DIFFERENTIAL EQUATIONS

FIRST ORDER DIFFERENTIAL EQUATIONS

DEFINITION

- A differential equation is an equation involving a differential coefficient i.e. $\frac{dy}{dx}$
- In this syllabus, we will only learn the first order
- To solve differential equation $\frac{dy}{dx}$, we integrate and find the equation y which satisfies the differential equation

Example	General Solution	Particular Solution
$\frac{dy}{dx} = x - 1$	$y = \int x - 1 \, dx$ $y = \frac{x^2}{2} - x + c$	$y = \frac{x^2}{2} - x + 3$ $y = \frac{x^2}{2} - x + 15$ $y = \frac{x^2}{2} - x - 30$

Example	General Solution	Particular Solution
$\frac{dy}{dx} = x - 3x^2$	$y = \int x - 3x^2 \, dx$ $y = \frac{x^2}{2} - \frac{3x^3}{3} + c$ $y = \frac{x^2}{2} - x^3 + c$	$y = \frac{x^2}{2} - x^3 + 7$ $y = \frac{x^2}{2} - x^3 - 1$

Example	General Solution	Particular Solution
$\frac{dy}{dx} = \sqrt{1-x}$	$y = \int \sqrt{1-x} \, dx$ $y = \int (1-x)^{\frac{1}{2}} \, dx$ $y = \frac{(1-x)^{\frac{3}{2}}}{\frac{3}{2}(-1)} + c$ $y = \frac{-2(1-x)^{\frac{3}{2}}}{3} + c$	$y = \frac{-2(1-x)^{\frac{3}{2}}}{3} + 20$ $y = \frac{-2(1-x)^{\frac{3}{2}}}{3} - 100$

SEPARATING THE VARIABLES

HOW DO WE KNOW IF IT CAN BE SEPARATED?

To put it simply, you separate the variables of y and x

$$xy^2\frac{dy}{dx} = 1$$

y on the left — x on the right

$$y^2\frac{dy}{dx} = \frac{1}{x}$$

$$y^2\ dy = \frac{1}{x}\ dx$$

$$\int y^2\ dy = \int \frac{1}{x}\ dx$$

To put it simply, you separate the variables of y and x

$$(xy + x)\frac{dy}{dx} = 3$$

$$x(y+1)\frac{dy}{dx} = 3$$

y on the left — x on the right

$$(y+1)\frac{dy}{dx} = \frac{3}{x}$$

$$(y+1)\ dy = \frac{3}{x}\ dx$$

$$\int (y+1)\ dy = \int \frac{3}{x}\ dx$$

To put it simply, you separate the variables of y and x

$$(y^2x - y)\frac{dy}{dx} = 5$$

$$y(yx - 1)\frac{dy}{dx} = 5$$

y on the left — x on the right

CANNOT BE SEPARATED

DIFFERENTIAL EQUATIONS

SOLVING FIRST ORDER DIFFERENTIAL EQUATIONS

METHOD

- Separate the variables x and y
- Integrate the left hand side in terms of y and the right hand side in terms of x

