# Concrete Pictorial Abstract

Singapore's Approach to Math Instruction


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https://sites.google.com/site/**singmathproject/**

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Will also post on NCTM website.

# Background

- 35 years teaching college students.
- 25 years working with K-12 teachers and their students.
- 15 years working with math textbooks from Singapore. Their students are top math performers in international studies.

# “Singapore” Textbooks in the US

Primary Mathematics

Math In Focus (Consulting Author)

Both published by Marshall Cavendish Education.

“The students are provided with the necessary learning experiences beginning with the **concrete** and **pictorial** stages, followed by the **abstract** stage to enable them to learn math meaningfully.”

# CPA Approach

- Based on the work of US Psychologist Jerome Bruner.

- “I shall call the three modes of representation mentioned earlier **enactive** representation, **iconic** representation, and **symbolic** representation. Their appearance in the life of the child is in that order, each depending upon the previous one for its development, yet all of them remaining more or less intact throughout life—…”

- Bruner: The Course of Cognitive Growth (1964)

## Bruner, J. S. (1960). *The Process of education.*

“ It has also been pointed out … that the method of discovery would be too time-consuming for presenting all of what a student must cover in mathematics.” (page 21)

# My Take

Concrete ↔ Abstract

It's a continuum

# This Talk

Examples from:

1. The Books.
2. Asian Classrooms.
3. My Classrooms.
4. Me.

# C→P→A

Concrete: ?

Pictorial: | | | | | | | |

Abstract: 8

# Abstraction

- Gives mathematics its power.
- But abstraction without understanding??
- Leads to confusion.

# MODEL DRAWING

# and

# C→P→A

The model drawing approach takes students from the concrete to the abstract stage via an intermediary pictorial stage.

Many complex problems are easier to solve with algebra….

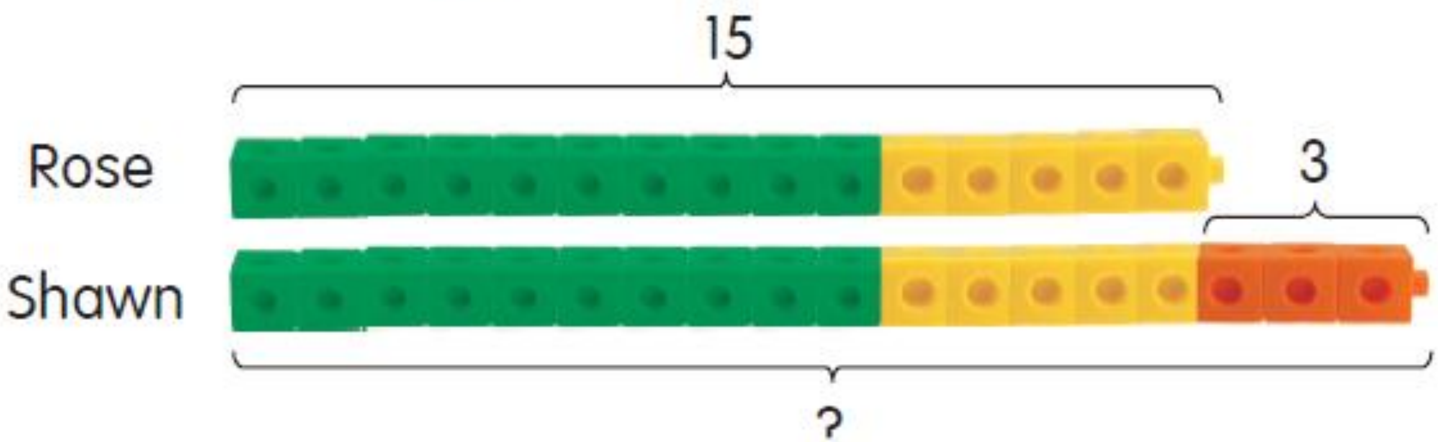
if you understand the algebra!

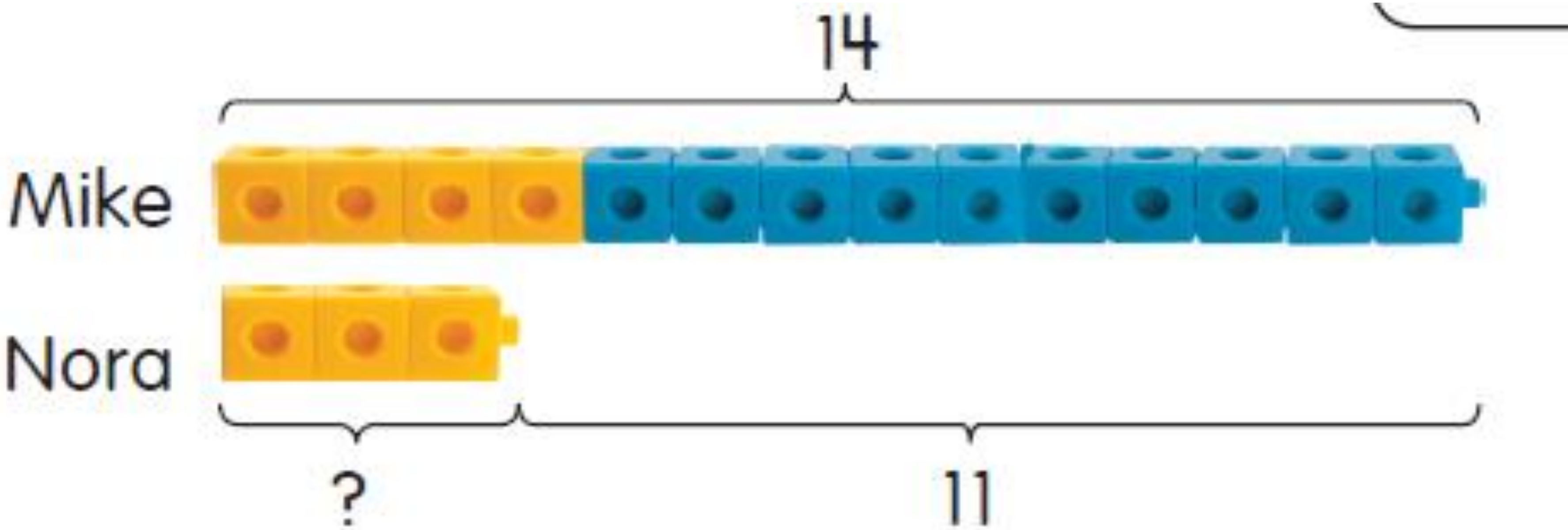
# Addition

Rose has 15 .

Shawn has 3 more  than Rose.

How many  does Shawn have?






$x + 11 = 14$

Learn

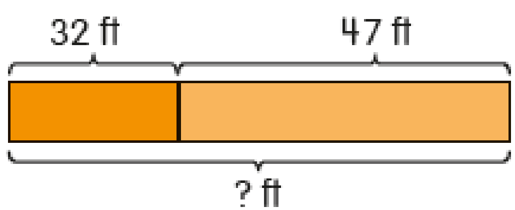
## You can use bar models to solve measurement problems.

Jenny walked 32 feet.
Then she turned right and walked 47 feet.

**a** How far did Jenny walk in all?

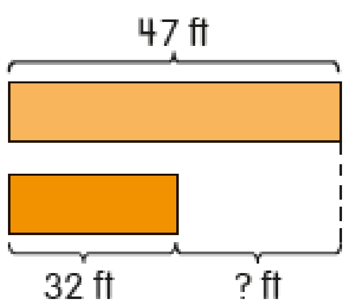
**b** In which direction did she walk farther, the first direction or the second direction?
How much farther?

**a**




32 + 47 = 79
Jenny walks 79 feet in all.

**b**




47 – 32 = 15
She walked 15 feet farther in the second direction.

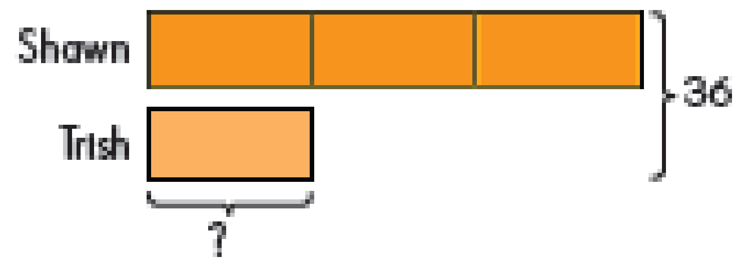
# Division

Learn **Use bar models to solve one-step division word problems.**

Shawn and Trish scored 36 goals in all.
Shawn scored 3 times as many goals as Trish.
How many goals did Trish score?

4 units ⟶ 36
1 unit ⟶ 36 ÷ 4 = 9

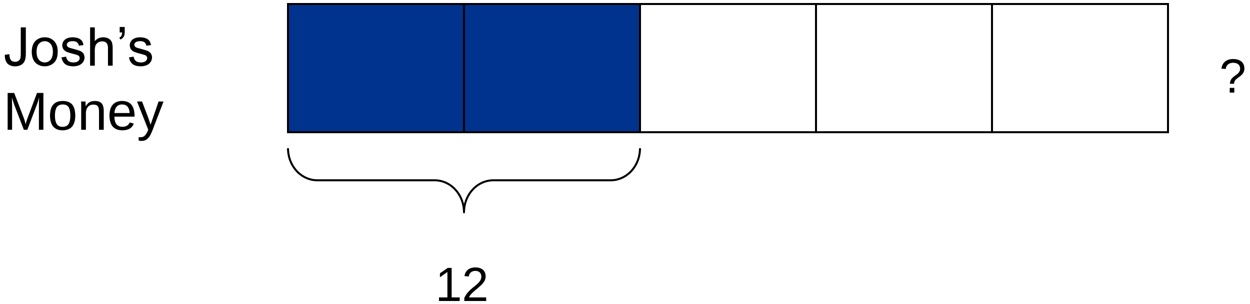
Trish scored 9 goals.




