Chapter 6

Control of Aircraft Motions

These notes provide a brief background in modern control theory and its application to the equations of motion for a flight vehicle. The description is meant to provide the basic background in linear algebra for understanding how modern tools for the analysis of linear systems work, and provide examples of their application to flight vehicle dynamics and control. The treatment includes a brief introduction to optimal control.

6.1 Control Response

6.1.1 Laplace Transforms and State Transition

So far, we have investigated only the response of a system to a perturbation, which corresponds to the *homogeneous* solution to the system of ordinary differential equations describing the system. In order to study the response of the system to *control input*, it is convenient to use Laplace transforms; see Section 6.7 for a brief review of Laplace transforms.

The Laplace transform of the function $y(t)$, assumed identically zero for $t < 0$, is

$$\mathcal{L}(y(t)) = Y(s) = \int_0^\infty y(t)e^{-st}\,\mathrm{d}t \tag{6.1}$$

and this operation can be applied to each component of a state vector to give the Laplace transform of the state vector

$$\mathcal{L}(\mathbf{x}(t)) = \begin{pmatrix} \mathcal{L}(x_1(t)) \\ \mathcal{L}(x_2(t)) \\ \dots \\ \mathcal{L}(x_n(t)) \end{pmatrix} = \begin{pmatrix} X_1(s)) \\ X_2(s)) \\ \dots \\ X_n(s)) \end{pmatrix} = \mathbf{X}(s) \tag{6.2}$$

Applying this operation to the terms of the (linear) state space equation (see Eq. (6.233) in Section 6.7 for the Laplace transform of the derivative of a function)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta \tag{6.3}$$

gives

$$-\mathbf{x}(0) + s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\eta(s)$$

or

$$[s\mathbf{I} - \mathbf{A}]\,\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\eta(s) \tag{6.4}$$

Assuming that the inverse $[s\mathbf{I} - \mathbf{B}]^{-1}$ exists, this can be written as

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\,[\mathbf{x}(0) + \mathbf{B}\eta(s)] \tag{6.5}$$

The matrix $[s\mathbf{I} - \mathbf{B}]^{-1}$ is called the *resolvent*, and its inverse Laplace transform is called the *transition matrix*

$$\mathbf{\Phi}(t) = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\} \tag{6.6}$$

Taking the inverse Laplace transform of Eq. (6.5) gives

$$\mathcal{L}^{-1}\left(\mathbf{X}(s)\right) = \mathbf{x}(t) = \mathcal{L}^{-1}\left([s\mathbf{I} - \mathbf{A}]^{-1}\right)\mathbf{x}(0) + \mathcal{L}^{-1}\left([s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B}\eta(s)\right) \tag{6.7}$$

The convolution theorem (see Eq. (6.251)) can be used to write the inverse Laplace transform of the product appearing in the second term on the right hand side of this equation as

$$\mathcal{L}^{-1}\left([s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B}\eta(s)\right) = \mathcal{L}^{-1}\left(\mathcal{L}(\mathbf{\Phi})\mathbf{B}\eta(s)\right) = \int_0^t \mathbf{\Phi}(t-\tau)B\eta(\tau)\,\mathrm{d}\tau \tag{6.8}$$

whence Eq. (6.7) can be written

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\eta(\tau)\,\mathrm{d}\tau \tag{6.9}$$

Thus, it is seen that the matrix $\mathbf{\Phi}$ "transitions" the state vector from its initial state $\mathbf{x}(0)$ to its state at a later time t, including the effects of control input through the convolution integral in the second term on the right-hand side.

6.1.2 The Matrix Exponential

A useful expression for the transition matrix for the case of *linear, time-invariant systems* – i.e., those systems that can be described by systems of differential equations of the form of Eqs. (6.3) in which the matrices **A** and **B** are constants, independent of time – can be written in terms of the so-called *matrix exponential*.

As motivation, recall that for the case of a single (scalar) equation

$$\dot{x} = ax \tag{6.10}$$

the Laplace transform gives

$$-x(0) + sX(s) = aX(s)$$

or

$$X(s) = \frac{1}{s-a}x(0) \tag{6.11}$$

and we can write

$$x(t) = \mathcal{L}^{-1}\left(\frac{1}{s-a}\right)x(0) \tag{6.12}$$

Now, recall that (see Eq. (6.239) in the Section 6.7)

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at} \tag{6.13}$$

so the solution to Eq. (6.10) is

$$x(t) = e^{at}x(0) \tag{6.14}$$

This is, of course, no surprise; we have simply determined the solution to the almost trivial Eq. (6.10) using the very powerful tool of Laplace transforms. But Eq. (6.14) shows us that, for the case of a single equation the transition matrix is simply

$$\mathbf{\Phi}(t) = e^{at}$$

Now, we can also write

$$\frac{1}{s-a} = \frac{1}{s(1-a/s)} = \frac{1}{s} + \frac{a}{s^2} + \frac{a^2}{s^3} + \frac{a^3}{s^4} + \cdots \tag{6.15}$$

and, since

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \quad \text{or} \quad \mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n \tag{6.16}$$

the series of Eq. (6.15) can be inverted, term by term, to give

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \frac{(at)^4}{4!} + \cdots \tag{6.17}$$

Now, it may seem that we've just taken the long way around to illustrate the usual power series representation of e^{at}. But our goal was to suggest that the *matrix analog* of Eq. (6.13) is

$$\mathcal{L}^{-1}\left([s\mathbf{I} - \mathbf{A}]^{-1}\right) = e^{\mathbf{A}t} \tag{6.18}$$

where the *matrix exponential* is understood to be defined as

$$e^{\mathbf{A}t} \equiv \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \frac{(\mathbf{A}t)^3}{3!} + \frac{(\mathbf{A}t)^4}{4!} + \cdots \tag{6.19}$$

To verify this conjecture, we note that the matrix analog of Eq. (6.15) is

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{\mathbf{A}^2}{s^3} + \frac{\mathbf{A}^3}{s^4} + \cdots \tag{6.20}$$

The validity of this equation can be verified by premultiplying by $s\mathbf{I} - \mathbf{A}$ to give

$$\mathbf{I} = \mathbf{I} - \frac{\mathbf{A}}{s} + \left(\frac{\mathbf{A}}{s} - \frac{\mathbf{A}^2}{s^2}\right) + \left(\frac{\mathbf{A}^2}{s^2} - \frac{\mathbf{A}^3}{s^3}\right) + \cdots$$

Successive terms on the right hand side cancel to give the identity $\mathbf{I} = \mathbf{I}$, so long as the series converges, which will be assumed here.

Now, taking the inverse Laplace transform of Eq. (6.20) gives

$$\mathcal{L}^{-1}\left([s\mathbf{I} - \mathbf{A}]^{-1}\right) = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \frac{(\mathbf{A}t)^3}{3!} + \cdots = e^{\mathbf{A}t} \tag{6.21}$$

Thus, we have shown that the state transition matrix for the general linear time-invariant system can be expressed as

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} \tag{6.22}$$

where the definition of the matrix exponential appearing here is taken to be Eq. (6.19). The numerical computation of the matrix exponential is not always a trivial task, especially if the matrix is large and ill-conditioned; but most software packages, such as MATLAB have standard routines that work well for most cases of interest.

Using Eq. (6.22), we can express the solution of the state space system as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\eta(\tau)\,\mathrm{d}\tau$$

or

$$\mathbf{x}(t) = e^{\mathbf{A}t}\left[\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\eta(\tau)\,\mathrm{d}\tau\right] \tag{6.23}$$

Some useful properties of the state transition matrix, which can be seen from its definition in terms of the matrix exponential are:

1. The transition matrix evaluated at $t = 0$ is the identity matrix; i.e.,

$$\mathbf{\Phi}(0) = \mathbf{I} \tag{6.24}$$

2. The transition matrix for the sum of two time intervals is the product of the individual transition matrices in either order; i.e.,

$$\mathbf{\Phi}(t_1 + t_2) = \mathbf{\Phi}(t_1)\mathbf{\Phi}(t_2) = \mathbf{\Phi}(t_2)\mathbf{\Phi}(t_1) \tag{6.25}$$

   This is equivalent to

$$e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}e^{\mathbf{A}t_2} = e^{\mathbf{A}t_2}e^{\mathbf{A}t_1} \tag{6.26}$$

   which can be verified directly by substitution into Eq. (6.19).

3. The relation

$$e^{-\mathbf{A}t} = \left[e^{\mathbf{A}t}\right]^{-1} \tag{6.27}$$

   can be verified by setting $t_2 = -t_1$ in Property 2, then using Property 1.

4. The commutativity property

$$\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A} \tag{6.28}$$

   can be verified directly by pre- and post-multiplying Eq. (6.19) by the matrix **A**.

5. The differentiation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\mathbf{A}t}\right) = \mathbf{A}e^{\mathbf{A}t} \tag{6.29}$$

   can be verified by differentiating Eq. (6.19) term by term.

6.2 System Time Response

The state vector solution for the *homogeneous* response of the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\eta \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\eta \end{aligned} \tag{6.30}$$

has been seen to be

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) \tag{6.31}$$

and hence

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) \tag{6.32}$$

We now consider the system response to several typical control inputs.

6.2.1 Impulse Response

For an impulsive input, we define

$$\eta(\tau) = \eta_0 \delta(\tau) \tag{6.33}$$

where $\eta_0 = [\delta_{1_0} \quad \delta_{2_0} \quad \dots \quad \delta_{p_0}]^T$ is a constant vector that determines the relative weights of the various control inputs and $\delta(t)$ is the *Dirac delta function*. Recall that the Dirac delta (or *impulse*) function has the properties

$$\begin{aligned} \delta(t-\tau) = 0 \quad \text{for} \quad t \neq \tau \\ \int_{-\infty}^{\infty} \delta(t-\tau)\, \mathrm{d}t = 1 \end{aligned} \tag{6.34}$$

These properties can be used to see that

$$\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\eta_0\delta(\tau)\,\mathrm{d}\tau = \mathbf{B}\eta_0 \tag{6.35}$$

so the system response to the impulsive input of Eq. (6.33) is seen to be

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{B}\eta_0 \\ \mathbf{y}(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{B}\eta_0 \end{aligned} \tag{6.36}$$

Note that since the vector $\mathbf{B}\eta_0$ can be interpreted as a specified initial perturbation $\mathbf{x}(0)$, we see that the system response to an impulsive input at $t = 0$ is equivalent to the homogeneous solution for the specified $\mathbf{x}(0) = \mathbf{B}\eta_0$.

6.2.2 Doublet Response

A doublet is the derivative of the delta function, so the system response to a doublet control input is simply the derivative of the analogous impulsive response. Thus, if

$$\eta(\tau) = \eta_0 \frac{\mathrm{d}\delta(\tau)}{\mathrm{d}\tau} \tag{6.37}$$

where $\eta_0 = [\delta_{1_0} \quad \delta_{2_0} \quad \dots \quad \delta_{p_0}]^T$ is a constant vector that determines the relative weights of the various control inputs and $\delta(t)$ is the *Dirac delta function*, the system response will be the derivative of the impulsive response given by Eq. (6.36), i.e.,

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{A}\mathbf{B}\eta_0 \\ \mathbf{y}(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{A}\mathbf{B}\eta_0 \end{aligned} \tag{6.38}$$

As for the impulsive response, here the vector $\mathbf{AB}\eta_0$ can be interpreted as a specified initial perturbation $\mathbf{x}(0)$, so we see that the system response to a doublet input at $t = 0$ is equivalent to the homogeneous solution for the specified initial perturbation $\mathbf{x}(0) = \mathbf{AB}\eta_0$.

6.2.3 Step Response

For a step input, we define

$$\eta(\tau) = \eta_0 H(\tau) \tag{6.39}$$

where the *Heaviside step function* is defined as

$$H(\tau) = \begin{cases} 0, & \text{for } \tau < 0 \\ 1, & \text{for } \tau \geq 0 \end{cases} \tag{6.40}$$

These properties can be used to see that

$$\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\eta_0 H(\tau)\,\mathrm{d}\tau = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\eta_0\,\mathrm{d}\tau = \left(\int_0^t e^{-\mathbf{A}\tau}\,\mathrm{d}\tau\right)\mathbf{B}\eta_0 \tag{6.41}$$

We can evaluate the integral in this expression by integrating the definition of the matrix exponential term by term to give

$$\begin{aligned} \int_0^t e^{-\mathbf{A}\tau}\,\mathrm{d}\tau &= \int_0^t \left(\mathbf{I} - \mathbf{A}\tau + \frac{(\mathbf{A}\tau)^2}{2!} - \frac{(\mathbf{A}\tau)^3}{3!}\cdots\right)\mathrm{d}\tau \\ &= \mathbf{I}t - \frac{\mathbf{A}t^2}{2!} + \frac{\mathbf{A}^2t^3}{3!} - \frac{\mathbf{A}^3t^4}{4!} + \cdots \\ &= \left(\mathbf{A}t - \frac{(\mathbf{A}t)^2}{2!} + \frac{(\mathbf{A}t)^3}{3!} - \frac{(\mathbf{A}t)^4}{4!} + \cdots\right)\mathbf{A}^{-1} \\ &= \left(\mathbf{I} - e^{-\mathbf{A}t}\right)\mathbf{A}^{-1} \end{aligned} \tag{6.42}$$

so the system response to a step input becomes

$$\begin{aligned} \mathbf{x}(t) &= \left[e^{\mathbf{A}t} - \mathbf{I}\right]\mathbf{A}^{-1}\mathbf{B}\eta_0 \\ \mathbf{y}(t) &= \mathbf{C}\left[e^{\mathbf{A}t} - \mathbf{I}\right]\mathbf{A}^{-1}\mathbf{B}\eta_0 + \mathbf{D}\eta_0 \end{aligned} \tag{6.43}$$

For a stable system,[1]

$$\lim_{t\to\infty} e^{\mathbf{A}t} = 0 \tag{6.44}$$

so Eq. (6.43) gives

$$\lim_{t\to\infty} \mathbf{x}(t) = -\mathbf{A}^{-1}\mathbf{B}\eta_0 \tag{6.45}$$

as the steady state limit for the step response.

