k}$.

Proof. For all $f\in C^\infty(M)$,

$$\frac{\partial}{\partial x_i}(f)=\frac{\partial}{\partial x_i}(f\circ\phi)=\frac{\partial}{\partial x_i}(f\circ\psi\circ h)=\sum\frac{\partial h_k}{\partial x_i}\circ\frac{\partial f\circ\psi}{\partial y_k}.$$

□

Therefore,

$$g_{ij}^{\phi}=g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=\sum_{k,\ell}\frac{\partial h_k}{\partial x_i}\frac{\partial h_\ell}{\partial x_j}g_{k\ell}^{\psi}.$$

As a matrix,

$$\left(g_{ij}^{\phi}\right)=(Dh)^T\left(g^{\psi}\right)(Dh).$$

Given a curve $\gamma: I\to M$, the

$$length(\gamma)=\int_I\sqrt{g(\gamma'(t),\gamma'(t))}dt.$$

If R is a region of M, so that $R\subseteq\phi(U)$ of a chart,

$$vol(R)=\int_{\phi^{-1}(R)}\sqrt{\det(g_{ij})}dx_1...dx_n.$$

If R is not contained in a single chart, use a partition of unity.

Example 2.4. On $\mathbb{R}^n$, for $X,Y\in T_x\mathbb{R}^n=\mathbb{R}^n$, $g_x(X,Y)=X\cdot Y$ is called the Eucliean metric.

If $M^n\subseteq\mathbb{R}^{n+k}$ is a manifold, $g_p(X,Y)=X\cdot Y$ is called induced metric.

$S^n=\left\{x\in\mathbb{R}^{n+1}:|x|=1\right\}\subseteq\mathbb{R}^{n+1}$ then g_{s^n} is the induced metric on S^n.

$B^n=\{x\in\mathbb{R}^n:|x|<1\}$ then $g_{\mathbb{H}^n}(X,Y)=\frac{4}{(1-|x|^2)^2}X\cdot Y$ is called the hyperbolic metric

Definition 2.5. $F:(M,g)\to(N,h)$ is an isometry if

(1) F is a local diffeomorphism
(2) $g(X,Y)=h((dF)(X),(dF)(Y))$ for all $X,Y\in\mathfrak{X}(M)$

Example 2.6. Any two curves of the same length are isometric.

$\{(x,y) : y > 0\}$ is isometric to a cone (locally)

Isometries of ($\mathbb{R}^n$, Euclidean Metric) are rigid motions

Isometries of $(S^n, g_{s^n}) = \mathcal{O}(n+1)$. For $A \in O(n+1)$,

$$Ax \cdot Ax = \left(A^T A\right) x \cdot x = x \cdot x = 1$$

for all $x \in S^n$. Therefore, A is an isometry of S^n.

Isometries of $(\mathbb{H}^n, g_{\mathbb{H}^n})$ are comformal maps from B^n to B^n. $F : B^n \to B^n$ is comformal if $(dF)_x(X) \cdot (dF)_x(Y) = \lambda(x) X \cdot Y$.

Definition 2.7. A connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ so that

(1) $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$ where $f \in C^\infty(M)$
(2) $\nabla_X (fY + Z) = X(f)Y + f\nabla_X Y + \nabla_X Z$

A connection is symmetric if $\nabla_X Y - \nabla_Y X = [X, Y]$.

A connection is compatible with metric g if for all $X, Y, Z \in \mathfrak{X}(M)$,

$$X\left(g(Y,Z)\right) = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right).$$

A connection compatible with g and symmetric is called a Levi-Civita connection.

In coordinates (U, φ),

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k},$$

where Γ is called the Christoffel symbols. If $X = \sum a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_j b_j \frac{\partial}{\partial x_j}$, then

$$\begin{aligned}
\nabla_X Y &= \sum_i a_i \nabla_{\frac{\partial}{\partial x_i}} \left(\sum_j b_j \frac{\partial}{\partial x_j}\right) = \sum_{i,j} a_i b_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} \\
&= \sum_{i,j,k} a_i b_j \Gamma_{i,j}^k \frac{\partial}{\partial x_k} + \sum_j X(b_j) \frac{\partial}{\partial x_j} \\
&= \sum_{i,j,k} a_i b_j \Gamma_{ij}^k \frac{\partial}{\partial x_k} + \sum_k X(b_k) \frac{\partial}{\partial x_k}
\end{aligned}$$

Suppose $\gamma : I \to M$ is a curve, and $Y(t) \in T_{\gamma(t)}M$. If γ has self intersection, $\gamma(t_1) = \gamma(t_2)$, Y may take different values there. $\nabla_{\gamma'(t)} Y$ is well-defined from the above expression, because the first term just need values. The second part, we just need that the first vector field X to be differentiable along the integral curves defined by Y.

Theorem 2.8. *Levi-Civita connections exist and are unique.*

Proof. We have

$$\begin{aligned}
X\left(\langle Y, Z\rangle\right) &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\
Y\left(\langle Z, X\rangle\right) &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\
Z\left(\langle X, Y\rangle\right) &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle
\end{aligned}$$

Now add the first two, and subtract the last, to get (using symmetricity)

$$X\left(\langle Y, Z\rangle\right) + Y\left(\langle Z, X\rangle\right) - Z\left(\langle X, Y\rangle\right) = 2\langle \nabla_X Y, Z\rangle + \langle Z, [Y, X]\rangle + \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle.$$

Therefore,

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle\right) - \langle Z, [Y, X]\rangle - \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle,$$

which defines it. For a chart (U, φ), then by combining symmetry and Christofell symbols,

$$\left\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle = \sum_{\ell} \Gamma_{ij}^{\ell} \cdot g_{k\ell} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij} \right).$$

If g^{ij} is $\left(g_{ij}^{-1}\right)_{ij}$, then

$$\Gamma_{ij}^{\ell} = \sum_{k} \frac{g^{k\ell}}{2} \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij} \right).$$

□

Example 2.9. On $(\mathbb{R}^n, \text{Euclidean})$, $D_X Y = (X(y_1), ..., X(y_n)) = (\langle X, Dy_1 \rangle, ..., \langle X, Dy_n \rangle)$ where $Y = (y_1, ..., y_n)$.

For $M^n \subseteq \mathbb{R}^{n+k}$, $X, Y \in \mathfrak{X}(M)$,

$$\nabla_X Y = (D_X Y)^T$$

induced connection (the T is the projection to the tangent space, as a subspace of $\mathbb{R}^{n+k}$).

Definition 2.10. Let (M, g) be a Riemannian manifold. A curve $\gamma : I \to M$ is a geodesic if $\nabla_{\gamma'(t)} \gamma'(t) = 0$ for all $t \in I$ (its acceleration is constant).

Example 2.11. On $\mathbb{R}^n$ with the Euclidean metric, a curve $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$ and so $\gamma'(t) = (\gamma_1'(t), ..., \gamma_n'(t))$. Then

$$\nabla_{\gamma'(t)} \gamma'(t) = D_{\gamma'(t)} \gamma'(t) = (\gamma_1''(t), ..., \gamma_n''(t)) = \gamma''(t).$$

Therefore, $\gamma \subseteq \mathbb{R}^n$ is a geodesic iff $\gamma''(t) = 0$ for all t. That is $\gamma(t) = a + tv$ for $a, v \in \mathbb{R}^n$.

Remark 2.12. If γ is a geodesic, then $g(\gamma'(t), \gamma'(t)) = |\gamma'(t)|^2$ is constant.

Proof. We have

$$\frac{\partial}{\partial t} (g(\gamma'(t), \gamma'(t))) = g(\nabla_{\gamma'} \gamma', \gamma') + g(\gamma', \nabla_{\gamma'} \gamma') = 0.$$

□

Example 2.13. Let $M^n \subseteq \mathbb{R}^{n+k}$ with induced metric. $\gamma : I \to M^n$ is a geodesic iff

$$0 = \nabla_{\gamma'} \gamma' = (D_{\gamma'} \gamma')^T = (\gamma'')^T$$

has no tangential acceleration.

Example 2.14. If $M^n = S^n \subseteq \mathbb{R}^{n+1}$. Let P be a 2-plane through the origin. Let $\gamma = S^n \cap P$ (a circle) is then a geodesic (if parametrized by constant speed).

Can check this via the connections. Or

(1) $\gamma'' \in P$ (all the other components are zero)
(2) $\gamma'' \cdot \gamma' = 0$. Since $|\gamma'(t)| = 1$, take derivatives, to get that this must be 0.
(3) $\eta(p)$= the normal vector to S^n at p is p at all points of S^n. $\eta(\gamma(t)) = \gamma \in P$

Then $\eta(\gamma(t))$ is parallel to $\gamma''(t)$. Therefore, $(\gamma'')^T = 0$ and so it's a geodesic.

Example 2.15. Let $M^2 = \left\{a^2x^2 + b^2y^2 + c^2z^2 = 1\right\} \subseteq \mathbb{R}^3$. If P is a coordinate plane (ie, the xy, yz or xz plane). Let $\gamma = P \cap M^2$, then γ is a geodesic.

Do a similar thing, and not that reflections are isometries (they are isometries of $\mathbb{R}^3$ and we are using induced metric).

Fact 2.16. *Every closed surface has infinite number of geodesics.*

Example 2.17. On $(\mathbb{H}^n, g_{\mathbb{H}^n}) = \left(B^n, \frac{4}{(1-|x|^2)^2} \delta_{ij}\right)$. Geodescis are planar circles that intersect $\partial B^n = S^n$ orthogonally. These have finite length for Euclidean metric, but it's infinite length for this metric, because the $\frac{1}{1-|x|^2}$ blows up as you approach the boundary.

(1) $\gamma(t) = tv$ for some v and $|t| < 1$. Then $\gamma(s) = \tanh sv$ for $s \in \mathbb{R}$.
If R is reflection across γ, then $R \in O(n)$. Hence, $R \in isom(\mathbb{H}^n)$. Now, is γ a geodesic. First, it's clear that $R\gamma = \gamma$.
Since R is an isometry, $(dR)(\nabla_{\gamma'}\gamma') = \nabla_{\gamma'}\gamma'$ (R is also linear... so $dR = R$). Hence, $D_{\gamma'}\gamma'$ is parallel to γ'. But these are always orthogonal, so $D_{\gamma'}\gamma' = 0$.

(2) If $\tilde{\gamma}$ is a circle intersecting ∂B^n orthogonally, then there exists $F \in conf(B^n) = isom(\mathbb{H}^n)$ such that $F(\gamma) = \gamma$ a curve from point 1.

Theorem 2.18. *Given $(p, v) \in TM$ for $v \in T_pM$, there exists U nbhd of $(\rho, v) \in TM$ and $\delta > 0$ so that for all $(x, z) \in U$, there exists unique geodesic $\gamma_{x,z} : (-\delta, \delta) \to M$ such that $\gamma_{x,z}(0) = x$ and $\gamma'_{x,z}(0) = z$.*

Proof. Let (V, φ) be a chart where $p = \varphi(0)$. Write $\gamma(t) = (x_1(t), ..., x_n(t))$. Then

$$\begin{aligned}
\nabla_{\gamma'(t)}\gamma'(t) &= \nabla_{\gamma'(t)}\left(\sum_{i=1}^n x'_i \frac{\partial}{\partial x_i}\right) = \sum_i x''_i \frac{\partial}{\partial x_i} + \sum_i x'_i \nabla_{\gamma'}(\frac{\partial}{\partial x_i}) \\
&= \sum_i x''_i \frac{\partial}{\partial x_i} + \sum_{ij} x'_i x'_j \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \sum_i x''_i \frac{\partial}{\partial x_i} + \sum_{i,j,k} \Gamma^k_{ij} x'_i x'_j \frac{\partial}{\partial x_k} \\
&= \sum_k \left(x''_k + \sum_{i,j} x'_i x'_j \Gamma^k_{ij}\right) \frac{\partial}{\partial x_k}.
\end{aligned}$$

Therefore, $\nabla_{\gamma'}\gamma' = 0$ iff

$$x''_k + \sum_{i,j} x'_i x'_j \Gamma^k_{ij} = 0, \quad (\star)$$

Need to solve the equations

$$\begin{cases} x''_k + \sum_{i,j} x'_i x'_j \Gamma^k_{ij} = 0 \\ \gamma(0) = x \\ \gamma'(0) = z \end{cases}$$

Need to reduce this to first order ODE. Let $x'_k = y_k$, then

$$y'_k = \sum_{i,j} y_i y_j \Gamma^k_{ij}(x_1, ..., x_n) = 0, \qquad x(0) = (x_1, ..., x_n), \qquad y(0) = (z_1, ..., z_n).$$

Let $W(x, y) = \left(y, \left(-y_i y_j \Gamma^k_{ij}(x)\right)_{k=1}^n\right)$ is a vector field, so the equation is now

$$\begin{cases} (x', y') = W(x, y) \\ (x(0), y(0)) = (x, z) \end{cases}.$$

By the theorem that says vector fields can be integrated and solved locally, with continuous dependence on parameters. $\square$

Remark 2.19. If $\tilde{\gamma}(t) = \gamma_{p,v}(\epsilon t)$, then $\tilde{\gamma}$ is also a geodesic, because $\tilde{\gamma}' = \epsilon\gamma'$ and $\tilde{\gamma}(0) = p$ with $\tilde{\gamma}'(0) = \epsilon v$. Therefore, $\tilde{\gamma} = \gamma_{\rho,\epsilon v}$.

Corollary 2.20. *Given $p \in M$, there exists $\epsilon > 0$ and U nbhd of p so that for all $q \in U$, $v \in T_pM$, $|v| < \epsilon$, then $\gamma_{q,v}(t)$ is defined for $|t| < 2$.*

Proof. Use theorem with $v = 0$. Use remark to shrink the interval to get $(-2, 2)$. $\square$

Definition 2.21. $\exp_p : B_\epsilon(0) \subseteq T_pM \to M$ given by $\exp_p(v) = \gamma_{p,v}(1)$ is called the exponential map.

Remark 2.22. If $\gamma(t) = \exp_p(tv)$ with $|t| < 1$, then $v = \gamma'(0) = d\left(\exp_p\right)_0 v$.

Lemma 2.23. *(Gauss Lemma). For all $p \in M$, $v, w \in T_pM$, (assuming everything is well-defined),*

$$\left\langle d\left(\exp_p\right)_v(v), d\left(\exp_p\right)_v(w)\right\rangle = \langle v, w\rangle$$

Not quite an isometry.

Proof.

(1) w parallel to v, then

$$d\left(\exp_p\right)_v(v) = \frac{d}{dt}\left(\exp_p(v+tv)\right)_{t=0} = \frac{d}{dt}\left(\exp_p((1+t)v)\right)_{t=0} = \gamma'(1)$$

where $\gamma(t) = \exp_p(tv)$. Hence,

$$\left\langle d\exp_p(v), d\exp_p(v)\right\rangle = \langle\gamma'(1),\gamma'(1)\rangle = \langle\gamma'(0),\gamma'(0)\rangle = \langle v,v\rangle$$

where the second equality is because geodesics have constant speed.

(2) Now, suppose $w \perp v$. Let $w(s)$ be a curve in T_pM such that $w(0) = v$, $w'(0) = w$ and $|w(s)| = |v|$ for all $|s| < \delta$. Let $F : (-\delta,\delta)\times(0,1) \to M$ by

$$F(s,t) = \exp_p(tw(s)).$$

Check that

$$\left[\frac{\partial F}{\partial s},\frac{\partial F}{\partial t}\right] = \frac{\partial^2 F}{\partial t\partial s} - \frac{\partial^2 F}{\partial s\partial t} = 0.$$

See Do Carmo for more careful discussion.

$$\begin{aligned}\frac{\partial F}{\partial s}(0,t) &= (d\exp)_{tv}(tw) \text{ here, } w'(0) = w\\ \frac{\partial F}{\partial t}(0,t) &= (d\exp)_{tv}(v) \text{ using } w(0) = v.\end{aligned}$$

Then

$$\frac{\partial}{\partial t}\left\langle (d\exp)_{tv}(tw), (d\exp)_{tv}(v)\right\rangle|_{(0,t)} = 0.$$

This is also equal to

$$\frac{\partial}{\partial t}\left\langle\frac{\partial F}{\partial s},\frac{\partial F}{\partial t}\right\rangle|_{(0,t)} = \left\langle\nabla_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial s},\frac{\partial F}{\partial t}\right\rangle + \left\langle\frac{\partial F}{\partial s},\nabla_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial s}\right\rangle.$$

The second of the last expression is 0 because $t\mapsto F(s,t)$ is a geodesic for all s.

$$\left\langle\nabla_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial s},\frac{\partial F}{\partial t}\right\rangle = \frac{\partial}{\partial s}\frac{\left\langle\frac{\partial F}{\partial t},\frac{\partial F}{\partial t}\right\rangle}{2} = \frac{\partial}{\partial s}\langle w(s),w(s)\rangle = 0.$$

because $t\mapsto\left\langle\frac{\partial F}{\partial t}(t,s),\frac{\partial F}{\partial t}(t,s)\right\rangle$ is constant, and equal to $\left\langle\frac{\partial F}{\partial t}(0,s),\frac{\partial F}{\partial t}(0,s)\right\rangle = \langle w(s),w(s)\rangle$. Therefore,

$$\begin{aligned}\left\langle (d\exp)_{tv}(w),(d\exp)_{tv}(v)\right\rangle &= \left\langle\frac{\partial F}{\partial t}(0,1),\frac{\partial F}{\partial s}(0,1)\right\rangle = \left\langle\frac{\partial F}{\partial t}(0,0),\frac{\partial F}{\partial s}(0,0)\right\rangle\\ &= \left\langle\frac{\partial F}{\partial t}(0,0),0\right\rangle = 0.\end{aligned}$$

Recall: for all $p\in M$, there exists $\epsilon > 0$ and a neighbourhood U of p such that for all $q\in U$,

$$\exp_q : B_\epsilon(0)\subseteq T_qM\to M$$

is well-defined, and

$$\frac{d}{dt}\exp_q(tv)|_{t=0} = v.$$

□

Proposition 2.24. *For all $p\in M$, there exists $\tilde{U}$ neighbourhood of p, $\epsilon > 0$ such that*

(1) *For all $q\in\tilde{U}$, $\exp_q : B_\epsilon(0)\subseteq T_qM\to M$ is a diffeomorphism onto its image*
(2) *$\tilde{U}\subseteq\exp_q(B_\epsilon(0))$ for all $q\in\tilde{U}$*

Definition 2.25. Neighbourhood like the one in the proposition are called totally normal neighbourhoods.

Remark 2.26. $\left(B_\epsilon(0), \exp_q\right)$ is a chart, has the property that the metric that $(g_{ij})(0) = Id = \delta_{ij}$, and $\partial_k g_{ij}(0) = 0$ for all $i, j, k = 1, ..., n$. That is, the chart is Euclidean up to first order at the origin. They are called normal coordinates.

Proof. $F : U \times B_\epsilon(0) \to M \times M$ given by $(x, v) \mapsto (x, \exp_x(v)) = \left(x, \exp_x\left(\sum_i v_i \frac{\partial}{\partial x_i}\right)\right)$.