Suppose Trish scored x goals.
Then Shawn scores 3x goals

$$x + 3x = 36$$
$$4x = 36$$
$$x = 9$$

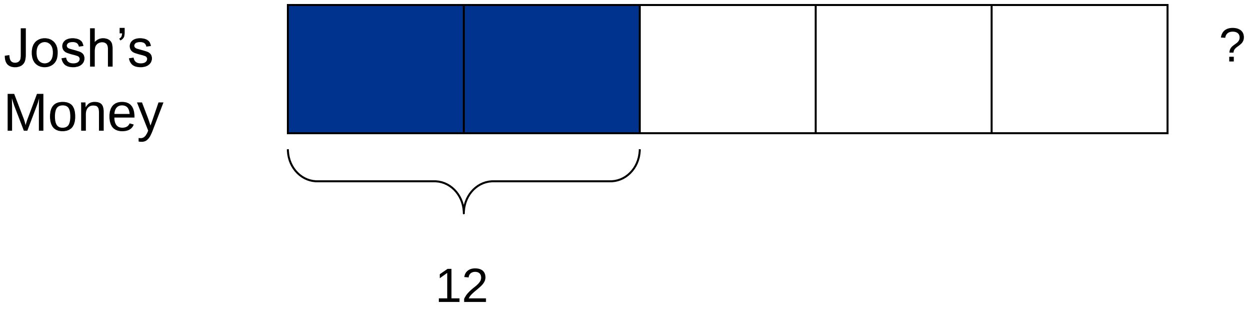
**Example** (grade 4): Josh spent 2/5 of his money on a present for his Dad. The present cost $12. How much money did he have at first?




2 units = 12

1 unit = 6

5 units = 30 Josh started with $30.