[1] The easiest way to see the validity of Eq. (6.44) is to realize that the response to an initial perturbation is shown by Eq. (6.23) to be equal to this matrix exponential times the initial perturbation. For a stable system, this must vanish in the limit as $t \to \infty$ for any initial perturbation.

6.2.4 Example of Response to Control Input

We here include two examples of aircraft response to control input. We examine the longitudinal response to both, impulsive and step, elevator input for the Boeing 747 in powered approach at $\mathbf{M} = 0.25$ and standard sea level conditions. This is the same equilibrium flight condition studied in the earlier chapter on unforced response. The aircraft properties and flight condition are given by

$$\begin{aligned} V &= 279.1 \text{ ft/sec}, \quad & \rho &= 0.002377 \text{ slug/ft}^3 \\ S &= 5,500. \text{ ft}^2, & \bar{c} &= 27.3 \text{ ft} \\ W &= 564,032. \text{ lb}, & I_y &= 32.3 \times 10^6 \text{ slug-ft}^2 \end{aligned} \tag{6.46}$$

and the relevant aerodynamic coefficients are

$$\begin{array}{lllll} \mathbf{C}_L = 1.108, & \mathbf{C}_D = 0.102, & \Theta_0 = 0 & & \\ \mathbf{C}_{L\alpha} = 5.70, & \mathbf{C}_{L\dot{\alpha}} = 6.7, & \mathbf{C}_{Lq} = 5.4, & \mathbf{C}_{LM} = 0 & \mathbf{C}_{L\delta_e} = 0.338 \\ \mathbf{C}_{D\alpha} = 0.66, & & & & \\ \mathbf{C}_{m\alpha} = -1.26, & \mathbf{C}_{m\dot{\alpha}} = -3.2, & \mathbf{C}_{mq} = -20.8, & \mathbf{C}_{mM} = 0 & \mathbf{C}_{m\delta_e} = -1.34 \end{array} \tag{6.47}$$

These values correspond to the following dimensional stability derivatives

$$\begin{array}{lllll} X_u = -0.0212, & X_w = 0.0466 & & & \\ Z_u = -0.2306, & Z_w = -0.6038, & Z_{\dot{w}} = -0.0341, & Z_q = -7.674 & Z_{\delta_e} = -9.8175 \\ M_u = 0.0, & M_w = -0.0019, & M_{\dot{w}} = -0.0002, & M_q = -0.4381 & M_{\delta_e} = -.5769 \end{array} \tag{6.48}$$

and the plant and control matrices are

$$\mathbf{A} = \begin{pmatrix} -0.0212 & 0.0466 & 0.0000 & -.1153 \\ -0.2229 & -0.5839 & 0.9404 & 0.0000 \\ 0.0150 & -0.5031 & -0.5015 & 0.0000 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0.0000 \\ -.0340 \\ -.5746 \\ 0.0000 \end{pmatrix} \tag{6.49}$$

when the state vector is chosen to be[2]

$$\mathbf{x} = \begin{pmatrix} u/u_0 & \alpha & q & \theta \end{pmatrix}^T \tag{6.50}$$

The response to an impulsive input is shown in Fig. 6.1. Both short period and phugoid modes are excited, and the phugoid is very lightly damped and persists for a long time. Ultimately, however, original equilibrium state will be restored, since impulsive input is equivalent to unforced response with a particular initial perturbation, as shown if Eq.(6.36).

The response to a one-degree step input is shown in Fig. 6.2. Both short period and phugoid modes are again excited, the short-period less than for the impulsive input as the step input has less high-frequency content. Since the phugoid is very lightly damped it again persists for a long time.

In this case, the system ultimately settles into a new equilibrium state, that given by Eq. (6.45) which, for this case, is found to be

$$\lim_{t\to\infty} \mathbf{x}(t) = -\mathbf{A}^{-1}\mathbf{B}\eta_0 = [0.0459 \quad -.0186 \quad 0.0 \quad -.0160]^T \tag{6.51}$$

[2]Note that this is the form introduced in Chapter 5 in which the velocities have been normalized by the equilibrium flight speed.

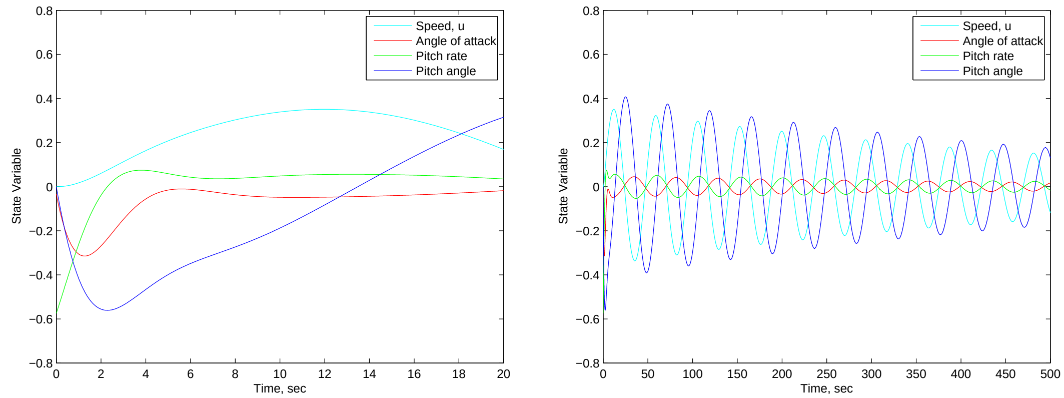


Figure 6.1: Response of Boeing 747 in powered approach at $\mathbf{M} = 0.25$ and standard sea level conditions to impulsive elevator input. Left plot is scaled to illustrate short-period response, and right plot is scaled to illustrate phugoid.

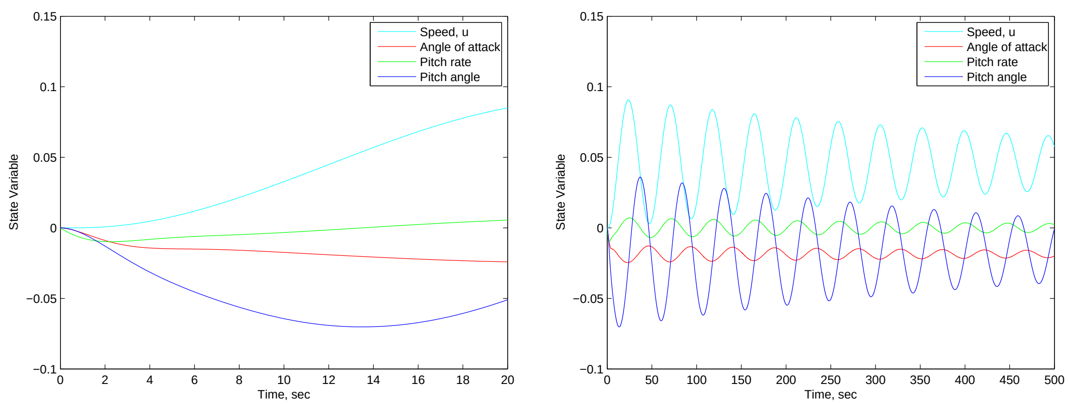


Figure 6.2: Response of Boeing 747 in powered approach at $\mathbf{M} = 0.25$ and standard sea level conditions to one-degree step elevator input. Left plot is scaled to illustrate short-period response, and right plot is scaled to illustrate phugoid.

for the one-degree value of η_0. The new equilibrium state corresponds to an increase in flight speed at a reduced angle of attack. Since the resulting lift coefficient is reduced, the pitch angle becomes negative – i.e., the aircraft has begun to descend.

The approximations of the preceding analysis are completely consistent with those made in our earlier study of *static* longitudinal control, where we found the control sensitivity to be

$$\left.\frac{\mathrm{d}\delta_e}{\mathrm{d}\mathbf{C}_L}\right)_{\text{trim}} = \frac{\mathbf{C}_{m\alpha}}{\Delta} \tag{6.52}$$

where

$$\Delta = -\mathbf{C}_{L\alpha}\mathbf{C}_{m\delta_e} + \mathbf{C}_{m\alpha}\mathbf{C}_{L\delta_e} \tag{6.53}$$

Thus, from the static analysis we estimate for a step input of one degree in elevator

$$\Delta\mathbf{C}_L = \frac{\delta_e}{\mathbf{C}_{m\alpha}/\Delta} = \frac{\pi/180}{(-1.26)/(7.212)} = -.100 \tag{6.54}$$

The asymptotic steady state of the dynamic analysis gives exactly the same result

$$\Delta\mathbf{C}_L = \mathbf{C}_{L\alpha}\alpha + \mathbf{C}_{L\delta_e} = 5.70(-.0186) + 0.338(\pi/180) = -.100 \tag{6.55}$$

This result illustrates the consistency of the static and dynamic analyses. Note, however, that if the dynamic analysis included compressibility or aeroelastic effects, the results would not have agreed exactly, as these effects were not taken into account in the static control analysis.

6.3 System Frequency Response

The *frequency response* of a system corresponds to its response to harmonic control input of the form

$$\begin{aligned}\eta(t) &= \eta_0 e^{\imath\omega t} H(t)\\ &= \eta_0\left(\cos\omega t + \imath\sin\omega t\right) H(t)\end{aligned} \tag{6.56}$$

where $H(t)$ is the Heaviside step function, see Eq. (6.40). This input corresponds to a sinusoidal oscillation of the control at frequency ω, and the system response consists of a start-up transient which ultimately evolves into an asymptotically steady-state harmonic response. Plots of the amplitude of the steady-state harmonic response as a function of the input frequency ω are known as *Bode plots*, and are useful for identifying resonant frequencies of the system.

Frequency response is an important element of classical control theory, and is the principal reason that Laplace transforms are such an important tool for control system designers. We will, however, constrain ourselves (at least for now) to dealing with response in the time domain, and not consider frequency response further.

6.4 Controllability and Observability

Two important properties of a system are its controllability and its observability. *Controllability* relates to the ability of the control input to influence all modes of the system. For a system having a single scalar control variable $\eta(t)$ the system is described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta(t) \tag{6.57}$$

The system response is related to the eigenvalues of the matrix $\mathbf{A}$, and these are invariant under a transformation of coordinates. The state vector can be transformed to *modal coordinates* by

$$\mathbf{v}(t) = \mathbf{P}^{-1}\mathbf{x}(t) \tag{6.58}$$

where $\mathbf{P}$ is the *modal matrix* of $\mathbf{A}$. That is, the *similarity transformation*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda} \tag{6.59}$$

transforms $\mathbf{A}$ to the *diagonal* matrix $\mathbf{\Lambda}$. If such a transformation exists,[3] then the state equations can be written as

$$\mathbf{P}\dot{\mathbf{v}} = \mathbf{A}\mathbf{P}\mathbf{v} + \mathbf{B}\eta(t) \tag{6.60}$$

or, after pre-multiplying by $\mathbf{P}^{-1}$,

$$\dot{\mathbf{v}} = \mathbf{\Lambda}\mathbf{v} + \mathbf{P}^{-1}\mathbf{B}\eta(t) \tag{6.61}$$

Since $\mathbf{\Lambda}$ is a diagonal matrix, this transformation has completely *decoupled* the equations; i.e., Eqs. (6.61) are equivalent to

$$\dot{v}_j = \lambda_j v_j + f_j\eta(t) \tag{6.62}$$

where f_j are the elements of the vector $\mathbf{P}^{-1}\mathbf{B}$. Thus, the evolution of each mode is independent of all the others, and the j-th mode is affected by the control so long as $f_j \neq 0$. In other words, all the modes are controllable so long as no element of $\mathbf{P}^{-1}\mathbf{B}$ is zero.

This same transformation process can be applied to the case when η is a vector – i.e., when there are multiple control inputs. In this case, all modes are controllable so long as at least one element in each *row* of the matrix $\mathbf{P}^{-1}\mathbf{B}$ is non-zero.

6.4.1 Controllability

For cases in which the plant matrix is not diagonalizable a more general procedure must be followed to determine whether the system is controllable. In these cases, we introduce the more specific definition of controllability:

> **Definition:** A system is said to be controllable if it is possible by means of an unconstrained controller to transfer the physical system between any two arbitrarily specified states in a finite time.

The requirement for controllability is well understood for linear, time-invariant systems. For such systems we can write

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta(t) \tag{6.63}$$

where we assume, for simplicity of presentation, that $\eta(t)$ represents a single control variable. Thus, if the state vector $\mathbf{x}$ has n elements, $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B}$ is an $n \times 1$ column vector.

[3] A diagonalizing transformation will exist if the matrix $\mathbf{A}$ has a complete set of linearly independent eigenvectors; in this case the modal matrix $\mathbf{P}$ will be non-singular and its inverse will exist. A sufficient condition for the matrix $\mathbf{A}$ to have a complete set of linearly independent eigenvectors is that its eigenvalues be real and distinct, but this condition is not necessary.

The system of Eqs. (6.63) has the response

$$\dot{\mathbf{x}}(t) = e^{\mathbf{A}t}\left[\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}t}\mathbf{B}\eta(\tau)\,\mathrm{d}\tau\right] \tag{6.64}$$

Since the states are arbitrary, we can choose the final state $\mathbf{x}(t) = 0$ with no loss of generality, in which case Eqs. (6.64) become

$$\mathbf{x}(0) = -\int_0^t e^{-\mathbf{A}t}\mathbf{B}\eta(\tau)\,\mathrm{d}\tau \tag{6.65}$$

Thus, the question of controllability reduces to whether a control law $\eta(\tau)$ exists that satisfies Eqs. (6.65) for every possible initial state $\mathbf{x}(0)$.