Then

$$(dF)_{(x,0)} = \begin{pmatrix} Id & Id \\ 0 & Id \end{pmatrix}.$$

Since

$$(dF)_{(x,0)}\left(\frac{\partial}{\partial x_i}, 0\right) = \frac{\partial}{\partial x_i} F(x, 0) = \frac{\partial}{\partial x_i}(x, x) = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right).$$

Similarly,

$$(dF)_{(x,0)}\left(0, \frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i} F\left(x, t\frac{\partial}{\partial x_i}\right) = \frac{d}{dt}\left(x, \exp_x(t\frac{\partial}{\partial x_i})\right)|_{t=0} = \left(0, \frac{\partial}{\partial x_i}\right).$$

For the second, Inverse function theorem, if we choose small ϵ and U, we get that $F : U \times B_\epsilon \to F(U \times B_\epsilon(0)) = V$ is a diffeomorphism. Choose $\tilde{U}$ such that $\tilde{U} \times \tilde{U} \subseteq V$ and this should be the one we want (check!). $\square$

Theorem 2.27. *Let U be a totally normal neighbourhood. Given $x, y \in U$, we have*

(1) *There exists a unqiue geodesic γ connecting x to y, with length$(\gamma) < \epsilon$*
(2) *If $\sigma : (0, 1) \to M$ is a curve with $\sigma(0) = x$ and $\sigma(1) = y$, then length$(\sigma) \geq$ length(γ)*
(3) *If $\sigma : (0, 1) \to M$ is a curve that is piecewise smooth, connects x to y with the same length as γ, then $\sigma = \gamma$*

Proof.

(1) Let $x, y \in U$, then $y = \exp_x(v)$ where $|v| < \epsilon$. $\gamma(t) = \exp_x(tv)$ is geodesic connecting x to y. If $\tilde{\gamma}$ is another geodesic, then let $\tilde{v} = \tilde{\gamma}'(0)$.

Claim. $|\tilde{v}| < \epsilon$.

Proof. We have

$$\epsilon > \text{length}(\tilde{\gamma}) = \int_0^1 |\tilde{\gamma}'(t)|\, dt = \int_0^1 |\tilde{\gamma}'(0)|\, dt = |\tilde{v}| \int_0^1 dt = |\tilde{v}|.$$

$\square$

Then, $\tilde{\gamma} = \exp_x(t\tilde{v})$ and $\tilde{\gamma}(1) = y$ and so $\exp_x(\tilde{v}) = \exp_x(v)$ and so $\tilde{v} = v$ by local diffeo.

(2) $\sigma \nsubseteq \exp_x(B_\epsilon(0))$. There exists $\bar{t}$ such that $\sigma(\bar{t}) = \exp_x(\bar{v})$ where $|\bar{v}| = \epsilon$, and $\sigma(t) \in \exp_x(B_\epsilon(0))$ for all $0 \leq t < \bar{t}$. Assume we proved the 2nd case, then

$$\sigma : (0, \bar{t}) \to M$$

connects x to $\exp_x(\bar{v})$ by the second case. We get if $\bar{\gamma}(t) = \exp_x(t\bar{v})$ then length$(\bar{\sigma}) \geq$ length$(\bar{\gamma}) = \int_0^1 |\bar{v}|\, dt = \epsilon$.
Second case: $\sigma \subseteq \exp_x\left(\bar{B}_\epsilon(0)\right)$.
$\sigma(t) = \exp_x(r(t)v(t))$ where $|v(t)| = |v|$ for all t. WLOG, assume $r(t) > 0$ for $0 < t \leq 1$. (this is polar coordinates).
Then $r(0) = 0$, $r(1) = 1$ and $v(1) = v$.
Check that length$(\sigma) \geq$ length$(\gamma) = |v|$. Then

$$\sigma'(t) = r'(d\exp_x)(v(t)) + d\exp_x(rv'(t)),$$

then by the Gauss lemma

$$\begin{aligned} \langle \sigma'(t), \sigma'(t) \rangle &= (r'(t))^2 |(d\exp_x)_{rv}(v(t))|^2 + 2r'r\langle (d\exp_x)_{rv} v, (d\exp_x)_{rv} v' \rangle + r^2 |d\exp_x(v)|^2 \\ &= |r'(t)|^2 |v(t)|^2 + 2r'r\langle v(t), v'(t) \rangle + (\cdot)^2 \\ &\geq |r'(t)|^2 |v|^2 \text{ by orthogonality} \end{aligned}$$

Thus,

$$\begin{aligned} length(\sigma) &= \int_0^1 |\sigma'(t)|\,dt \geq \int_0^1 |r'(t)|\cdot|v|\,dt = |v|\int_0^1 |r'(t)|\,dt \\ &\geq |v|\int_0^1 r'(t)dt = |v|\,(r(1)-r(0)) = |v| = \text{length}(\gamma). \end{aligned}$$

□

Theorem 2.28. *(Cartan) If (M,g) closed Riemannian manifold, and $\pi_1(M)\neq 0$, then there exists closed geodesic $\gamma:\,S^1\to M$ smooth.*

Remark 2.29. Theorem is true even if $\pi_1(M)=0$ (hard, by Birkhoff).

M^n is closed is crucial (in $\mathbb{R}^n$, there are no closed geodesics)

Open question does $\left(S^3,g\right)$ admit two distinct closed geodesics (not nec the same metric)

Proof. Let $[\sigma]\in\pi_1(M)$. Let

$$\ell = \inf\{\text{length}(\gamma):\,\gamma\in[\sigma]\}.$$

$\ell>0$ because if $\gamma:\,S^1\to M$ is a curve with length$(\gamma)<<1$, it can be contracted to a point. (HWK proves this)

Let $\gamma_i\in[\sigma]$ such that length$(\gamma_i)\to\ell$. Want "convergent" subsequence. Reduce from space of curves (∞ dimensional) to space of points in M (finite dimensional). Suppose $\gamma_i:\,S^1\to M$ have

$$|\gamma_i'(t)| = \text{costant in } t.$$

Given δ_0 small, break S^1 into n-intervals $I_1,...,I_N$ such that length$(\gamma_i\,|_{I_j})<\delta_0$ for all i large and $j\in 1,...,N$. Consider $\bar{\gamma}_{i,j}$ unique geodesic connecting the end points of $\gamma_i\left(I_j\right)$. Set $\bar{\gamma}_i$ to be the piece-wise geodesic.

$$\text{dist}\left(\bar{\gamma}_i(0),\gamma_i(0)\right)<2\delta_0$$

for all σ. Hence, (by the hwk again) $\bar{\gamma}_i\in[\sigma]$. Let $\Omega_i=\{\bar{\gamma}_i(\sigma_0),...,\bar{\gamma}_i(\sigma_{N-1})\}\subseteq M$ (the end points). There exists $\Omega=\{\rho_0,\rho_1,...,\rho_{N-1}\}$ such that $\Omega_i\to\Omega$. $\bar{\gamma}$ geodesic connecting them. Then

$$\bar{\gamma}_i\to\bar{\gamma}$$

and $\bar{\gamma}\in[\sigma]$.

$$\ell\leq\text{length}(\bar{\gamma})=\lim(\bar{\gamma}_i)\leq\lim\gamma_i=\ell.$$

Therefore, the length is exactly ℓ and it has to be smooth. □

3. Curvature

(M,g) a Riemannian manifold.

Definition 3.1. $R:\,\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$,

$$R(X,Y)Z=\nabla_X\left(\nabla_Y Z\right)-\nabla_Y\left(\nabla_X Z\right)-\nabla_{[X,Y]}Z.$$

- R is bilinear in all of its entries.

$$R\left(X+fT,Y\right)Z=R\left(X,Y\right)Z+fR\left(T,Y\right)Z$$

for all $f\in C^\infty(M)$. Similarly,

$$R\left(X,Y+fT\right)Z=R\left(X,Y\right)Z+fR(X,T)Z$$

Finally,

$$R\left(X,Y\right)\left(Z+fT\right)=R\left(X,Y\right)Z+fR(X,Y)T.$$

Proof. Check the third.

$$\begin{aligned}
R(X,Y)(fT) &= \nabla_X\nabla_Y f(T) - \nabla_Y\nabla_X(fT) - \nabla_{[X,Y]}(fT) \\
&= \nabla_X\left(Y(f)T + f\nabla_Y T\right) - \nabla_Y\left(X(f)T + f\nabla_X T\right) - [X,Y](f)T - f\nabla_{[X,Y]}T \\
&= X\left(Y(f)\right)T + Y(f)\nabla_X T + X(f)\nabla_Y T + f\nabla_X\nabla_Y T - Y\left(X(f)\right)T - X(f)\nabla_Y T \\
&- Y(f)\nabla_X T - f\nabla_Y\nabla_X T - [X,Y](f)T - f\nabla_{[X,Y]}T \\
&= f\left(\nabla_X\nabla_Y T - \nabla_Y\nabla_X T - \nabla_{[X,Y]}T\right) = fR(X,Y)(T).
\end{aligned}$$

□

Definition 3.2. The curvature tensor is $R:\mathfrak{X}(M)\times\mathfrak{X}(M)\times X(M)\times\mathfrak{X}(M)\to C^\infty(M)$ given by

$$R(X,Y,Z,W) = g\left(R(X,Y)Z,W\right).$$

Supose $T:\underbrace{\mathfrak{X}(M)\times...\times\mathfrak{X}(M)}_{r}\to C^\infty(M)$ is a r-tensor, if T is linear in all its entries.

Key property: $T(X_1,...,X_r)(p)$ only depends on $X_1,...,X_r$ at the point p (not of T)

Check: suffices to see that if $X_j(p)=0$, then $T\left(X_1,...,X_r\right)(p)=0$. Take $j=1$. $X_1=\sum a_i\frac{\partial}{\partial x_i}$ where $a_i(p)=0$ for all i. Then

$$T\left(X_1,...,X_r\right)(p) = \sum_i^n a_i T\left(\frac{\partial}{\partial x_1},X_2,...,X_r\right)(p) = 0.$$

Example 3.3. $T:\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to C^\infty(M)$ given by $T(X,Y,Z)=\langle\nabla_X Y,Z\rangle$ is not a tensor. (not linear in Y).

In coordinates, the curvature tensor is given by:

$$\begin{aligned}
R_{i,j,k,\ell} &= R\left(\frac{\partial}{\partial x_i},...,\frac{\partial}{\partial x_\ell}\right) = \left\langle \nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k} - \nabla_{\left[\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right]}\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_\ell}\right\rangle \\
&= \left\langle \nabla_{\partial x_i}\left(\sum_p \Gamma^p_{jk}\frac{\partial}{\partial x_k}\right) - \nabla_{\partial x_j}\left(\Gamma^m_{ik}\frac{\partial}{\partial x_m}\right),\frac{\partial}{\partial x_\ell}\right\rangle \text{ here, } \left[\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right]=0 \\
&= \sum_p \partial_i\Gamma^p_{jk}g_{p\ell} - \sum_p \partial_j\Gamma^p_{ik}g_{p\ell} + \sum_{p,s}\Gamma^p_{jk}\Gamma^s_{ip}g_{s\ell} - \sum \Gamma^m_{ik}\Gamma^s_{jm}g_{s\ell}
\end{aligned}$$

Properties:

1. $R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)=0$ The first Bianchi identity
2. $R(X,Y,Z,W)=-R(Y,X,Z,W)$
3. $R(X,Y,Z,W)=R(Z,W,X,Y)$
4. $R(X,Y,Z,W)=-R(X,Y,W,Z)$

Proof. The first follows from Jacobi identity. Will check that

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y+[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

The second follows from the fact that $R(X,Y)Z=-R(Y,X)Z$.

$$\begin{aligned}
R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W) &= 0 \\
R(Y,Z,W,X)+R(Z,W,Y,X)+R(W,Y,Z,X) &= 0 \\
R(Z,W,X,Y)+R(W,X,Z,Y)+R(X,Z,W,Y) &= 0 \\
R(W,X,Y,Z)+R(X,Y,W,Z)+R(Y,W,X,Z) &= 0.
\end{aligned}$$

Add all of these up. .... then use last property.

For the last, suffice to check that $R(X,Y,Z,Z)=0$ for all X,Y,Z. Use identity above with $Z+W$ instead of Z. Use $\left\{\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n}\right\}$ are normal coordinates. Then $g_{ij}(0)=\delta_{ij}$ and $\nabla_{\partial x_i}\partial x_j(0)=0$. Need to check that $R_{i,j,n,n}=0$. WLOG, $\frac{\partial}{\partial x_n}(p)=\frac{z}{|z|}(p)$ (choose k).

$$\begin{aligned} R_{i,j,n,n} &= \left\langle \nabla_{\partial_i}\nabla_{\partial_j}\partial_n,\partial_n\right\rangle - \left\langle \nabla_{\partial_j}\nabla_{\partial_i}\partial_n,\partial_n\right\rangle \\ &= \partial_i\left\langle \nabla_{\partial_j}\partial_n,\partial_n\right\rangle - \left\langle \nabla_{\partial_j}\partial_n,\nabla_{\partial_i}\partial_n\right\rangle - \partial_j\left\langle \nabla_{\partial_i}\partial_n,\partial_n\right\rangle + \left\langle \nabla_{\partial_i}\partial_n,\nabla_{\partial_j}\partial_n\right\rangle \\ &= \partial_i\left\langle \nabla_{\partial_j}\partial_n,\partial_n\right\rangle - \partial_j\left\langle \nabla_{\partial_i}\partial_n,\partial_n\right\rangle \\ &= \partial_i\partial_j\frac{g_{nn}}{2} - \partial_j\partial_i\frac{g_{nn}}{2} = 0, \end{aligned}$$

because $\partial_i\partial_j=\partial_j\partial_i$. □

Definition 3.4. Given Q a 2-plane in T_pM, sectional curvative of $Q=K(Q)(p)=R(e_1,e_2,e_2,e_1)$ where $\{e_1,e_2\}$ is an orthonormal basis of Q. Or,

$$\frac{R(u,v,v,u)}{|u\wedge v|^2}$$

if $Q=span\{u,v\}$. Here, $|u\wedge v|^2=|u|^2|v|^2-\langle u,v\rangle^2$. (Professor originally defined this as $R(u,v,u,v)$, which is wrong).

Lemma 3.5. *The sectional curvature determine the curvature tensor.*

Proof. Choose coordinates (φ,U).

$$\begin{aligned} f(\alpha,\beta) &= R\left(\partial_{x_i}+\alpha\partial_{x_k},\partial_{x_j}+\beta\partial_{x_\ell},\partial_{x_i}+\alpha\partial_{x_k},\partial_{x_j}+\beta\partial_{x_\ell}\right) \\ &- R\left(\partial_{x_i}+\alpha\partial_{x_\ell},\partial_{x_j}+\beta\partial_{x_k},\partial_{x_i}+\alpha\partial_{x_\ell},\partial_{x_j}+\beta\partial_{x_\ell}\right). \end{aligned}$$

If $Q^1_{\alpha,\beta}=span\left\{\partial_{x_i}+\alpha\partial_{x_k},\partial_{x_j}+\beta\partial_{x_\ell}\right\}$ then

$$f(\alpha,\beta)=K\left(Q^1_{\alpha,\beta}\right)\left|(\partial_{x_i}+\alpha\partial_{x_k})\wedge(\partial_{x_k}+\beta\partial_{x_\ell})\right|^2-...$$

Check that

$$\frac{\partial^2 f}{\partial\alpha\partial\beta}=6R_{i,j,k,\ell}.$$

Therefore, f determine R. Therefore, sectional curvatures determine R. □

Corollary 3.6. *If $K(Q)=c(p)$ for all $p\in M$, $Q\subseteq T_pM$ (depends only on the point, not the plane) then*

$$R(X,Y,Z,W)=c(p)\left(\langle X,W\rangle\langle Y,Z\rangle-\langle X,Z\rangle\langle Y,W\rangle\right).$$

Example 3.7.

(1) For $(\mathbb{R}^n,\text{Euclidean})$ then $R(X,Y,Z,W)=0$ and so sectional curvatures are zero.
(2) $(\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n,\text{Euclidean})$ also has sectional curvature zero.
(3) If $(M,\mathfrak{g})=(S^n,\mathfrak{g}_{sphere})$ or $(\mathbb{H}^n,\mathfrak{g}_{\mathbb{H}^n})$, then for every $x,y\in M$ and every Q_x,Q_y 2-planes $\subseteq T_xM$ and T_yM then there is an isometry $A\in Isom(M^n)$ such that $Ax=y$ and $dA(Q_x)=Q_y$. Therefore, the sectional curvature is the same at every point. They are ±1 respectively.

Definition 3.8. If T is a r-tensor, then ∇T is a $(r+1)$-tensor, given by

$$(\nabla T)(X,Y_1,...,Y_r)=X(T(Y_1,...,Y_r))-\sum_{i=1}^n T(Y_1,...,\nabla_X Y_i,...,Y_r).$$

We write $\nabla_X T(...)=\nabla T(X,...)$. This last term is so that Leibniz rule holds.

Example 3.9. $\nabla g\equiv 0$ because $\nabla_X g(Y,Z)=X(g(Y,Z))-g(\nabla_X Y,Z)-g(Y,\nabla_X Z)=0$, so it is always parallel.

Proposition 3.10. *Second Bianchi identity.*

$$\nabla_X R(Y,Z,T,W)+\nabla_Y R(Z,X,T,W)+\nabla_Z R(X,Y,T,W)=0$$

for all $X, Y, T, W, Z \in \mathfrak{X}(M)$.

Proof. Use normal coordinates $\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right\}$. Check this for a fixed point, 0. We know $g_{ij}(0) = \delta_{ij}$ and $\nabla_{\partial_i}\partial_j(0) = 0$.

(1) $\nabla_{\partial_k} R_{i,j,\ell,m} = \partial_k R_{i,j,\ell,m} - R \star \nabla_{\partial_a}\partial_b(0)$. (the second term is curvature times some Christopher symbols. This is zero, so...)

$$\begin{aligned} &= \partial_k \left(\left\langle \nabla_{\partial_i}\nabla_{\partial_j}\partial_\ell, \partial_m\right\rangle - \left\langle \nabla_{\partial_j}\nabla_{\partial_i}\partial_\ell, \partial_m\right\rangle\right)(0) \\ &= \left\langle \nabla_{\partial_k}\nabla_{\partial_i}\nabla_{\partial_j}\partial_\ell, \partial_m\right\rangle - \left\langle \nabla_{\partial_k}\nabla_{\partial_j}\nabla_{\partial_i}\partial_\ell, \partial_m\right\rangle \text{ skipped zero terms} \end{aligned}$$

(2) $\nabla_{\partial_k}\nabla_{\partial_i}\left(\nabla_{\partial_i}\partial_\ell\right) = \nabla_{\partial_i}\nabla_{\partial_k}\left(\nabla_{\partial_i}\partial_\ell\right) + R\left(\partial_k, \partial_i\right)\left(\nabla_{\partial_i}\partial_\ell\right)$ (another term of Lie bracket, which is 0 for all points, not just at 0). Notice that the curvature term is once again 0. This is abusing the fact that the 0 point is really Euclidean again.

These two facts implies Bianchi identity. □

Why the identities. $X \in \mathfrak{X}(M)$, $\{\phi_t\}$ such that $\frac{\partial \phi_t}{\partial t} = X(\phi_t)$. The Jacobi identity says

$$\left(\phi_t\right)_\star [Y, Z] = \left[\left(\phi_t\right)_\star Y, \left(\phi_t\right)_\star Z\right].$$

Differentiate the both sides, to get Lie derivative

$$\mathfrak{L}_X [Y, Z] = \left[\mathfrak{L}_X Y, Z\right] + \left[Y, \mathfrak{L}_X Z\right].$$

Where, $\mathfrak{L}_X W = [X, W]$. So we get Jacobi identity from this.