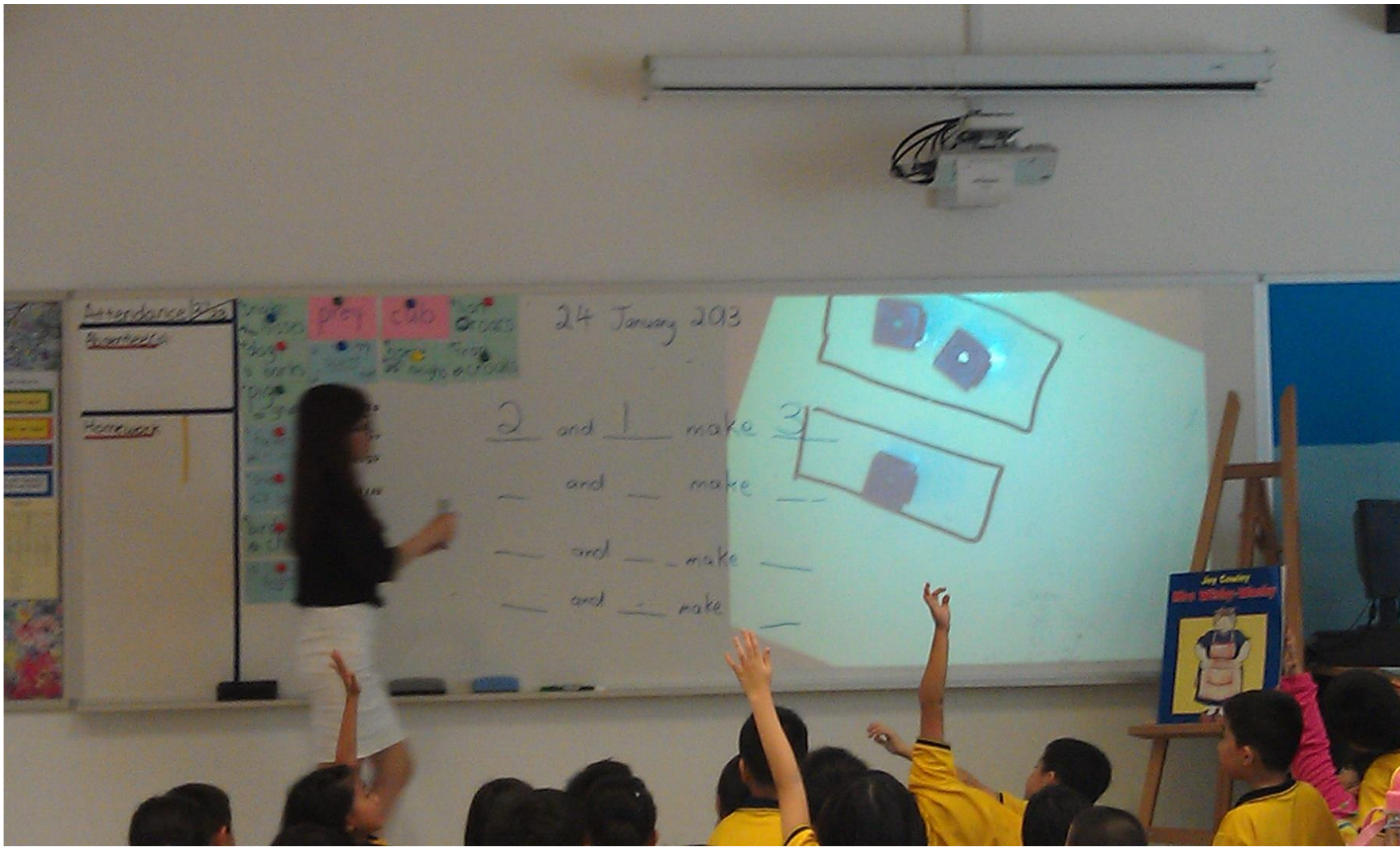

2 units = 12 $\qquad \frac{2}{5}x = 12$

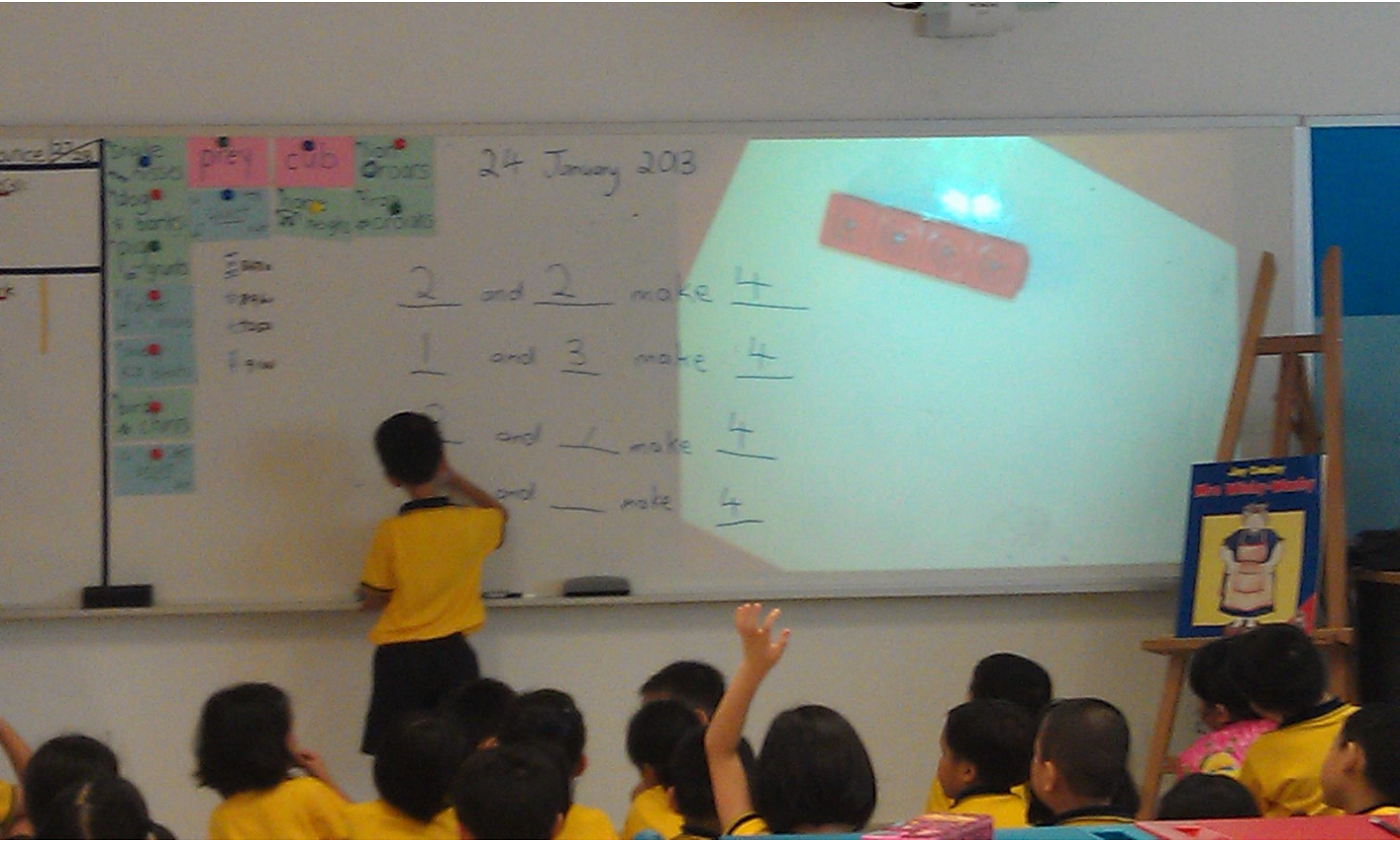
1 unit = 6 $\qquad \frac{1}{5}x = 6$

5 units = 30. $\qquad x = 30$

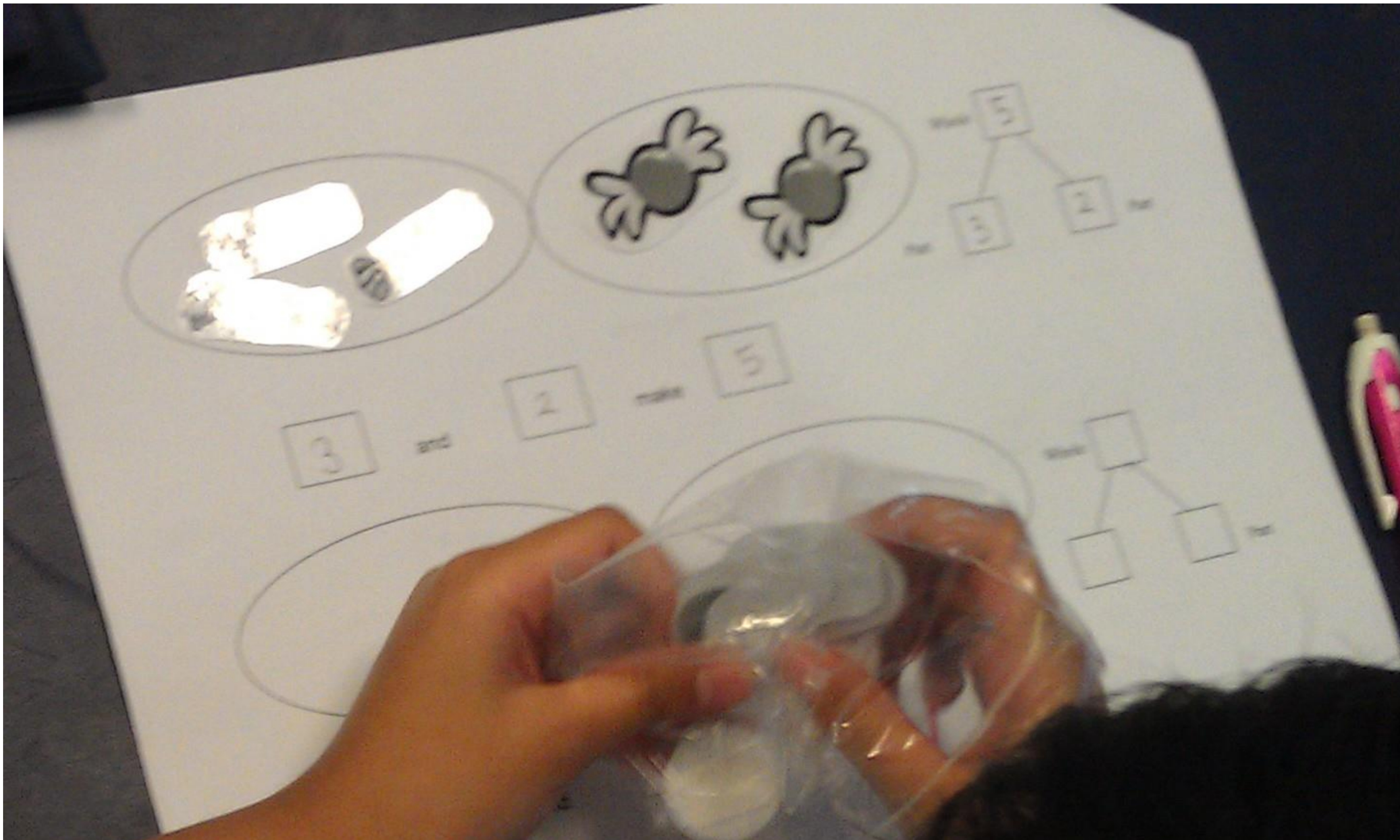
Suppose Josh started with x dollars.