Recall that the Cayley-Hamilton theorem tells us that, if the characteristic equation of the plant matrix is

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0 \tag{6.66}$$

then we also have

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + a_{n-2}\mathbf{A}^{n-2} + \cdots + a_1\mathbf{A} + a_0\mathbf{I} = 0 \tag{6.67}$$

This equation can be used to represent any polynomial in the matrix $\mathbf{A}$ as a polynomial of order $n-1$. In particular, it can be used to represent the matrix exponential as the *finite* sum

$$\begin{aligned} e^{-\mathbf{A}\tau} &= \mathbf{I} - \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} - \frac{\mathbf{A}^3\tau^3}{3!} + \cdots \\ &= \sum_{k=0}^{n-1} f_k(\tau)\mathbf{A}^k \end{aligned} \tag{6.68}$$

The actual process of determining the coefficient functions $f_k(\tau)$ might be very difficult and tedious, but for our purposes here we don't need to determine these coefficient functions explicitly, we only need to believe that such a representation is always possible.

Using Eq. (6.68) allows us to write the controllability requirement as

$$\mathbf{x}(0) = -\int_0^t \sum_{k=0}^{n-1} f_k(\tau)\mathbf{A}^k\mathbf{B}\eta(\tau)\,\mathrm{d}\tau \tag{6.69}$$

or

$$\mathbf{x}(0) = -\sum_{k=0}^{n-1} \mathbf{A}^k\mathbf{B}\int_0^t f_k(\tau)\eta(\tau)\,\mathrm{d}\tau \tag{6.70}$$

Now, for any $f_k(\tau)$ and $\eta(\tau)$ we can write

$$\int_0^t f_k(\tau)\eta(\tau)\,\mathrm{d}\tau = g_k \qquad \text{for } k = 0, 1, 2, \ldots, n-1 \tag{6.71}$$

so Eqs.(6.70) can be written as

$$\begin{aligned}
\mathbf{x}(0) &= -\sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{B} g_k \\
&= -\mathbf{B}g_0 - \mathbf{A}\mathbf{B}g_1 - \mathbf{A}^2\mathbf{B}g_2 - \cdots - \mathbf{A}^{n-1}\mathbf{B}g_{n-1} \\
&= -\begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_{n-1} \end{pmatrix}
\end{aligned} \tag{6.72}$$

This is a system of equations for the vector $\mathbf{g} = [g_0 \quad g_1 \quad g_2 \quad \ldots \quad g_{n-1}]^T$ of the form

$$\mathbf{V}\mathbf{g} = -\mathbf{x}(0) \tag{6.73}$$

which will have a solution for any arbitrarily chosen $\mathbf{x}(0)$ if the *controllability matrix*

$$\mathbf{V} = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} \tag{6.74}$$

has full rank n.

Our analysis here has assumed there is only a single (scalar) control variable, but the analysis follows through with no essential change in the case when the control variable $\eta(\tau)$ is a p-element vector. In this case $\mathbf{g}$ will be an $n \cdot p \times 1$ vector, and the corresponding controllability matrix will have the same form as in Eq. (6.74), but since each element there has the same shape as $\mathbf{B}$ – an $n \times p$ matrix – the controllability matrix will have n rows and $n \cdot p$ columns. The controllability criterion still requires that the rank of this matrix be n.

Generally, elevator control alone is sufficient to control all longitudinal modes, and *either* rudder or aileron control is sufficient to control all lateral/directional modes.

It should be noted that controllability alone says nothing about the *quality* of the control, since arbitrarily large control input was assumed to be available. So, it is still important to look at specific control responses and/or sensitivities to determine if sufficient control action is available to achieve desired motions without saturating the controls.

Example

We consider the example of the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta(t) \tag{6.75}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.76}$$

The characteristic equation of the plant matrix is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 3\lambda + 2 = 0 \tag{6.77}$$

whose roots are $\lambda = -1, -2$.

The eigenvectors of $\mathbf{A}$ are thus determined from

$$(\mathbf{A} - \lambda_1 \mathbf{I})\, \mathbf{u}_1 = 0$$

which gives

$$u_{1_1} + u_{1_2} = 0$$

whence

$$\mathbf{u}_1 = [1 \quad -1]^T$$

Similarly,

$$(\mathbf{A} - \lambda_2 \mathbf{I})\, \mathbf{u}_2 = 0$$

gives

$$2u_{2_1} + u_{2_2} = 0$$

whence

$$\mathbf{u}_2 = [1 \quad -2]^T$$

The modal matrix of $\mathbf{A}$ and its inverse are then

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{P}^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \tag{6.78}$$

The characteristic variables are thus

$$\mathbf{v} = \mathbf{P}^{-1}\mathbf{x} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ -x_1 - x_2 \end{pmatrix} \tag{6.79}$$

and

$$\mathbf{PAP}^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \tag{6.80}$$

and

$$\mathbf{P}^{-1}\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6.81}$$

The canonical form of the equations describing the system can thus be written

$$\dot{\mathbf{v}} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{v} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \eta(t) \tag{6.82}$$

Both modes are thus seen to be controllable.

Alternatively, since

$$\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{6.83}$$

the controllability matrix is

$$\mathbf{V} = [\mathbf{B} \quad \mathbf{AB}] = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix} \tag{6.84}$$

The determinant of the controllability matrix $\det(\mathbf{V}) = -1$, is non-zero, so its rank must be 2, and the system is again seen to be controllable.

Note that if the control matrix is changed to

$$\mathbf{B} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{6.85}$$

then

$$\mathbf{P}^{-1}\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{6.86}$$

and the second mode is seen to be uncontrollable. Equivalently, since we now have

$$\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6.87}$$

the controllability matrix becomes

$$\mathbf{V} = [\mathbf{B} \quad \mathbf{AB}] = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \tag{6.88}$$

The determinant of the controllability matrix $\det(\mathbf{V}) = 0$, is now zero, so its rank must be less than 2, and the modified system is again seen to be uncontrollable.

6.4.2 Observability

The mathematical *dual* of controllability is *observability*, which is defined according to:

> **Definition:** A system is observable at time t_0 if the output history $\mathbf{y}(t)$ in the time interval $[t_0, t_f]$ is sufficient to determine $\mathbf{x}(t_0)$.

It can be shown, by a process analogous to that of the preceding section, that, for linear, time-invariant systems, observability is guaranteed when the rank of the *observability matrix*

$$\mathbf{U} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{6.89}$$

is equal to n. Note that if the output vector $\mathbf{y}$ has q elements, the observability matrix will have $q \cdot n$ rows and n columns.

6.4.3 Controllability, Observability, and Matlab

Once the plant matrix $\mathbf{A}$, the control matrix $\mathbf{B}$, and the output matrix $\mathbf{C}$ have been defined, the Matlab function

$$\mathbf{V} = \mathit{ctrb}(\mathbf{A},\mathbf{B})$$

determines the controllability matrix **V**, and the MATLAB function

$$\mathbf{U} = obsv(\mathbf{A},\mathbf{C})$$

determines the observability matrix **U**. The rank of either of these matrices can then be determined using the MATLAB function *rank*.

6.5 State Feedback Design

A feedback control system can be designed within the state-variable framework to provide a specific eigenvalue structure for the closed-loop plant matrix. Consider the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\eta \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{6.90}$$

It can be shown that, if the system is controllable it is possible to define a linear control law to achieve any desired closed-loop eigenvalue structure. For a single-input system, a linear control law is given by

$$\eta = -\mathbf{k}^T\mathbf{x} + \eta' \tag{6.91}$$

where η' is the control input in the absence of feedback, and **k** is a vector of feedback gains. The block diagram of this system is illustrated in Fig. 6.3.

Introducing the control law into the state equation system gives

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\left[\eta' - \mathbf{k}^T\mathbf{x}\right] \\ &= \left[\mathbf{A} - \mathbf{B}\mathbf{k}^T\right]\mathbf{x} + \mathbf{B}\eta' \\ &= \mathbf{A}^*\mathbf{x} + \mathbf{B}\eta' \end{aligned} \tag{6.92}$$

where the plant matrix describing the behavior of the closed-loop system

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{k}^T \tag{6.93}$$

is called the *augmented matrix* for the system.

For cases in which the plant matrix **A** of the system has undesirable eigenvalues, the augmented matrix $\mathbf{A}^*$ can be made to have more desirable eigenvalues by proper choice of the elements of the feedback gain vector **k**. Note that the effect of state-variable feedback can be interpreted as modifying the plant matrix of the system; i.e., the effect of the feedback can be interpreted as effectively changing the properties of the system – the aerodynamic stability derivatives – to achieve more desirable response characteristics.

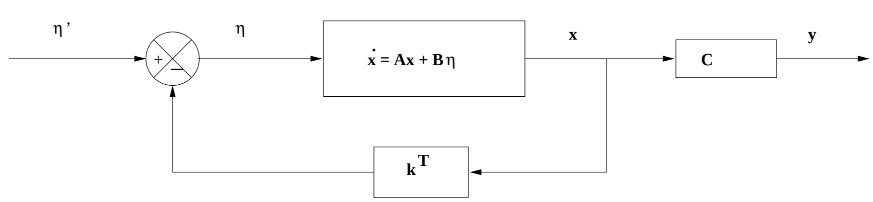


Figure 6.3: Block diagram for system with state-variable feedback.

Example: State Feedback Design

Given the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\eta \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{6.94}$$

with

$$\mathbf{A} = \begin{pmatrix} -3 & 8 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{6.95}$$

we wish to use state-variable feedback to provide closed-loop response having

$$\omega_n = 25 \text{ sec}^{-1} \quad \text{and} \quad \zeta = 0.707 \tag{6.96}$$

Note that the characteristic equation of the original plant matrix is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-3-\lambda)(-\lambda) = \lambda^2 + 3\lambda = \lambda(\lambda+3) = 0 \tag{6.97}$$

so the original system has one neutrally stable eigenvalue.

First, the controllability of the system is verified. For this system the controllability matrix is

$$\mathbf{V} = [\mathbf{B} \quad \mathbf{AB}] = \begin{pmatrix} 0 & 32 \\ 4 & 0 \end{pmatrix} \tag{6.98}$$

so, $\det(\mathbf{V}) = -128$, whence $\mathbf{V}$ has full rank so the system is controllable. The general form of the augmented matrix is

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{k}^T = \begin{pmatrix} -3 & 8 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix} = \begin{pmatrix} -3 & 8 \\ -4k_1 & -4k_2 \end{pmatrix} \tag{6.99}$$

The characteristic equation of the augmented matrix $\mathbf{A}^*$ is then

$$\det(\mathbf{A}^* - \lambda\mathbf{I}) = (-3-\lambda)(-4k_2-\lambda) + 32k_1 = \lambda^2 + (3+4k_2)\lambda + 32k_1 + 12k_2 = 0 \tag{6.100}$$

Since the desired system response corresponds to the characteristic equation

$$\begin{aligned} \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 &= 0 \\ \lambda^2 + 2(0.707)(25)\lambda + (25)^2 &= 0 = \lambda^2 + 35.35\lambda + 625 \end{aligned} \tag{6.101}$$

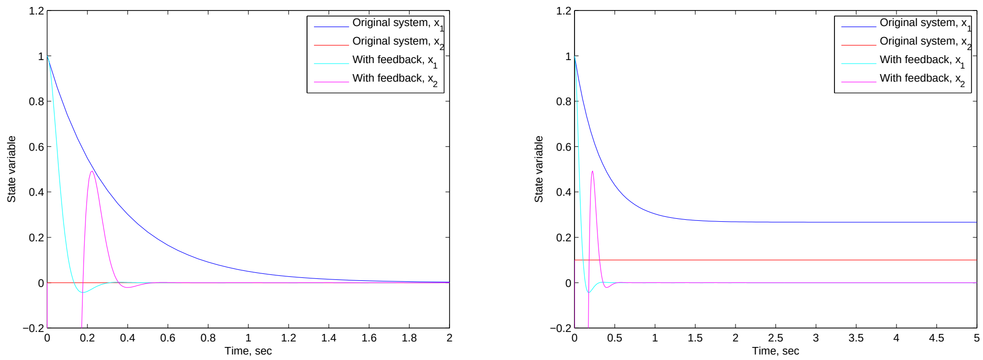
a comparison of Eqs. (6.100) and (6.101) shows that we must choose the elements of the gain vector such that

$$\begin{aligned} 3 + 4k_2 &= 35.35 \\ 32k_1 + 12k_2 &= 625 \end{aligned}$$

or

$$\begin{aligned} k_2 &= \frac{35.35 - 3}{4} = 8.09 \\ k_1 &= \frac{625 - 12(8.09)}{32} = 16.5 \end{aligned} \tag{6.102}$$

The response of the original system and the closed-loop response are compared in Fig. 6.4 for two different initial perturbations. Figure 6.4(a) illustrates the response when the neutrally stable mode is not excited. Figure 6.4(b) illustrates the response when the neutrally stable mode is excited; in this case the original system never returns to the original equilibrium state. But, in both cases the closed-loop system returns quickly, with minimal overshoot, to the original equilibrium state. Note, however, that significant excitation of x_2 is required in the latter case, even for the first initial condition.



(a) $\mathbf{x}(0) = [1.0 \quad 0.0]$ (b) $\mathbf{x}(0) = [1.0 \quad 0.1]$

Figure 6.4: Response of linear, second-order system, showing effect of state variable feedback. Original system has $\lambda_1 = -3.0$, $\lambda_2 = 0.0$. Modified system has $\omega_n = 25$ sec^{-1} and $\zeta = 0.707$. (a) $\mathbf{x}(0) = [1.0 \quad 0.0]^T$; (b) $\mathbf{x}(0) = [1.0 \quad 0.1]^T$.

6.5.1 Single Input State Variable Control

When the control variable is a single scalar, the feedback gains are uniquely determined by the locations of the roots of the characteristic equation of the augmented matrix. In this case, the algorithm of Bass & Gura (see, e.g., [3]) can be used to determine the elements of the gain vector.