First and second Bianchi gives (maybe $-t$)

$$g_t = \left(\phi_t\right)^\star g$$

and so $\phi_t : (M, g_t) \to (M, g)$ is an isometry. So $R(g_t) = \left(\phi_t\right)^\star R(g)$. Differentiate both with respect to t. Then

$$\frac{d}{dt} R(g_t) = \frac{d}{dt}\left(\phi_t\right)^\star R(g) = \mathfrak{L}_X\left(R(g)\right) = \nabla R \star X + R \star \nabla X$$

(RHS is some expression like that). On the other hand, if we let $h = \frac{dg_t}{dt}$.

$$\frac{d}{dt} R(g_t) = (DR)(h) = \partial^3 h + R \star h = \nabla R \star X + R \star \nabla X.$$

The RHS of the first expression will be true for all R. The second has to be only specific to our R, giving us different expressions.

Pick $p \in M$ and choose X such that $X(p) = 0$ and $\nabla_{\partial_i} X_j(0) = \delta_{ij}$. Then ∇R is zero, and $R \star \nabla X$ will be some expression depending on R. This will give first Bianchi identity.

Choose X such that $X(p) = e_1$, then $\left(\nabla_{\partial_i} X\right)(p) = 0$. Do the same thing to get second Bianchi identity.

Definition 3.11. Define $Ric(g) : T_pM \times T_pM \to \mathbb{R}$ given by

$$Ric(X, Y) = \sum_{i=1}^n R\left(X, e_i, e_i, Y\right)(p)$$

where $\{e_i\}_{i=1}^n$ some o.n. basis of T_pM. This is called Ricci Tensor.

Fact 3.12. *Ric is symmetric. That is,* $Ric(X, Y) = Ric(Y, X)$ *and bilinear.*

Definition 3.13. Scalar Curvature $s(g) : M \to \mathbb{R}$

$$s(g)(x) = \sum_{i=1}^n Ric\left(e_i, e_i\right)(x) = \sum_{i,j=1}^n R\left(e_i, e_j, e_j, e_i\right)$$

where e_i and e_j are in the same o.n. basis.

Example 3.14. Let M be a surface. $K(x) = K(T_xM)$ called the Gaussian curvature (only one choice of a 2-plane). Then

$$R(X,Y,Z,W) = K(x)(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle).$$

When we do

$$Ric(X,Y) = K(x)\langle X,Y\rangle.$$

The scalar curvature is then

$$S(g)(x) = 2K(x)$$

(the trace).

Theorem 3.15. *Cartan Theorem. If (M^n,g) is closed has constant sectional curvature. That is, $K(P^2)(x) = c_0$ is constant for all $P^2 \subseteq T_xM$ and $x \in M$. Then the universal cover $\left(\tilde{M}^n,\tilde{g}\right)$ of (M^n,g) is isometric to*

$$\begin{cases} (S^n, g_{round}) & c_0 = 1 \\ (\mathbb{R}^n, Euclid) & c_0 = 0 \\ (\mathbb{H}^n, g_{\mathbb{H}^n}) & c_0 = -1 \end{cases}.$$

Basically, if $c_0 \neq 0,1,-1$, we increase or decrease the sphere or $\mathbb{H}^n$ to make the curvature go down or up. Therefore, M is S^n/Γ, $\mathbb{R}^n/\Gamma$ or $\mathbb{H}^n/\Gamma$.

Theorem 3.16. *When $n=2$, have uniformization theorem. $\left(M^2,g\right)$ closed surface. Then there exists $u \in C^\infty(M)$ such that $K\left(e^{2u}g\right) = c_0$ constant.*

Combine this with the above theorem, to get that if M^2 is orientable,

$$\left(M^2,g\right) = \begin{cases} S^2 & c_0 = 1 \\ \mathbb{T}^2 & c_0 = 0 \\ \mathbb{H}^2/\Gamma & c_0 = -1 \end{cases}.$$

Proof. $g_t = e^{2tu}g$ for $0 \le t \le 1$. Then $g_0 = g$ and $g_1 = \bar{g}$. Then

$$\begin{aligned} \frac{d}{dt}K(g_t)dV_{g_t} &= \frac{d}{dt}K(g_t)dV + K(g_t)\frac{d}{dt}V_{g_t} \\ &= \frac{d}{dt}K(g_t)dV + K(g_t)2udV_{g_t} \\ &= div\,(....)\,dV_{g_t}. \end{aligned}$$

We know that $dV_{g_t} = e^{2tu}dV$ and so $\frac{d}{dt}dV_{g_t} = 2udVg_t$. This gives second line.

This implies that

$$\frac{d}{dt}\int K(g_t)dV_{g_t} = \int div\,(\cdot)\,dV_t = 0.$$

See assignment 3 to see what it is the divergence of. □

Remark 3.17. The LHS without the $\frac{d}{dt}$ is Gauss-Bonet (for M closed, $\int_M K(g)dV_g = 2\pi\chi(M)$), which gives you the genus ($\chi = 2-2g$ if compact orientable).

$$\int K(g)dV_g = \int K(\bar{g})d\bar{V} = \begin{cases} 4\pi & \text{if } c_0 = 1 \\ 0 & \text{if } c_0 = 0 \\ 4\pi(1-\delta) & \text{if } c_0 = -1 \end{cases}.$$

Definition 3.18. Suppose we have constant sectional metric.

- g is Einstein if $Ric(g) = \Lambda g$
- g has constant scalar curvature if $S(g) = constant$

Awesome fact: when $n = 3$, Einstein implies constant sectional curvature.

Theorem 3.19. *Perelman. If (M^3, g) is closed and simply connected, then there exists $\bar{g}$ an Einstein metric on M^3. Hence, has constant sectional curvature, and so*

$$M^3 = \begin{cases} S^3 \\ \mathbb{R}^3 \\ \mathbb{H}^3 \end{cases}.$$

By closed, $M^3 = S^3$.

Constant scalar curvature $\Longleftarrow$ Einstein $\Leftarrow$ constant sectional curvature.

When $n = 2$, they are all the same. When $n = 3$, have backwards of the second arrow.

Example 3.20. $(S^1 \times S^2, d\sigma^2 + g_{S^2})$ has universal cover $\mathbb{R} \times S^2$, which is not the above. This is constant, but not Einstein.

$(S^2 \times S^2, g_{S^2} + g_{S^2})$ does not admit constant scalar curvature, cuz simply connected. This is Einstein not constant sectional curvature.

Open question. Is there an Eistein metric on S^n, which is not standard.

3.1. Relation of Parallel transport and curvature.

Definition 3.21. Parallel Transport. $\gamma : I \to M$, smooth curve. For all $V : I \to TM$, $V(t) \in T_{\gamma(t)}M$. V is parallel transport of $V(0)$, if $\nabla_{\gamma'} V = 0$.

Lemma 3.22. *V is unique.*

Proof. Choose (U, ψ) a chart. $\gamma(t) = (x_1(t), ..., x_n(t))$. $\gamma'(t) = \sum_i x_i' \frac{\partial}{\partial x_i}$. Suppose $V(0) = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, with $V(t) = \sum_i a_i(t) \frac{\partial}{\partial x_i}$. Then

$$\begin{aligned} 0 &= \nabla_{\gamma'} V = \sum_i a_i'(t) \frac{\partial}{\partial x_i} + \sum a_j \sum_i \nabla_{\gamma'} \frac{\partial}{\partial x_i} \\ &= \sum_k a_k' \frac{\partial}{\partial x_k} + \sum_{i,j} a_i x_j' \Gamma_{i,j}^k \frac{\partial}{\partial x_k}. \end{aligned}$$

This says

$$\begin{cases} a_k' + \sum_{i,j} a_i x_j' \Gamma_{ij}^k = 0 \\ a_k(0) = a_k \end{cases}$$

is a linear ODE, so solvable. $\square$

Example 3.23. $(\mathbb{R}^n, \text{Euclid})$, take a triangle. transport V along the triangle, then we get V again. This does not work on the sphere. Get picture from ADAN

Let $u, v \in T_pM$. Consider $\diamond$ the parallelogram generated by u and v. Let $P_t : T_pM \to T_pM$ by $P_t(X)$ =parallel transport of X along the scaled parallelogram $\exp_p(t\diamond)$. (Need to scale to be small enough to apply $\exp_p$). Take coordinate vectors to be safe.

Theorem 3.24. $P_t(x) = x + t^2 R(u, v)x + O(t^3)$.

3.2. Jacobi vector fields.

Let $p, q \in M^n$. Let $\Omega_{p,q} = \{\gamma : [0, 1] \to M, \gamma(0) = p, \gamma(1) = q\}$.

$$T_\gamma \Omega_{p,q} = \left\{X : [0, 1] \to TM : X(t) \in T_{\gamma(t)}M, X(0) = X(1) = 0\right\}.$$

Consider $F : [0,1] \times (-\epsilon, \epsilon) \to M$ with $F(t,s) = \exp_{\gamma(t)}(sX(t))$. We have $F(0,s) = \gamma(0)$ and $F(1,s) = \gamma(1)$ for all $|s| < \epsilon$. Then writing $\gamma_s = F(\cdot, s)$, then

$$X(t) = \frac{d}{ds}\gamma_s(t) = \frac{\partial F}{\partial s}(t,s).$$

$\mathscr{C}_{p,q} = \{\gamma \in \Omega_{p,q} : \gamma \text{ is a geodesic}\}$. What is $T_\gamma \mathscr{C}_{p,q}$?

$$T_\gamma \mathscr{C}_{p,q} = \{X \in T_\gamma \Omega_{p,q} : \nabla_{\gamma'} \nabla_{\gamma'} X + R(X, \gamma')\gamma' = 0\}.$$

We usually write it as $X'' + R(X, \gamma')\gamma' = 0$.

Let $(\gamma_s)_{|s|<\epsilon}$ be a one parameter family of geodesics with $\gamma_0 = \gamma$. Let $\gamma_s = F(\cdot, s)$. Set $X(t) = \frac{\partial F}{\partial s}(t, 0)$.

$$0 = \nabla_{\gamma_s'} \gamma_s' = \nabla_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial t}$$

for all s. Then

$$\begin{aligned} 0 &= \nabla_{\frac{\partial F}{\partial s}} \nabla_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial t} = \nabla_{\frac{\partial F}{\partial t}} \nabla_{\frac{\partial F}{\partial s}} \frac{\partial F}{\partial t} + R\left(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial t} \\ &= \nabla_{\frac{\partial F}{\partial t}} \nabla_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial s} + R\left(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial t} \\ &= \nabla_{\gamma'} \nabla_{\gamma'} X + R(X, \gamma')\gamma' \end{aligned}$$

which is what we wanted. Here, we have a lot of commutator terms that we ignored, because we can pick nice parametrizations.

Definition 3.25. γ a geodesic. $\mathfrak{J}$ is a Jacobi vector field if $\mathfrak{J}'' + R(\mathfrak{J}, \gamma')\gamma' = 0$ along γ.

Proposition 3.26.

(1) *Jacobi vector fields along γ with $\mathfrak{J}(0) = x$ and $\mathfrak{J}'(0) = Y$ exists and are unique*
(2) *If $\mathfrak{J}(0) = 0$ then $\mathfrak{J}(t) = \left(d\exp_p\right)_{t\gamma'(0)}(tY)$*
(3) *γ' and $t\gamma'$ are Jacobi vector fields*

Example 3.27. On (S^n, g_{S^n}) choose $p \in S^n$ where $u, v \in T_p S^n$ with $u \cdot v = 0$ and $|u| = |v| = 1$.

$$F : (0, \pi) \times (-\epsilon, \epsilon) \to S^n$$

by

$$F(t,s) = \cos tp + \sin t \frac{u + sv}{\sqrt{1+s^2}}.$$

Let $\mathfrak{J}(t) = \frac{\partial F}{\partial s}(t, 0)$. Can check that $\mathfrak{J}''(t) + \mathfrak{J}(t) = 0$ for all t. Thus,

$$R(\mathfrak{J}, \gamma')\gamma' = \mathfrak{J}.$$

Differentiate again. We knew the curvature of the sphere is constant, so taking $\frac{d}{dt}$, we get

$$R(\mathfrak{J}', \gamma')\gamma' = \mathfrak{J}'.$$

$\mathfrak{J}'(0) = v$ and $\gamma'(0) = u$ and so sectional curvature of *span* $\{u, v\} = 1$.

Example 3.28. On $\mathbb{H}^n$, we need

$$F(t,s) = \cosh tp + (\sinh t)\frac{u + sv}{\sqrt{1+s^2}} \in \mathbb{H}^3.$$

Argue similarly, implies that sectional curvature of $\mathbb{H}^n = -1$.

Example 3.29. Assume (M^n, g) has constant sectional curvature $\bar{K}$. Then $\mathfrak{J}(t)$ Jacobi vector field along γ with $\mathfrak{J}(0) = 0$ and $\mathfrak{J}'(0) = W$ is given by $\mathfrak{J}(t) = K(t)W(t)$ where $W(t)$ is parallel transport of W along γ.

$$K(t) = \begin{cases} \frac{\sin\sqrt{\bar{K}}t}{\sqrt{\bar{K}}} & \text{if } \bar{K} > 0 \\ t & \text{if } \bar{K} = 0 \\ \frac{\sinh\sqrt{\bar{K}}t}{\sqrt{\bar{K}}} & \text{if } \bar{K} < 0 \end{cases}$$

Proof. If $\mathfrak{J}(t) = K(t)W(t)$, then $\mathfrak{J}''(t) = K''(t)W = -\bar{K}\mathfrak{J}(t)$. Hence, $\mathfrak{J}'' + \bar{K}\mathfrak{J} = 0$. Use uniqueness. □

Remark 3.30. Get picture from Adan

Proof. of proposition

(1) γ geodesic $\{e_1, ..., e_n\}$ an orthonormal basis of $T_{\gamma(0)}M$ with $e_1 = \gamma'(0)$. $\{e_i(t)\}_{i=1}^n$ parallel transport of $\{e_i\}$. By uniqueness, $\gamma'(t) = e_1(t)$. Check that $\{e_1(t)\}$ is an orthonormal frame. Suppose $\mathfrak{J}(t) = \sum_i^n a_i(t)e_i(t)$.

$$\mathfrak{J}'(t) = \nabla_{\gamma'}\mathfrak{J}(t) = \sum_i a_i'(t)e_1(t) + \sum_i a_i(t)e_1'(t) = \sum a_1'(t)e_1(t).$$

Meanwhile,

$$\mathfrak{J}''(t) = \sum_i a_i''(t)e_i(t).$$

Then

$$\begin{aligned} 0 &= \mathfrak{J}'' + R(\mathfrak{J}, \gamma')\gamma' = \sum_i a_i''(t)e_i(t) + \sum_j a_j R(e_j, e_1)e_1 \\ &= \sum a_i'' e_i + \sum_{i,j} a_j \langle R(e_j, e_1)e_1, e_i\rangle e_i \\ &= \sum_i \left(a_i'' + \sum_j a_j R_{j11i} \right) e_i. \end{aligned}$$

This is a second order linear ODE equation, so has unique solution after specifying initial speed $a_i(0)$ and $a_i'(0)$.

(2) $F : [0,1] \times (-\epsilon, \epsilon) \to M$ given by

$$F(t,s) = \exp_p(t(\gamma'(0) + sy)).$$

Then $\gamma_s = F(\cdot, s)$ is a geodesic for all $|s| < \epsilon$. So $\mathfrak{J}(t) = \frac{\partial F}{\partial s}(t, 0)$ is a Jacobi vector field. Just need $\mathfrak{J}(0) = 0$ and $\mathfrak{J}'(0) = y$. Note that

$$\mathfrak{J}(t) = \left(d\exp_p\right)_{t\gamma'(0)}(ty).$$

$\mathfrak{J}(0) = \left(d\exp_p\right)_0(v) = v.$

$$\begin{aligned} \mathfrak{J}'(0) &= \nabla_{\gamma'(t)}\mathfrak{J}(t) = \nabla_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial s} = \nabla_{\frac{\partial F}{\partial t}}(d\exp)_{t\gamma'(0)}(ty) \\ &= \nabla_{\frac{\partial F}{\partial t}}\left(t\left(d\exp_{t\gamma'(0)}(y)\right)\right) \\ &= (d\exp)_{t\gamma'(0)}(y) + t\nabla_{\frac{\partial F}{\partial t}}\left(d\exp_p\right)_{t\gamma'(0)}(y). \end{aligned}$$

At $t = 0$, we get $y + 0(blah) = y$.

□

Definition 3.31. $\gamma(t_0)$ is conjugate point to $\gamma(0)$ along γ, if there exists $\mathfrak{J}$ a Jacobi vector field with $\mathfrak{J}(0) = \mathfrak{J}(t_0) = 0$.

Remark 3.32. This means that $\gamma(t_0)$ is conjugate to $\gamma(0)$ iff $t_0\gamma'(0)$ is a critical point for $\exp_{\gamma(0)}$. That is, $\ker(d\exp)_{t_0\gamma'(0)} \neq 0$. (by proposition).

If γ is a geodesic, $\mathfrak{J}(0) = 0$, $\mathfrak{J}'(0) = W$, then

$$\mathfrak{J}(t) = (d\exp)_{tv}(tW)$$

where $\gamma(t) = \exp(tv)$.

Remark 3.33. $\gamma(t_0)$ conjugates to $\gamma(0) \iff \ker\left(d\exp\right)_{t_0v} \neq \{0\} \iff \exp_p$ has a critical point at t_0v.

Lemma 3.34. *Fix x, there exists $\epsilon > 0$ such that if $y \in M$, there is $z \in M$ so that*

(1) $z \in \exp_x\left(\partial B_\epsilon\right)$
(2) $d(x,y) = d\left(x,z\right) + d\left(z,y\right)$

Proof. Choose z so that

$$d\left(z,y\right) = d\left(\exp(\partial B_\epsilon), y\right)$$

then $d\left(x,y\right) \le d\left(x,z\right) + d\left(z,y\right)$.

Take γ connecting x to y so that length(γ) is almost $d(x,y)$. There exists E so that $\gamma(E) \in \exp\left(\partial B_\epsilon\right)$. Since

$$\begin{aligned}\text{length}(\gamma) &= \text{length}(\gamma(0,E)) + \text{length}\left(\gamma(E,1)\right)\\ &\ge \epsilon + d(z,y) = d(x,z) + d(z,y).\end{aligned}$$

□

Theorem 3.35. *(Hopf-Rinow Theorem). TFAE:*

(1) *There exists $p \in M$ such that $\exp_p : T_pM \to M$ is defined everywhere*
(2) *Bounded and closed sets of M are compact*
(3) *Cauchy sequences converge*
(4) $\exp_q$ *is defined on T_qM for all $q \in M$ (geodesically complete)*

They also all imply that for all q, there exists γ geodesic connecting p to q where length$(\gamma) = d(p,q)$.

Proof. $1 \implies 5$. Given p, q. By the lemma, we can get a $z \in \exp_p\left(\partial B_\epsilon\right)$ satisfying.... Let $\gamma(t)$ be the geodesic of unit speed with $\gamma(\epsilon) = z$. Let $d = d(p,q)$ and

$$I = \left\{t \in [0,d] : \, d\left(\gamma(t),p\right) = d - t\right\}.$$

$I \neq \emptyset$ because $\epsilon \in I$. Let $T = \sup I$. Need to show that $T = d$ (which implies $d\left(\gamma(T),q\right) = 0$, so γ connects to q). I is closed by continuity. Suppose $T < d$. Apply the lemma, so

$$d\left(\gamma(T),q\right) = d\left(\gamma(T),z'\right) + d\left(z',q\right).$$

Want to argue that $z' = \gamma\left(T + \epsilon'\right)$. This will give the contradiction. Let σ be the curve along γ to $\gamma(T)$, and then to z', a broken geodesic.

Claim. length$(\sigma) = d\left(p,z'\right)$ which implies that σ is a simple geodesic and so $\sigma = \gamma$.