# Grade 1 - Number Bonds

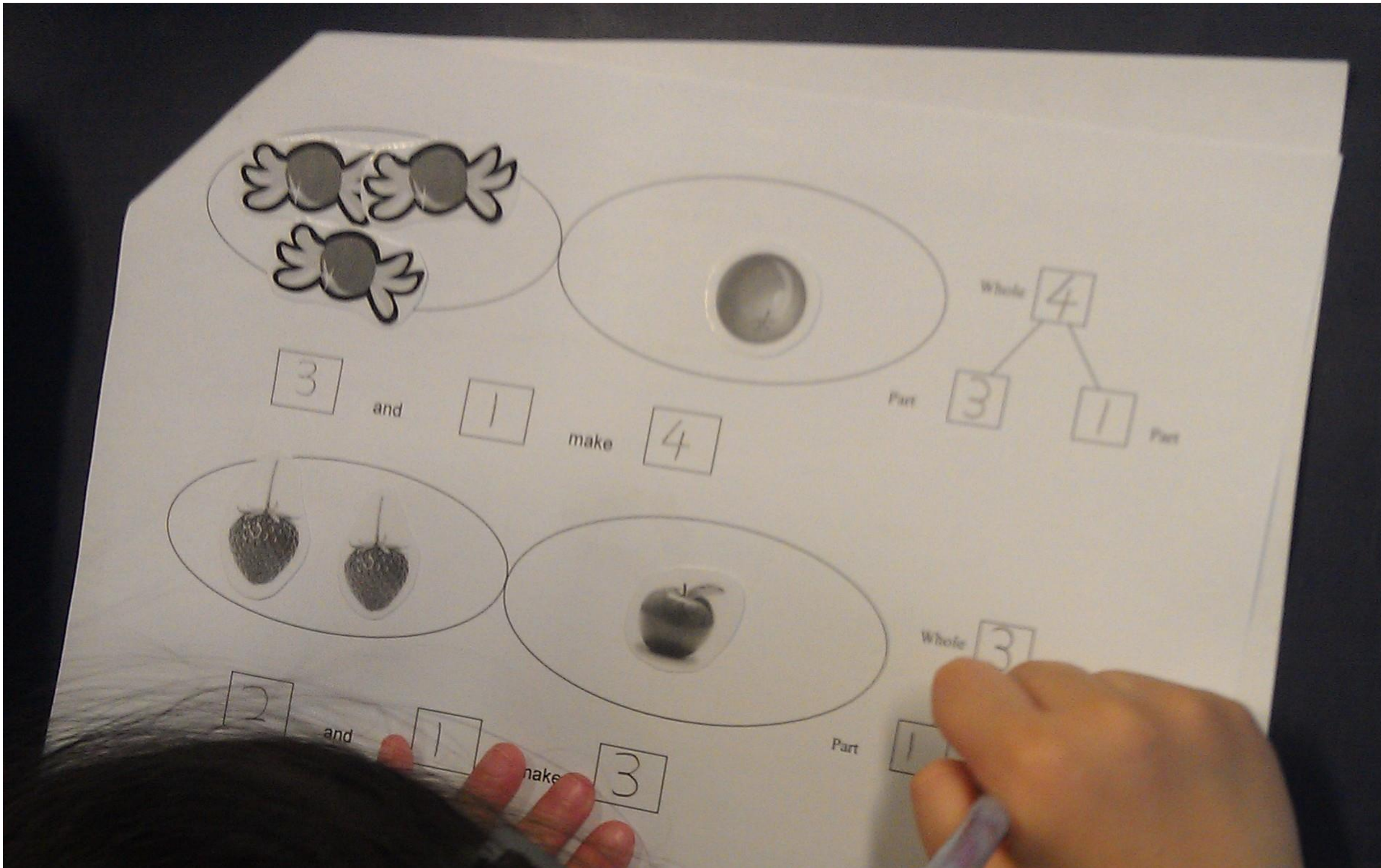



prey
cub
2 and 2 make 4
1 and 3 make 4

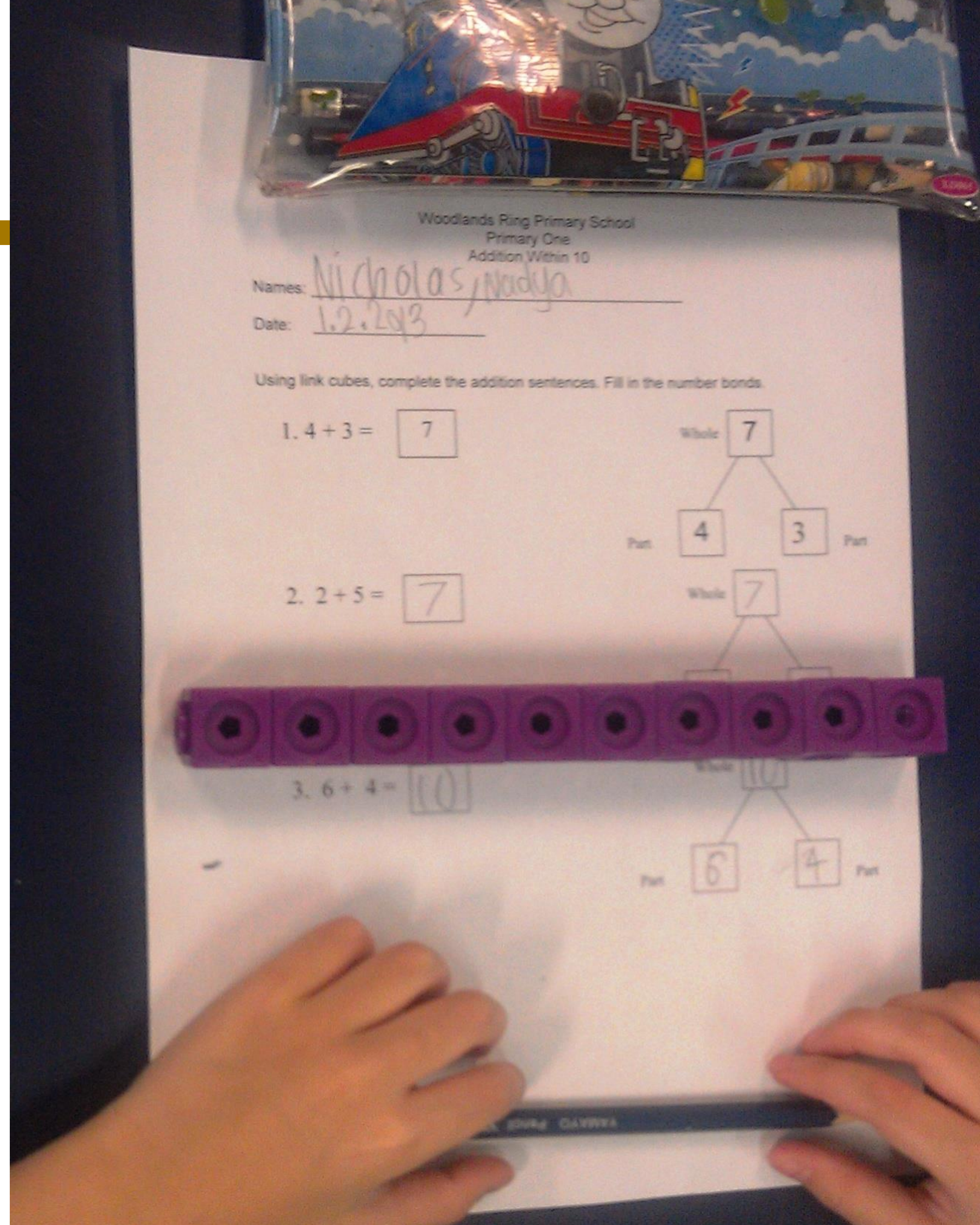
# Number Bonds



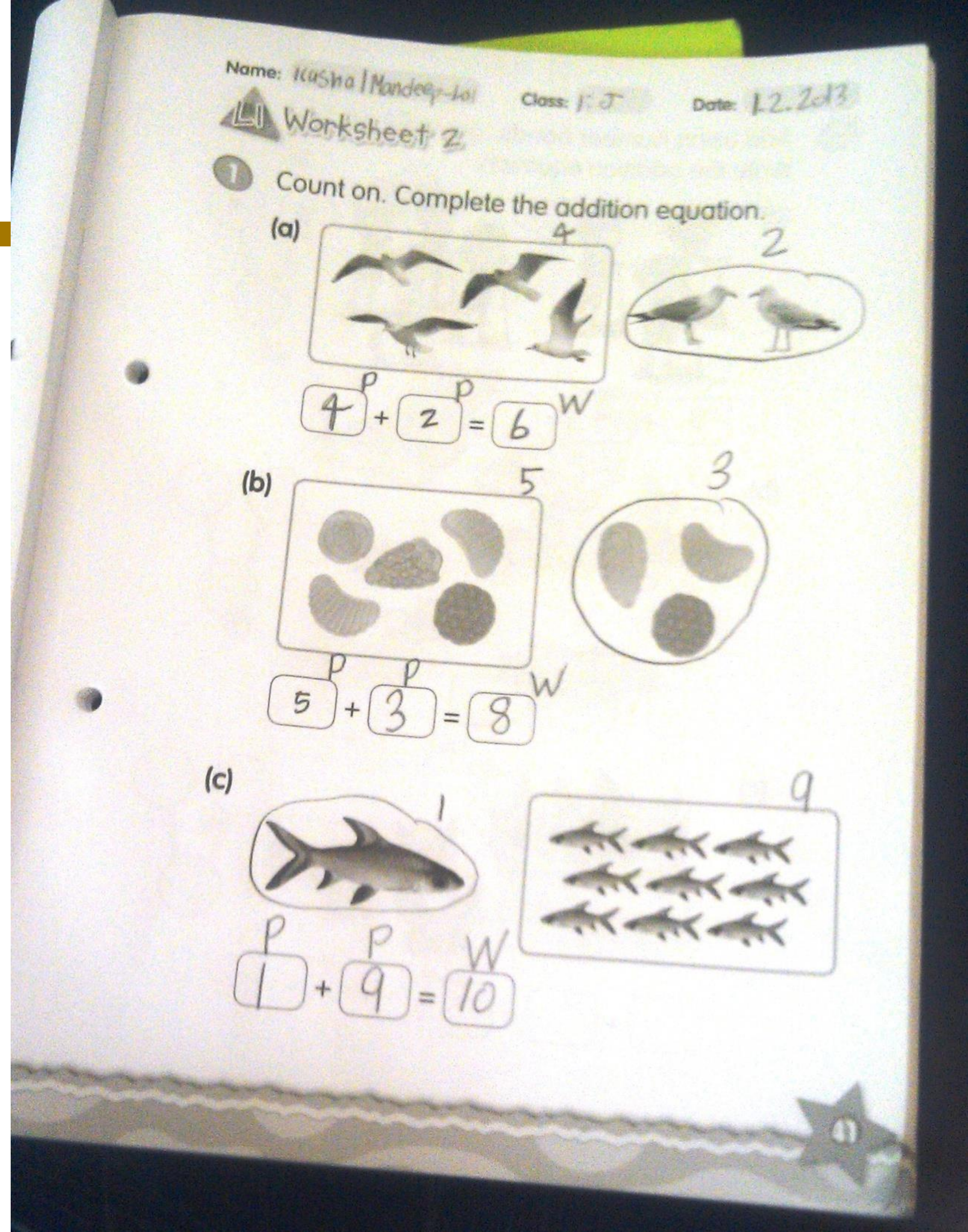
# Number Bonds



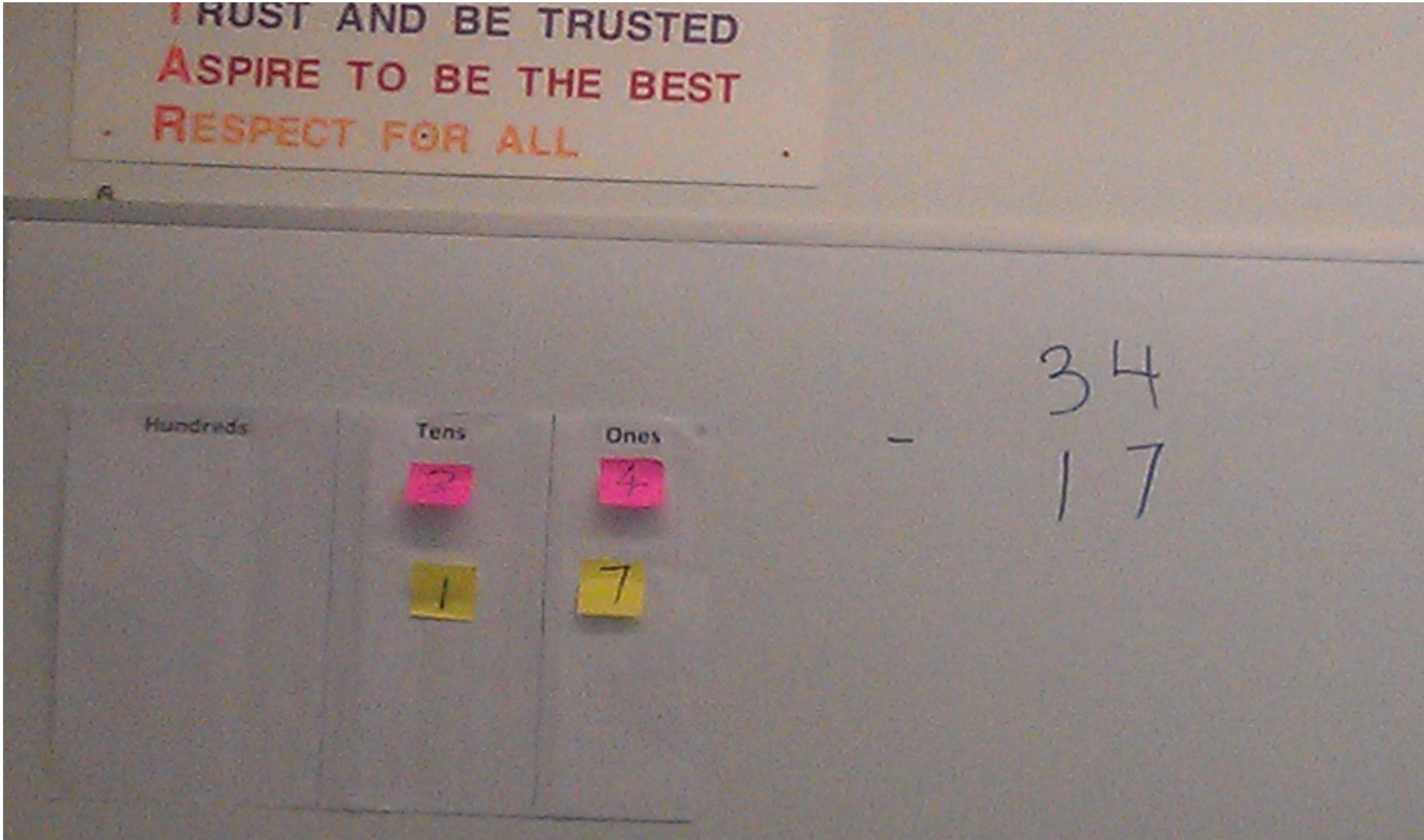