We describe the procedure for the system described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta \tag{6.103}$$

with the control law

$$\eta = -\mathbf{k}^T\mathbf{x} + \eta' \tag{6.104}$$

It is desirable to have the plant matrix in the (first) *companion form*,

$$\mathbf{A} = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{6.105}$$

where it is clear from direct calculation of the determinant of $\mathbf{A} - \lambda\mathbf{I}$ that the elements a_i are the coefficients of the characteristic equation

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0 \tag{6.106}$$

of the plant matrix $\mathbf{A}$. Note that the homogeneous equations corresponding to the plant matrix of

Eq. (6.105) are of the form

$$\begin{aligned}
\dot{x}_1 &= -a_1x_1 - a_2x_2 - a_3x_3 - \cdots - a_{n-1}x_{n-1} - a_nx_n \\
\dot{x}_2 &= x_1 \\
\dot{x}_3 &= x_2 \\
\dot{x}_4 &= x_3 \\
&\cdots \\
\dot{x}_n &= x_{n-1}
\end{aligned} \tag{6.107}$$

so the system is equivalent to the single higher-order equation

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_1\frac{\mathrm{d}^{n-1}y}{\mathrm{d}t^{n-1}} + a_2\frac{\mathrm{d}^{n-2}y}{\mathrm{d}t^{n-2}} + \cdots + a_{n-1}\frac{\mathrm{d}y}{\mathrm{d}t} + a_n y = f(t) \tag{6.108}$$

where $y = x_n$. Thus, when the equations are in companion form, the control matrix takes the special form

$$\mathbf{B} = [1 \quad 0 \quad 0 \quad \cdots \quad 0]^T \tag{6.109}$$

Now, as we have seen, when the control law of Eq. (6.104) is substituted into the Eqs. (6.103), the equations take the form

$$\dot{\mathbf{x}} = \mathbf{A}^*\mathbf{x} + \mathbf{B}\eta' \tag{6.110}$$

where

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{k}^T \tag{6.111}$$

is the augmented matrix. Because of the special form of the control matrix when the equations are in companion form, the augmented matrix takes the form

$$\mathbf{A}^* = \begin{pmatrix}
-a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 & \cdots & -a_{n-1} - k_{n-1} & -a_n - k_n \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
 & & & \cdots & & \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix} \tag{6.112}$$

The characteristic equation of the augmented matrix can thus be computed directly as

$$\lambda^n + (a_1 + k_1)\,\lambda^{n-1} + (a_2 + k_2)\,\lambda^{n-2} + \cdots (a_{n-1} + k_{n-1})\,\lambda + (a_n + k_n) = 0 \tag{6.113}$$

Now, once the *desired* eigenvalues $\bar{\lambda}_i$ have been established, the characteristic equation of the *desired* augmented matrix can also be computed directly from

$$\begin{gathered}
\left(\lambda - \bar{\lambda}_1\right)\left(\lambda - \bar{\lambda}_2\right)\left(\lambda - \bar{\lambda}_3\right)\cdots\left(\lambda - \bar{\lambda}_n\right) = 0 \\
\lambda^n + \bar{a}_1\lambda^{n-1} + \bar{a}_2\lambda^{n-2} + \bar{a}_3\lambda^{n-3} + \cdots + \bar{a}_{n-1}\lambda + \bar{a}_n = 0
\end{gathered} \tag{6.114}$$

and the desired gains are determined by equating the coefficients in Eqs. (6.113) and (6.114):

$$a_i + k_i = \bar{a}_i\,, \text{ for } i = 1, 2, \ldots, n$$

or

$$k_i = \bar{a}_i - a_i\,, \text{ for } i = 1, 2, \ldots, n \tag{6.115}$$

Transformation to Companion Form

In order to use the results of the preceding section for a general system, we need to be able to transform an arbitrary plant matrix $\mathbf{A}$ to its (first) companion form

$$\bar{\mathbf{A}} = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{6.116}$$

That is, it is necessary to find the matrix $\mathbf{T}$ such that

$$\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \tag{6.117}$$

where $\bar{\mathbf{A}}$ has the desired form illustrated in Eq. (6.116). It is convenient to represent the needed matrix as the product of two simpler matrices

$$\mathbf{T} = \mathbf{R}\mathbf{S} \tag{6.118}$$

so that

$$\bar{\mathbf{A}} = \mathbf{R}\mathbf{S}\mathbf{A}\mathbf{S}^{-1}\mathbf{R}^{-1} \tag{6.119}$$

where the intermediate transformation

$$\tilde{\mathbf{A}} = \mathbf{S}\mathbf{A}\mathbf{S}^{-1} \tag{6.120}$$

takes the matrix to the (second) *companion form*

$$\tilde{\mathbf{A}} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 0 & 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 0 & 1 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-3} \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix} \tag{6.121}$$

We first show that the intermediate transformation

$$\tilde{\mathbf{A}} = \mathbf{S}\mathbf{A}\mathbf{S}^{-1} \tag{6.122}$$

is achieved when $\mathbf{S}$ is chosen to be the inverse of the controllability matrix $\mathbf{V}$, defined in Eq. (6.74). Thus, we must show that

$$\mathbf{S}^{-1}\tilde{\mathbf{A}} = \mathbf{A}\mathbf{S}^{-1} \tag{6.123}$$

or

$$\mathbf{V}\tilde{\mathbf{A}} = \mathbf{A}\mathbf{V} \tag{6.124}$$

For a single-input system, the controllability matrix takes the form

$$\mathbf{V} = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \cdots & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} \tag{6.125}$$

where $\mathbf{b}$ is an n-vector and, for $\tilde{\mathbf{A}}$ in the (second) *companion form*, we have

$$\mathbf{V}\tilde{\mathbf{A}} = \begin{bmatrix}\mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \cdots & \mathbf{A}^{n-1}\mathbf{b}\end{bmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 1 & \cdots & 0 & -a_{n-3} \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} & \cdots & -a_n\mathbf{b} - a_{n-1}\mathbf{A}\mathbf{b} \cdots - a_1\mathbf{A}^{n-1}\mathbf{b}\end{bmatrix} \tag{6.126}$$

The Cayley-Hamilton Theorem can be used to express the final column in the above matrix as

$$\left(-a_n\mathbf{I} - a_{n-1}\mathbf{A} - a_{n-2}\mathbf{A}^2 - \cdots - a_1\mathbf{A}^{n-1}\right)\mathbf{b} = \mathbf{A}^n\mathbf{b} \tag{6.127}$$

Thus,

$$\mathbf{V}\tilde{\mathbf{A}} = \mathbf{A}\begin{bmatrix}\mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \cdots & \mathbf{A}^{n-1}\mathbf{b}\end{bmatrix} = \mathbf{A}\mathbf{V} \tag{6.128}$$

as was to be shown.

For the final transformation, we require

$$\bar{\mathbf{A}} = \mathbf{R}\tilde{\mathbf{A}}\mathbf{R}^{-1} \tag{6.129}$$

to have the desired form, or

$$\mathbf{R}^{-1}\bar{\mathbf{A}} = \tilde{\mathbf{A}}\mathbf{R}^{-1} \tag{6.130}$$

The required matrix $\mathbf{R}^{-1}$ has the form

$$\mathbf{R}^{-1} = \begin{pmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 1 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} \\ 0 & 0 & 1 & a_1 & \cdots & a_{n-4} & a_{n-3} \\ 0 & 0 & 0 & 1 & \cdots & a_{n-5} & a_{n-4} \\ & & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \mathbf{W} \tag{6.131}$$

which can be verified by noting that

$$\mathbf{R}^{-1}\bar{\mathbf{A}} = \begin{pmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 1 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} \\ 0 & 0 & 1 & a_1 & \cdots & a_{n-4} & a_{n-3} \\ 0 & 0 & 0 & 1 & \cdots & a_{n-5} & a_{n-4} \\ & & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 1 & a_1 & a_2 & \cdots & a_{n-2} & 0 \\ 0 & 1 & a_1 & \cdots & a_{n-3} & 0 \\ 0 & 0 & 1 & \cdots & a_{n-4} & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & a_1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{6.132}$$

while

$$\tilde{\mathbf{A}}\mathbf{R}^{-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 1 & \cdots & 0 & -a_{n-3} \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix} \begin{pmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 1 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} \\ 0 & 0 & 1 & a_1 & \cdots & a_{n-4} & a_{n-3} \\ 0 & 0 & 0 & 1 & \cdots & a_{n-5} & a_{n-4} \\ & & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 1 & a_1 & a_2 & \cdots & a_{n-2} & 0 \\ 0 & 1 & a_1 & \cdots & a_{n-3} & 0 \\ 0 & 0 & 1 & \cdots & a_{n-4} & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & a_1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} = \mathbf{R}^{-1}\bar{\mathbf{A}} \tag{6.133}$$

as was to be shown.

Now, we have seen for the system in companion form

$$\dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{B}}\eta \tag{6.134}$$

subject to the control law

$$\eta = -\bar{\mathbf{k}}^T\mathbf{z} + \eta' \tag{6.135}$$

the roots of the augmented matrix are driven to those of the characteristic equation

$$\lambda^n + \bar{a}_1\lambda^{n-1} + \bar{a}_2\lambda^{n-2} + \cdots + \bar{a}_{n-1}\lambda + \bar{a}_0 = 0 \tag{6.136}$$

by the gain vector having elements

$$\bar{k}_i = \bar{a}_i - a_i \tag{6.137}$$

where a_i are the coefficients of the characteristic equation of the original (open-loop) plant matrix.

The system of Eqs.(6.134) in companion form can be related back to the original system by introducing the transformation

$$\mathbf{z} = \mathbf{T}\mathbf{x} \tag{6.138}$$

to give

$$\mathbf{T}\dot{\mathbf{x}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\mathbf{T}\mathbf{x} + \bar{\mathbf{B}}\eta \tag{6.139}$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{T}^{-1}\bar{\mathbf{B}}\eta \tag{6.140}$$

The control law then transforms as

$$\eta = -\mathbf{k}^T\mathbf{x} = -\mathbf{k}^T\mathbf{T}^{-1}\mathbf{z} = -\bar{\mathbf{k}}^T\mathbf{z} \tag{6.141}$$

whence

$$\mathbf{k}^T\mathbf{T}^{-1} = \bar{\mathbf{k}}^T \tag{6.142}$$

or

$$\mathbf{k} = \mathbf{T}^T\bar{\mathbf{k}} \tag{6.143}$$

Finally, since

$$\mathbf{T} = \mathbf{RS} = \mathbf{W}^{-1}\mathbf{V}^{-1} = (\mathbf{VW})^{-1} \tag{6.144}$$

we can write

$$\mathbf{k} = \left[(\mathbf{VW})^{-1}\right]^T \bar{\mathbf{k}} \tag{6.145}$$

where the matrices $\mathbf{V}$ and $\mathbf{W}$ are defined in Eqs. (6.125) and (6.131), respectively. Equation (6.145), known as the Bass-Gura formula, gives the gain matrix for the original state space (in which the plant matrix is $\mathbf{A}$), in terms of the coefficients of the desired characteristic equation, given by Eq. (6.137).

Example of Single-Variable Feedback Control

We here present an example of single-variable feedback control used to stabilize the Dutch Roll mode of the Boeing 747 aircraft in powered approach at sea level. We saw in an earlier chapter that the Dutch Roll mode for this flight condition was very lightly damped, so we will use state-variable feedback to increase the damping ratio of this mode to $\zeta = 0.30$, while keeping the undamped natural frequency of the mode, and the times to damp to half amplitude of the rolling and spiral modes, unchanged.

For the Boeing 747 powered approach condition (at $\mathbf{M} = 0.25$, standard sea-level conditions), the relevant vehicle parameters are

$$\begin{aligned} &W = 564,032 \text{ lbf} \qquad b = 195.7 \text{ ft} \qquad u_0 = 279.1 \text{ ft/sec} \\ &I_x = 14.3 \times 10^6 \text{ slug ft}^2, \quad I_z = 45.3 \times 10^6 \text{ slug ft}^2, \quad I_{xz} = -2.23 \times 10^6 \text{ slug ft}^2 \end{aligned} \tag{6.146}$$

and the relevant aerodynamic derivatives are

$$\begin{array}{lllll} \mathbf{C}_{y\beta} = -.96 & \mathbf{C}_{yp} = 0.0 & \mathbf{C}_{yr} = 0.0 & \mathbf{C}_{y\delta_r} = 0.175 & \mathbf{C}_{y\delta_a} = 0 \\ \mathbf{C}_{l\beta} = -.221 & \mathbf{C}_{lp} = -.45 & \mathbf{C}_{lr} = 0.101 & \mathbf{C}_{l\delta_r} = 0.007 & \mathbf{C}_{l\delta_a} = 0.0461 \\ \mathbf{C}_{n\beta} = 0.15 & \mathbf{C}_{np} = -.121 & \mathbf{C}_{nr} = -.30 & \mathbf{C}_{n\delta_r} = -.109 & \mathbf{C}_{n\delta_a} = 0.0064 \end{array} \tag{6.147}$$

These values correspond to the following dimensional stability derivatives

$$\begin{array}{lllll} Y_v = -0.0999, & Y_p = 0.0, & Y_r = 0.0 & Y_{\delta_r} = 5.083 & Y_{\delta_a} = 0.0 \\ L_v = -0.0055, & L_p = -1.0994, & L_r = 0.2468 & L_{\delta_r} = 0.0488 & L_{\delta_a} = 0.3212 \\ N_v = 0.0012, & N_p = -.0933, & N_r = -.2314 & N_{\delta_r} = -.2398 & N_{\delta_a} = 0.0141 \end{array} \tag{6.148}$$

and the dimensionless product of inertia factors

$$i_x = -.156, \qquad i_z = -.0492 \tag{6.149}$$

Using these values, the plant matrix is found to be

$$\mathbf{A} = \begin{pmatrix} -0.0999 & 0.0000 & 0.1153 & -1.0000 \\ -1.6038 & -1.0932 & 0.0 & 0.2850 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.4089 & -.0395 & 0.0 & -.2454 \end{pmatrix} \tag{6.150}$$

when the state vector is defined as[4]

$$\mathbf{x} = \begin{pmatrix} \beta & p & \phi & r \end{pmatrix}^T \tag{6.151}$$

[4]Note that this is the form introduced in Chapter 5 in which the sideslip velocity has been normalized by the equilibrium flight speed.