$$f(y)\frac{dy}{dx} = g(x)$$

$$\int f(y)\ dy = \int g(x)\ dx$$

Example 1: Find the general solution of the differential equations

1.(a) $$\frac{dy}{dx} = \frac{1}{x}$$

$$1\ dy = \frac{1}{x}\ dx$$

$$\int 1\ dy = \int \frac{1}{x}\ dx$$

$$y = \ln x + c$$

Example 1: Find the general solution of the differential equations

1.(b) $\frac{dy}{dx} = \text{cosec } y$

$$\frac{1}{\text{cosec } y}\, dy = 1\, dx$$

$$\frac{1}{\frac{1}{\sin y}}\, dy = 1\, dx$$

$$\sin y\, dy = 1\, dx$$

$$\int \sin y\, dy = \int 1\, dx$$

$$-\cos y = x + c$$

Example 1: Find the general solution of the differential equations

1.(c) $\frac{dy}{dx} = 5y$

$$\frac{1}{y}\, dy = 5\, dx$$

$$\int \frac{1}{y}\, dy = \int 5\, dx$$

$$\ln y = 5x + c$$

Example 1: Find the general solution of the differential equations

1.(d) $y(1+x)\frac{dy}{dx} - 2(1+y^2) = 0$

$$y(1+x)\frac{dy}{dx} = 2(1+y^2)$$

$$\frac{y}{(1+y^2)}\frac{dy}{dx} = \frac{2}{(1+x)}$$

$$\frac{y}{(1+y^2)}\, dy = \frac{2}{(1+x)}\, dx$$

$$\int \frac{y}{(1+y^2)}\, dy = \int \frac{2}{(1+x)}\, dx$$

$$\frac{\not{y}\ln(1+y^2)}{2\not{y}} = 2\ln(1+x) + c$$

Example 1: Find the general solution of the differential equations

1.(e) $x\frac{dy}{dx} = y^2$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{1}{y^2}\, dy = \frac{1}{x}\, dx$$

$$\int y^{-2}\, dy = \int \frac{1}{x}\, dx$$

$$\frac{y^{-1}}{-1} = \ln x + c$$

$$-\frac{1}{y} = \ln x + c$$

Example 2: Solve the following differential equations:

2.(a) $\frac{dy}{dx} = 3x^2 + 4x^3$

$$1\, dy = 3x^2 + 4x^3\, dx$$

$$\int 1\, dy = \int 3x^2 + 4x^3\, dx$$

$$y = \frac{3x^3}{3} + \frac{4x^4}{4} + c$$

$$y = x^3 + x^4 + c$$

$y = 1\,,\ x = 1$

$$y = x^3 + x^4 + c$$

$$1 = 1^3 + 1^4 + c$$

$$1 = 2 + c$$

$$c = 2 - 1$$

$$c = 1$$

$$y = x^3 + x^4 + 1$$

Example 2: Solve the following differential equations:

2.(b) $\frac{dy}{dx} = 5y$

$$\frac{1}{y}\, dy = 5\, dx$$

$$\int \frac{1}{y}\, dy = \int 5\, dx$$

$$\ln y = 5x + c$$

$y = 2\,,\ x = 0$

$$\ln y = 5x + c$$

$$\ln 2 = 5(0) + c$$

$$c = \ln 2$$

$$\ln y = 5x + \ln 2$$

$$y = e^{5x + \ln 2}$$

$$y = e^{5x}e^{\ln 2}$$

$$y = 2e^{5x}$$

Example 2: Solve the following differential equations:

2.(c) $x\frac{dy}{dx} = y^2$

$$\frac{1}{y^2}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{y^2}\,dy = \int \frac{1}{x}\,dx$$

$$\int y^{-2}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{y^{-1}}{-1} = \ln x + c$$

$$-\frac{1}{y} = \ln x + c$$

$x = 1\ ,\ y = 1$

$$-\frac{1}{y} = \ln x + c$$

$$-\frac{1}{1} = \ln 1 + c$$

$$-1 = 0 + c$$

$$c = -1$$

$$-\frac{1}{y} = \ln x - 1$$

$$y = \frac{-1}{\ln x - 1}$$

Example 2: Solve the following differential equations:

2.(d) $\frac{dy}{dx} = \frac{\sin^2 x}{y^2}$

$$y^2\,dy = \sin^2 x\,dx$$

$$\int y^2\,dy = \int \sin^2 x\,dx$$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$\frac{y^3}{3} = \int \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$$

$$\frac{y^3}{3} = \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c$$

$$\frac{y^3}{3} = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

$y = 1\ ,\ x = 0$

$$\frac{y^3}{3} = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

$$\frac{1^3}{3} = \frac{1}{2}(0) - \frac{1}{4}\sin 2(0) + c$$

$$\frac{1}{3} = 0 - 0 + c$$

$$c = \frac{1}{3}$$

$$\frac{y^3}{3} = \frac{1}{2}x - \frac{1}{4}\sin 2x + \frac{1}{3}$$

Example 2: Solve the following differential equations:

2.(e) $(x+1)\frac{dy}{dx} = y$

$$\frac{1}{y}\,dy = \frac{1}{(x+1)}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{(x+1)}\,dx$$

$$\ln y = \ln(x+1) + c$$

$y = 1\ ,\ x = 1$

$$\ln y = \ln(x+1) + c$$

$$\ln 1 = \ln(1+1) + c$$

$$0 = \ln 2 + c$$

$$c = -\ln 2$$

$$\ln y = \ln(x+1) - \ln 2$$

DIFFERENTIAL EQUATIONS

APPLICATION ON DIFFERENTIAL EQUATIONS

APPLICATION ON DIFFERENTIAL EQUATIONS

- Examples are real life
- Rate of change of a quantity Q is **proportional** to the value Q

$$\frac{dQ}{dt}\ \alpha\ Q$$

- Rate of change is taken with respect to time

$$\frac{dQ}{dt}$$

APPLICATION ON DIFFERENTIAL EQUATIONS

- Natural Growth
- Natural Decay
- Newton's Law of Cooling
- Rate of Liquid Leaking

APPLICATION ON DIFFERENTIAL EQUATIONS

NATURAL GROWTH
AND
NATURAL DECAY

(I) Natural Growth

If Q is increasing with time, it can be expressed as differential equation

$$\frac{dQ}{dt} \; \alpha \; Q \quad \Rightarrow \quad \frac{dQ}{dt} = kQ$$

Solve the first order differential equation:

Q to the left ..... t to the right

$$\frac{1}{Q}\,dQ = k\,dt$$

$$\int \frac{1}{Q}\,dQ = \int k\,dt$$

$$\ln Q = kt + c$$

$$Q = e^{kt+c}$$

$$Q = e^{kt}e^{c}$$

$$Q = Ae^{kt}$$

Rate of increase of number of cells of yeast is proportional to the number of cells

(II) Natural Decay

If the rate of decrease of Q is proportional to Q,

$$\frac{dQ}{dt} \; \alpha \; Q \quad \Rightarrow \quad \frac{dQ}{dt} = -kQ$$

Solve the first order differential equation:

Q to the left ..... t to the right

$$\frac{1}{Q}\,dQ = -k\,dt$$

$$\int \frac{1}{Q}\,dQ = \int -k\,dt$$

$$\ln Q = -kt + c$$

$$Q = e^{-kt+c}$$

$$Q = e^{-kt}e^{c}$$

$$Q = Ae^{-kt}$$

METHOD

- Interpret the word problem into a differential equation
- Solve the first order of differential equation
- Find the value of k and A (or c)
- Answer the question

Example 2(I): The rate of increase of a colony of bacteria is proportional to the number of bacteria present at any particular time. If initial number of bacteria is 100 and 4 hrs later there are 1350, then find **k** and **A**.

Let bacteria be B.

$$\frac{dB}{dt} \; \alpha \; B$$

$$\frac{dB}{dt} = kB$$

k is positive because it is a rate of increase

$$\frac{1}{B}\,dB = k\,dt$$

$$\int \frac{1}{B}\,dB = \int k\,dt$$

$$\ln B = kt + c$$

$$B = e^{kt+c}$$

$$B = Ae^{kt}$$

Example 2(I): The rate of increase of a colony of bacteria is proportional to the number of bacteria present at any particular time. If initial number of bacteria is 100 and 4 hrs later there are 1350, then find k and A.

$$B = Ae^{kt}$$

At $t = 0$, $B = 100$
At $t = 4$, $B = 1350$

k and A ?

$$100 = Ae^{k0}$$

$$100 = Ae^{0}$$

$$A = 100$$

$$1350 = Ae^{4k}$$

$$1350 = 100e^{4k}$$

$$\frac{1350}{100} = e^{4k}$$

$$e^{4k} = 1.35$$

$$4k = \ln(1.35)$$

$$k = \frac{\ln(1.35)}{4}$$

$$k = 0.0750$$

Example 2(2): The rate of increase of a population of tigers is proportional to the population at any particular instant of time. If initial number of tigers is 400 and if after 2 years there are 480, how many tigers will there be after 5 years?