# Multiple Representations

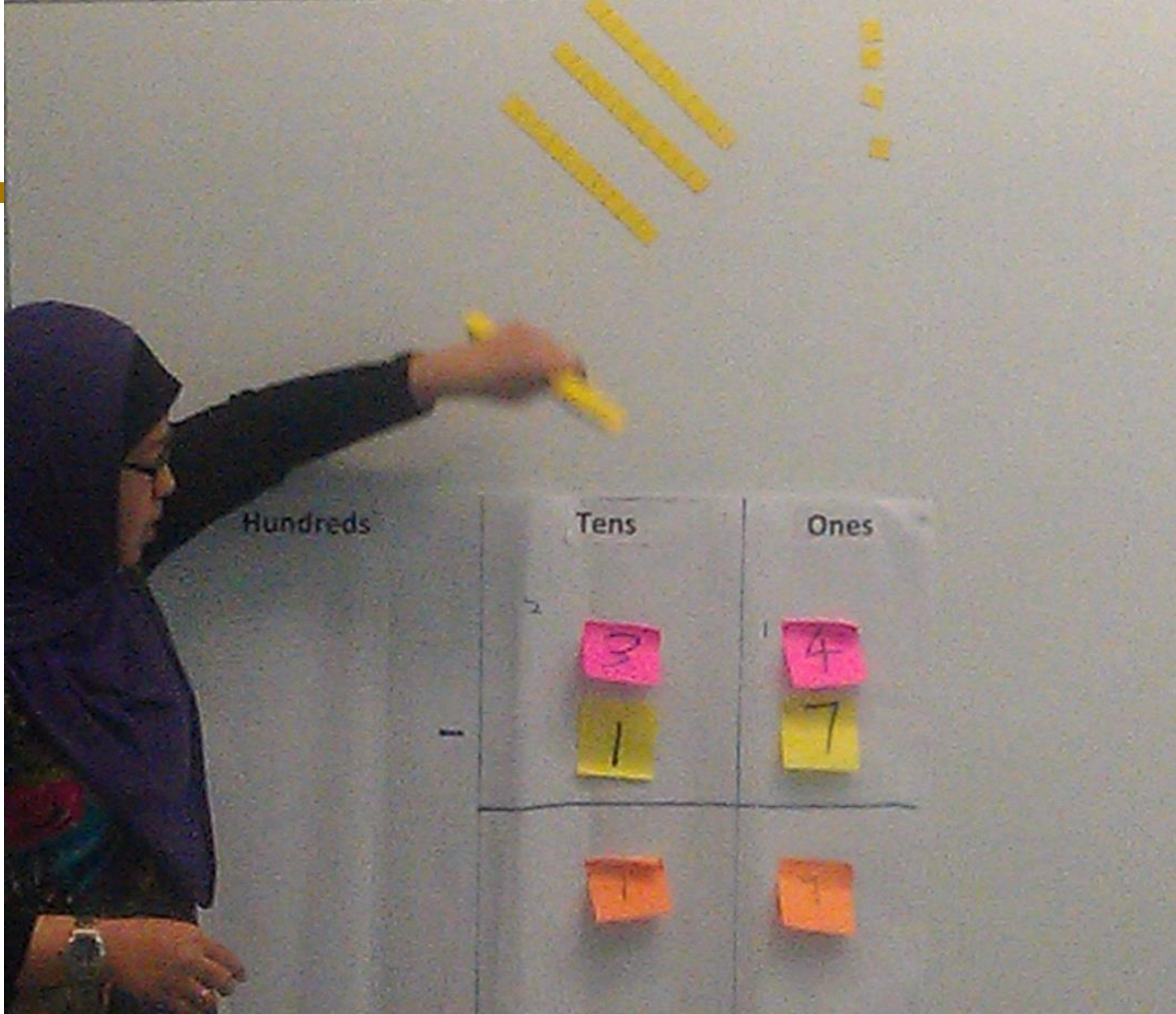


# Progression to Abstract



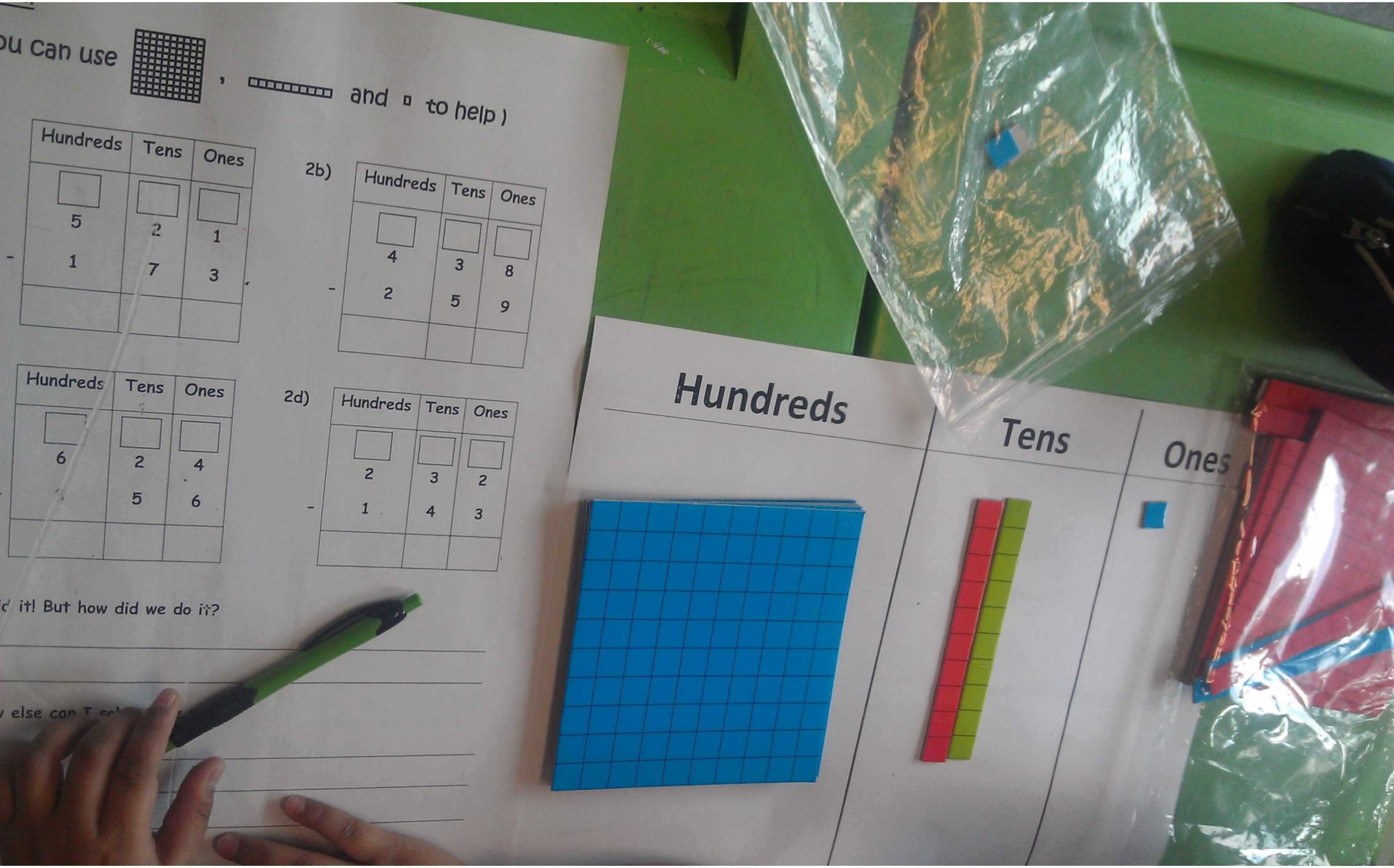
# Grade 2


TRUST AND BE TRUSTED
ASPIRE TO BE THE BEST
RESPECT FOR ALL
Hundreds
Tens
Ones
3
4
1
7
34
- 17