Proof. We know $d\left(\gamma,q\right) = d - T$ and $d\left(\gamma(T),q\right) = d\left(z',q\right) + d\left(z',\gamma(T)\right)$. Hence,

$$\begin{aligned}d\left(p,z'\right) &\ge d\left(p,q\right) - d\left(q,z'\right) = d - d\left(z',q\right)\\ &= d - \left(d - T - \epsilon'\right) = T + \epsilon' = \text{length}(\sigma).\end{aligned}$$

Therefore, length$(\sigma) = d\left(p,z'\right)$ □

□

Definition 3.36. Given (M^n, g) connected, define

$$d(p,q) = \inf\left\{\text{length}(\gamma) : \, \gamma \text{ connects } p \text{ to } q\right\}.$$

If σ is a piecewise smooth curve so that

- σ connects p to q

- $d(p,q) = \text{length}(\sigma)$

then σ is a geodesic (hence, completely smooth).

Corollary 3.37. *If M is compact (no boundary), then it's geodesically complete. If $M^n \subseteq \mathbb{R}^{n+k}$ is closed and complete, then it's geodesically complete.*

Example 3.38. of not complete, just take M-{a point}. Then can't extend over the point. Or Orbifold.

Theorem 3.39. *(M,g) simply connected, COMPLETE and constant sectional curvature, then (M,g) is isometric to*

$$\begin{cases} (\mathbb{R}^n, \mathit{Euclid}) & k=0 \\ (S^n, g_{S^n}) & k=1 \\ (\mathbb{H}^n, g_{\mathbb{H}^n}) & k=-1 \end{cases}$$

Remark 3.40. If g has constant sectional curvature $K \neq 0$, then $\frac{g}{|K|}$ has constant curvature ∓ 1 (so can scale to assume the above).

Proof. Suppose $k=0$. Want to show $g\left((d\exp)_v(X), (d\exp)_v(Y)\right) = \langle X, Y\rangle$ for all $X, Y \in T_pM$ and $v \in T_pM$. Let γ be geodesic corresponding to v. If X, Y are parallel to v, then this is just Gauss lemma. Suppose $X_1, X_2 \perp v$. Let $X_i(t)$ be the parallel transport of X_i along γ. $\mathfrak{J}_i(t)$ Jacobi vector fields with $\mathfrak{J}_i(0) = 0$ and $\mathfrak{J}_i'(0) = X_i(0)$.

$\mathfrak{J}_i(t) = tX_i(t)$ for all t because, in this case, the Jacobi vector field equation is $L'' = 0$ and both sides satisfy the equation and $(tX_i(0))'' = (X_i(t))' = 0$. They are both 0 at $t=0$ and derivative at 0 is the same.

$$\begin{aligned} g\left((d\exp)_v(X_1), (d\exp)_v(X_2)\right) &= g\left(\mathfrak{J}_1(1), \mathfrak{J}_2(2)\right) = g\left(X_1(1), X_2(1)\right) \\ &= g\left(X_1(0), X_2(0)\right) = \langle X_1, X_2\rangle \end{aligned}$$

When $k=-1$, the candidate for the isometry is then $F = \overline{\exp} \circ A \circ \exp^{-1}$ where exp is the exponential on M, $\overline{\exp}$ is that of $\mathbb{H}^n$ and A is $T_pM \to T_{\bar{p}}\mathbb{H}^n$.

$$\bar{g}\left((dF)_x(X_1), (dF)_x(X_2)\right) = g\left(X_1, X_2\right)$$

for all $X_1, X_2 \in T_xM$ and all $x \in M$. Let $\gamma = \exp(tv)$, and check that $\bar{\gamma} = \overline{\exp}\,(tAv)$. Once again, if parallel, this is again by Gauss lemma. Assume $X_1, X_2 \perp \gamma'(1)$. Let $X_i(t)$ be the parallel transport of X_i along γ. Let $\mathfrak{J}_i(t)$ be Jacobi vector fields with $\mathfrak{J}_i(0) = 0$ and $\mathfrak{J}_i'(0) = X_i(0)$. Then

$$\mathfrak{J}_i(t) = \sinh t X_i(t)$$

because both satisfies $X'' - X = 0$.

$$\mathfrak{J}_i(1) = (d\exp)_v\left(X_i(0)\right)$$

and combining the identities, we get

$$X_i = \frac{d\exp\left(X_i(0)\right)}{\sinh(1)}.$$

Hence,

$$(dF)(X_1) \quad = \quad (d\overline{\exp})dA(d\exp)^{-1}(X_1) = d\overline{\exp}dA\frac{X_1(0)}{\sinh(1)},$$

and so

$$\begin{aligned} \bar{g}\left((dF)(X_1), (dF)(X_2)\right) &= \frac{\bar{g}\left(d\overline{\exp}\left(dA(X_1(0))\right), d\overline{\exp}\left(dA(X_2(0))\right)\right)}{\sinh^2 1} \\ &= \bar{g}\left(\overline{dA\left(X_1(0)\right)(1)}, \overline{dA\left(X_2(0)\right)(1)}\right) \\ &= \bar{g}\left(dA(X_1(0), dA\left(X_2(0)\right)\right) \\ &= \bar{g}\left(X_1(0), X_2(0)\right) = g\left(X_1, X_2\right). \end{aligned}$$

where the first equality is by undo-ing what we do with hyperbolic metric.

$k=1$. We have the diagram (see Adan or Nacho). $F=\exp\circ A\overline{\exp}^{-1}$.

Claim. $F: S^n-\{\text{south pole}\}\to M^n$ is an isometry

Proof. $\bar{X}_i\in T_yS^n$ and P_t is parallel transport along γ or $\bar{\gamma}$.

They obey the identity

$$(d\exp)_{tv}(tZ)=\sin tP_t(Z).$$

$$\begin{aligned} X_i &= (dF)\left(\bar{X}_i\right)=d\exp dA\left(d\overline{\exp}\right)^{-1}\left(\bar{X}_i\right)=d\exp dA\left(d\overline{\exp}\right)^{-1}\left(P_1\left(\bar{X}_i(0)\right)\right)\\ &= d\exp dA\left(\frac{\bar{X}_i(0)}{\sin 1}\right)=\frac{1}{\sin 1}d\exp(dA)\bar{X}_i(0)\\ &= P_1\left(dA\left(\bar{X}_i(0)\right)\right)\end{aligned}$$

Since

$$\begin{aligned} g\left(dF(X_1),dF(X_2)\right) &= g\left(P_1\left(dA(\bar{X}_1(0))\right),P_1\left(dA(\bar{X}_2(0))\right)\right)\\ &= g\left(dA\left(\bar{X}_i(0)\right),dA\left(\bar{X}_2(0)\right)\right)=\left\langle\bar{X}_1(0),\bar{X}_2(0)\right\rangle\\ &= \left\langle P_1\left(\bar{X}_1(0)\right),P_1\left(\bar{X}_2(0)\right)\right\rangle=\left\langle\bar{X}_1,\bar{X}_2\right\rangle.\end{aligned}$$

where the last inequality is because A was chosen to be an isometry. ☐

☐

Theorem 3.41. *Hadamard Theorem. If (M^n,g) with sectional curvature ≤ 0 (not necessarily constant) and M^n simply connected, then $M^n\cong\mathbb{R}^n$ (diffeo, not isometric)*

Corollary 3.42. *If closed (M^n,g) has sectional curvature ≤ 0, then $\pi_k(M^n)=0$ if $k\ge 2$.*

Proof. If $\left(\tilde{M},g\right)$ is universal cover, then

$$\pi_k(M^n)=\pi_k\left(\tilde{M}\right)=\pi_k(\mathbb{R}^n)=0.$$

☐

If (M^n,g) has section curvature ≤ 0, then all topology is in $\pi_1(M^n)$.

Corollary 3.43. *$S^n,\mathbb{CP}^n$ or $S^n\times M^k$, $\mathbb{CP}^n\times M^k$ have no metric with non-positive sectional curvature.*

Theorem 3.44. *(Gao-Yau). S^3 has metric with negative Ricci curvature.*

Remark 3.45. This is because for non-surfaces Ricci curvature and sectional curvature are not the same. Also, will contradict Gauss-Bonnet with averages non-positive.

Proof. Idea: There exists a map $\gamma: S^1\to S^3$ the figure eight knot, so that $S^3-\gamma$ has a complete hyperbolic metric. Fill in a metric in tubular neighbourhood of the knot (which is a torus), so that the metric is complete, and the Ricci is always negative. ☐

Theorem 3.46. *(Locham) Every (M^n,g) with $n\ge 3$ has metric with Ricc <0.*

Theorem 3.47. *Every (M^n,g) with $n\ge 3$, has a metric with constant negative scalar curvature.*

Conjecture 3.48. *If n is large enough, can we put an Eisenstein metric with negative constant on any manifold.*

Proof. of Hadamard theorem.

(1) $\exp : T_pM^n \to M^n$ is a local diffeo

Suppose $\ker (d\exp)_v \neq \emptyset$. Then there exists $\mathfrak{J}$ a Jacobi vector field along $\gamma(t) = \exp(tv)$ where $\mathfrak{J}(0) = \mathfrak{J}(1) = 0$. This is because critical points of exponential are critical points, so they define Jacobi vector fields. Let $f(t) = \langle \mathfrak{J}(t), \mathfrak{J}(t) \rangle$. Then

$$f''(t) = (2\langle \mathfrak{J}', \mathfrak{J} \rangle)' = 2\langle \mathfrak{J}'', \mathfrak{J} \rangle + 2\langle \mathfrak{J}', \mathfrak{J}' \rangle .$$

Since $\mathfrak{J}'' + R(\mathfrak{J}, \mathfrak{J}')\mathfrak{J}' = 0$, we get

$$\begin{aligned} f''(t) &= -2R(\mathfrak{J}, \mathfrak{J}', \mathfrak{J}', \mathfrak{J}) + 2|\mathfrak{J}'|^2 \\ &= -|\mathfrak{J} \wedge \mathfrak{J}''|^2 K(span(\mathfrak{J}, \mathfrak{J}'\}) + 2|\mathfrak{J}'|^2 \\ &\geq 2|\mathfrak{J}'|^2 \geq 0. \end{aligned}$$

This means that f is a convex function. If $f'' \geq 0$, C^2 then $f \leq \max\{f(0), f(1)\}$. (draw a picture). But $f(0) = 0$ and $f(1) = 0$, so

$$|\mathfrak{J}(t)|^2 = 0$$

for all t and so $\mathfrak{J} = 0$.

(2) Show that exp is a convering map

Let $\bar{g} = (\exp)^\star g$. $\bar{g}$ is complete, because $t \mapsto \exp(tv)$ is a globally defined geodesic. exp is an isometry with respect to $\bar{g}$. Then exp map has unique lifting property.

These two facts plus the fact that $\pi_1(M) = 0$ implies that exp is a diffeomorphism. □

4. First and Second variation of energy

For $p, q \in M^n$,

$$\Omega_{p,q} = \{\gamma : [a, b] \to M, \gamma(a) = p, \gamma(b) = q, \text{ piece-wise smooth}\}.$$

Let $L : \Omega_{p,q} \to \mathbb{R}$ given by $L(\gamma) = \text{length}(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

Let the energy $E : \Omega_{p,q} \to \mathbb{R}$ is $E(\gamma) = \int_a^b |\gamma'(t)|^2\, dt$.

Lemma 4.1. *Then $L(\gamma)^2 \leq (b-a)\, E(\gamma)$ for all γ, with equality iff $|\gamma'(t)|$ is constant for all t*

Proof. We have

$$|L(\gamma)|^2 = \left(\int_a^b |\gamma'(t)|\, dt\right)^2 \leq (b-a)\int_a^b |\gamma'(t)|^2\, dt.$$

□

Lemma 4.2. *If $C : (a, b) \to M$ is a minimizing geodesic with $C \in \Omega_{p,q}$, then $E(\gamma) \geq E(C)$ for all $\gamma \in \Omega_{p,q}$. With equality iff γ is a geodesic*

Remark 4.3. Just because length(γ) = length(C) does not imply $|\gamma'(t)|$=constant.

Proof. We get

$$\begin{aligned} E(C) &= \int_a^b |C'(t)|^2\, dt = (b-a)\, L(C)^2 \\ &\leq \frac{1}{b-a} L(\gamma)^2 \leq E(\gamma). \end{aligned}$$

If equality holds, then γ is length minimizing and constant speed. Therefore, it's a geodesic. □

What are the critival points, say ∇E (but this is infinite dimensional).

If $\gamma \in \Omega_{p,q}$ and smooth on $[t_i, t_{i+1}]$, then

$$\nabla E = -2\nabla_{\gamma'}\gamma' - 2\sum_i \left(\gamma'(t_i^+) - \gamma'\left(t_i^-\right)\right)\delta_{\gamma(t_i)}.$$

$F:(-\epsilon,\epsilon)\times[a,b]\to M$ a C^2-map on $(-\epsilon,\epsilon)\times[t_i,t_{i+1}]$ and continuous everywhere, where $F(0,\cdot)=\gamma$. Let $\gamma_s=F(s,\cdot)$, and $V(t)=\frac{\partial F}{\partial s}(0,t)$

$$\frac{1}{2}\frac{\partial}{\partial s}E(\gamma_s)\mid_{s=0}=-\int_a^b\langle V,\nabla_{\gamma'}\gamma'\rangle\,dt-\sum_i\left(\gamma'(t_i^+)-\gamma'(t_i^-)\right)V(t_i)-\langle V(0),\gamma'(0)\rangle+\langle V(1),\gamma'(1)\rangle.$$

Proof. We have

$$\begin{aligned}\frac{d}{ds}E(\gamma_s) &= \frac{d}{ds}\int_a^b|\gamma_s'(t)|^2\,dt=\frac{d}{ds}\int_a^b\left|\frac{\partial F}{\partial t}(s,t)\right|^2dt\\ &= \int_a^b 2\left\langle\frac{\partial^2F}{\partial s\partial t}(0,t),\frac{\partial F}{\partial t}(0,t)\right\rangle dt=\int_a^b 2\left\langle\frac{\partial^2F}{\partial t\partial s},\frac{\partial F}{\partial t}\right\rangle dt\\ &= \int_a^b 2\partial_t\left\langle\frac{\partial F}{\partial s},\frac{\partial F}{\partial t}\right\rangle dt-\int_a^b 2\left\langle\frac{\partial F}{\partial s}(0,t),\nabla_{\frac{\partial F}{\partial s}}\frac{\partial F}{\partial t}\right\rangle dt\\ &= -2\int_a^b\langle v(t),\nabla_{\gamma'}\gamma'\rangle\,dt+2\sum_i\left\langle v(t_{i+1}),\gamma'(t_{i+1}^-)\right\rangle-2\sum_i\left\langle v(t_i),\gamma'(t_i^+)\right\rangle\end{aligned}$$

□

For $\gamma\in\Omega_{p,q}$, $T_\gamma\Omega_{p,q}$ is

$$\{V:(0,1)\to TM: V\text{ is continuous, }V(0)=V(1)=0,\ V(t)\in T_\gamma M\text{ piece-wise smooth}\}.$$

Additionally,

$$dE=-\nabla_{\gamma'}\gamma'-\sum_i\left(\gamma'(t_i^+)-\gamma'(t_i^-)\right)\delta_{\gamma(t_n)}.$$

That is for all $X\in T_\gamma\Omega_{p,q}$,

$$dE(X)=-\int_0^1\langle\nabla_{\gamma'}\gamma',X\rangle-\sum_i\left\langle\gamma'(t_i^+)-\gamma'(t_i^-),X(t_i)\right\rangle.$$

Lemma 4.4. *γ is a critical point (ie, $dE_\gamma=0$) iff γ is a geodesic.*

Proof. Backwards is clear.

Choose $\phi:(0,1)\to\mathbb{R}$, $\phi\geq0$ and $\phi(t_i)=0$. Let $X=\phi\nabla_{\gamma'}\gamma'\in T_\gamma\Omega_{p,q}$. Then

$$0=(dE)(X)=-\int_0^1\phi\left|\nabla_{\gamma'}\gamma'\right|^2dt$$

and so $\nabla_{\gamma'}\gamma'$ is 0 almost everywhere. Hence, γ' is a broken geodesic.

Now choose $X\in T_\gamma\Omega_{p,q}$ such that $X(t_i)=\gamma'(t_i^+)-\gamma'(t_i^-)$. Then

$$0=dE(X)=-\sum\left\langle\gamma'(t_i^+)-\gamma'(t_i^-),X(t_i)\right\rangle=-\sum_i\left|\gamma'(t_i^+)-\gamma'(t_i^-)\right|^2.$$

Therefore, $\gamma'(t_i^+)=\gamma'(t_i^-)$ for all i. Hence, γ is a single geodesic. □

Lemma 4.5. *γ geodesic. $V\in T_\gamma\Omega_{p,q}$ with V smooth.*

$$d^2E_\gamma(V,V)=-2\int_0^1\langle V''+R(V,\gamma')\gamma',V\rangle\,dt.$$

Remark 4.6. $V\in T_\gamma\Omega_{p,q}$ then $F:(-\epsilon,\epsilon)\times(0,1)\to M$ such that $F(0,\cdot)=\gamma.V(t)=\frac{\partial F}{\partial s}(0,t)$, $F(s,0)=p$, $F(s,1)=q$ for all s.

Then $\gamma_s=F(s,\cdot)$.

$$\frac{d}{ds}E(\gamma_s)\mid_{s=0}=?$$

Proof. We have $\gamma_s' = \frac{\partial F}{\partial s}$ and $\nabla_{\gamma'}\gamma' = \frac{\partial^2 F}{\partial^2 t}(0,t) = 0$,

$$\begin{aligned}
d^2 E_\gamma(V,V) &= \frac{d^2}{(ds)^2} E\left(\gamma_s\right)_{s=0} = \frac{d^2}{(ds)^2}\int_0^1 \left|\gamma_s'(t)\right|^2 dt \\
&= \frac{d}{ds}\int_0^2 2\left\langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t}\right\rangle = 2\int_0^1 \left\langle \frac{\partial^3 F}{\partial s \partial s \partial t}, \frac{\partial F}{\partial t}\right\rangle + 2\left\langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial^2 F}{\partial s \partial t}\right\rangle \\
s = 0 \quad &= 2\int_0^1 \left\langle \frac{\partial^3 F}{\partial s \partial t \partial s}, \frac{\partial F}{\partial t}\right\rangle + 2\left\langle \frac{\partial}{\partial t}V, \frac{\partial}{\partial t}V\right\rangle \\
&= 2\int_0^1 \left\langle \frac{\partial^3 F}{\partial t \partial^2 s}, \frac{\partial F}{\partial t}\right\rangle + \left\langle R\left(\partial_s F, \partial_t F\right)\partial_s F, \partial_t F\right\rangle - 2\left\langle V, V''\right\rangle dt \\
&= 2\left\langle \frac{\partial^2 F}{(\partial s)^2}(0,1), \gamma'(1)\right\rangle - 2\left\langle \frac{\partial^2 F}{(\partial s)^2}(0,0), \gamma'(0)\right\rangle - 2\int \left\langle \frac{\partial^2 F}{\partial s \partial s}, \frac{\partial^2 F}{(\partial t)^2}\right\rangle \\
&+ \left\langle R\left(\partial_s F, \partial_t F\right)\partial_s F, \partial_t F\right\rangle - 2\left\langle V, V''\right\rangle dt \\
\text{all others are } 0 \quad &= \int_0^1 \left\langle R\left(\partial_s F, \partial_t F\right)\partial_s F, \partial_t F\right\rangle - 2\left\langle V, V''\right\rangle dt \\
&= -2\int_0^2 \left\langle V'' + R(V,\gamma')\gamma', V\right\rangle dt
\end{aligned}$$

□

Remark 4.7. This is like the Jacobi relation.

$$d^2 E\left(V,V\right) = -\int \left\langle \mathfrak{L}V, V\right\rangle dt$$

where $\mathfrak{L}V = V'' + R\left(V,\gamma'\right)\gamma'$ Jacobi operator

If (M,g) has section curvature < 0, then

$$\left(d^2 E\right)(V,V) = \int_0^1 \left|V'\right|^2 - R\left(V,\gamma',\gamma',V\right) dt > 0$$

Theorem 4.8. *If (M^n,g) closed has sectional curvature < 0, then close geodesics are unqiue in their homotopy class.*

That is, $\pi_1(M) = \{$closed geodesics$\}$.