Let tigers be Q.

$$\frac{dQ}{dt} \ \alpha \ Q$$

$$\frac{dQ}{dt} = kQ$$

k is positive because it is a rate of increase

$$\frac{1}{Q}\,dQ = k\,dt$$

$$\int \frac{1}{Q}\,dQ = \int k\,dt$$

$$\ln Q = kt + c$$

$$Q = e^{kt+c}$$

$$Q = Ae^{kt}$$

Example 2(2): The rate of increase of a population of tigers is proportional to the population at any particular instant of time. If initial number of tigers is 400 and if after 2 years there are 480, how many tigers will there be after 5 years?

$$Q = Ae^{kt}$$

At $t = 0$, $Q = 400$
At $t = 2$, $Q = 480$

At $t = 5$, $Q = ?$

$$400 = Ae^{k0}$$

$$400 = Ae^{0}$$

$$A = 400$$

$$480 = Ae^{2k}$$

$$480 = 400e^{2k}$$

$$\frac{480}{400} = e^{2k}$$

$$e^{2k} = 1.2$$

$$2k = \ln(1.2)$$

$$k = \frac{\ln(1.2)}{2}$$

$$Q = 400e^{\frac{\ln(1.2)}{2}t}$$

$$Q = 400e^{\frac{\ln(1.2)}{2}(5)}$$

$$Q = 630.97$$

$$Q = 631 \text{ tigers}$$

Example 2(3): The number of insects in a population t days after the start of observations is denoted by N. The variation in the number of insects is modelled by a differential equation of the form

$$\frac{dN}{dt} = kN\cos(0.02t)$$

where k is a constant and N is taken to be a continuous variable. It is given that $N = 125$ where t = 0.

(i) Solve the differential equation, obtaining a relation between N, k and t.

$$\frac{dN}{dt} = kN\cos(0.02t)$$

$$\frac{1}{N}\,dN = k\cos(0.02t)\,dt$$

$$\int \frac{1}{N}\,dN = \int k\cos(0.02t)\,dt$$

$$\ln N = k\frac{\sin(0.02t)}{0.02} + c$$

$$N = e^{k\frac{\sin(0.02t)}{0.02} + c}$$

$$N = e^{k\frac{\sin(0.02t)}{0.02}}e^{c}$$

$$N = Ae^{k\frac{\sin(0.02t)}{0.02}}$$

At $t = 0, N = 125$

$$125 = Ae^{k\frac{\sin(0.02(0))}{0.02}}$$

$$125 = Ae^{0}$$

$$A = 125$$

$$N = 125e^{k\frac{\sin(0.02t)}{0.02}}$$

Example 2(3): **(ii)** Given also that N = 166 when t = 30, find the value of k.

$$N = 125e^{k\frac{\sin(0.02t)}{0.02}}$$

At $t = 30$, $N = 166$

$$166 = 125e^{k\frac{\sin(0.02(30))}{0.02}}$$

$$166 = 125e^{k\frac{\sin(0.6)}{0.02}}$$

$$\frac{166}{125} = e^{k\frac{\sin(0.6)}{0.02}}$$

$$1.328 = e^{k\frac{\sin(0.6)}{0.02}}$$

$$\ln 1.328 = k\frac{\sin(0.6)}{0.02}$$

calculator in radians!

$$k = \frac{(\ln 1.328)(0.02)}{\sin 0.6}$$

$$k = 0.0100479$$

$$k = 0.0100$$

Example 2(3): **(iii)** Obtain an expression for N in terms of t, and find the least value of N predicted by this model.

$$N = 125e^{0.0100\frac{\sin(0.02t)}{0.02}}$$

$$k = 0.0100479$$

$$k = 0.0100$$

$$-1 \leq \sin t \leq 1$$

$$-1 \leq \sin(0.02t) \leq 1$$

$$-1(0.0100) \leq 0.0100\sin(0.02t) \leq 1(0.0100)$$

$$\frac{-1(0.0100)}{0.02} \leq 0.0100\frac{\sin(0.02t)}{0.02} \leq \frac{1(0.0100)}{0.02}$$

$$-0.502 \leq 0.0100\frac{\sin(0.02t)}{0.02} \leq 0.502$$

$$N = 125e^{-0.502}$$

$$N = 75.66$$

$$N = 76$$

APPLICATION ON DIFFERENTIAL EQUATIONS

NEWTON'S LAW OF COOLING

NEWTON'S LAW OF COOLING

The rate of cooling of a body is proportional to the difference between it's temperature and the temperature of the surroundings