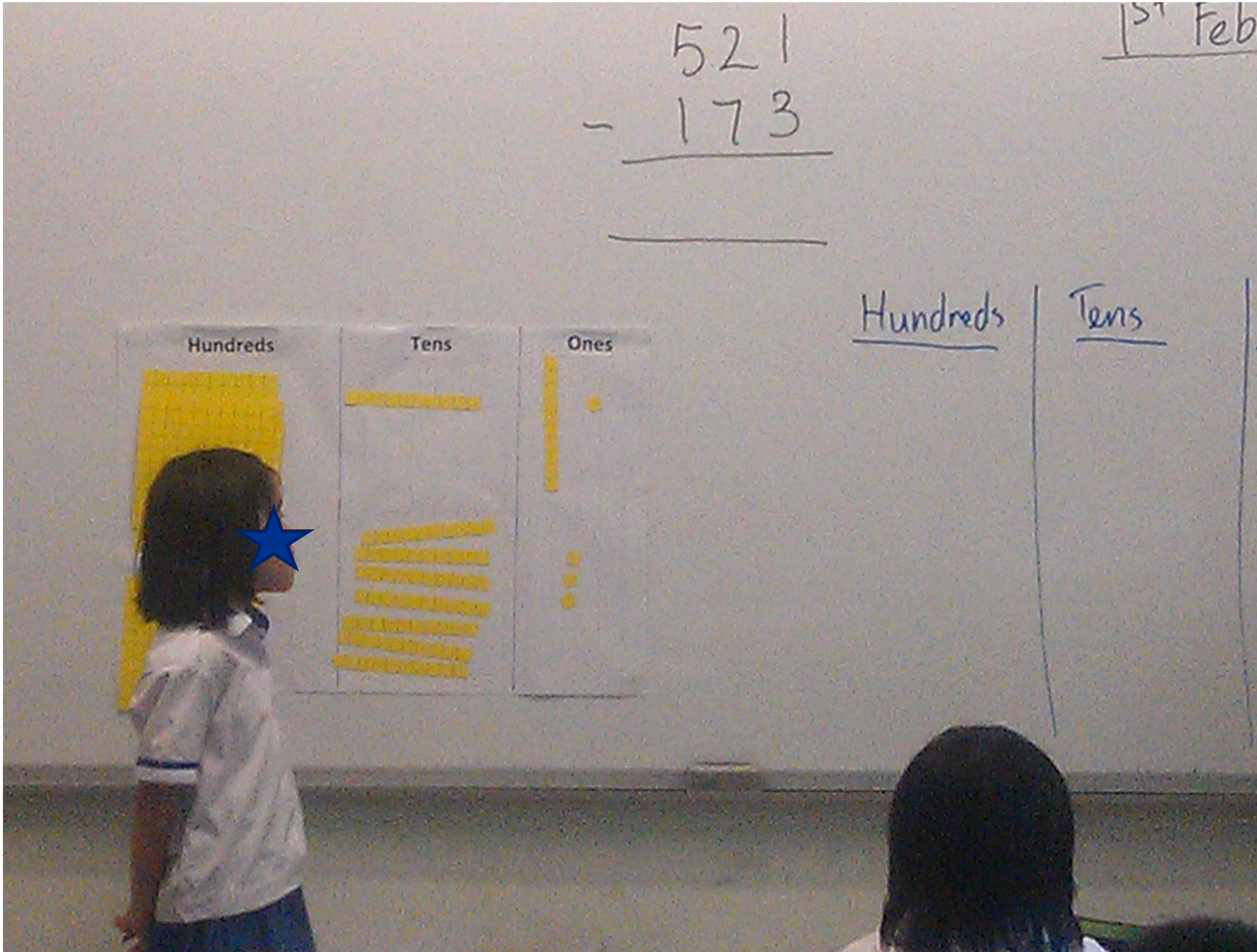

Hundreds
Tens
Ones
3
4
1
7


SUCCESS IN OUR PURSUIT
TRUST AND BE TRUSTED
ASPIRE TO BE THE BEST
RESPECT FOR ALL
Tens
Ones
3
4
1
7

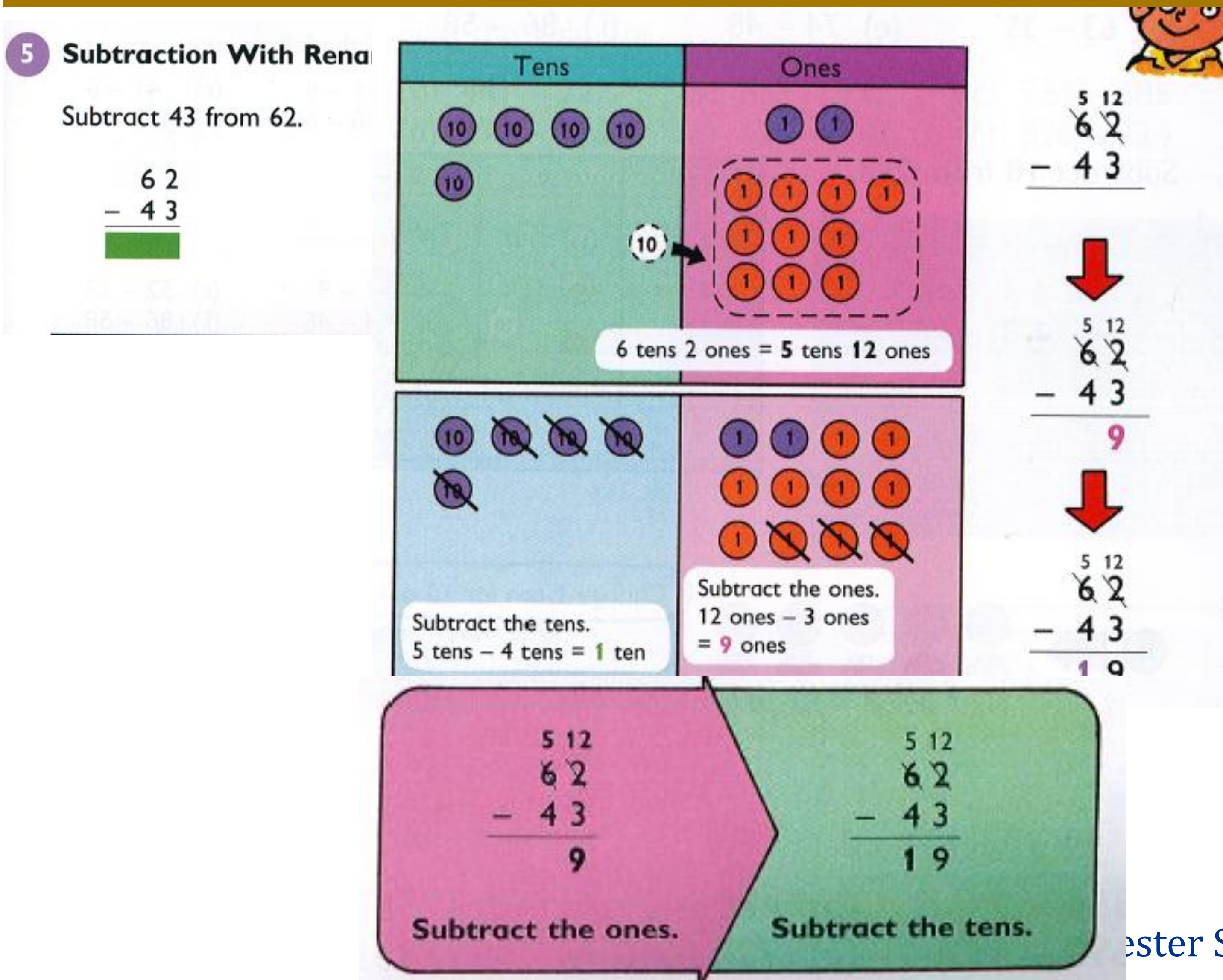

ASPIRE TO BE THE BEST
RESPECT FOR ALL
Tens
Ones
3
4
1
7



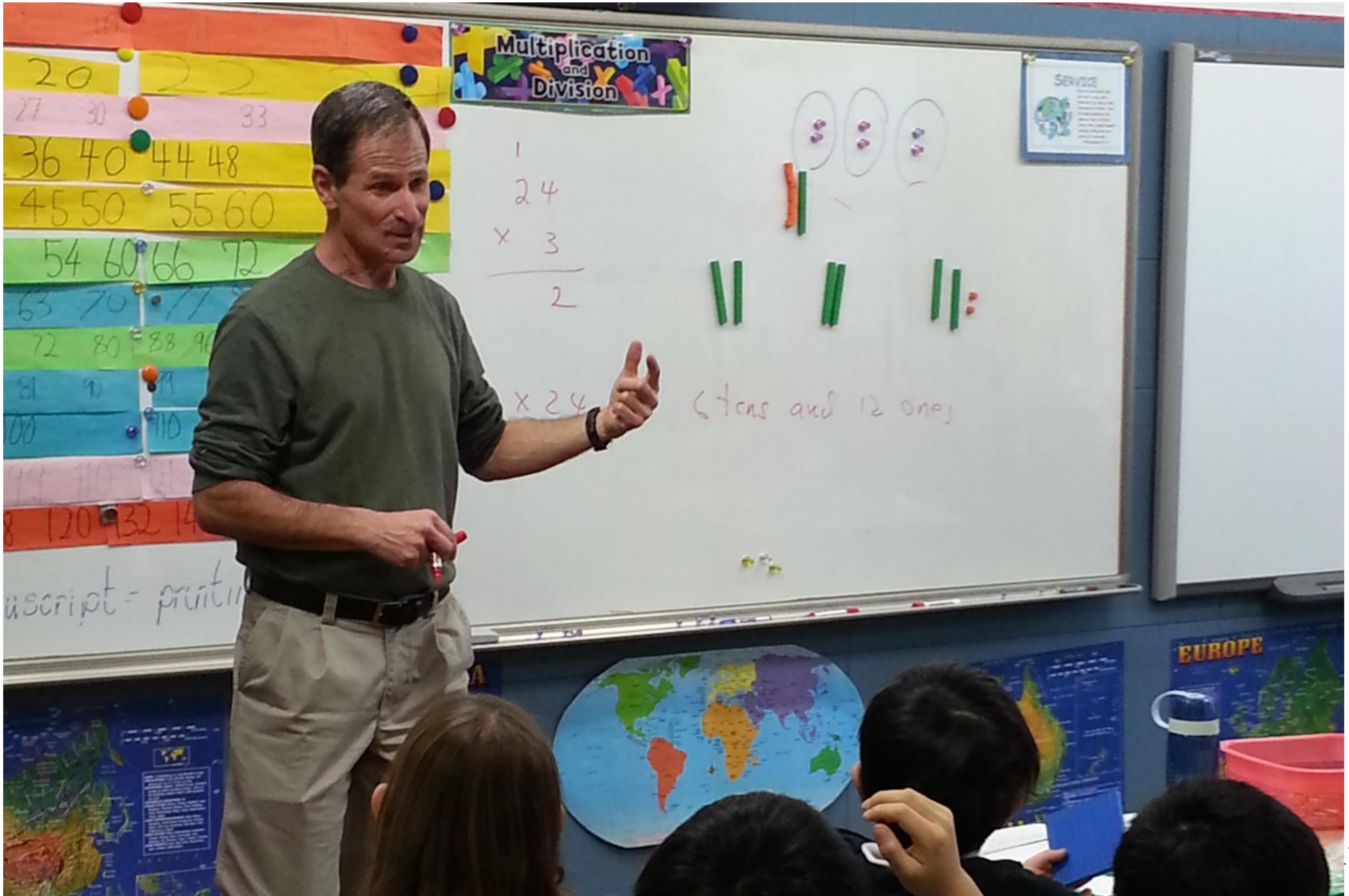

Worcester State University


521
- 173
Hundreds
Tens
Ones
Hundreds
Tens

# Still Concrete; but More Abstract

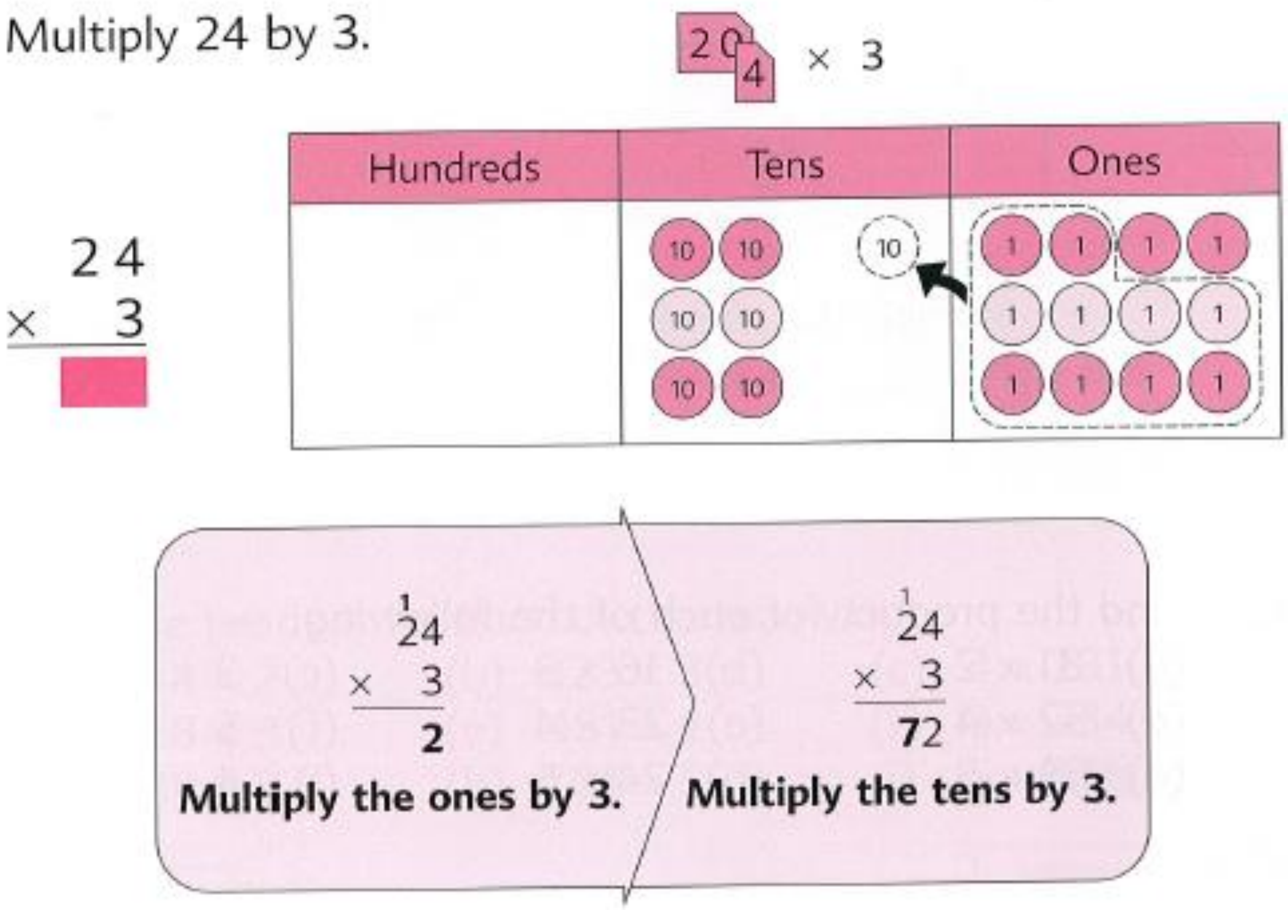


# Seoul – Grade 3




ty

# Same Problems with Chips



# Fractions

- Where we lose too many students.
- They don’t understand what a fraction is.
- They learn algorithms without first learning meaning.

# Common Core – Grade 3

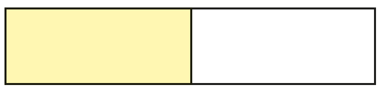
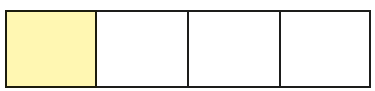
CCSS.Math.Content.3.NF.A.1 Understand a fraction 1/*b* as the quantity formed by 1 part when *a* whole is partitioned into *b* equal parts; understand a fraction *a*/*b* as the quantity formed by a parts of size 1/*b*.