Remark 4.9. For a torus, take the rings going up down. The above theorem does not apply.

With negative curvature, geometric objects tend to be unique (geodesics, harmonic maps, unit volume metric with constant scalar metrics in conformal classes)

Proof. Convexity.

Suppose γ_0, γ_1 are two closed geodesics, homotopic. There exists $H : (0,1)\times S^1 \to M$ such that $H(0,\cdot) = \gamma_0$ and $H(1,\cdot) = \gamma_1$.

Claim. There exists $F : (0,1)\times S^1 \to M$ homotopy between γ_0 and γ_1' such that $s \to F(\theta,s) = \gamma_\theta(s)$ is a geodesic.

Proof. $H : (0,1)\times\mathbb{R} \to \tilde{M}$ the lift to universal cover. H is $\pi_1(M)$-equivariant.

For all $\theta \in \mathbb{R}$, there exists unqiue geodesic γ_0 connecting $\gamma_0(\theta)$ to $\gamma_1(\theta)$. Define F in this way. γ_θ unique implies that F is smooth and $\pi_1(M)$-equivariant.

Set $\gamma_s = F(s,\cdot)$ and $f(s) = E(\gamma_s)$. Then $f'(0) = f'(1) = 0$ because γ_0, γ_1 are geodesics.

$$f''(s) \geq 0 \text{ (think you need >0)}$$

because the sectional curvature is < 0. Thus, there's only one critical point, which is a contradiction. □

□

Theorem 4.10. *(Bonnet-Myers) If (M^n,g) with Ric $\geq \frac{n-1}{r^2}g$ then diam $(M^n,g) \leq \pi r$.*

Remark 4.11. Here $Ric \geq \epsilon g$ means that $Ric - \epsilon g$ is positive definite. That is, $Ric(X,X) \geq \epsilon |X|^2$ for all $X \in T_pM$.

- If $S^n(r) = 2\left\{|x|^2 \leq r^2 : x \in \mathbb{R}^{n+1}\right\}$
- $Ric = \frac{n-1}{r^2} g_{S^n(r)}$

Proof. Strategy: If γ is length minimizing, then γ is absolute minimum for E.

$$d^2E(X,X) \geq 0$$

for all X. Want to choose γ a minimizing geodesic realizing the diameter. Use geometry to find X vector field such that $\left(d^2E\right)(X,X) < 0$, which will be a contradiction.

Assume $diam(M,g) = \ell > \pi r$. Let p, q have distance $> \pi r$ and we can connect them by a geodesic γ.

$\{e_i\}_{i=0}^{n-1}$ orthonormal basis of T_pM and $e_0 = \gamma'(0)$. $\{e_i(t)\}_{i=0}^{n-1}$ parallel transport along γ.

$$X_i(t) = \sin(\pi t)\, e_i(t)$$

for $i = 1, ..., n-1$. Then

$$\begin{aligned} \left(d^2E\right)(X_i, X_i) &= \int_0^1 |X_i'(t)|^2 - R(X_i, \gamma', \gamma', X_i)\, dt \\ &= \int_0^1 \pi^2 \cos^2(\pi t) - \left(\sin^2 \pi t\right) R(e_i, \gamma', \gamma', e_i)\, dt \end{aligned}$$

Now, average,

$$\begin{aligned} \sum_{i=1}^{n-1} \left(d^2E(X_i, X_i)\right) &= \int_0^1 (n-1)\pi^2 \cos^2(\pi t) - \sin^2(\pi t) Ric(\gamma', \gamma')\, dt \\ &\leq \int_0^1 (n-1)\,\pi^2 \cos^2(\pi t) - \sin^2(\pi t)\frac{|\gamma'(t)|^2}{r^2} dt \\ &= \int_0^1 (n-1)\pi^2 \cos^2(\pi t) - (n-1)\sin^2(\pi t)\frac{\ell^2}{r^2} dt \\ &< \int_0^1 (n-1)\,\pi^2 \cos^2(\pi t) - (n-1)\pi^2 \sin^2(\pi) dt = 0 \end{aligned}$$

where $\ell = length(\gamma) > 2\pi$. Therefore, there exists X_j such that $\left(d^2E\right)(X_j, X_j) < 0$. $\square$

Theorem 4.12. *(Bonnet-Myers) M a complete metric. If $Ric \geq \frac{n-1}{r^2} g$ then $diam(M,g) \leq \pi r$.*

Corollary 4.13. *If (M^n, g) with $Ric \geq \epsilon g$ for $\epsilon > 0$ then $vol(M^n) < \infty$ and $|\pi_1(M)| < \infty$*

Proof. $\left(\tilde{M}^n, g\right)$ universal cover of (M^n, g).

$$diam\left(\tilde{M}^n, g\right) < \infty \implies vol(M^n, g) < \infty.$$

Then $|\pi_1(M)| < \infty$ and so there's finitely many covers, because

$$vol(M^n) = \frac{vol\left(\tilde{M}^n\right)}{|\pi_1(M)|}.$$

$\square$

Example 4.14.

(1) There is no metric on parabloid which is complete and κ=Gaussian curvature $\geq \epsilon$ (because $Ric(g) = K(g)g$ for a surface).
(2) $S^1 \times S^2$, $\mathbb{T}^n \times M^k$ can not have a metric with positive Ricci curvature.

Remark 4.15. Every three manifold has a negative Ricci curvature.

For surfaces, we know almost everything about Gaussian curvature, except for the following.

- Nirenberg. Pick $f : S^2 \to \mathbb{R}^+$, is there a metric g on S^2 so that $K(g) = f$?

If f is constant, this is easy. There are obstructions. This is possible for negative metric and any surface of genus higher than one.

Theorem 4.16. *(Hamilton) If $\left(M^3, g\right)$ with Ric > 0, then $M^3 \cong S^3/\Gamma$.*

Theorem 4.17. *$\left(M^4, g\right)$ has Ric > 0 iff it admits a metric with positive scalar curvature.*

$n > 5$, no idea.

Theorem 4.18. *(Synge-Weinstein) If (M^n, g) with positive sectional curvature then if n is even, then*

$$\begin{cases} \pi_1(M) = 0 & \textit{if } M^n \textit{ is orientable} \\ \pi_1(M) = \mathbb{Z}/2\mathbb{Z} & \textit{else} \end{cases}$$

if n is odd, then M^n is orientable.

Remark 4.19. Just need to prove the orientable case. For non-orientable, we always have a two cover that's orientable. That must be simply-connected by the first case.

Conjecture 4.20. *(Hopf) $S^2 \times S^2$ has no metric with positive sectional curvature.*

Conjecture 4.21. *If $\left(M^4, g\right)$ complete with sectional curvature > 0, then $M^4 = S^4$ or $\mathbb{CP}^2$.*

Theorem 4.22. *(Cheeger) For each integer $n \in \mathbb{N}$, there are only finitely many diffeomorphism types of manifolds (M^n, g) with $\pi_1(M^n) = 0$ and having positive sectional curvature.*

- Let M^4 be the orientable covers of $\mathbb{RP}^2 \times \mathbb{RP}^2$, then M^4 has no metric with positive sectional curvature (because $\mathbb{RP}^2 \times \mathbb{RP}^2$ has fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$ so M^4 has fundamental group $\mathbb{Z}_2$ which contradicts the theorem)

What about negative sectional curvature? When $n = 3$, $M^3 \cong \mathbb{H}^3/\Gamma$. When $n \geq 4$ there are way too many to have classification

Proof. Assume n even and M^n orientable. Suppose $\pi_1(M^n) \neq 0$, then there exists $\gamma : S^1 \to M$ length minimizing geodesic (which is also energy minimizing).

$$T_\gamma \Omega = \left\{V : S^1 \to M \text{ so that } V \text{ smooth, } V(t) \in T_{\gamma(t)} M\right\}.$$

$V \in T_\gamma \Omega$ then

$$\left(d^2 E\right)_\gamma (V, V) = \int_{S^1} |V'|^2 - R(V, \gamma', \gamma', V)\, d\theta.$$

We know

$$\left(d^2 E\right)(V, V) \geq 0$$

for all $V \in T_\gamma \Omega$, because γ is absolute minimum for E.

Let $V \in T_{\gamma(0)} M$ and set $V(t)$ to be the parallel transport along γ. $\langle V, \gamma'(0)\rangle = 0$ iff $\langle V(\theta), \gamma'(\theta)\rangle = 0$ for all θ. Then

$$\left(d^2 E\right)(V, V) = \int_{S^1} 0 - |V \wedge \gamma'| \text{ sectional span}\{V, \gamma'\}\, d\theta < 0.$$

Problem: $V(0)$ might be different from $V(2\pi)$.

Let $Q = (\gamma'(0))^\perp = \left\{V \in T_{\gamma(0)} M : V \cdot \gamma'(0) = 0\right\}$. $P : Q \to Q$ with $P(V)$ is parallel transport along γ. We want V such that $P(V) = V$. P is isometry, $\dim Q = n - 1$ is odd, $\det P = 1$. The real eigenvalues of P are ± 1. There are odd numbers of them. Therefore, $\lambda = 1$ must be an eigenvalue, and so there exists V such that $P(V) = V$.

n-odd, M^n non-orientable the $\pi_1(M) \neq 0$ so there exists $\gamma : S^1 \to M$ length minimizing geodesic. Argue as before and get $P : Q \to Q$ so that

$$\dim Q = n - 1 = \text{even}.$$

$\det P = -1$ (not orientable). P isometry. The number of real eigenvalues is even, and so $\det P = -1$ and so $\lambda = 1$ is an eigenvalue. □

5. Cut Locus

$P \in M^n$, $V \in T_pM$, $|V| = 1$. Let $\gamma(t) = \exp(tv)$. Let

$$t_0(p) = \sup \left\{ t : \gamma_{[0,t]} \text{ is length minimizing} \right\}.$$

Also

$$cut(p) = \{ \exp(t_0(v)v) : |v| = 1, v \in T_pM \}.$$

$$U_p = \{ tv : |v| = 1, v \in T_pM, 0 < t < t_0(v) \} \subseteq T_pM.$$

$cut(p) = \exp(\partial U_p)$.

Example 5.1.

(1) (S^n, g), take p to be the north pole. Then $U_p = B_\pi(0)$ and $cut(p) = \{\text{south pole}\}$.
(2) $(\mathbb{RP}^n, g)$, take the north pole again, it stops being length minimizing when we hit the equator. So $U_p = B_{\pi/2}(0)$, and the cut locus is the whole "half-equator", which is $\mathbb{RP}^{n-1}$
(3) Torus, draw as a rectangle with p being the center. Then the U_p is the same rectangle (the fundamental domain).

Fact 5.2.

(1) $\exp_p : \overline{U_p} \to M$ *is surjective.*
(2) $\exp_p : U_p \to M$ *is a global diffeo (needs proof)*
(3) ∂U_p *has measure zero in* T_pM

Proof. Of 3. $t_0 : S^{n-1} \subseteq T_pM \to \mathbb{R}^+$ is continuous. Then

$$\partial U_p = \left\{ (t_0(v), v) : v \in S^{n-1} \right\}$$

(written in polar coordinates). Fubini theorem in polar coordinates imply that we have measure zero. □

Proposition 5.3. *$q \in Cut(p)$ if one of the following holds.*

(1) *q is conjugate to p along a geodesic*
(2) *there are two length minimizing geodesics connecting p to q*

Example 5.4. Only b occurs: take $\mathbb{RP}^n$

both a and b, use S^2

Only a, take an ellipse, and pull out one side a bit more to have a unique geodesic.

Proof. $q \in cut(p)$. Suppose a does not happen. Then $q = \gamma(t_0)$ where $\gamma(t) = \exp(tv)$. a not happening means there exist Λ a neighbourhood of t_0v where exp is a diffeo.

There exists σ_i length minimizing geodesic connecting p to $\gamma(t_0 + \frac{1}{i})$ (no longer in the cut locus). $|\sigma_i'(0)| = 1$. Pass to a subsequence, so that $\sigma_i'(0) \to u \in T_pM$. If $u \neq v$ we have found a second geodesic connecting p to q, contradicting the local diffeo.

If $u = v$, we will be contradicting the local diffeo property. Because we would have two points in Λ that maps to $\gamma(t_0 + \frac{1}{i})$ when i sufficiently large.

Assume a. So there is a Jacobi vector field $\mathfrak{J}$ with $\mathfrak{J}(0) = \mathfrak{J}(t_0) = 0$. Choose $\epsilon > 0$ small. Want to show that

$$d(p, \gamma(t_0 + \epsilon)) < t_0 + \epsilon.$$

$X \in T_\gamma \Omega_{p,\gamma(t_0+\epsilon)}$. $dE_\gamma(X) = 0$ (geodesic). Want $d^2 E_\gamma(X,X) < 0$ (this will mean our length is less than the less than $t_0 + \epsilon$).

Suppose X does not exists, so $\left(d^2 E_\gamma\right)(X,X) \geq 0$ for all X. Extend $\mathfrak{J}$ a bit more, to $t_0 + \epsilon$ by making it 0 between t and $t_0 + \epsilon$. But

$$\left(d^2 E\right)(\mathfrak{J},\mathfrak{J}) = -\int \langle \mathfrak{J}, \mathfrak{J}'' + R(\mathfrak{J},\gamma)\gamma\rangle \, dt = 0$$

by being a Jacobi vector field. This implies that $\mathfrak{J} \in T_\gamma \Omega_{p,\gamma(t_0+\epsilon)}$. This implies that $\mathfrak{J}$ is a minimizer of $d^2 E_\gamma$. By general PDE principle, $\mathfrak{J}$ is smooth (from being a minimizer). But it can't be smooth (only continuous) by the way we extended it. (By the uniqueness of ODE to get the Jacobi vector field, if it's zero in a neighbourhood, it should be the identically zero). Finish the proof with assignment problem.

Assume b. Let γ and σ be two length minimizing geodesics. Let $|v| = 1$ be the vector field coor to γ. Let $t_0 = d(p,q)$.

Let η minimizing geodesic connecting $\sigma(t_0+\epsilon)$ to $\gamma(t_0+\epsilon)$. Hence,

$$\mathit{length}\,(\eta) < d\left(\sigma(t_0+\epsilon), q\right) + d\left(q, \gamma(t_0+\epsilon)\right) = 2\epsilon.$$

Then

$$d\left(p, \gamma(t_0+\epsilon)\right) \leq d\left(p, \sigma(t_0+\epsilon)\right) + d\left(\sigma(t_0+\epsilon), \gamma(t_0+\epsilon)\right) < t_0 - \epsilon + 2\epsilon = t_0 + \epsilon.$$

□

Last time: (M^n, g) a manifold, $q \in Cut(p)$ iff two minimizing geodesics connecting p to q OR q conjugate to p.

Fact 5.5. *Given $K \in \mathbb{R}$, g_K constant curvature metric, with curvature k*

$$\mathit{vol}\left(B_r^K(0)\right) = \begin{cases} \left(\frac{1}{\sqrt{k}}\right)^{n-1} \int_0^1 \sin^{n-1}\left(\sqrt{k}s\right) ds & k > 0 \\ \int_0^r s^{n-1} ds & k = 0 \\ \left(\frac{1}{\sqrt{-k}}\right)^{n-1} \int_0^1 \sinh^{n-1}\left(\sqrt{-k}s\right) & k < 0 \end{cases}.$$

Theorem 5.6. *(Gromov-Bishop) Assume (M^n, g) has $Ric(g) \geq (n-1)kg$ for some $k \in \mathbb{R}$. Then $\frac{vol(B_r(p))}{vol(B_r^k(0))} \leq \frac{vol(B_s(p))}{vol(B_s^k(0))}$ for $s < r$. With equality iff $(B_r(p), g) \equiv \left(B_r^k(0), g_k\right)$.*

Example 5.7. Get from ADAN

Remark 5.8. Can assume $k = 0, -1$, or 1.

Making $s \to 0$,

$$\mathit{vol}\left(B_r(p)\right) \leq \mathit{vol}\left(B_r^k(0)\right)$$

for all $r \geq 0$ if $Ric(g) \geq (n-1)kg$.

False if we replace Ric by scal meaning $scal(g) \geq n(n-1)$ does not imply $\mathit{vol}\,(M) \leq \mathit{vol}\,(S^n)$.

Note that if $Ric \geq (n-1)g$ then Bonnet-Myers imply that $\mathit{diam}\,(M^n) \leq \pi$. Volume comparison implies that

$$\mathit{vol}\,(M^n) = \mathit{vol}\,(B_\pi(p)) \leq \mathit{vol}\left(B_\pi^{S^n}(0)\right) = \mathit{vol}\,(S^n)$$

and with equality iff $(M^n, g) \equiv (S^n, g_{S^n})$.

Contradiction for scalar, is use things like $S^2 \times S^1(r)$, the scalar curvature is 2, $\mathit{vol}\left(M^3\right) = 3\pi^2 r$.

Course: Peter Li Lectures in Geometric Analysis (chapter 1 and 2).

6. Isometric Immersions

Let $\Sigma^{n-1} \subseteq (M^n, g)$

Definition 6.1. The second fundamental form is

$$A : T_p\Sigma \times T_p\Sigma \to \left(T_p\Sigma\right)^\perp$$

defined by

$$A\left(X,Y\right)=\left(\nabla_{X}Y\right)^{\perp}.$$

Let (U,ϕ) chart so that $\phi(0)=p$ and $\phi\left(x_{1},...,x_{n-1},0\right)\subseteq\Sigma$.

$$A\left(\frac{\partial\phi}{\partial x_{i}},\frac{\partial\phi}{\partial x_{j}}\right)=\left(\nabla_{\partial_{x_{i}}\phi}\frac{\partial\phi}{\partial x_{j}}\right)^{\perp}=\left(\nabla_{\partial_{j}\phi}\partial_{i}\phi\right)^{\perp}=A\left(\frac{\partial\phi}{\partial x_{j}},\frac{\partial\phi}{\partial x_{i}}\right)$$

so it is symmetric.

Additionally,

$$A\left(fX,Y\right)=fA\left(X,Y\right)$$

so A is a symmetric bilinear 2-tensor.

Let η be a unit normal to Σ^{n-1} near p. Let $B:T_{p}\Sigma\to T_{p}\Sigma$ given by $B(X)=\nabla_{X}\eta$ (check that $\nabla_{X}\eta\cdot\eta=0$).

$$\begin{aligned}\left\langle A\left(\partial_{i},\partial_{j}\right),\eta\right\rangle &= \left\langle\left(\nabla_{\partial_{i}}\partial_{j}\right)^{\perp},\eta\right\rangle=\left\langle\nabla_{\partial_{i}}\partial_{j},\eta\right\rangle=\partial_{i}\left\langle\partial_{j},\eta\right\rangle-\left\langle\partial_{j},\nabla_{\partial_{i}}\eta\right\rangle\\ &= -\left\langle B\left(\partial_{i}\right),\partial_{j}\right\rangle.\end{aligned}$$

Definition 6.2. The mean curvature of Σ^{n-1} is $H=Tr(A)=\sum_{i=1}^{n-1}A\left(e_{i},e_{i}\right)$ where $\{e_{i}\}_{i=1}^{n-1}$ is an orthonormal basis of $T_{p}\Sigma$.