Let θ be the temperature of the body
Let θ_0 be the temperature of the surroundings

$$\frac{d\theta}{dt} \;\alpha\; (\theta - \theta_0) \quad \Rightarrow \quad \frac{d\theta}{dt} = k(\theta - \theta_0)$$

$$\frac{1}{\theta - \theta_0}\, d\theta = k\, dt$$

$$\int \frac{1}{\theta - \theta_0}\, d\theta = \int k\, dt$$

$$\ln(\theta - \theta_0) = kt + c$$

Example 3(1): A cup of coffee, originally at $90^\circ C$ is left to cool in a room at a constant $20^\circ C$. After 5 minutes, the temperature is $80^\circ C$. Find how long the coffee takes to cool to $60^\circ C$.

At $t = 0$ Initial Body temperature $90^\circ C$
Surrounding temperature $20^\circ C$

$$\frac{d\theta}{dt} \;\alpha\; (\theta - \theta_0) \quad \Rightarrow \quad \frac{d\theta}{dt} = k(\theta - \theta_0)$$

$$\frac{1}{\theta - \theta_0}\, d\theta = k\, dt$$

$$\int \frac{1}{\theta - \theta_0}\, d\theta = \int k\, dt$$

$$\ln(\theta - \theta_0) = kt + c$$

$$\ln(90 - 20) = k(0) + c$$

$$c = \ln 70$$

$$\ln(\theta - \theta_0) = kt + \ln 70$$

At $t = 5, \theta = 80^\circ C$

$$\ln(80 - 20) = k(5) + \ln 70$$

$$\ln 60 = 5k + \ln 70$$

$$5k = \ln 60 - \ln 70$$

$$k = \frac{\ln 60 - \ln 70}{5}$$

$$k = -0.0308$$

$$\ln(\theta - \theta_0) = -0.0308t + \ln 70$$

Example 3(1): A cup of coffee, originally at $90^\circ C$ is left to cool in a room at a constant $20^\circ C$. After 5 minutes, the temperature is $80^\circ C$. Find how long the coffee takes to cool to $60^\circ C$.

$$\ln(\theta - \theta_0) = -0.0308t + \ln 70$$

At $t = ?, \theta = 60^\circ C$

Surrounding temperature $20^\circ C$

$$\ln(60 - 20) = -0.0308t + \ln 70$$

$$\ln 40 = -0.0308t + \ln 70$$

$$0.0308t = \ln 70 - \ln 40$$

$$t = \frac{\ln 70 - \ln 40}{0.0308}$$

$$t = 18.2$$

Example 3(2): A liquid is heated in an oven kept at a constant $180^\circ C$. It is assumed that the rate of increase in the temperature of the liquid is proportional to $(180^\circ C - \theta)$ where θ is the temperature of the liquid at time t minutes. If the temperature rises from $0^\circ C$ to $120^\circ C$ in 5 minutes, find the temperature of the liquid after a further 5 minutes.

At $t = 0$ Initial Body temperature $0^\circ C$
Surrounding temperature $180^\circ C$