(i.e., is based on sideslip *angle* rather than sideslip *velocity*).

The roots of the characteristic equation are found to be the same as those in Chapter 5 of the class notes:

$$\begin{aligned} \lambda_{\mathrm{DR}} &= -.08066 \pm \imath\, 0.7433 \\ \lambda_{\mathrm{roll}} &= -1.2308 \\ \lambda_{\mathrm{spiral}} &= -.04641 \end{aligned} \tag{6.152}$$

The undamped natural frequency and damping ratio of the Dutch Roll mode are thus

$$\omega_{n_{\mathrm{DR}}} = 0.7477 \text{ sec}^{-1} \quad \text{and} \quad \zeta_{\mathrm{DR}} = 0.1079 \tag{6.153}$$

The times to damp to half amplitude for the rolling and spiral modes are seen to be

$$t_{1/2_{\mathrm{roll}}} = 0.56 \text{ sec} \quad \text{and} \quad t_{1/2_{\mathrm{spiral}}} = 14.93 \text{ sec} \tag{6.154}$$

respectively.

We now determine the gains required, using *rudder control only*, to increase the damping ratio of the Dutch Roll mode to $\zeta = 0.30$, while keeping the other modal properties fixed.

The original plant matrix is the same as that in Eq. (6.150), and its characteristic equation is given by

$$\lambda^4 + 1.4385\lambda^3 + 0.8222\lambda^2 + 0.7232\lambda + 0.0319 = 0 \tag{6.155}$$

The characteristic equation of the desired system is

$$\begin{aligned} (\lambda - \lambda_{\mathrm{roll}})(\lambda - \lambda_{\mathrm{spiral}})(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)_{\mathrm{DR}} &= 0 \\ (\lambda + 1.2308)(\lambda + 0.04641)(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2) &= 0 \\ (\lambda^2 + 1.2772\lambda + 0.05712)(\lambda^2 + 2(0.30)(0.7477)\lambda + (0.7477)^2) &= 0 \\ (\lambda^2 + 1.2772\lambda + 0.05712)(\lambda^2 + 0.4486\lambda + 0.5591) &= 0 \\ \lambda^4 + 1.7258\lambda^3 + 1.1891\lambda^2 + 0.7396\lambda + 0.0319 &= 0 \end{aligned} \tag{6.156}$$

and by construction the roots will be the same as for the original system, except the damping ratio for the Dutch Roll mode will be increased to

$$\zeta = 0.30$$

Comparing the coefficients in the characteristic Eqs. (6.155) and (6.156), the gain vector in the *companion form* space is seen to be

$$\begin{aligned} \bar{\mathbf{k}} &= [1.7258 \quad 1.1891 \quad 0.7396 \quad 0.0319]^T - [1.4385 \quad 0.8222 \quad 0.7232 \quad 0.0319]^T \\ &= [0.2873 \quad 0.3669 \quad 0.0164 \quad 0.0000]^T \end{aligned} \tag{6.157}$$

The control matrix, assuming rudder-only control, is

$$\mathbf{B} = [0.0182 \quad 0.0868 \quad 0.0 \quad -.2440]^T \tag{6.158}$$

and the gain vector in the original state vector space required to achieve the desired augmented matrix is

$$\mathbf{k} = \left[(\mathbf{VW})^{-1}\right]^T \bar{\mathbf{k}} = [0.1383 \quad 0.0943 \quad 0.1250 \quad -1.1333]^T \tag{6.159}$$

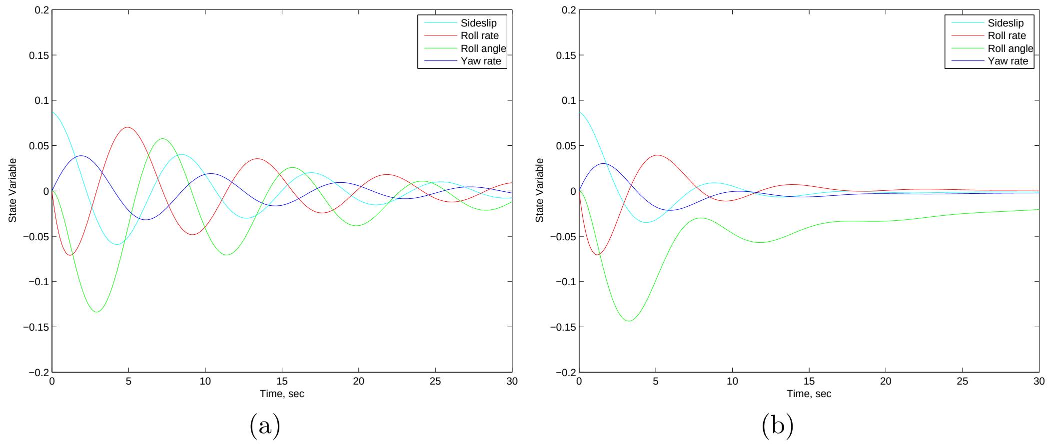


Figure 6.5: Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M} = 0.25$; response to 5 degree (0.08727 radian) perturbation in sideslip. (a) Original open-loop response; (b) Closed loop response with Dutch Roll damping ratio changed to $\zeta = 0.30$ using rudder state-variable feedback.

where

$$\mathbf{V} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \mathbf{A}^3\mathbf{B}] \tag{6.160}$$

is the controllability matrix and

$$\mathbf{W} = \begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.161}$$

where the element a_i is the coefficient of λ^{4-i} in the characteristic equation of the original system.

The augmented plant matrix for the closed-loop system is

$$\mathbf{A}^* = \begin{pmatrix} -0.1024 & -.0017 & 0.1130 & -.9794 \\ -1.6158 & -1.1014 & -.0109 & 0.3834 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.4427 & -.0165 & 0.0305 & -.5220 \end{pmatrix} \tag{6.162}$$

Comparing the augmented plant matrix of Eq. (6.162) with that for the original (open-loop) system in Eq. (6.150), we see that by far the largest change is in the $a_{4,4}$ element, indicating that the effective value of *yaw damping* has more than doubled. We saw from our approximate analysis that yaw damping had a stabilizing effect on both, the spiral and Dutch Roll modes.

The response of the closed-loop system to a 5 degree perturbation in sideslip angle is compared to that of the original open-loop system in Fig. 6.5. The Dutch Roll response of the closed-loop system is seen, as expected, to be much more heavily damped than that of the original system.

We next determine the gains required, using *aileron control only*, to increase the damping ratio of the Dutch Roll mode to $\zeta = 0.30$, while keeping the other modal parameters unchanged. The original plant matrix, its characteristic equation, and the characteristic equation of the desired system are

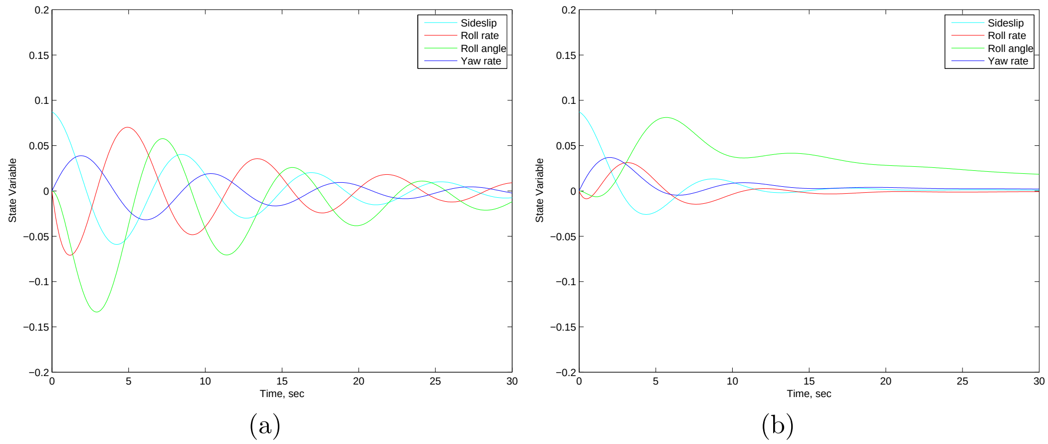


Figure 6.6: Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M} = 0.25$; response to 5 degree (0.08727 radian) perturbation in sideslip. (a) Original open-loop response; (b) Closed loop response with Dutch Roll damping ratio changed to $\zeta = 0.30$ using aileron state-variable feedback.

all the same as in the previous exercise, so the gain matrix for the *companion form* system is also unchanged. The control matrix, however, is now that for aileron-only control, and is given by

$$\mathbf{B} = [0.0000 \quad 0.3215 \quad 0.0000 \quad -.0017]^T \tag{6.163}$$

The gain vector in the original state vector space required to achieve the desired augmented matrix is then

$$\mathbf{k} = \left[(\mathbf{VW})^{-1}\right]^T \bar{\mathbf{k}} = [-3.5417 \quad 0.8715 \quad 0.6746 \quad -4.0504]^T \tag{6.164}$$

The response of the closed-loop system to a 5 degree perturbation in sideslip angle is compared with that of the original system in Fig. 6.6. As for the case of rudder-only control, the closed-loop response is seen to be much more heavily damped than that of the open-loop system.

The augmented plant matrix for the closed-loop system in this case is

$$\mathbf{A}^* = \begin{pmatrix} -0.0999 & 0.0000 & 0.1153 & -1.0000 \\ -0.4651 & -1.3734 & -.2169 & 1.5873 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.4027 & -.0380 & .0012 & -.2525 \end{pmatrix} \tag{6.165}$$

Comparing this plant matrix with that for the original (open-loop) system in Eq. (6.150), we see that by far the largest changes are in the $a_{2,1}$ and $a_{2,4}$ elements. The effective dihedral effect has been reduced to less than 30% of its original value, while the effective *roll-due-to-yaw rate* has been increased by more than a factor of five. Thus, it seems that the control algorithm has stabilized the Dutch Roll mode by reducing the effective dihedral effect; then, in order to not increase the spiral mode stability it has effectively increased the (positive) roll-due-to-yaw rate derivative.

6.5.2 Multiple Input-Output Systems

For multiple input-output systems having p controls, the feedback control law has the form

$$\eta = -\mathbf{K}\mathbf{x} + \eta' \tag{6.166}$$

where $\mathbf{K}$ is the $p \times n$ *gain matrix*. Thus, there are now $p \times n$ gains to be specified, but there are still only n eigenvalues to be specified.

This additional flexibility can be used to configure the control system in a more optimal way if the control engineer understands the system well enough to make intelligent choices for how to allocate the gains. But, even for the single-input system, it is not always clear what is the best placement for the eigenvalues of the augmented matrix. Clearly, more stability is desirable for the less stable modes, but too much stability can result in a system that requires great effort from the pilot to achieve required maneuvers. Equations (6.137) and (6.145) indicate that more control effort will be required as the roots of the augmented matrix are moved further and further to the left of those of the original plant matrix. Also, it is generally important that the closed-loop frequency response not be increased too much to avoid exciting modes that have not been modeled, such as those arising from structural deformation due to aeroelasticity.

6.6 Optimal Control

As has been seen in the previous sections, use of the Bass-Gura procedure often is difficult, or results in sub-optimal performance for a variety of reasons. These include:

1. The best choice of desired placement for the eigenvalues of the augmented matrix is not always obvious;
2. Particular eigenvalue placement may require more control input than it available; this can result in saturation of the control action, which introduces non-linearity and can even result in instability;
3. For multiple input-output systems, we need to develop strategies for deciding on how to allocate the gains among the $n \times p$ elements, since we have only n eigenvalues to place;
4. The process may not be controllable; i.e., if the rank of the controllability matrix $\mathbf{V}$ is less than n, the method fails since Eq. (6.145) requires determination of the inverse of $\mathbf{V}$.

All these points argue for a control design strategy that, in some sense, optimizes the gain matrix for stabilizing a given system. This is the goal of what has come to be called *optimal control*.

6.6.1 Formulation of Linear, Quadratic, Optimal Control

The *optimal control* of the linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta(t) \tag{6.167}$$

is defined as the control vector $\eta(t)$ that drives the state from a specified initial state $\mathbf{x}(t)$ to a desired final state $\mathbf{x}_d(t_f)$ such that a specified performance index

$$J = \int_t^{t_f} g(\mathbf{x}(\tau), \eta(\tau), \tau)\, \mathrm{d}\tau \tag{6.168}$$

is minimized. For *quadratic* optimal control, the performance index is specified in the form

$$g = \mathbf{x}^T\mathbf{Q}\mathbf{x} + \eta^T\mathbf{R}\eta \tag{6.169}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are symmetric, positive-definite matrices, and the performance index becomes

$$J = \int_t^{t_f} \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \eta^T\mathbf{R}\eta\right)\, \mathrm{d}\tau \tag{6.170}$$

If the control law is assumed to be linear, i.e., of the form

$$\eta = -\mathbf{K}\mathbf{x} + \eta' \tag{6.171}$$

then the determination of the gain matrix $\mathbf{K}$ that minimizes J is called the linear quadratic regulator (LQR) problem. For this control law the closed-loop response of the system to a perturbation is given by

$$\dot{\mathbf{x}} = [\mathbf{A} - \mathbf{B}\mathbf{K}]\,\mathbf{x} = \mathbf{A}^*\mathbf{x} \tag{6.172}$$

where

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{K} \tag{6.173}$$

is the *augmented* plant matrix.