Understand a fraction 1/*3* as the quantity formed by 1 part when *a* whole is partitioned into *3* equal parts; understand a fraction *2*/*3* as the quantity formed by 2 parts of size 1/*3*.

# MIF – 2B

Learn **Name fractional parts.**

How do you read fractional parts?

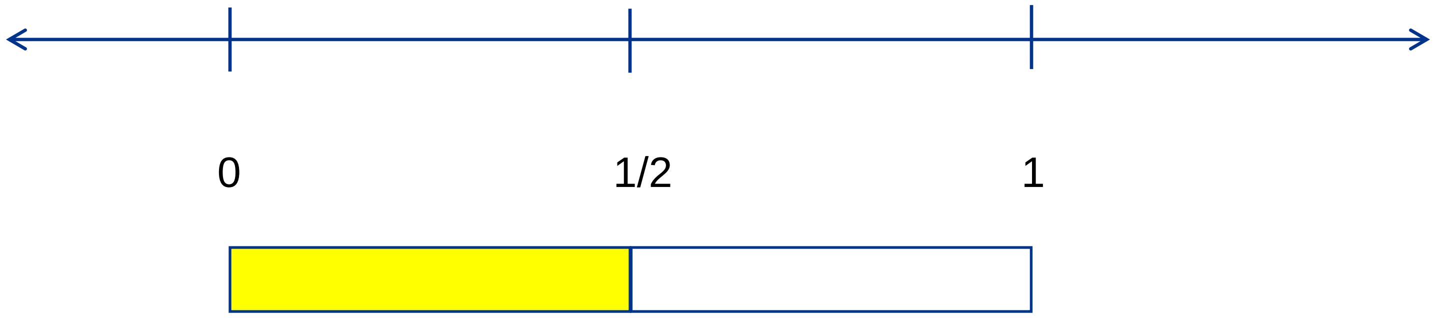
| Fraction | Read As |
|---|---|
|  $\frac{1}{2}$ | one-half |
|  $\frac{1}{3}$ | one-third |
|  $\frac{1}{4}$ | one-quarter or one-fourth |

$\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ are unit fractions.

A **unit fraction** names one of the equal parts of a whole.

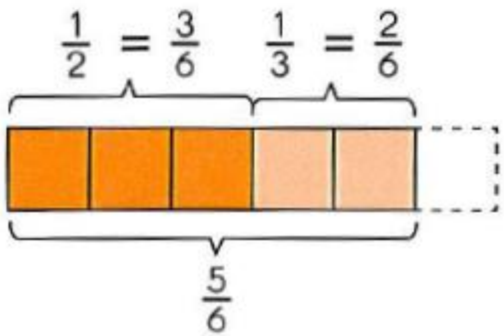
# Common Core – Grade 3

CCSS.Math.Content.3.NF.A.2 Understand a fraction as a number on the number line; represent fractions on a number line diagram.

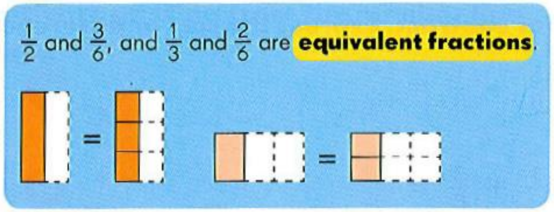


# Adding Fractions

$$\frac{1}{2} + \frac{1}{3} = ?$$




$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6}$$
$$= \frac{5}{6}$$

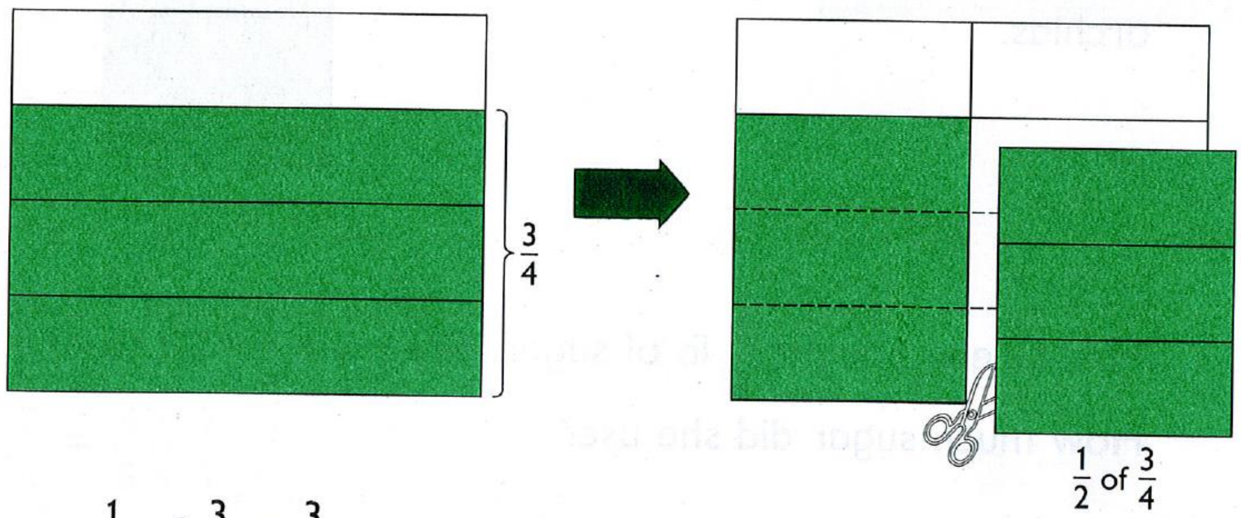


# Multiplying Fractions

Color $\frac{3}{4}$ of a rectangle.

Cut out $\frac{1}{2}$ of the colored parts.

What fraction of the rectangle is cut out?