Geometric meaning of H. Let $\Sigma^{n-1}\subseteq M^{n}$. Given $X\in\mathfrak{X}(M)$, let $\{\phi_{t}\}_{t\in\mathbb{R}}$ be the flow.

$$\Sigma_{t}=\phi_{t}(\Sigma)$$

for $\phi_{t}:\Sigma\to M$. dV_{t} be volume for Σ_{t}.

Fact 6.3. *$\frac{d}{dt}dV_{t}\mid_{t=0}=div_{\Sigma}XdV_{0}=\sum_{i=1}^{n-1}\left\langle D_{e_{i}}X,e_{i}\right\rangle dV_{0}$ where $\{e_{i}\}$ is an orthonormal basis of $T_{p}\Sigma$.*

Lemma 6.4. *$div_{\Sigma}X=div_{\Sigma}X^{T}-\langle H,X\rangle$. Thus, if X is parallel to η then $div_{\Sigma}X=-\langle H,X\rangle$.*

Proof. We have

$$\begin{aligned}div_{\Sigma}X &= div_{\Sigma}X^{T}+div_{\Sigma}X^{\perp}=div_{\Sigma}X^{T}+\sum_{i}\left\langle D_{e_{i}}X^{\perp},e_{i}\right\rangle\\ &= div_{\Sigma}X^{T}+\sum_{i}e_{i}\left\langle X^{\perp},e_{i}\right\rangle-\sum_{i}\left\langle X,\left(\nabla_{e_{i}},e_{i}\right)^{\perp}\right\rangle\\ &= div_{\Sigma}X^{T}-\sum_{i}\left\langle X,A\left(e_{i},e_{i}\right)\right\rangle\end{aligned}$$

∎

If Σ^{n-1} is closed ($\partial\Sigma=\emptyset$), then

$$\begin{aligned}\frac{d}{dt}vol\left(\Sigma_{t}\right)\mid_{t=0} &= \frac{d}{dt}\int_{\Sigma_{t}}dV_{t}\mid_{t=0}=\int_{\Sigma}\left(div_{\Sigma}X^{T}-\langle H,X\rangle\right)dV\\ &= -\int_{\Sigma}\langle H,X\rangle\,dV\end{aligned}$$

that is, $H=-\nabla Vol$ (gradient of the volume in some sense). Philosphy, mean curvature points towards the direction that decreases the area.

Consider g_{k} for $k=0,1,-1$. Then

$$H\left(\partial B_{r}(0)\right)=\begin{cases}(n-1)\frac{\cos r}{\sin r} & \text{if }k=1\\ \frac{n-1}{r} & \text{if }k=0\\ (n-1)\frac{\cosh r}{\sinh r} & \text{if }k=-1\end{cases}.$$

Note: if η is unit normal to Σ^{n-1}, let $H(\Sigma)=\langle H,\eta\rangle\in C^{\infty}(\Sigma)$ (not longer vectors, just the magnitude).

When $k=0$,

$$H\left(S^{n-1}(r)\right)=\frac{n-1}{r}$$

When $n=2$, $k=1$ (draw picture). $H>0$ then $=0$ then <0.

When $n=2$ and $k=-1$, $r\to\infty$, $H\left(\partial B_r\right)\to 1$.

Definition 6.5. First variation formula.

$$\frac{d}{dt}dV_t=\left(div_\Sigma X^T-\langle H,X\rangle\right)dV.$$

6.1. **Ricatti equation (2nd variation formula).** $\Sigma^{n-1}\subseteq M^n$, with η the unit normal vector.

$\phi_t:\Sigma\to M$ with $\phi_t(x)=\exp_x\left(t\eta(x)\right)$.

$$H\left(\Sigma_t\right)=\langle H(\Sigma_t),\eta\rangle$$

and

$$\frac{d}{dt}\mid_{t=0}H\left(\Sigma_t\right)=-\left|A\right|^2-Ric\left(\eta,\eta\right).$$

where for ℓ_i eigenvalues of A,

$$|A|^2=\sum_{i=1}^{n-1}\ell_i^2\geq\frac{\left(\sum_{i=1}^{n-1}\ell_i\right)^2}{n-1}=\frac{(TrA)^2}{n-1}=\frac{H^2}{n-1}.$$

Hence,

$$\frac{d}{dt}H\leq\frac{-H^2}{n-1}-Ric\left(\eta,\eta\right)$$

is also called Ricatti equation.

Theorem 6.6. *If $Ric\geq(n-1)kg$, then we have*

$$\frac{vol\left(B_r(p)\right)}{vol\left(B_r^k(0)\right)}\quad\text{is monotonically decreasing.}$$

Proof. $\exp_p:T_pM\to M$. Outside the cut locus,

$$\left(\exp_p\right)^\star(g)=dr^2+\mathfrak{J}(r,\theta)g_{S^{n-1}}$$

by Gauss lemma, with polar coordinates. In particular,

$$(\exp)^\star\left(g^k\right)=dr^2+\begin{cases}\sin^{n-1}(r) & k=1\\ r^{n-1} & k=0\\ \sinh^{n-1}(r) & k=-1\end{cases}\cdot g_{S^{n-1}}.$$

Let $\phi_r:S^{n-1}\to M$ by $\phi_r(\theta)=\exp_p(r\theta)$. Then $S_r=\phi_r\left(S^{n-1}\right)$.

$$dVol_{S^r}=\mathfrak{J}(r,\theta)dVol_{S^{n-1}}.$$

First variation, implies that

$$\frac{d}{dr}dVol_{S^r}=H\left(r,\theta\right)dVol_{S^r}$$

The LHS is

$$\partial_r\mathfrak{J}dVol_{S^{n-1}}=\frac{\partial_r\mathfrak{J}}{\mathfrak{J}}dVol_{S^r}.$$

Therefore, $H(r,\theta)\mathfrak{J}(r,\theta)=\partial_r\mathfrak{J}(r,\theta)$. We also have

$$\partial_rH\leq-\frac{H^2}{n-1}-Ric(\eta,\eta)\leq-\frac{H^2}{n-1}-(n-1)k.$$

Moreover, this holds with equality if we have constant curvature. □

Notation: $K=-1,0,1$. $\bar{g}=g_K$ =constant sectional curvature metric

Theorem 6.7. *(Gromov-Bishop) Assume* (M^n, g) *with* $Ric \geq (n-1)gK$ *then* $\frac{vol(B_r(p))}{vol(\bar{B}_r(0))} \leq \frac{vol(B_s(p))}{vol(\bar{B}_s(0))}$*, if* $s < r$ *and equality gives rigidity.*

Proof. First part was done last time, with small typo.

Take $\exp_p : T_pM \to M$, $(\exp_p)^\star g = dr^2 + f^2(r,\theta)\, g_{S^{n-1}}$. $S_r = \exp_p(\partial B_r(0))$, and $dV_{sr} = \mathfrak{J}(r,\theta)\, ds^{n-1}$ ($\mathfrak{J} \neq f$ was the thing that's wrong. In fact, $\mathfrak{J} = f^{n-1}$). Have two equations

$$\begin{aligned} \partial_r \mathfrak{J} &= H(r,\theta)\mathfrak{J} \qquad (1) \\ \partial_r H &\leq -\frac{H^2}{n-1} - Ric(\eta,\eta) \leq -\frac{H^2}{n-1} - K(n-1) \qquad (2). \end{aligned}$$

We know $\partial_r f = \frac{H}{n-1} f$ then

$$f'' = \frac{H'}{n-1} f + \frac{H}{n-1} f' \leq -\frac{H^2}{(n-1)^2} f - Kf + \frac{H^2}{(n-1)^2} f \leq -Kf.$$

Then $f'' + Kf \leq 0$.

$$\bar{f} = \begin{cases} r & \text{if } k = 0 \\ \sin r & \text{if } k = 1 \\ \sinh r & \text{if } k = -1 \end{cases}$$

then $\bar{f}'' + K\bar{f} = 0$. We also know that $f(0,\theta) = 0$ (volume of a point). From the equation, $f'(0,\theta) = 1$ (think locally, all functions are Euclidean). Let

$$\phi = f'\bar{f} - \bar{f}'f$$

then

$$\begin{aligned} \phi' &= f''\bar{f} + f'\bar{f}' - \bar{f}''f - \bar{f}'f' \\ &\leq -Kf\bar{f} + Kf\bar{f} = 0. \end{aligned}$$

Since $\phi(0) = 0$, so $\phi \leq 0$ for all r that makes sense. Then

$$f'\bar{f} - \bar{f}'f \leq 0 \implies \left(\frac{f}{\bar{f}}\right)' \leq 0$$

which implies that

$$\left(\frac{\mathfrak{J}}{\bar{\mathfrak{J}}}\right)' \leq 0 \implies \frac{\mathfrak{J}}{\bar{\mathfrak{J}}} \text{ is monotone.}$$

$S(\theta)$ =cut point of $t \mapsto \exp(t\theta)$ for some $\theta \in S^{n-1}$. Then

$$\mathrm{vol}(B_r(p)) = \int_{S^{n-1}} \int_0^{\min\{r(\theta),s\}} \mathfrak{J}(s,\theta)\, ds\, d\theta.$$

Want to use Fubini, so set

$$\mathfrak{J}^+(r,\theta) = \begin{cases} \mathfrak{J}(r,\theta) & \text{if } r < r(\theta) \\ 0 & \text{else} \end{cases}$$

then we still have $\frac{\mathfrak{J}^+}{\bar{\mathfrak{J}}} \searrow$.

$$\begin{aligned} \mathrm{vol}(B_r(p)) &= \int_{S^{n-1}} \int_0^r \mathfrak{J}^+(s,\theta)\, ds\, d\theta = \int_0^r \int_{S^{n-1}} \mathfrak{J}^+(r,\theta)\, d\theta\, ds \\ &= \int_0^r \left(\int_{S^{n-1}} \frac{\mathfrak{J}^+(s,\theta)}{\bar{\mathfrak{J}}(s)} d\theta \right) \bar{\mathfrak{J}}(s)\, ds \\ &= \mathrm{vol}(S^{n-1}) \int_0^r ⨍_{S^{n-1}} \frac{\mathfrak{J}^+(r,\theta)}{\bar{\mathfrak{J}}(\theta)} d\theta\, \bar{\mathfrak{J}}(s)\, ds. \end{aligned}$$

Call the $\int_0^r \frac{\mathfrak{J}^+(r,\theta)}{\bar{\mathfrak{J}}(\theta)} d\theta$ by $q(s)$. Then

$$\frac{\mathrm{vol}\left(B_r(p)\right)}{\mathrm{vol}\left(\bar{B}_r(0)\right)} = \frac{\mathrm{vol}\left(S^{n-1}\right)\int_0^r q(s)\bar{\mathfrak{J}}(s)ds}{\mathrm{vol}(S^{n-1})\int_0^r \bar{\mathfrak{J}}(s)ds} = \frac{\int_0^r q(s)\bar{\mathfrak{J}}(s)ds}{\int_0^r \bar{\mathfrak{J}}(s)ds}.$$

Let $\nu = \bar{\mathfrak{J}}ds$ (new measure), then this is just

$$= \frac{\int_0^r q d\nu}{\int_0^r d\nu}$$

is monotone, because this is just the average of a montone function q which is monotone. □

Ricatti-Equation.

Proof. Let $\Sigma^{n-1} \subseteq M^n$, $\phi_t : \Sigma \to M$ by $x \mapsto \exp_t(t\eta_x)$ and $\Sigma_t = \phi_t(\Sigma)$ then

$$\partial_t H = -\left|A\right|^2 - Ric\left(\eta, \eta\right).$$

Will prove this. First,

$$A : T\Sigma \times T\Sigma \to (TZ)^{\perp}$$

by $A\left(X, Y\right) = \left(\nabla_X Y\right)^{\perp}$ and $B : T\Sigma \to T\Sigma$ by $B(X) = \nabla_X \eta$.

$$H = TrB = \sum_{\ell=1}^{n-1} \left\langle B(e_i), e_i\right\rangle$$

where $\{e_i\}$ an o.n.b of $T_p\Sigma$.

$$\partial_\eta H = \sum_i \left\langle \left(\nabla_\eta B\right)(e_i), e_i\right\rangle.$$

Choose $\{x_i\}_{i=1}^{n-1}$ coordinates near $p \in \Sigma$.

$$\begin{aligned}
\left(\nabla_\eta B\right)\left(\frac{\partial \phi_t}{\partial e_i}\right) &= \left(\nabla_{\frac{\partial\phi}{\partial t}} B\right)\left(\frac{\partial \phi}{\partial x_i}\right) = \nabla_{\frac{d\phi}{\partial t}}\left(B(\frac{\partial\phi}{\partial x_i})\right) - B\left(\nabla_{\frac{\partial\phi}{\partial t}}\frac{\partial\phi}{\partial x_i}\right) \\
&= \nabla_{\frac{\partial\phi}{\partial t}}\nabla_{\frac{\partial\phi}{\partial x_i}}\frac{\partial\phi}{\partial t} - B\left(\nabla_{\frac{\partial\phi}{\partial t}}\frac{\partial\phi}{\partial t}\right) \\
&= \nabla_{\partial x_i}\left(\nabla_{\partial_t}\frac{\partial}{\partial t}\right) + R\left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x_i}\right)\frac{\partial\phi}{\partial t} - B\left(B(\frac{\partial\phi}{\partial t})\right) \\
&= R\left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x_i}\right)\frac{\partial\phi}{\partial t} - B^2\left(\frac{\partial\phi}{\partial x_i}\right) \\
&= R\left(\eta, \frac{\partial\phi}{\partial x_i}\right)\eta - B^2\left(\frac{\partial\phi}{\partial x_i}\right).
\end{aligned}$$

THen

$$\begin{aligned}
\sum_{i=1}^{n-1}\left\langle\left(\nabla_\eta B\right)(e_i), e_i\right\rangle &= \sum_{i=1}^{n-1} R\left(\eta, e_i, \eta, e_i\right) - \sum_{i=1}^{n-1}\left\langle B^2(e_i), e_i\right\rangle \\
&= -\sum_{i=1}^{n-1} R(\eta, e_i, e_i, \eta) - TrB^2 \\
&= -Ric\left(\eta,\eta\right) - \left|B^2\right| = -Ric(\eta,\eta) - \left|\eta\right|^2.
\end{aligned}$$

□

6.2. **Applications.**

Theorem 6.8. *(Cheng) If $Ric \geq (n-1)g$ and $diam(M^n) = \pi$ then $(M^n, g) = (S^n, g_{S^n})$.*

Proof. Let p, q be such that $d(p,q) = diam = \pi$. Then $B_{\frac{\pi}{2}}(p) \cap B_{\frac{\pi}{2}}(q) = \emptyset$.

$$\text{vol}\left(B_{\frac{\pi}{2}}(p)\right) \geq \frac{\text{vol}\left(\bar{B}_{\frac{\pi}{2}}(p)\right)}{\text{vol}\left(\bar{B}_{\pi}(0)\right)}\text{vol}\left(B_{\pi}(0)\right) = \frac{\text{vol}\left(B_{\pi}(p)\right)}{2} = \frac{vol\left(M\right)}{2}.$$

Similarly,

$$\text{vol}\left(B_{\frac{\pi}{2}}(q)\right) \geq \frac{\text{vol}(M)}{2}.$$

Therefore,

$$\text{vol}(M) \geq \text{vol}\left(B_{\frac{\pi}{2}}(p)\right) + \text{vol}\left(B_{\frac{\pi}{2}}(q)\right) \geq \text{vol}(M).$$

Since equalities must hold, so

$$\left(B_{\pi}(p), g\right) = \left(S^n - \{q\}, g_{S^n}\right)$$

and so $(M^n, g) = (S^n, g_{S^n})$. ☐

Let G be a finitely generated by $S = \{a_1, ..., a_N\}$ (assume $S = S^{-1}$).

$$\begin{aligned} N(k) &= \#\{g \in G : \, g = b_1 ... b_k, \, b_i \in S\} \\ &= \#\text{words of length } k. \end{aligned}$$

Theorem 6.9. *(Milnor) If $Ric \geq 0$, and $\pi_1(M)$ is finitely generated (ie. if M^n is closed), then $N(k) \leq Ck^n$ for all $k \in \mathbb{N}$ (polynomial growth, of order dimension of the manifold).*

Remark 6.10. Counter example: surface of infinitely generated

Conjecture 6.11. *(Milnor) $Ric \geq 0$ then $\pi_1(M)$ is finitely generated.*

Gromov: this is true if $sec \geq 0$.

Proof. $\tilde{M}^n$ universal cover, $\pi_1 \subseteq isom\left(\tilde{M}\right)$.

There exists $r > 0$, $B_r(0) \cap B_r\left(\phi(p)\right) = \emptyset$ for all $\phi \in \pi_1(M)$ iff $B_r(\phi(p)) \cap B_r\left(\psi(p)\right)) = \emptyset$ for all $\phi \neq \psi$.

Set $L = \max\{d\left(p, \phi(p)\right) : \, \phi \in S\}$ (S the generating set).

If $\phi \in N(k)$ then $d\left(p, \phi(p)\right) \leq kL$.

$B_r(\phi(p)) \subseteq B_{kL+r}(p)$ for all $\phi \in N(k)$.

By first and fourth fact, $N(k)\text{vol}\left(B_r(p)\right) = \sum_{\phi \in N(k)} \text{vol}\left(B_r(\phi(p)\right) \leq \text{vol}\left(B_{kL+r}(p)\right)$. By the theorem, this is $\leq (kL + r)^n$. ☐

Theorem 6.12. *(Milnor) If (M^n, g) is closed, and sectional is < 0 then $\pi_1(M)$ has exponential growth.*

Remark 6.13. Thus, $\mathbb{T}^n$ has no metric with sectional < 0

There exists simply connected manifolds with $Ric = 0$ (K^3-surface)

Theorem 6.14. *(Preissman) M^n closed, $sec < 0$, then every abelian group of $\pi_1(M)$ is $\mathbb{Z}$ (ie. no $\mathbb{Z}^2 \subseteq \pi_1(M)$). (Will prove this at the end)*

7. Lacplacian Comparison

Recall that for $f \in C^\infty(M)$, $\Delta f = div\left(\nabla f\right) = \sum_i \langle \nabla_{e_i} \nabla f, e_i \rangle$ where $\{e_i\}_{i=1}^n$ is an o.n. basis.