We usually are interested in cases for which the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{K}$ are independent of time, but the development here is easier if we allow the augmented matrix $\mathbf{A}^*$ to vary with time. In this case, we cannot express the solution to Eq. (6.172) in terms of a matrix exponential, but we can still express it in terms of the general state transition matrix $\mathbf{\Phi}^*$ as

$$\mathbf{x}(\tau) = \mathbf{\Phi}^*(\tau, t)\mathbf{x}(t) \tag{6.174}$$

Equation (6.174) simply implies that the state of the system at any time τ depends linearly on the state at any other time t. When the control law of Eq. (6.171) is substituted into the performance index of Eq. (6.170) and Eq. (6.174) is used to express the evolution of the state variable, the quantity to be minimized becomes

$$\begin{aligned} J &= \int_t^{t_f} \mathbf{x}^T(\tau)\left[\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}\right]\mathbf{x}(\tau)\,\mathrm{d}\tau \\ &= \int_t^{t_f} \mathbf{x}^T(t)\mathbf{\Phi}^{*T}(\tau, t)\left[\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}\right]\mathbf{\Phi}^*(\tau, t)\mathbf{x}(t)\,\mathrm{d}\tau \\ &= \mathbf{x}^T(t)\left(\int_t^{t_f} \mathbf{\Phi}^{*T}(\tau, t)\left[\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}\right]\mathbf{\Phi}^*(\tau, t)\,\mathrm{d}\tau\right)\mathbf{x}(t) \end{aligned} \tag{6.175}$$

or

$$J = \mathbf{x}^T(t)\mathbf{S}\mathbf{x}(t) \tag{6.176}$$

where

$$\mathbf{S}(t, t_f) = \int_t^{t_f} \mathbf{\Phi}^{*T}(\tau, t)\left[\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}\right]\mathbf{\Phi}^*(\tau, t)\,\mathrm{d}\tau \tag{6.177}$$

Note that, by its construction, the matrix $\mathbf{S}$ is symmetric, since the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ are both symmetric.

The simple appearance of Eq. (6.176) belies the complexity of determining $\mathbf{S}$ from Eq. (6.177). In fact, if we had to use the latter equation to determine the matrix $\mathbf{S}$, we would face an almost hopeless task. Our expression of the solution in terms of the general state transition matrix seems to have resulted in a simple expression for the integral we wish to minimize, but it is almost impossible to develop a useful expression for the state transition matrix, itself, in general. Instead, in order to find the gain matrix $\mathbf{K}$ that minimizes J, it is convenient to find a differential equation that the matrix $\mathbf{S}$ satisfies. To this end, we note that since

$$J = \int_t^{t_f} \mathbf{x}^T(\tau)\mathbf{L}\mathbf{x}(\tau)\,\mathrm{d}t \tag{6.178}$$

where

$$L = \mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K} \tag{6.179}$$

we can write

$$\frac{\mathrm{d}J}{\mathrm{d}t} = -\left.\mathbf{x}^T(\tau)\mathbf{L}\mathbf{x}(\tau)\right|_{\tau=t} = -\mathbf{x}^T(t)\mathbf{L}\mathbf{x}(t) \tag{6.180}$$

But, from differentiating Eq. (6.176), we have

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \dot{\mathbf{x}}^T(t)\mathbf{S}(t,t_f)\mathbf{x}(t) + \mathbf{x}^T(t)\dot{\mathbf{S}}(t,t_f)\mathbf{x}(t) + \mathbf{x}^T(t)\mathbf{S}(t,t_f)\dot{\mathbf{x}}(t) \tag{6.181}$$

and, substituting the closed-loop differential equation, Eq. (6.172), for $\dot{\mathbf{x}}$ gives

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \mathbf{x}^T(t)\left[\mathbf{A}^{*T}\mathbf{S}(t,t_f) + \dot{\mathbf{S}}(t,t_f) + \mathbf{S}(t,t_f)\mathbf{A}^*(t)\right]\mathbf{x}(t) \tag{6.182}$$

Thus, we have two expressions for the derivative $\mathrm{d}J/\mathrm{d}t$: Eqs. (6.180) and (6.182). Both are quadratic forms in the initial state $\mathbf{x}(t)$, which must be *arbitrary*. The only way that two quadratic forms in $\mathbf{x}$ can be equal for any choice of $\mathbf{x}$ is if the underlying matrices are equal; thus, we must have

$$-\mathbf{L} = \mathbf{A}^{*T}\mathbf{S} + \dot{\mathbf{S}} + \mathbf{S}\mathbf{A}^*$$

or

$$-\dot{\mathbf{S}} = \mathbf{S}\mathbf{A}^* + \mathbf{A}^{*T}\mathbf{S} + \mathbf{L} \tag{6.183}$$

Equation (6.183) is a first-order differential equation for the matrix $\mathbf{S}$, so it requires a single initial condition to completely specify its solution. We can use Eq. (6.177), evaluated at $t = t_f$ to give the required condition

$$\mathbf{S}(t_f, t_f) = 0 \tag{6.184}$$

Once a gain matrix $\mathbf{K}$ has been chosen to close the loop, the corresponding performance of the system is given by Eq. (6.176), where $\mathbf{S}(t,t_f)$ is the solution of Eq. (6.183), which can be written in terms of the original plant and gain matrices as

$$-\dot{\mathbf{S}} = \mathbf{S}\left(\mathbf{A} - \mathbf{B}\mathbf{K}\right) + \left(\mathbf{A}^T - \mathbf{K}^T\mathbf{B}^T\right)\mathbf{S} + \mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K} \tag{6.185}$$

Our task, then, is to find the gain matrix $\mathbf{K}$ that makes the solution to Eq. (6.185) as small as possible – in the sense that the quadratic forms (Eq. (6.176)) associated with the matrix $\mathbf{S}$ are minimized. That is, we want to find the matrix $\hat{\mathbf{S}}$ for which

$$\hat{J} = \mathbf{x}^T\hat{\mathbf{S}}\mathbf{x} < \mathbf{x}^T\mathbf{S}\mathbf{x} \tag{6.186}$$

for any arbitrary initial state $\mathbf{x}(t)$ and every matrix $\mathbf{S} \neq \hat{\mathbf{S}}$.

We will proceed by assuming that such an optimum exists, and use the calculus of variations to find it. The minimizing matrix $\hat{\mathbf{S}}$ must, of course, satisfy Eq. (6.185)

$$-\dot{\hat{\mathbf{S}}} = \hat{\mathbf{S}}\left(\mathbf{A} - \mathbf{B}\hat{\mathbf{K}}\right) + \left(\mathbf{A}^T - \hat{\mathbf{K}}^T\mathbf{B}^T\right)\hat{\mathbf{S}} + \mathbf{Q} + \hat{\mathbf{K}}^T\mathbf{R}\hat{\mathbf{K}} \tag{6.187}$$

and any *non*-optimum gain matrix, and its corresponding matrix $\mathbf{S}$, can be expressed as

$$\begin{aligned} \mathbf{S} &= \hat{\mathbf{S}} + \mathbf{N} \\ \mathbf{K} &= \hat{\mathbf{K}} + \mathbf{Z} \end{aligned} \tag{6.188}$$

Substituting this form into Eq. (6.185) and subtracting Eq. (6.187) gives

$$-\dot{\mathbf{N}} = \mathbf{N}\mathbf{A}^* + \mathbf{A}^{*^T}\mathbf{N} + \left(\hat{\mathbf{K}}^T\mathbf{R} - \hat{\mathbf{S}}\mathbf{B}\right)\mathbf{Z} + \mathbf{Z}^T\left(\mathbf{R}\hat{\mathbf{K}} - \mathbf{B}^T\hat{\mathbf{S}}\right) + \mathbf{Z}^T\mathbf{R}\mathbf{Z} \tag{6.189}$$

where

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{K} = \mathbf{A} - \mathbf{B}\left(\hat{\mathbf{K}} + \mathbf{Z}\right) \tag{6.190}$$

Note that Eq. (6.189) has exactly the same form as Eq. (6.183) with

$$\mathbf{L} = \left(\hat{\mathbf{K}}^T\mathbf{R} - \hat{\mathbf{S}}\mathbf{B}\right)\mathbf{Z} + \mathbf{Z}^T\left(\mathbf{R}\hat{\mathbf{K}} - \mathbf{B}^T\hat{\mathbf{S}}\right) + \mathbf{Z}^T\mathbf{R}\mathbf{Z} \tag{6.191}$$

so its solution must be of the form of Eq. (6.177)

$$\mathbf{N}(t, t_f) = \int_t^{t_f} \mathbf{\Phi}^{*^T}(\tau, t)\mathbf{L}\mathbf{\Phi}^*(\tau, t)\,\mathrm{d}\tau \tag{6.192}$$

Now, if $\hat{J}$ is a minimum, then we must have

$$\mathbf{x}^T\hat{\mathbf{S}}\mathbf{x} \leq \mathbf{x}^T\left(\hat{\mathbf{S}} + \mathbf{N}\right)\mathbf{x} = \mathbf{x}^T\hat{\mathbf{S}}\mathbf{x} + \mathbf{x}^T\mathbf{N}\mathbf{x} \tag{6.193}$$

and this equation requires that the quadratic form $\mathbf{x}^T\mathbf{N}\mathbf{x}$ be positive definite (or, at least, positive semi-definite). But, if $\mathbf{Z}$ is sufficiently small, the linear terms in $\mathbf{Z}$ (and $\mathbf{Z}^T$) in Eq. (6.191) will dominate the quadratic terms in $\mathbf{Z}^T\mathbf{R}\mathbf{Z}$, and we could easily find values of $\mathbf{Z}$ that would make $\mathbf{L}$, and hence $\mathbf{N}$, negative definite. Thus, *the linear terms in Eq. (6.191) must be absent altogether*. That is, for the gain matrix $\hat{\mathbf{K}}$ to be optimum, we must have

$$\hat{\mathbf{K}}^T\mathbf{R} - \hat{\mathbf{S}}\mathbf{B} = 0 = \mathbf{R}\hat{\mathbf{K}} - \mathbf{B}^T\hat{\mathbf{S}} \tag{6.194}$$

or, assuming that the weighting matrix $\mathbf{R}$ is not singular,

$$\hat{\mathbf{K}} = \mathbf{R}^{-1}\mathbf{B}^T\hat{\mathbf{S}} \tag{6.195}$$

Equation (6.195) gives the optimum gain matrix $\hat{\mathbf{K}}$, once the matrix $\hat{\mathbf{S}}$ has been determined. When this equation is substituted back into Eq. (6.187) we have

$$-\dot{\hat{\mathbf{S}}} = \hat{\mathbf{S}}\mathbf{A} + \mathbf{A}^T\hat{\mathbf{S}} - \hat{\mathbf{S}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\hat{\mathbf{S}} + \mathbf{Q} \tag{6.196}$$

This equation, one of the most famous in modern control theory, is called the *matrix Riccati equation*, consistent with the mathematical nomenclature that identifies an equation with a quadratic nonlinearity as a Riccati equation. The solution to this equation gives the matrix $\hat{\mathbf{S}}$ which, when substituted into Eq. (6.195), gives the optimum gain matrix $\hat{\mathbf{K}}$.

Because of the quadratic nonlinearity in the Riccati equation, it is necessary, except in a few very special cases, to solve it numerically. Since the matrix $\hat{\mathbf{S}}$ is symmetric, Eq. (6.196) represents $n(n+1)/2$ coupled, first-order equations. Since the "initial" condition is

$$\hat{\mathbf{S}}(t_f, t_f) = 0 \tag{6.197}$$

the equation must be integrated *backward* in time, since we are interested in $\hat{\mathbf{S}}(t, t_f)$ for $t < t_f$.

When the control interval $[t, t_f]$ is finite, the gain matrix $\mathbf{K}$ will generally be time-dependent, even when the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$, and $\mathbf{R}$ are all constant. But, suppose the control interval is *infinite*, so that we want to find the gain matrix $\hat{\mathbf{K}}$ that minimizes the performance index

$$J_\infty = \int_t^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \eta^T\mathbf{R}\eta\right)\, \mathrm{d}\tau \tag{6.198}$$

In this case, integration of Eq. (6.196) backward in time will either grow without limit or converge to a *constant* matrix $\bar{\mathbf{S}}$. If it converges to a limit, the derivative $\dot{\hat{\mathbf{S}}}$ must tend to zero, and $\bar{\mathbf{S}}$ must satisfy the *algebraic* equation

$$0 = \bar{\mathbf{S}}\mathbf{A} + \mathbf{A}^T\bar{\mathbf{S}} - \bar{\mathbf{S}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}} + \mathbf{Q} \tag{6.199}$$

and the optimum gain in the steady state is given by

$$\bar{\mathbf{K}} = \mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}} \tag{6.200}$$

The single *quadratic* matrix Eq. (6.199) represents $n(n+1)/2$ coupled scalar, quadratic equations, so we expect there will be $n(n+1)$ different (symmetric) solutions. The nature of these solutions is, as one might expect, connected with issues of controllability and observability – and a treatment of these issues is beyond the scope of our treatment here. But, for most design applications, it is enough to know that

1. If the system is asymptotically stable; or

2. If the system defined by ($\mathbf{A}$, $\mathbf{B}$) is *controllable*, and the system defined by ($\mathbf{A}$, $\mathbf{C}$), where the weighting matrix $\mathbf{Q} = \mathbf{C}^T\mathbf{C}$, is observable,

then the algebraic Riccati equation has an unique positive definite solution $\bar{\mathbf{S}}$ that minimizes J_∞ when the control law

$$\eta = -\bar{\mathbf{K}}\mathbf{x} = -\mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}}\mathbf{x} \tag{6.201}$$

is used.[5]

[5] It should be understood that there are still $n(n+1)$ symmetric solutions; the assertion here is that, of these multiple solutions, one, and only one, is *positive definite*.

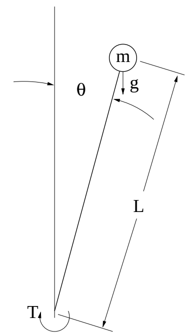


Figure 6.7: Inverted pendulum affected by gravity g and control torque T.