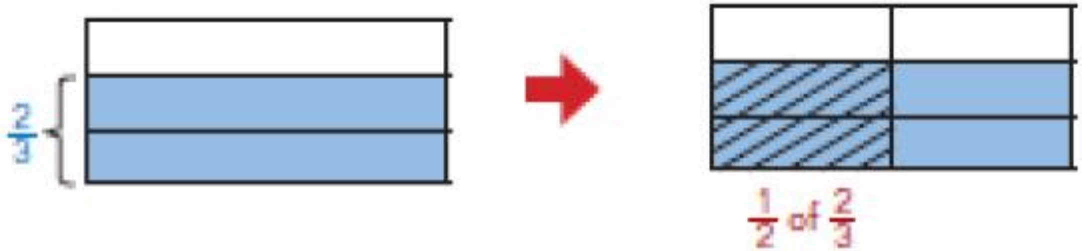

$\frac{1}{2}$ of $\frac{3}{4} = \frac{3}{8}$

# Multiplying Fractions

Find $\frac{1}{2} \times \frac{2}{3}$.

Margie drew a rectangle and colored $\frac{2}{3}$ of it blue.

She then drew stripes over $\frac{1}{2}$ of the colored parts.

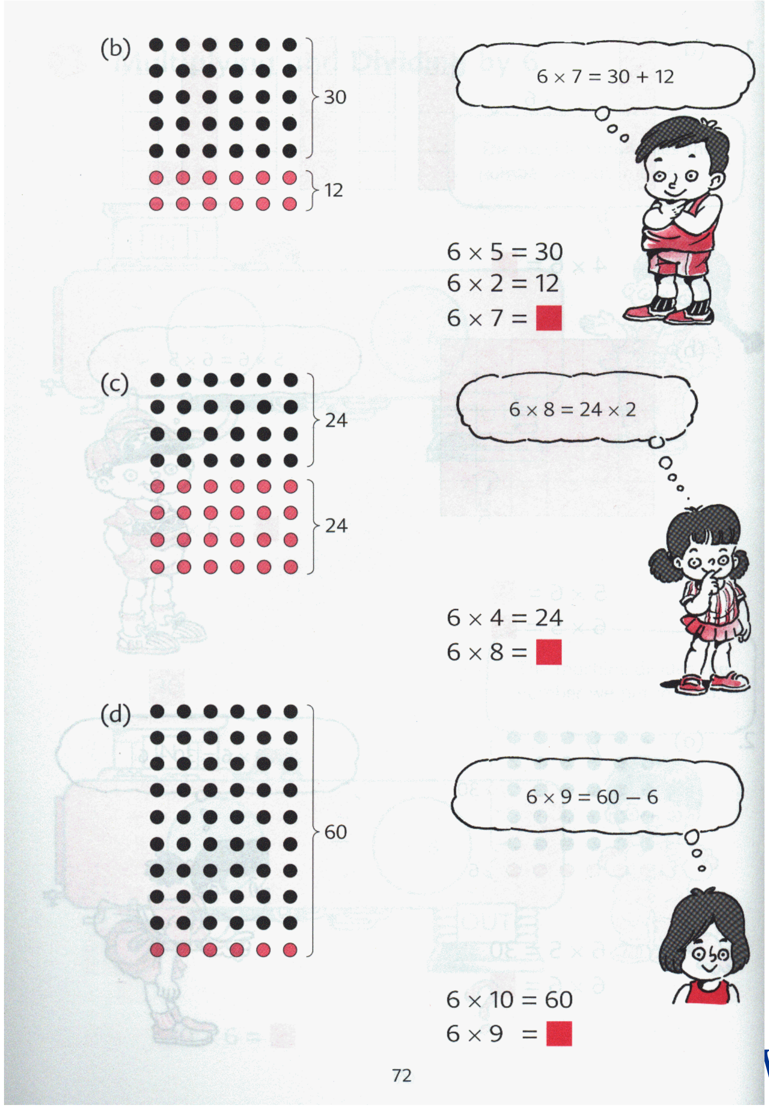



$$\frac{1}{2} \times \frac{2}{3} = \frac{1}{2} \text{ of } \frac{2}{3}$$

$$= \frac{2}{6}$$ ← Number of parts with stripes / ← Total number of parts

$$= \frac{1}{3}$$

# Multiplication



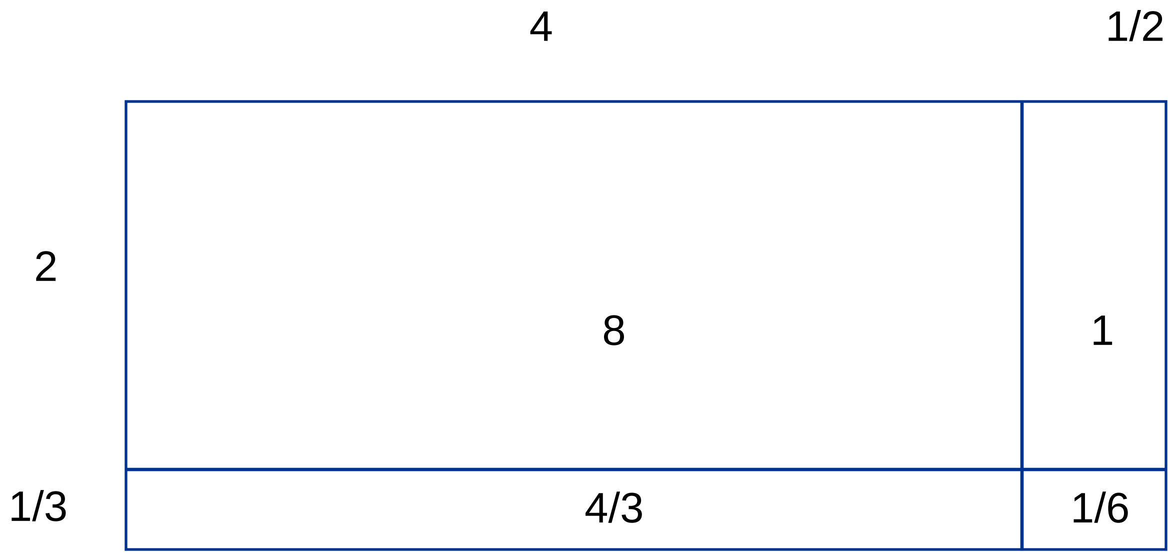
# Multiplication

- 23 x 45

| | 40 | 5 |
|---|---|---|
| 20 | 800 | 100 |
| 3 | 120 | 15 |

# Multiplication

- 2 1/3 x 4 1/2

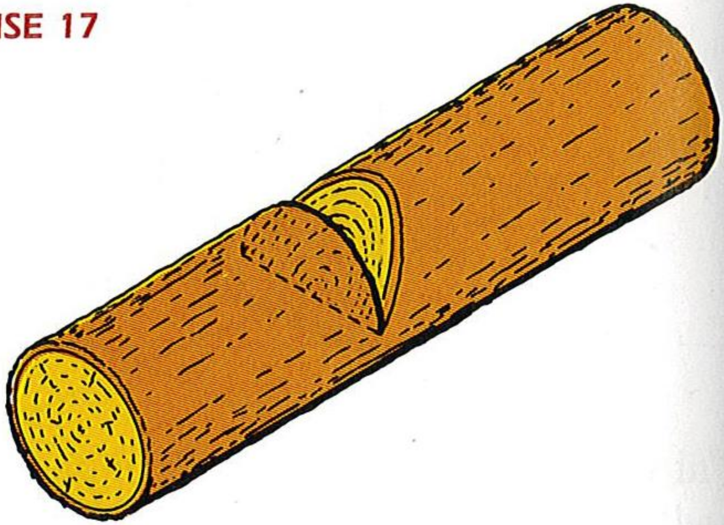


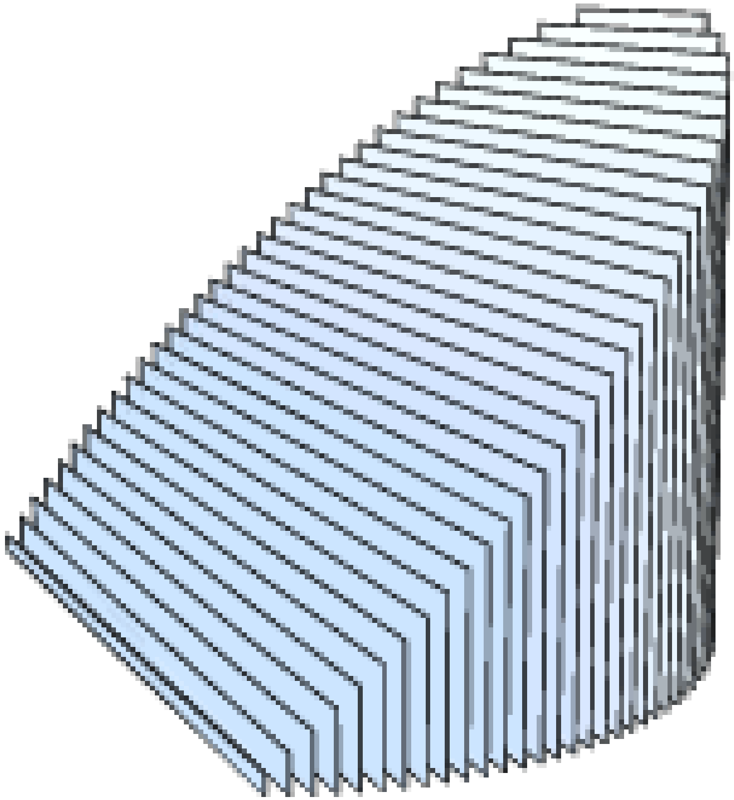
# Calculus

Source: Swokowski: *Calculus – Classic Edition*

17 A log having the shape of a right circular cylinder of radius $a$ is lying on its side. A wedge is removed from the log by making a vertical cut and another cut at an angle of 45°, both cuts intersecting at the center of the log (see figure). Find the volume of the wedge.

EXERCISE 17





$$\int_0^a A(x)\,dx$$

# Bruner: *The Process of Education*

*The teacher is not only a communicator but a model. Somebody who does not see anything beautiful or powerful about mathematics is not likely to ignite others with a sense of the intrinsic excitement of the subject.*