Definition 7.1. If f is continuous, then we say $\Delta f \leq h$ in the weak sense for $h \in C^0(M)$, if for all $\phi \in C_c^\infty(M)$ and $\phi \geq 0$, we have

$$\int_M f \Delta \phi dV \leq \int_M h \phi dV$$

Theorem 7.2. *If (M^n, g) has $Ric \geq (n-1)Kg$ for $K = 0, -1, 1$ then if we set $r(x) = d(x,p)$ for fixed $p \in M$, then in the weak sense,*

$$\Delta r \leq \begin{cases} (n-1)\frac{\cos r}{\sin r} & \text{if } k = 1 \\ \frac{n-1}{r} & \text{if } k = 0 \\ (n-1)\frac{\cosh r}{\sinh r} & \text{if } k = -1 \end{cases}.$$

Proof. $\exp_p : T_pM \to M$, then $\exp^\star g = dr^2 + f^2(r,\theta)g_{S^{n-1}}$. $\mathfrak{J}(r,\theta) = f^{n-1}(r,\theta)$. We had

(1) $\partial_r \mathfrak{J} = H(r,\theta)\mathfrak{J}(r,\theta)$
(2) If $\bar{\mathfrak{J}}, \bar{H}$ are the corresponding quantities for constant curvature, then

$$\partial_r \mathfrak{J}\bar{\mathfrak{J}} - \partial_r \bar{\mathfrak{J}}\mathfrak{J} \leq 0$$

Then

$$H(r,\theta) = \frac{\partial_r \mathfrak{J}}{\mathfrak{J}} \leq \frac{\partial_r \bar{\mathfrak{J}}}{\bar{\mathfrak{J}}} = \bar{H} = \begin{cases} (n-1)\frac{\cos r}{\sin r} & \text{if } k = 1 \\ \frac{n-1}{r} & \text{if } k = 0 \\ (n-1)\frac{\cosh r}{\sinh r} & \text{if } k = -1 \end{cases}.$$

Claim. $\Delta r = H(r,\theta)$

Proof. We know that

$$\Delta r = div\,(\nabla r) = div\left(\frac{\partial}{\partial r}\right).$$

We know that $\frac{\partial}{\partial r} = \gamma'(r)$ where $\gamma(t) = \exp_p(t\theta)$. Let $\{e_i\}_{i=1}^{n-1}$ be an orthonormal basis for $T_{(r,\theta)}\partial B_r(p)$. Then

$$\begin{aligned} \Delta r &= \sum_i \left\langle \nabla_{e_i}\frac{\partial}{\partial r}, e_i \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle \text{ second is 0 by geodesics} \\ &= \sum_i e_i \left\langle \frac{\partial}{\partial r}, e_i \right\rangle - \sum_i \left\langle \frac{\partial}{\partial r}, \nabla_{e_i} e_i \right\rangle. \end{aligned}$$

The first term is zero, because we can choose the basis to be always orthogonal. By Gauss lemma, the second is

$$-\sum_i \langle \eta, A(e_i,e_i)\rangle = -\langle \eta, H\rangle = H(r,\theta).$$

We know that mean curvature always points inwards, so the last equality is just from changing the exterior normal to the interior normal. □

Then $\Delta r = H(r,\theta) \leq \bar{H} = \Delta \bar{r}$ and so this theorem holds where things are smooth.

Check that in the weak sense, for $\phi \in C_c^0(M)$ with $\phi \geq 0$, let $r(\theta)$ be the cut point, then

$$\begin{aligned} \int_M \Delta \bar{r}\phi dV &= \int_M \bar{H}\phi dV = \int_{S^{n-1}} \int_0^{r(\theta)} \bar{H}(r)\phi \mathfrak{J}(r,\theta) dr d\theta \\ &\geq \int_{S^{n-1}} \int_0^{r(\theta)} H(r,\theta)\mathfrak{J}(r,\theta)\phi dr d\theta = \int_{S^{n-1}} \int_0^{r(\theta)} \partial_r \mathfrak{J}\phi dr d\theta \\ &= \int_{S^{n-1}} \int_0^{r(\theta)} \partial_r(\mathfrak{J}\phi) - \mathfrak{J}\partial_r \phi dr d\theta \\ &= \int_{S^{n-1}} \mathfrak{J}\left(r(\theta),\theta\right)\phi - \int_{S^{n-1}} \int_0^{r(\theta)} \mathfrak{J}\left\langle \nabla\phi, \nabla r\right\rangle dr d\theta \\ &\geq -\int_M \left\langle \nabla\phi, \nabla r\right\rangle dV = \int_M \nabla\phi r dV \end{aligned}$$

where the last inequality is because $\mathfrak{J} \geq 0$ and $\phi \geq 0$, and things are bounded in the second term. □

Theorem 7.3. *(Splitting theorem, Cheeger-Gromoll) Assume (M^n, g) with $Ric(g) \geq 0$ and containing a line. Then*

$$M^n = N^{n-1} \times \mathbb{R}, \quad g = g_N + dr^2$$

and $Ric(g) \geq 0$ in N^{n-1}.

Remark 7.4.

(1) GET PICTURE FROM ADAN OR NACHO
(2) (Schoen-Yau, Liu) If M^3 non-compact, has $Ric \geq 0$, then either $M^3 \cong \mathbb{R}^3$, or $M^3 = N^2 \times \mathbb{R}$ isometrically, $g = g_{N^2} + dt^2$.
(3) If M^n has two ends and $Ric \geq 0$, then $M^n = N^{n-1} \times \mathbb{R}$ isometrically. Here, two ends means that there exists K compact, contained in M such that $M - K = M_1 \amalg M_2$ (the pieces are not compact). For example: (GET PICTURE FROM ADAN/NACHO). Basically, an infinite cylinder, or torus that extends infinitely to the left or right.
(4) (Soul Theorem) (Cheeger-Fromoll, Perelman). If (M^n, g) has sectional ≥ 0, then there exists K a totally geodesic compact set (soul) so that M^n is diffeomorphic to normal bundle of K.
If $sec > 0$ at some point and (M^n, g) is non-compact, then $K = \{pt\} \subseteq M^n \cong \mathbb{R}^n$.

If $M^n = N^{n-1} \times \mathbb{R}$, then there exists $f : M^n \to \mathbb{R}$ where $f^{-1}(t) = N^{n-1} \times \{t\}$ and $Hess(f) = 0$.

Converse is true, if there exists such an f non-constant, then $M^n = N^{n-1} \times \mathbb{R}$ with $g = g_{N^{n-1}} + dt^2$, where $N^{n-1} = f^{-1}(0)$ (the map is let $X = \nabla f$, which gives $\{\phi_t\}_{t\in\mathbb{R}}$ diffeos. Show $\phi : f^{-1}(0) \times \mathbb{R} \to M^n$ with $\phi(x,t) = \phi_t(x)$ is an isometry).

This f should be seen as the distance to $\gamma(\infty)$.

Definition 7.5. Given γ a line, the Busemann function is $b_\gamma : M \to \mathbb{R}$ given by

$$b_\gamma(x) = \lim_{t\to\infty} \left(t - d\left(\gamma(t), x\right)\right).$$

Example 7.6. If $M^n = \mathbb{R}^n$ with $\gamma(t) = te_1$ then check that $b_\gamma(x) = x \cdot e_1 = x_1$.

In $\mathbb{H}^n$, $b_\gamma^{-1}(t)$ are horospheres. (Get picture).

Basic properties:

(1) If $s \leq t$, then $s - d\left(\gamma(s), x\right) \leq t - d\left(\gamma(t), x\right)$. The LHS is

$$t - d\left(\gamma(t), \gamma(s)\right) - d\left(\gamma(s), x\right) = t - (t-s) - d\left(\gamma(s), x\right)$$

so this is Lipschitz and so the limit exists.

(2) Also,

$$|b_\gamma(x) - b_\gamma(y)| = \lim_{t\to\infty} |b_\gamma\left(x, \gamma(t)\right) - b_\gamma\left(x, \gamma(t)\right)| \leq d(x,y)$$

so the Lipschitz constant is just 1

(3) Also,

$$b_\gamma\left(\gamma(s)\right) = \lim_{t\to\infty}\left(t - d\left(\gamma(s), \gamma(t)\right)\right) = \lim_{t\to\infty}\left(t - (t-s)\right) = s.$$

(4) We have,

$$b_\gamma^+ = \lim_{t\to\infty}\left(t - d\left(\gamma(t), x\right)\right), \quad b_\gamma^- = \lim_{t\to\infty}\left(t - d\left(\gamma(-t), x\right)\right)$$

where the $-$ is distance to $\gamma(-\infty)$.

Proof. Since $Ric \geq 0$, weakly, we have

$$\Delta b_\gamma^+ = \lim_{t\to\infty}\left(-\Delta d_{\gamma(t)}\right) \geq -\lim_{t\to\infty}\frac{n-1}{d\left(\gamma(t), x\right)} \geq 0.$$

Similarly,

$$\Delta\left(b_\gamma^+ + b_\gamma^-\right) \geq 0.$$

Additionally,

$$\left(b_\gamma^+ + b_\gamma^-\right)(\gamma(s)) = s - s = 0.$$

Everywhere else,

$$\begin{aligned} b_\gamma^+(x) + b_\gamma^-(x) &= \lim_{t\to\infty}\left(2t - d\left(\gamma(t) - x\right) - d\left(\gamma(-t) - x\right)\right) \\ &\leq \lim\left(2t - d\left(\gamma(t), \gamma(-t)\right)\right) = \lim(2t - 2t) = 0. \end{aligned}$$

Therefore, it achieves its maximal. By maximal principal, we must get that $b_\gamma^+ + b_\gamma^-$ is constantly zero.

Therefore, $\Delta b_\gamma^+ = -\Delta b_\gamma^-$, and so $\Delta b_\gamma^+ = 0$. This means that it is a harmonic function, so it has to be smooth.

Bochman formula.

$$\Delta\left|\nabla b_\gamma^+\right|^2 = 2\left|\nabla^2 b_\gamma^+\right|^2 + Ric\left(\nabla b_\gamma^+, \nabla b_\gamma^+\right).$$

After proving this, we will get that $Hess\left(b_\gamma^+\right) = 0$. □

Theorem 7.7. *Maximal Principle. If f is a function on U (bounded open set) such that*

1. $\Delta f \geq 0$ *(in $W^{1,2}$)*
2. *f assumes maximal at interior point*

then it is contstant. In particular, $f \leq \sup_{\partial U} f$.

Proof. Assume $f \in C^2$ (see other sources for weaker condition). There exists $p \in U$ and r small such that

$$f(p) = \max f = I \text{ and } \inf_{\partial B_r(p)} f < I.$$

1. Suppose $\Delta f > 0$ in U.
   But $\Delta f(p) \leq 0$ because p is a global maximum, contradiction.
2. Let $V = \{x \in \partial B_r(p) : f(x) = I\}$.

   Claim. There exists $h \in C^\infty(\bar{U})$ such that $\Delta h > 0$, $h(p) = 0$ and $h < 0$ on V

   Proof. $h = e^{\alpha\phi} - 1$. Get picture for description of ϕ. Basically, ϕ is constructed to be always increasing, so $|D\phi| > 0$ on $\bar{B}_r$.

   $h < 0$ on V because $\phi < 0$ on V.

   $$\begin{aligned} \Delta h &= \Delta e^{\alpha\phi} = \sum_i \partial_i\left(\alpha\partial_i\phi e^{\alpha\phi}\right) = \alpha^2\left|D\phi\right|^2 e^{\alpha\phi} + \alpha D\phi e^{\alpha\phi} \\ &= \alpha e^{\alpha\phi}\left(\alpha\left|D\phi\right|^2 + \Delta\phi\right) \end{aligned}$$

   if $\alpha \gg 1$, then $\Delta h > 0$. □

   Set $f_\delta = f + \delta h$. Then $\Delta f_\delta > 0$. Then for all $\delta \ll 1$, $\sup_{\partial B_r} f_\delta < I$. $f_\delta(p) = I$ so

   $$\sup_{\bar{B_r}} f_\delta \geq I > \sup_{\partial B_r} f_\delta$$

   but f_δ has interior maximuum which is impossible.

□

Recall: we had

Theorem 7.8. *$Ric \geq 0$ + line implies $M^n = N^{n-1} \times \mathbb{R}$*

Proof. Recall: stuff about b_γ^+ and b_γ^-.

By the maximum principle, $b_\gamma^+ + b_\gamma^- \equiv 0$ and so $\Delta b_\gamma^+ \equiv 0$ and so $b_\gamma^+ = 0$.

$\left|\nabla b_\gamma^+\right| \leq 1$, $\left|b_\gamma^+(x) - b_\gamma^-(y)\right| \leq a(x,y)$. Since $b_\gamma^+(\gamma(t)) = t$ we get $\left|\nabla b_\gamma^+\right| = 1$ on γ.

$$\Delta\left|\nabla b_\gamma^+\right|^2 = 2\left\langle \nabla\Delta b_\gamma, \nabla b_\gamma\right\rangle + 2\left|\nabla^2 b_\gamma\right|^2 + 2Ric\left\langle \nabla b_\gamma, \nabla b_\gamma\right\rangle.$$

Will prove this later. By harmonic, the first term is zero. Since $Ric \geq 0$, $\Delta\left|\nabla b_\gamma^+\right|^2 \geq 0$. By maximal principle, $\left|\nabla b_\gamma^+\right| \equiv 1$ and so $\left|\nabla^2 b_\gamma^+\right| \equiv 0$. □

8. De Rham Cohomology

Definition 8.1. $\Lambda^k(M)$ the space of k-forms on M. Have $d^k : \Lambda^k(M) \to \Lambda^{k+1}(M)$, with $H^k(M) = \ker(d^k)/\mathrm{im}\left(d^{k-1}\right)$.

If (M^n, g) has a metric, have a metric on $\Lambda^k(M)$ where we declare that

$$e_{i_1}^\star \wedge ... \wedge e_{i_k}^\star$$

is an o.n.b. Then

$$(\alpha, \beta) = \int_M \langle \alpha, \beta\rangle\, dV$$

on $\Lambda^k(M)$. Define $\delta^k : \Lambda^{k+1}(M) \to \Lambda^k(M)$ be the L^2-adjoint of d^k. That is, $(\delta\alpha, \beta) = (\alpha, d\beta)$ for all $\alpha \in \Lambda^{k+1}(M)$ and $\beta \in \Lambda^k(M)$. Then

$$\Delta_{dR}\alpha = \delta d\alpha + d\delta\alpha$$

where $\Delta_{dR} : \Lambda^k \to \Lambda^k$. Define

$$\mathcal{H}^k(M) = \left\{\alpha \in \Lambda^k(M) : \Delta_{dR}\alpha = 0\right\}.$$

Example 8.2. Let $f \in C^\infty(M) = \Lambda^0(M)$. Then

$$\Delta_{dR} f = \delta df = -div\left((df)^\#\right) = -div(\nabla f) = -\Delta f.$$

Claim. $\alpha \in \Lambda^1(M)$, $\delta\alpha = -div(\alpha^\#)$. (see next remark for definition.

Proof. $\phi \in C^\infty(M)$, then $(d\phi)^\# = \nabla\phi$.

$$\begin{aligned}\int(\delta\alpha)\,\phi &= (\delta\alpha, \phi) = (\alpha, d\phi) = \int\langle\alpha, d\phi\rangle = \int\left\langle\alpha^\#, D\phi\right\rangle \\ &= \int div\left(\alpha^\#\phi\right) - \phi div\left(\alpha^\#\right) = -\int \phi div(\alpha^\#)\end{aligned}$$

for all ϕ. □

Theorem 8.3. *(Hodge)* $\mathcal{H}^k(M) \cong H^k(M)$.

If $\alpha \in \Lambda^1(M) = (TM)^\star$, then there exists a unique $\alpha^\# \in \mathfrak{X}(M)$ such that $\alpha(Z) = \left\langle\alpha^\#, Z\right\rangle$ for all $Z \in \mathfrak{X}(M)$.

$$\Delta\alpha^\# = Tr\nabla^2\alpha^\#.$$

Question: relate $\Delta\alpha^\#$ with $(\Delta_{dR}\alpha)^\#$.

Theorem 8.4. *(Bocher)* $\Delta\alpha^\# = -(\Delta_{dR}\alpha)^\# + Ric\left(\alpha^\#\right)$. *That is,*

$$\left\langle\Delta\alpha^\#, Z\right\rangle = -\left\langle(\Delta_{dR}\alpha)^\#, Z\right\rangle + Ric\left(\alpha^\#, Z\right)$$

for all $Z \in \mathfrak{X}(M)$.

Proof. (U, ϕ) chart so that $\left\{\frac{\partial\phi}{\partial x_i} = \partial_i\right\}$ give normal coordinates at p ($\langle\partial_i, \partial_j\rangle = \delta_{ij}$ and $\nabla_i j = 0$).

We have two identities for $\alpha \in \Lambda^1(M)$.

$$(d\alpha)(\partial_i \wedge \partial_j) = \left\langle\nabla_{\partial_i}\alpha^\#, \partial_j\right\rangle - \left\langle\nabla_{\partial_j}\alpha^\#, \partial_i\right\rangle$$

for all i, j (this is hwk). For $\beta \in \Lambda^2(M)$,

$$\begin{aligned}(\delta\beta)(\partial_i)(p) &= -\sum_k \partial_k\left(\beta(\partial_k,\partial_i)\right) - \beta\left(\nabla_{\partial_k}\partial_k,\partial_i\right) - \beta\left(\partial_k,\nabla_{\partial_k}\partial_i\right) \\ &= -\sum_k \partial_k\left(\beta(\partial_k,\partial_i)\right)(p).\end{aligned}$$

We have $\Delta_{dR}\alpha = \delta d\alpha + d\delta\alpha$, so

$$\begin{aligned}\delta d\alpha_p(\partial_i) &= -\sum_k \partial_k\left(d\alpha\left(\partial_k,\partial_i\right)\right) = -\sum_k \partial_k\left(\left\langle \nabla_{\partial_k}\alpha^\#,\partial_i\right\rangle - \left\langle \nabla_{\partial_i}\alpha^\#,\partial_k\right\rangle\right) \\ &= -\sum_k \left(\left\langle \nabla_{\partial k}\nabla_{\partial k}\alpha^\#,\partial_i\right\rangle + \left\langle \nabla_{\partial k}\alpha^\#,\nabla_{\partial_k}\partial_i\right\rangle - \left\langle \nabla_{\partial_k}\nabla_{\partial_i}\alpha^\#,\partial_k\right\rangle - \left\langle \nabla_{\partial_i}\alpha^\#,\nabla_{\partial_k}\partial_i\right\rangle\right) \\ &= -\left\langle \Delta\alpha^\#,\partial_i\right\rangle + \sum_k \left\langle \nabla_{\partial_k}\nabla_{\partial_k}\alpha^\#,\partial_i\right\rangle.\end{aligned}$$

Meanwhile,

$$\begin{aligned}(d\delta\alpha)_p(\partial_i) &= \partial_i(\delta\alpha)(p) = -\partial_i\left(\sum_k \left\langle \nabla_{e_k}\alpha^\#,e_k\right\rangle\right) \\ &= -\sum_k \left\langle \nabla_{\partial_i}\nabla_{e_k}\alpha^\#,e_k\right\rangle - \sum_k \left\langle \nabla_{e_k}\alpha^\#,\nabla_{\partial_i}e_k\right\rangle \\ &= -\sum_k \left\langle \nabla_{\partial_i}\nabla_{\partial_k}\alpha^\#,\partial_k\right\rangle,\end{aligned}$$

here e_i is a o.n. frame in the neighbourhood of p.

$$\begin{aligned}(\Delta_{dR}\alpha)(\partial_i) &= -\left\langle \Delta\alpha^\#,\partial_i\right\rangle + \sum_k \left\langle \nabla_{\partial_k}\nabla_{\partial_i}\alpha^\#,\partial_k\right\rangle - \sum_k \left\langle \nabla_{\partial_i}\nabla_{\partial_k}\alpha^\#,\partial_k\right\rangle \\ &= -\left\langle \Delta\alpha^\#,\partial_i\right\rangle + \sum_k R\left(\partial_k,\partial_i,\alpha^\#,\partial_k\right) \\ &= -\left\langle \Delta\alpha^\#,\partial_i\right\rangle + \sum_k R\left(\partial_i,\partial_k,\partial_k,\alpha^\#\right) \\ &= -\left\langle \Delta\alpha^\#,\partial_i\right\rangle + Ric\left(\partial_i\alpha^\#\right).\end{aligned}$$

The rest follows. ∎

Fact 8.5. *Identities:*

- *If $X \in \mathfrak{X}(M)$, then*

$$\begin{aligned}\Delta|X|^2 &= \sum_i \partial_i\partial_i|X|^2 = 2\sum_i \partial_i\left(\langle \nabla_{\partial_i}X,X\rangle\right) = 2\sum_i \left\langle \nabla^2_{\partial_i\partial_i}X,X\right\rangle + 2\sum_i \langle \nabla_{\partial_i}X,\nabla_{\partial_i}X\rangle \\ &= 2\langle \Delta X,X\rangle + 2|\nabla X|^2.\end{aligned}$$

Corollary 8.6. *For $\alpha \in \Lambda^1(M)$,*

$$\Delta|\alpha|^2 = -2\langle \Delta_{dR}\alpha,\alpha\rangle + 2\left|\nabla\alpha^\#\right|^2 + 2Ric\left(\alpha^\#,\alpha^\#\right).$$

Proof. We have

$$\begin{aligned}\Delta|\alpha|^2 &= \Delta\left|\alpha^\#\right|^2 = 2\left\langle \Delta\alpha^\#,\alpha^\#\right\rangle + 2\left|\nabla\alpha^\#\right|^2 \\ &= -2\langle \Delta_{dR}\alpha,\alpha\rangle + 2Ric\left(\alpha^\#,\alpha^\#\right) + 2\left|\nabla\alpha^\#\right|^2.\end{aligned}$$

∎

Corollary 8.7. $\Delta|\nabla f|^2 = 2\langle \Delta\nabla f,\nabla f\rangle + 2\left|\nabla^2 f\right|^2 + 2Ric(\nabla f,\nabla f)$.