6.6.2 Example of Linear, Quadratic, Optimal Control

We consider here the application of linear, quadratic optimal control to an example that is simple enough that we can carry our the analysis in closed form, illustrating the concepts of the preceding section. We consider using optimal control to stabilize an inverted pendulum. The equation of motion for an inverted pendulum near its (unstable) equilibrium point, as illustrated in Fig. 6.7 is

$$mL^2\ddot{\theta} = mgL\sin\theta + T = mgL\theta + T \tag{6.202}$$

where m is the mass of the pendulum, L is the pendulum length, g is the acceleration of gravity, and T is the externally-applied (control) torque; the second form of the right-hand side assumes the angle θ is small.

If we introduce the angular velocity $\omega = \dot{\theta}$ as a second state variable, Eq. (6.202) can be written in the standard state variable form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\theta\\ \omega\end{pmatrix} = \begin{pmatrix}0 & 1\\ \gamma & 0\end{pmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}\tau \tag{6.203}$$

where $\gamma = g/L$ and $\tau = T/(mL^2)$ are reduced gravity and input torque variables.

Now, we seek the control law that minimizes the performance index

$$J_\infty = \int_t^\infty \left(\theta^2 + \frac{\tau^2}{c^2}\right)\,\mathrm{d}t' \tag{6.204}$$

where c is a parameter that determines the relative weighting of control input and angular deviation in the penalty function. It is clear that this performance index corresponds to

$$\mathbf{Q} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \quad \text{and} \quad \mathbf{R} = \frac{1}{c^2} \tag{6.205}$$

If we define the elements of the matrix $\bar{\mathbf{S}}$ to be

$$\bar{\mathbf{S}} = \begin{pmatrix}s_1 & s_2\\ s_2 & s_3\end{pmatrix} \tag{6.206}$$

then the optimum gain matrix is

$$\bar{\mathbf{K}} = \mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}} = c^2\begin{bmatrix}0 & 1\end{bmatrix}\begin{pmatrix}s_1 & s_2\\ s_2 & s_3\end{pmatrix} = \begin{bmatrix}c^2 s_2 & c^2 s_3\end{bmatrix} \tag{6.207}$$

which is seen to be independent of the element s_1.

The terms needed for the algebraic Riccati equation

$$0 = \bar{\mathbf{S}}\mathbf{A} + \mathbf{A}^T\bar{\mathbf{S}} - \bar{\mathbf{S}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}} + \mathbf{Q} \tag{6.208}$$

are

$$\bar{\mathbf{S}}\mathbf{A} = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} s_2\gamma & s_1 \\ s_3\gamma & s_2 \end{pmatrix} \tag{6.209}$$

$$\mathbf{A}^T\bar{\mathbf{S}} = \begin{pmatrix} 0 & \gamma \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} = \begin{pmatrix} s_2\gamma & s_3\gamma \\ s_1 & s_2 \end{pmatrix} \tag{6.210}$$

and

$$\bar{\mathbf{S}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}} = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} c^2 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} = c^2 \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} \tag{6.211}$$

Thus, the Riccati equation is

$$0 = \begin{pmatrix} s_2\gamma & s_1 \\ s_3\gamma & s_2 \end{pmatrix} + \begin{pmatrix} s_2\gamma & s_3\gamma \\ s_1 & s_2 \end{pmatrix} - c^2 \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.212}$$

which is equivalent to the three scalar equations

$$\begin{aligned} 0 &= 2s_2\gamma - c^2 s_2^2 + 1 \\ 0 &= s_1 + s_3\gamma - c^2 s_2 s_3 \\ 0 &= 2s_2 - c^2 s_3^2 \end{aligned} \tag{6.213}$$

These equations are simple enough that we can solve them in closed form. The first of Eqs. (6.213) gives

$$s_2 = \frac{\gamma \pm \sqrt{\gamma^2 + c^2}}{c^2} \tag{6.214}$$

and the third of Eqs. (6.213) gives

$$s_3 = \pm\frac{1}{c}\sqrt{2s_2} \tag{6.215}$$

Since the elements of $\bar{\mathbf{S}}$ must be real, s_2 must be positive (or s_3 would be complex). Thus, we must choose the positive root in Eq. (6.203). Further, the second of Eqs. (6.213) gives

$$s_1 = c^2 s_2 s_3 - \gamma s_3 = s_3\sqrt{\gamma^2 + c^2} \tag{6.216}$$

Thus, elements s_1 and s_3 have the same sign which, for $\bar{\mathbf{S}}$ to be positive definite, must be positive. Thus,

$$\begin{aligned} s_2 &= \frac{\gamma + \sqrt{\gamma^2 + c^2}}{c^2} \\ s_3 &= \frac{1}{c}\sqrt{2s_2} = \frac{\sqrt{2}}{c^2}\left[\gamma + \sqrt{\gamma^2 + c^2}\right]^{1/2} \end{aligned} \tag{6.217}$$

represents the unique solution for the corresponding elements for which $\bar{\mathbf{S}}$ is positive definite.

Thus, the gain matrix is seen to be

$$\mathbf{K} = \begin{bmatrix} c^2 s_2 & c^2 s_3 \end{bmatrix} = \begin{bmatrix} \gamma + \sqrt{\gamma^2 + c^2} & \sqrt{2}\left[\gamma + \sqrt{\gamma^2 + c^2}\right]^{1/2} \end{bmatrix} \tag{6.218}$$

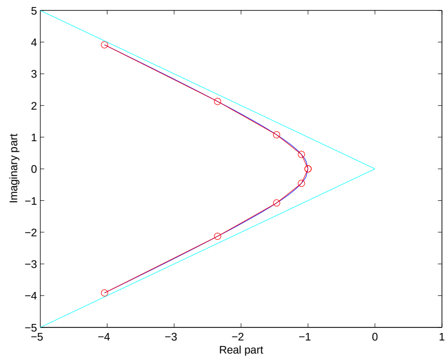


Figure 6.8: Locus of roots of characteristic equation of augmented plant matrix for inverted pendulum. Axes are scaled to give roots in units of γ. Open symbols represent roots at values of $c/\gamma = 0, 1, 10, 100, 1000$, with real root corresponding to $c/\gamma = 0$. Cyan lines represent asymptotes of root positions in the limit of large c/γ.

The augmented matrix is then given by

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\bar{\mathbf{K}} = \begin{pmatrix} 0 & 1 \\ -\sqrt{\gamma^2 + c^2} & -\sqrt{2}\left[\gamma + \sqrt{\gamma^2 + c^2}\right]^{1/2} \end{pmatrix} \tag{6.219}$$

and its characteristic equation is

$$\lambda^2 + \sqrt{2}\left[\gamma + \sqrt{\gamma^2 + c^2}\right]^{1/2} \lambda + \sqrt{\gamma^2 + c^2} = 0 \tag{6.220}$$

which has roots

$$\lambda = \frac{\sqrt{2}}{2}\left[-\sqrt{\gamma + \bar{\gamma}} \pm \imath\sqrt{\bar{\gamma} - \gamma}\right] \tag{6.221}$$

where we have introduced

$$\bar{\gamma} = \sqrt{\gamma^2 + c^2} \tag{6.222}$$

The locus of these roots is plotted in Fig. 6.8 as the weighting factor c is varied over the range $0 < c < 10^3$.

Note that as c/γ becomes large, $\bar{\gamma}$ becomes large relative to γ, so

$$\lim_{c/\gamma \to \infty} \lambda = -\sqrt{\frac{\bar{\gamma}}{\sqrt{2}}}\,(1 \pm \imath) \tag{6.223}$$

Thus, as c becomes large, the damping ratio of the system approaches a constant value of

$$\zeta = \frac{1}{\sqrt{2}}$$

while the undamped natural frequency increases as

$$\omega_n = \sqrt{\bar{\gamma}} \approx \sqrt{c}$$

Large values of c correspond to a performance index in which the weighting of the control term is small compared to that of the deviations in state variables – i.e., to a situation in which we are willing to spend additional energy in control to maintain very small perturbations of the state from its equilibrium position.

On the other hand, as c becomes small, the weighting of the control term in the performance index becomes large compared to that of the state variables. This is consistent with the fact that the gains in Eq. (6.218)

$$K_1 = \gamma + \sqrt{\gamma^2 + c^2}$$
$$K_2 = \sqrt{2}\left[\gamma + \sqrt{\gamma^2 + c^2}\right]^{1/2}$$

decrease monotonically with c. In the limit $c = 0$, however, the gains remain finite, with

$$\lim_{c \to 0} K_1 = 2\gamma$$
$$\lim_{c \to 0} K_2 = 2\sqrt{\gamma}$$

since *some* control is necessary to stabilize this, otherwise unstable, system.

6.6.3 Linear, Quadratic, Optimal Control as a Stability Augmentation System

We here present an example of the application of linear, quadratic optimal control to stabilize the motion of the Boeing 747 aircraft in powered approach at $\mathbf{M} = 0.25$ at standard sea level conditions. This is the same aircraft and flight condition for which we used the Bass-Gura procedure to design a feedback control system to stabilize the lateral/directional modes in Section. 6.5.1. In that earlier section, we determined the gains for specific placement of the eigenvalues of the associated augmented matrix using only one control, either rudder or ailerons, at a time.

Here, we apply linear, quadratic, optimal control to minimize the steady state performance index

$$J_\infty = \int_t^\infty \left(\mathbf{x}^T \mathbf{Q} \mathbf{x} + \frac{1}{c^2} \eta^T \mathbf{R} \eta \right) \mathrm{d}\tau \tag{6.224}$$

where, as in the previous simple example, c is a parameter that determines the relative weights given to control action and perturbations in the state variable in the penalty function. For lateral/directional motions at this flight condition, the plant matrix is given by Eq. (6.150), while the control matrix is the union of the two vectors given in Eqs. (6.158) and (6.163)

$$\mathbf{B} = \begin{pmatrix} 0.0182 & 0.0868 & 0.0000 & -.2440 \\ 0.0000 & 0.3215 & 0.0000 & -.0017 \end{pmatrix}^T \tag{6.225}$$

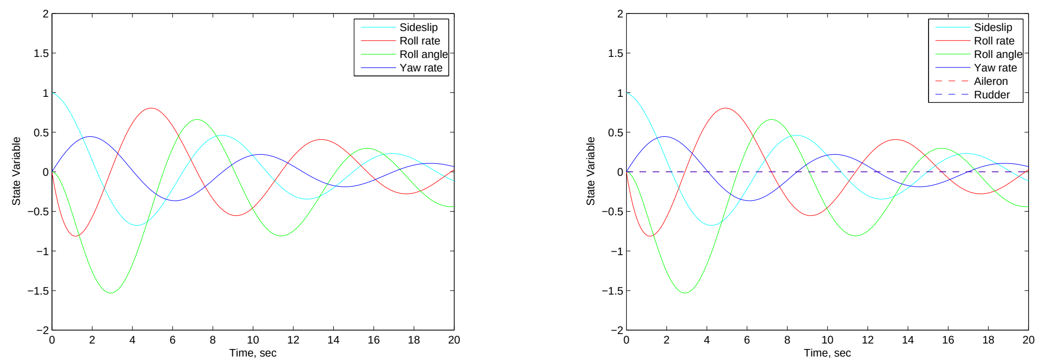
where the control vector is

$$\eta = [\delta_r \quad \delta_a]^T \tag{6.226}$$

The weighting matrices in the performance index are taken to be simply

$$\mathbf{Q} = \mathbf{I} \quad \text{and} \quad \mathbf{R} = \mathbf{I} \tag{6.227}$$

where $\mathbf{Q}$ is a 4×4 matrix and $\mathbf{R}$ is a 2×2 matrix.



(a) No feedback (b) $c = 0.001$

Figure 6.9: Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M} = 0.25$; response to unit perturbation in sideslip. (a) Original open-loop response; (b) Optimal closed loop response with performance parameter $c = 0.001$.

The MATLAB function

[S, L, G] = care(A,B,Q,R,T,E);

is used to solve the generalized matrix Riccati equation

$$\mathbf{E}^T\mathbf{SA} + \mathbf{A}^T\mathbf{SE} - \left(\mathbf{E}^T\mathbf{SB} + \mathbf{T}\right)\mathbf{R}^{-1}\left(\mathbf{B}^T\mathbf{SE} + \mathbf{T}^T\right) + \mathbf{Q} = 0 \tag{6.228}$$

which, with the additional input matrices are defined as

T = zeros(size(B));

and

E = eye(size(A));

reduces to Eq. (6.199). In addition to the solution matrix S, the MATLAB function *care* also returns the gain matrix

$$\mathbf{G} = \mathbf{R}^{-1}\left(\mathbf{B}^T\mathbf{SE} + \mathbf{T}^T\right) \tag{6.229}$$

and the vector

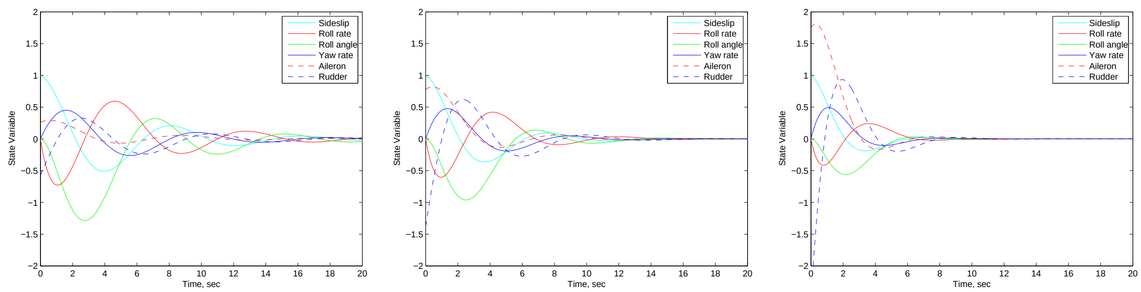
L = eig(A - BG, E)

containing the eigenvalues of the augmented matrix.