$$\frac{d\theta}{dt} \;\alpha\; (180^\circ C - \theta)$$

$$\frac{d\theta}{dt} = k(180 - \theta)$$

$$\frac{1}{(180 - \theta)}\, d\theta = k\, dt$$

$$\int \frac{1}{(180 - \theta)}\, d\theta = \int k\, dt$$

$$-\ln(180 - \theta) = kt + c$$

$$-\ln(180 - 0) = k(0) + c$$

$$c = -\ln(180)$$

$$-\ln(180 - \theta) = kt - \ln 180$$

At $t = 5,\ \theta = 120$

$$-\ln(180 - 120) = k(5) - \ln 180$$

$$-\ln 60 = 5k - \ln 180$$

$$5k = \ln 180 - \ln 60$$

$$k = \frac{\ln 180 - \ln 60}{5}$$

$$k = 0.220$$

$$-\ln(180 - \theta) = 0.220t - \ln 180$$

Example 3(2): A liquid is heated in an oven kept at a constant $180^\circ C$. It is assumed that the rate of increase in the temperature of the liquid is proportional to $(180^\circ C - \theta)$ where θ is the temperature of the liquid at time t minutes. If the temperature rises from $0^\circ C$ to $120^\circ C$ in 5 minutes, find the temperature of the liquid after a further 5 minutes.

Further 5 minutes $\quad t = 10 \quad \theta = ?$

$$-\ln(180 - \theta) = 0.220t - \ln 180$$

$$-\ln(180 - \theta) = 0.220(10) - \ln 180$$

$$\ln(180 - \theta) = -0.220(10) + \ln 180$$

$$180 - \theta = e^{-0.220(10) + \ln 180}$$

$$\theta = 180 - e^{-0.220(10) + \ln 180}$$

$$\theta = 180 - 20$$

$$\theta = 160$$

Example 3(3): The temperature of a quantity of liquid at time t is θ. The liquid is cooling in an atmosphere whose temperature is constant and equal to A. The rate of decrease of θ is proportional to the temperature difference $(\theta - A)$. Thus θ and t satisfy the differential equation.

$$\frac{d\theta}{dt} = -k(\theta - A)$$

where k is a constant.

(i) Find, in any form, the solution to this differential equation, given that $\theta = 4A$ when t = 0

$$\frac{d\theta}{dt} = -k(\theta - A)$$

$$\frac{1}{(\theta - A)}\, d\theta = -k\, dt$$

$$\int \frac{1}{(\theta - A)}\, d\theta = \int -k\, dt$$

$$\ln(\theta - A) = -kt + c$$

$$\ln(4A - A) = -k(0) + c$$

$$c = \ln 3A$$

$$\ln(\theta - A) = -kt + \ln 3A$$

Example 3(3): (ii)Given also that $\theta = 3A$ when t=1, show that $k = \ln \frac{3}{2}$

$$\boxed{\ln(\theta - A) = -kt + \ln 3A}$$

$$\boxed{\text{At } t = 1, \quad \theta = 3A}$$

$$\ln(3A - A) = -k(1) + \ln 3A$$

$$\ln 2A = -k + \ln 3A$$

$$k = \ln 3A - \ln 2A$$

$$k = \ln \frac{3A}{2A}$$

$$\boxed{k = \ln \frac{3}{2}}$$

Example 3(3): (iii) Find θ in terms of A when t = 2, expressing your answer in its simplest form.

$$\boxed{\ln(\theta - A) = \left(-\ln \frac{3}{2}\right) t + \ln 3A}$$

At $t = 2$ $\quad \ln(\theta - A) = \left(-\ln \frac{3}{2}\right)(2) + \ln 3A$

$$\theta - A = e^{\left(-\ln \frac{3}{2}\right)(2) + \ln 3A}$$

$$\theta = e^{\left(-\ln \frac{3}{2}\right)(2) + \ln 3A} + A$$

$$\theta = e^{\left(-\ln \frac{3}{2}\right)(2)} e^{\ln 3A} + A$$

$$\theta = e^{\left(-2\ln \frac{3}{2}\right)}(3A) + A$$

$$\theta = e^{\left(\ln\left(\frac{3}{2}\right)^{-2}\right)}(3A) + A$$

$$\theta = e^{\left(\ln \frac{4}{9}\right)}(3A) + A$$

$$\theta = \frac{4}{9}(3A) + A$$

$$\theta = \frac{4}{3}(A) + A \qquad \boxed{\theta = \frac{7}{3}A}$$

APPLICATION ON DIFFERENTIAL EQUATIONS

RATE OF LIQUID LEAKING

ON THE BOARD :D