Proof. Let $\alpha = df$. Then

$$\begin{aligned}\Delta |\nabla f|^2 &= -2\langle \Delta_{dR} df, df\rangle + 2\left|\nabla^2 f\right|^2 + 2Ric(\nabla f, \nabla f)\\ &= -2\langle \nabla\Delta_{dR} f, \nabla f\rangle + 2\left|\nabla^2 f\right|^2 + 2Ric(\nabla f, \nabla f).\end{aligned}$$

□

Corollary 8.8. *If (M^n, g) closed with $Ric \geq 0$, then*

1. *If $Ric > 0$ then $b_1(M) = \dim H^1(M) = 0$*
2. *If $Ric \geq 0$ then $b_1(M) \leq n$ and with equality iff $(M^n, g) = (\mathbb{T}^n, g_{flat})$.*

Proof. Suppose there exists $\alpha \in \Lambda^1(M)$, $\Delta_{dR}\alpha = 0$. The first corollary implies that

$$\Delta|\alpha|^2 = 2\left|\nabla\alpha^{\#}\right|^2 + Ric\left(\alpha^{\#}, \alpha^{\#}\right) > 0$$

which is not possible by maximal principal. □

Remark 8.9. If $(\mathbb{T}^n, g)$ with $Ric \geq 0$ then g is flat.

Theorem 8.10. *(Schoen-Yau). There does not exists g on $\mathbb{T}^n$ with $S(g) \geq 0$ and g not flat. ,*

Theorem 8.11. *If (M^n, g) is closed with $Ric \geq 0$, then*

1. *$Ric > 0$ then $b_1(M) = 0$*
2. *$Ric \geq 0$ then $b_1(M) \leq n$ and with equality iff $(M^n, g) = (\mathbb{T}^n, g_{flat})$*

(Betti number)

Proof. Recall that

$$\mathcal{H}^1(M) = \left\{\alpha \in \Lambda^1(M) : \Delta_{dR}\alpha = 0\right\}$$

and $\mathcal{H}^1(M) \cong H^1(M, \mathbb{R})$ and $b^1(M) = \dim H^1(M)$.

If α is harmonic, $\Delta_{dR}\alpha = 0$,

$$\Delta|\alpha|^2 = 2\left|\nabla\alpha^{\#}\right| + 2Ric\left(\alpha^{\#}, \alpha^{\#}\right)$$

this is called teh Bochner formula.

For $\alpha \in \mathcal{H}^1(M)$ and $Ric \geq 0$,

$$\Delta|\alpha|^2 \geq 2\left|\nabla\alpha^{\#}\right|^2 \geq 0$$

and so $|\alpha|$ =constant and $\nabla\alpha^{\#} = 0$ by maximal principle.

Let $\Phi : \mathcal{H}^1(M) \to T_pM$ given by $\alpha \mapsto \alpha^{\#}(p)$. Since $\nabla\alpha^{\#} = 0$, this means that Φ is injective, and so $b_1(M) \leq n$. If $b_1(M) = n$ then let $\{\alpha_i\}_{i=1}^n$ be a linearly independent parallel vectors $\left(\nabla\alpha_i^{\#} = 0\right)$ (do this at a point, by parallel, they are linearly independent everywhere. Similarly, orthonormal at one point will carry to everywhere). Get orthonormal basis $\{e_i\}_{i=1}^n$ (everywhere) which is parallel. Therefore, R the curvature tensor is zero. Hence, M^n is covered by $(\mathbb{R}^n, g_{flat})$. Since M is closed, $(M^n, g) \equiv (\mathbb{T}^n, g_{flat})$. □

Theorem 8.12. *If (M^n, g) closed has $Ric < 0$ then $\#(Isom(M, g)) < \infty$.*

Remark 8.13. $Isom(\mathbb{H}^n, g_{\mathbb{H}^n}) = conf\left(S^{n-1}\right)$ so closed is important.

Similarly, $Isom(S^n, g_{S^n}) = O(n+1)$ so $Ric < 0$ is important.

Proof. (Myers-Steenrod) $Isom(M, g)$ is a compact Lie Group and

$$T_{id}Isom(M^n, g) = \{X \in \mathfrak{X}(M) : \mathfrak{L}_X g = 0\}.$$

If $\{\phi_t\}_{|t|\leq\epsilon} \subseteq Isom(M, g)$ with $\phi_0 = id$ then

$$X = \frac{d\phi_t}{dt}\,|_{t=0},\ \phi_t^{\star} g = g$$

and $0 = \frac{d}{dt}(\phi_t)^\star g = L_X g$. Just need to show that there are no killing vector fields (the Lie group is a zero dimensional manifold, then by compactness, it's finite).

Suffices to see that if $X \in \mathfrak{X}(M)$ is killing, then $X = 0$. $L_X g = 0$ iff for (U, ψ) a chart, then

$$\left\langle \nabla_{\frac{\partial \psi}{\partial x_i}} X, \frac{\partial \psi}{\partial x_j} \right\rangle + \left\langle \nabla_{\frac{\partial \psi}{\partial x_j}} X, \frac{\partial \psi}{\partial x_i} \right\rangle = 0$$

for all i, j. First, notice that if $j = i$, then each part is 0. Write $\partial_i = \frac{\partial \psi}{\partial x_i}$.

Normal coordinates at p, then

$$\langle \Delta X, \partial_k \rangle (p) \quad = \quad \left\langle \sum_i \nabla_{\partial_i} \nabla_{\partial_i} X, \partial_k \right\rangle (p) = -\left\langle \sum_i \nabla_{\partial_i} \nabla_{\partial_k} X, \partial_i \right\rangle (p)$$

(because of normal coordinates from $\nabla_{\partial_k} \partial_i = 0$ and use the killing property). This is

$$\begin{aligned} &= -\left\langle \sum_i \nabla_{\partial_k} \nabla_{\partial_i} X, \partial_i \right\rangle (p) - \sum_i R(\partial_i, \partial_k, X, \partial_i) \\ &= -\sum_i \partial_k \langle \nabla_{\partial_i} X, \partial_i \rangle + \sum_i \langle \nabla_{\partial_i} X, \nabla_{\partial_k} \partial_i \rangle - Ric(X, \partial_k) \\ &= -Ric(X, \partial_k) \end{aligned}$$

because $\langle \nabla_{\partial_i} X, \partial_i \rangle = 0$.

$$\Delta |X|^2 = 2\langle \Delta X, X \rangle + 2|\nabla X|^2 = -2Ric(X, X) + 2|\nabla X|^2 \geq 0$$

By maximal principle, $|X| = \textit{constant}$ and $X \equiv 0$. □

Theorem 8.14. *Preissman Theorem. (M^n, g) closed with sectional < 0 then Γ abelian subgroup of $\pi_1(M)$ then $\Gamma \cong \mathbb{Z}$.*

Remark 8.15.

(1) There does not exists $(\mathbb{T}^n, g)$ with $sec < 0$ (since $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$)
(2) Lawson-Yau. If $(\mathbb{T}^n, g)$ with sectional ≤ 0 then g is flat.

Proof. Assume $\alpha, \beta \in \pi_1$ with $\alpha \star \beta = \beta \star \alpha$.

(1) There exists $F : \mathbb{T}^2 \to M^n$ which is homotopically non-trivial.

There exists H homotopy between $\alpha \star \beta$ and $\beta \star \alpha$. $H : [0,1] \times [0,1] \to M$ by

$$H(s,0) = H(s,1) = \alpha(s) \text{ and } H(0,t) = H(1,t) = \beta(t).$$

Then $H_\star : \pi_1(S^1, S^1) \to \pi_1(M)$ is injective.

Will come back to this proof □

$$C^\infty\left((N^k, h), (M^n, g)\right) = \left\{f : N^k \to M^n \text{ smooth}\right\} \text{ with } k \leq n.$$

$$E(f) = \int_N |df|^2 \, dh$$

Definition 8.16. Critical points for E are called Harmonic maps.

Example 8.17.

(1) $N^k = S^1$, $E(f) = \int_{S^1} |f'|^2 \, d\theta$ harmonic is geodesic.
(2) $N^k = S^1 \times S^1$ and $f : S^1 \times S^1 \to M$, $f(\theta_1, \theta_2) = f(\theta_1)$ then f harmonic iff $t \mapsto f(t)$ is a geodesic.
(3) $f : S^1 \times S^1 \to S^1 \times S^1 \times S^1$ by $f(\theta_1, \theta_2) = (\theta_1, \theta_2, \theta_1)$ then f is harmonic
(4) If $M^n = \mathbb{R}^n$ and $f : N^k \to \mathbb{R}^n$ is harmonic, iff $\Delta_N f_i = 0$ for all i

Theorem 8.18. *(Eels-Sampson) If section $(M^n) < 0$ then every homotopic non-trivial map $F : N^k \to M^n$ is homotopic to a harmonic map.*

Remark 8.19. There exists $f : \mathbb{RP}^2 \to S^2$ which is homotopically non-trivial set has no harmonic map in its homotopy class.

Question: can one remove $sect < 0$? When $k = 2$ and N is orientable, it is known that one can.

Remark 8.20. (Hartman) Under $sect(M^n) < 0$, harmonic maps are unique in their homotopy class and energy minimizing.

Let $f : N^k \to M^k$ be harmonic, with $(\Delta f)^T = 0$.

$$\Delta |df|^2 = 2 |\nabla df|^2 + 2 \sum_{i=1}^{k} \langle df(Ric^N(e_i)), df(e_i) \rangle - \sum_{i,j=1}^{k} R^M (df(e_i), df(e_j), df(e_j), df(e_i))$$

where $\{e_i\}_{i=1}^k$ o.n.b for $T_p N^k$. Here, $Ric\,(Z)$ is the unique vector such that

$$\langle Ric(Z), Y \rangle = Ric\,(Z, Y)$$

for all Y.

Recall, if $f \in C^\infty(N, \mathbb{R})$ with $\Delta f = 0$, then

$$\Delta |\nabla f|^2 = 2 \left|\nabla^2 f\right|^2 + 2Ric\,(\nabla f, \nabla f)$$

(special case of the above. CHECK! Diagonalize Ricci and use basis).

Corollary 8.21. *Assume $sec(M^n) < 0$. There does not exists $f : S^k \to M^n$ harmonic.*

Proof. We have $Ric^{S^k}(e_i) = (n-1)e_i$,

$$\Delta |\nabla f|^2 = 2 |\nabla f|^2 + 2(n-1)(df)^2 - \sum_i R^M(\cdot) \geq 2 |\nabla df|^2 + 2(n-1) |df|^2 .$$

Maximal principal implies that $|df|^2$ =constant and so $|df|^2 \equiv 0$. ☐

Corollary 8.22. *For $f : \mathbb{T}^2 \to M$ harmonic, then image of f is a closed geodesic.*

Proof. This is when $Ric^N = 0$.

$$\Delta |df|^2 = 2 |\nabla df|^2 - R\,(df(e_1), df(e_2), df(e_2), df(e_1)) \geq 0.$$

Maximal principal says that $|df|$ =constant and $\nabla df \equiv 0$. And $R\,(df(e_1), df(e_2), df(e_2), df(e_1)) = 0$, so $df(e_1)$ is colinear with $df(e_2)$ and so the image of f is one dimensional (a curve) which is also a geodesic. ☐

Proof. Application to proof of Preissman Theorem, there exists γ geodesic such that $\alpha = \gamma^k$ and $\beta = \gamma^k$ then $\langle \alpha, \beta \rangle \cong \mathbb{Z}$. ☐

Recall:

Theorem 8.23. *(Hadamard?) If (M^n, g) is closed, with $\pi_1(M) \neq 0$, then there exists a closed geodesic.*

Question: (Poincare, 1905). Is there a closed geodesicin $\left(S^2, g\right)$ for any g. He gave two approaches:

(1) Say there is a functional $F : \Omega \to \mathbb{R}$, does F need to have a critical point (not true on $\mathbb{R}$, like $x \mapsto x$)
(2) Must a geodesic always close?

Birkhof: (1912) Every $\left(S^2, g\right)$ has a closed geodesic.

(90's) (Franks, Bangert) Every $\left(S^2, g\right)$ has infinitely many closed geodesics.

Toy problem: Suppose Ω is compact, and finite dimensional, then there are at least two critical points (min and max). Question: does it need to have a third?

Claim 8.24. If $\pi_1(\Omega) \neq 0$, then there exists a third critical point.

Proof. Take $[\gamma] \neq 0$ in $\pi_1(\Omega)$. For each $\sigma : S^1 \to \Omega$ in $[\gamma]$, take

$$\inf_{\sigma \in [\gamma]} \max_{p \in S^1} F \circ \sigma(p) = W$$

is a critical point. (Get picture). □

Claim 8.25. There exists p such that $\nabla F(p) = 0$ and $F(p) = W$.

Proof. Choose $\{\sigma_i\} \subseteq [\gamma]$ such that $\max F \circ \sigma_i = F(\sigma_i(\theta_i)) \to W$. Pass to a subsequence, $\sigma_i(\theta_i) \to p$. Is $\nabla F(p) = 0$? Suppose not, consider $(\phi_t)_{|t| \leq \epsilon}$ where $\frac{\partial \phi_t}{\partial t} = -\nabla F(\phi_t)$. Then there exists $\delta > 0$ so that

$$\max F(\phi_\epsilon \sigma \sigma_i) \leq \max(F \circ \sigma_i) - \delta \to W - \delta.$$

But the

$$W \leq LHS$$

by definition. □

Proof. of Birchoff.

$$\Omega^L = \left\{ \gamma : S^1 \to S^2 \subseteq \mathbb{R}^3 \mid \int_{S^1} |\gamma|^2 + |\gamma'(0)|^2 \, d\theta < L \right\}.$$

(Note, this is with respect to the Euclidea embedding, not the w/e metric).

Sobolov. If $\gamma \in \Omega^L$ then γ is continuous. If $(\gamma_i) \subseteq \Omega^L$, there is $\gamma \in \Omega^L$ such that $\gamma_i \to \gamma$ in $W^{1/2}$ and in C^0.

$E : \Omega^L \to \mathbb{R}$, $E(\gamma) = \int_{S^1} \sqrt{\langle \gamma'(0), \gamma'(0) \rangle_g} d\theta$.

$\pi_1(\Omega^L) \neq 0$?, $\pi_1(\Omega^L) = \pi_2(S^2) = \mathbb{Z}$.

$\gamma : [-1, 1] \to \Omega^L$, by $\gamma(t) = S^2 \cap \{x_3 = t\}$. Let

$$[\gamma] = \{\sigma : (-1, 1) \to \Omega^1 : \sigma \text{ homotopic to } \gamma \text{ and } \sigma(-1), \sigma(1) \text{ are points}\}$$

Remark. If $\sigma \in [\gamma]$, $\sigma : (-1, 1) \times S^1 \to S^2$, $\hat{\sigma} : S^2 \to S^2$ which is homotopically non-trivial.

Claim. There exists $c_0 > 0$ such that if $\sigma \in [\gamma]$,

$$\max_{|t| \leq 1} E(\sigma(-1)) \geq c_0$$

Proof. Assume $E(\sigma(t)) \ll 1$, all $|t| \leq 1$. Fix $\bar{\theta} \in S^1$.

$$W = \inf_{\sigma \in [\gamma]} \max_{|t| \leq 1} E(\sigma(t)) \geq c_0.$$

There exists $D : \Omega^L \to \Omega^L$ such that

(1) $E(D(\gamma)) \leq E(\gamma)$
(2) $E(D(\gamma)) = E(\gamma)$ implies that γ is a geodesic
(3) $D(\gamma)$ is homotopic to γ
(4) D is continuous weakly in $W^{1,2}$. If $\gamma_i \to \gamma$, then $E(D(\gamma_i)) \to E(D(\gamma))$
(5) Let $\mathfrak{L} = \{\gamma : S^1 \to (S^2, g) : \text{ geodesic}\} \subseteq \Omega^L$, then for all $\epsilon > 0$, there exists δ so that $d(\gamma, \mathfrak{L}) \geq \epsilon$ implies $E(D(\gamma)) \leq E(\gamma) - \delta$.

Assume the existence of D. Choose $\sigma_i \in [\gamma]$, $\max E \circ \sigma_i \to W$. $D(\sigma_i) \in [\gamma]$ and $E(D \circ \sigma_i) \leq E(\sigma_i)$.

$$W \leq \max_{|t| \leq 1} E(D \circ \sigma_i) \leq \max E(\sigma_i) \to W.$$

There exists t_i, so that $E(D \circ \sigma_i(t_i)) = \max E(D \circ \sigma_i) \to W$. Then

$$|E(D \circ \sigma_i(t_i)) - E(\sigma_i(t_i))| \to 0$$

as $i \to \infty$. Fourth property implies that $d(\gamma_i(t_0), \mathfrak{L}) \to 0$. In particular, $\mathfrak{L} \neq 0$.

This D is the following. Let $\sigma \in \Omega^L$. Divide S^1 into $I_1, ..., I_{2N}$ all of the same legnth. If $N \gg 1$ then $length\left(\sigma(I_j)\right) \ll 1$ for all j, independent of σ (just on L). σ^1 replacement of σ by geodesics connecting endpoints of $\sigma(I_j \cup I_{j+1})$ where j is odd, $E(\sigma^1) \leq E(\sigma)$.

σ^2, do same process to σ^1, but on $\sigma^1\left(I_j \cup I_{j+1}\right)$, where j is even.

Set $D(\sigma) = \sigma^2$ if $D(\sigma) = \sigma$, the σ is a geodesic. $\square$

$\square$

(Rademacher) For a generic set of metrics, $\left(M^4, g\right)$, there are infinitely many closed geodescs.

(Gronoll-Meyers, Gromov) For all manifolds but a finite set $(S^n, \mathbb{CP}^n, ...)$, they have infinitely many closed geodesics.

Conjecture 8.26. *Every (S^n, g) has infinitely many closed geodesics.*

Conjecture 8.27. *(Yau) Every $\left(M^3, g\right)$ has infinitely many closed minimal surfaces.*

Theorem 8.28. *(Pitts, Schoen-Simon) Every (M^n, g) has 1 minimal hypersurface (codimension 1).*

Theorem 8.29. *(-N) Every (M^n, g) with $Ric(g) > 0$ has inifnitely many minimal surfaces.*

Theorem 8.30. *Every (M^n, g) for generic metrics g, there are infinitely many minimal surfaces.*