For small values of the parameter c, the control action is weighted heavily in the performance index. Figure 6.9 compares the open-loop response to a unit perturbation in sideslip to the closed-loop system response for a value of $c = 0.001$. This value penalizes control input so heavily that the open-loop and closed-loop responses are virtually identical. This is a quite different result from that for the simple example of Section 6.6.2, and results from the fact that this system is *stable*, so the natural (un-forced) return of the system to equilibrium is optimal when control action is heavily penalized.

Figure 6.10 shows the closed-loop system response for values of $c = 0.5, 1.0,$ and 2.0, respectively. Also plotted in these figures are the time histories of control response required to stabilize the motions, calculated as

$$\eta = -\mathbf{Kx} \tag{6.230}$$



(a) $c = 0.50$ (b) $c = 1.0$ (c) $c = 2.0$

Figure 6.10: Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M} = 0.25$; response to unit perturbation in sideslip illustrating effect of varying weighting parameter c. Optimal closed-loop responses with (a) $c = 0.50$; (b) $c = 1.0$; and (c) $c = 2.0$. Control deflections required to stabilize the motions are also shown.

This control response is calculated in MATLAB simply by defining the matrices $\mathbf{C}$ and $\mathbf{D}$ defining the output response as

$$C = \text{-}G;$$

and

$$D = zeros(2);$$

and then adding the output variables

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\eta \tag{6.231}$$

to the plots. It is seen in the plots that, as c is increased the motion becomes more heavily damped, but at the cost of significantly greater control input.

The role of the parameter c can be seen more clearly if we examine the behavior of the individual terms in the performance index J_∞. Figure 6.11 plots the quadratic forms $\mathbf{x}^T\mathbf{Q}\mathbf{x}$ and $\eta^T\mathbf{R}\eta$ as functions of time for the three values of c illustrated in Fig. 6.10. For greater clarity in the figure, *minus* the control term is plotted. Thus, for each value of c the optimal control strategy selects the gains that minimize the net area between the two curves. Three trends resulting from increasing c are evident in the figure: (1) the value of J_∞ – i.e., the area between the two curves – decreases; (2) the return of the system to its equilibrium state is more rapid and heavily damped; (3) most of the improvement happens for modest increases in the value of c, with continued increases requiring ever larger control inputs for relatively little further improvement in response.

Finally, we illustrate the behavior of the roots of the augmented equation as the parameter c is increased. Figure 6.12 shows the locations of the roots in the complex plane for selected values of c. Note that, in this range of values, all roots move to the left as c is increased until the Dutch Roll mode becomes critically damped at a value of approximately $c = 9.973$, as indicated by the joining of the roots on the real axis. With further increase in c, one of the Dutch Roll roots moves to the right. The value of c required to achieve critical damping of the Dutch Roll mode generally corresponds to much larger values of c than would ever be used in a practical system.

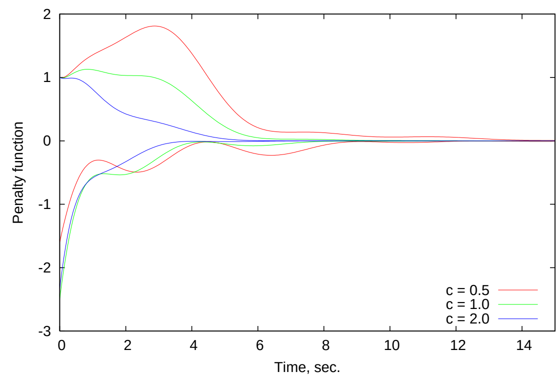


Figure 6.11: Penalty functions in performance index for optimal control solution; Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M} = 0.25$. Upper curves are $\mathbf{x}^T\mathbf{Q}\mathbf{x}$, and lower curves are $-\eta^T\mathbf{R}\eta$, as functions of time for response to unit perturbation in sideslip angle β, so the area between the curves is equal to the performance index J_∞.

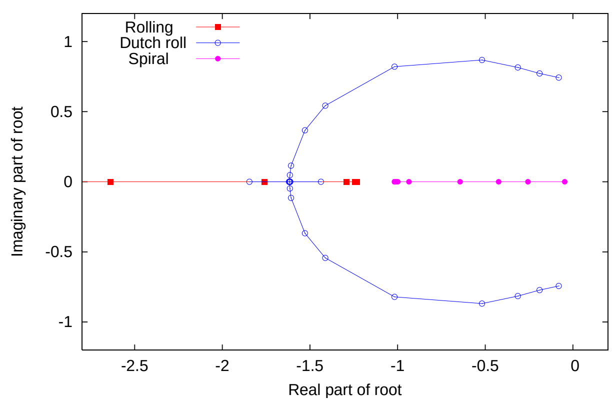


Figure 6.12: Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M} = 0.25$; locus of roots of characteristic equation of augmented matrix as control weighting parameter c is increased. Symbols represent root locations for $c = 0.001, 0.5, 1.0, 2.0, 5.0, 8.0, 9.0, 9.7, 9.76, 9.7727, 9.7728, 10.0$; as c is increased, all roots move to the left (except for one of the Dutch Roll roots after that mode becomes critically damped between $9.7227 < c < 9.7228$).

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6.7 Review of Laplace Transforms

The Laplace transform of the function $f(t)$, assumed identically zero for $t < 0$, is defined as

$$\mathcal{L}\left[f(t)\right] = F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t \tag{6.232}$$

The Laplace transform $F(s)$ of the function $f(t)$ can be shown to exist, for sufficiently large s, when [4]:

1. The function $f(t)$ is continuous or piecewise continuous in every finite interval $t_1 \leq t \leq T$, where $t_1 > 0$;
2. The function $t^n|f(t)|$ is bounded near $t = 0$ for some number $n < 1$; and
3. The function $e^{-s_0 t}|f(t)|$ is bounded for large values of t, for some number s_0.

All the functions we normally deal with in stability and control problems satisfy these conditions.

6.7.1 Laplace Transforms of Selected Functions

We here review the Laplace transforms of several important functions.

Laplace Transform of a Derivative

If $F(s)$ is the Laplace transform of the function $f(t)$, then the Laplace transform of the derivative $\mathrm{d}f/\,\mathrm{d}t$ can be determined as

$$\begin{aligned}\mathcal{L}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = \int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}t}e^{-st}\,\mathrm{d}t = fe^{-st}\big|_0^\infty + \int_0^\infty fse^{-st}\,\mathrm{d}t &= -f(0) + s\int_0^\infty fe^{-st}\,\mathrm{d}t \\ &= -f(0) + sF(s)\end{aligned} \tag{6.233}$$

Heaviside Step Function

The Heaviside step function is defined as

$$H(t-\tau) = \begin{cases} 0, & \text{for } t < \tau \\ 1, & \text{for } t \geq \tau \end{cases} \tag{6.234}$$

The Laplace transform of $H(t)$ is thus

$$H(s) = \int_0^\infty e^{-st}H(t)\,\mathrm{d}t = \int_0^\infty e^{-st}\,\mathrm{d}t = -\left.\frac{e^{st}}{s}\right|_0^\infty = \frac{1}{s} \tag{6.235}$$

Dirac Delta Function

The Dirac delta function is defined by the properties

$$\begin{aligned}\delta(t-\tau) = 0 \quad \text{for} \quad t \neq \tau \\ \int_{-\infty}^\infty \delta(t-\tau)\,\mathrm{d}t = 1\end{aligned} \tag{6.236}$$

The Laplace transform of $\delta(t)$ is thus

$$\delta(s) = \int_0^\infty e^{-st}\delta(t)\,\mathrm{d}t = e^0 = 1 \tag{6.237}$$

The function $f(t) = t$

The Laplace transform of $f(t) = t$ is

$$F(s) = \int_0^\infty te^{-st}\,\mathrm{d}t = -\left.\frac{te^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\,\mathrm{d}t = -\left.\frac{e^{-st}}{s^2}\right|_0^\infty = \frac{1}{s^2} \tag{6.238}$$

The function $f(t) = e^{-at}$

The Laplace transform of $f(t) = e^{-at}$ is

$$F(s) = \int_0^\infty e^{-at} e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-(a+s)t}\,\mathrm{d}t = -\left.\frac{e^{-(a+s)t}}{a+s}\right|_0^\infty = \frac{1}{s+a} \tag{6.239}$$

The Trigonometric Functions, $\cos\omega t$ and $\sin\omega t$

The Laplace transform of $f(t) = \cos\omega t$ is determined as follows. Since

$$\begin{aligned} \int_0^\infty \cos\omega t e^{-st}\,\mathrm{d}t &= -\left.\frac{\cos\omega t e^{-st}}{s}\right|_0^\infty - \omega\int_0^\infty \sin\omega t\frac{e^{-st}}{s}\,\mathrm{d}t \\ &= \frac{1}{s} - \frac{\omega}{s}\int_0^\infty \sin\omega t e^{-st}\,\mathrm{d}t = \frac{1}{s} + \frac{\omega}{s}\left[\frac{\sin\omega t e^{-st}}{s} - \omega\int_0^\infty \cos\omega t\frac{te^{-st}}{s}\,\mathrm{d}t\right] \\ &= \frac{1}{s} - \frac{\omega^2}{s^2}\int_0^\infty \cos\omega t e^{-st}\,\mathrm{d}t \end{aligned} \tag{6.240}$$

we have

$$\left(1 + \frac{\omega^2}{s^2}\right)\int_0^\infty \cos\omega t e^{-st}\,\mathrm{d}t = \frac{1}{s} \tag{6.241}$$

whence

$$F(s) = \int_0^\infty \cos\omega t e^{-st}\,\mathrm{d}t = \frac{1}{s\left(1+\frac{\omega^2}{s^2}\right)} = \frac{s}{s^2+\omega^2} \tag{6.242}$$

Similarly, to determine the Laplace transform of $f(t) = \sin\omega t$, since

$$\begin{aligned} \int_0^\infty \sin\omega t e^{-st}\,\mathrm{d}t &= -\left.\frac{\sin\omega t e^{-st}}{s}\right|_0^\infty + \frac{\omega}{s}\int_0^\infty \cos\omega t e^{-st}\,\mathrm{d}t \\ &= \frac{\omega}{s}\int_0^\infty \cos\omega t e^{-st}\,\mathrm{d}t = \frac{\omega}{s}\left[\frac{1}{s} - \frac{\omega}{s}\int_0^\infty \sin\omega t e^{-st}\,\mathrm{d}t\right] \end{aligned} \tag{6.243}$$

we have

$$\left(1 + \frac{\omega^2}{s^2}\right)\int_0^\infty \sin\omega t e^{-st}\,\mathrm{d}t = \frac{\omega}{s^2} \tag{6.244}$$

whence

$$F(s) = \int_0^\infty \sin\omega t e^{-st}\,\mathrm{d}t = \frac{\omega}{s^2\left(1+\frac{\omega^2}{s^2}\right)} = \frac{\omega}{s^2+\omega^2} \tag{6.245}$$

The Attenuation Rule

Exponentially damped harmonic functions appear often in linear system dynamics, so the following *attenuation rule* is useful.

If $F(s)$ is the Laplace transform of $f(t)$, then the Laplace transform of $e^{-at}f(t)$ is

$$\begin{aligned}\mathcal{L}\left[e^{-at}f(t)\right] = \int_0^\infty e^{-at}f(t)e^{-st}\,\mathrm{d}t &= \int_0^\infty e^{-(s+a)t}f(t)\,\mathrm{d}t \\ &= \int_0^\infty e^{-s't}f(t)\,\mathrm{d}t = F(s') = F(s+a)\end{aligned} \tag{6.246}$$

Thus, since Eq. (6.242) gives

$$\mathcal{L}\left[\cos\omega t\right] = \frac{s}{s^2+\omega^2} \tag{6.247}$$

we have

$$\mathcal{L}\left[e^{-at}\cos\omega t\right] = \frac{s+a}{(s+a)^2+\omega^2} \tag{6.248}$$

Also, since Eq. (6.245) gives

$$\mathcal{L}\left[\sin\omega t\right] = \frac{\omega}{s^2+\omega^2} \tag{6.249}$$

we have

$$\mathcal{L}\left[e^{-at}\sin\omega t\right] = \frac{\omega}{(s+a)^2+\omega^2} \tag{6.250}$$

The Convolution Integral

The convolution integral

$$\mathcal{L}^{-1}\left[F(s)G(s)\right] = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau \tag{6.251}$$

where $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$, respectively, can be verified formally as follows.

From the definition of the Laplace transform,

$$\begin{aligned}F(s)G(s) &= \int_0^\infty e^{-sv}f(v)\,\mathrm{d}v\int_0^\infty e^{-su}g(u)\,\mathrm{d}u \\ &= \int_0^\infty\int_0^\infty e^{-s(v+u)}f(v)g(u)\,\mathrm{d}v\,\mathrm{d}u \\ &= \int_0^\infty g(u)\left(\int_0^\infty e^{-s(v+u)}f(v)\,\mathrm{d}v\right)\mathrm{d}u\end{aligned} \tag{6.252}$$

Then, with the change of variable

$$v+u=t \tag{6.253}$$

we have

$$\int_0^\infty e^{-s(v+u)}f(v)\,\mathrm{d}v = \int_u^\infty e^{-st}f(t-u)\,\mathrm{d}t \tag{6.254}$$

so

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \left(\int_u^\infty e^{-st} f(t-u) g(u)\,\mathrm{d}t \right) \mathrm{d}u \\ &= \int_0^\infty \left(\int_0^t e^{-st} f(t-u) g(u)\,\mathrm{d}u \right) \mathrm{d}t \\ &= \int_0^\infty e^{-st} \left(\int_0^t f(t-u) g(u)\,\mathrm{d}u \right) \mathrm{d}t = \mathcal{L}\left[\int_0^t f(t-u) g(u)\,\mathrm{d}u \right] \end{aligned} \tag{6.255}$$

which was to be proved. The interchange of order of integration in this last step can be shown to be legitimate, by appropriate limiting procedures, when the Laplace transforms of $f(t)$ and $g(t)$ exist.