Complex Analysis: Interesting Problems

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Contents

1 Preface

This document is a collection of problems I have worked on in Complex Analysis. Most of the questions are either directly from, or a derivative of, questions from (in order of frequency) Ahlfors, Stein, Conway, Rudin and Cartan. As a notice, there likely will be mistakes in the solutions in this document. However, please feel free to use this document to hopefully help in your understanding of Complex Analysis.

I recently came across this description of a course taught by Professor John Roe, which provides a wonderful overview of the material taught in a first course in Complex Analysis, and just some of its general applications. I thought I'd add this description here for anyone interested in learning Complex Analysis in the near future.

(In Complex Analysis) We study the behavior of differentiable complex-valued functions $f(z)$ of a complex variable z. The key idea in an introductory course is that complex differentiability is a much more restrictive condition than real differentiability. In fact, complex-differentiable functions are so rigid that the entire behavior of such a function is completely determined if you know its values even on a tiny open set. One understands these rigidity properties by making use of contour integration - integration along a path in the complex plane.

The theory gains its force because there are plenty of interesting functions to which it applies. All the usual functions - polynomials, rational functions, exponential, trigonometric functions, and so on - are differentiable in the complex sense. Very often, complex analysis provides the solution to "real variable" problems involving these functions; as someone said, "The shortest path between two real points often passes through the complex domain." Moreover, complex analysis is a key tool for understanding other "higher transcendental functions" such as the Gamma function, the Zeta function, and the elliptic functions, which are important in number theory and many other parts of mathematics. A secondary aim of this course is to introduce you to some of these functions.

One of the surprises of complex analysis is the role that topology plays. Simple questions like "do I choose the positive or negative sign with the square root" turn out to have surprisingly subtle answers, rooted in the notion of the fundamental group of a topological space (which you will be looking at in the Topology and Geometry course parallel to this). These topological notions eventually culminate in the notion of a Riemann surface as the correct global context for complex analysis. We will not develop this idea fully, but we will discuss 'multiple-valued functions' and their branch points; again, we will try to illustrate how these exotic-sounding concepts help in doing practical calculations.

Also, I'd strongly recommend watching the following three (non-technical) videos to garner a little motivation for studying Complex Analysis:

1. The Riemann Hypothesis
2. Visualizing the Riemann zeta function and analytic continuation
3. Why Complex Numbers are Awesome

2 Complex numbers and the complex plane

2a) Complex roots

Find all values of $\sqrt{\frac{1-i\sqrt{3}}{2}}$ *and all 11th roots of* $\frac{1-i\sqrt{3}}{2}$ *in the form* $re^{i\theta}$.

We recall if $z^n = a$ then if $a = re^{i\varphi}$, then $z = r^{\frac{1}{n}} e^{\frac{\varphi}{n} + k\frac{2\pi}{n}}$. Applying these forumulas yield: $\sqrt{\frac{1-i\sqrt{3}}{2}} = e^{\frac{5\pi}{6}+k\pi}$, $k = 1, 2$. For the second computation: $\sqrt[11]{\frac{1-i\sqrt{3}}{2}} = e^{\frac{5\pi}{33}+m\frac{2\pi}{11}}$, where $m = 0, \ldots, 10$.

2b) Connectedness & connected components

Let Ω *be an open set in* $\mathbb{C}$ *and* $x \in \Omega$. *The connected component (or simply the component) of* Ω *containing* z *is the set* $\mathbb{C}_z$ *of all points* w *in* Ω *that can be joined to* z *by a curve entirely contained in* Ω.

1. *Check first that* $\mathbb{C}_z$ *is open and connected. Then, show that* $w \in \mathbb{C}_z$ *defines an equivalence relation, that is: (i)* $z \in \mathbb{C}_z$*, (ii)* $w \in \mathbb{C}_z \implies z \in \mathbb{C}_w$*, and (iii) if* $w \in \mathbb{C}_z$ *and* $z \in \mathbb{C}_\zeta$*,then* $w \in \mathbb{C}_\zeta$.

   Thus Ω *is the union of all its connected components, and two components are either disjoint or coincide.*

2. *Show that* Ω *can have only countably many distinct connected components.*

3. *Prove that if* Ω *is the complement of a compact set, then* Ω *has only one unbounded component.*

[Hint: For (b), one would otherwise obtain an uncountable number of disjoint open balls. Now, each ball contains a point with rational coordinates. For (c), note that the complement of a large disc containing the compact set is connected.]

1. *Proof.* We recall that any subset of finite topological space is connected $\iff$ it is path-connected. So, naturally this is true for $\mathbb{C}^n$ and any subset Ω.

   To show $\mathbb{C}_z$ is open, we note that since Ω is open, $\forall x \in \Omega$, $\exists \epsilon_x$ s.t. $\forall y$ if $d(x, y) < \epsilon_x \implies y \in \Omega$ where $d(\cdot, \cdot)$ is our standard metric in the complex plane. Therefore, $\forall w \in \mathbb{C}_z$ since by definition, $w \in \Omega$ and Ω is open, $\implies \exists \epsilon_w$ to satisfy the openness definition, so $\mathbb{C}_z$ is open.

   To show connectedness, since $\forall w \in \mathbb{C}_z$ $\exists$ a curve lying in Ω which connects w to z (which is a path), and due to the equivalence of path-connectedness and connectedness in this topological space, $\mathbb{C}_z$ being connected is immediate.

   Since we now know $\forall w \in \mathbb{C}_z$, $\exists$ a path, we may formalize this path as g_w, s.t. $g_w : [0, 1] \to \mathbb{C}_z$ and where $g_w(0) = z$ and $g_w(1) = w$. Formalizing the path in this fashion makes trivial (i). For (ii), we already have our path g_w, thus, we consider $g_z \equiv g_w([1, 0])$ and hence if $w \in \mathbb{C}_z$, $\exists g_w$ which, $\implies g_z$ exists and hence $z \in \mathbb{C}_w$.

   For (iii), if $w \in \mathbb{C}_z$ and $z \in \mathbb{C}_\zeta$, it $\implies \exists\ g_w$ and f_z where $g_w : [\frac{1}{2}, 1] \to \mathbb{C}_z$, $f_z : [0, \frac{1}{2}] \to \mathbb{C}_\zeta$ where $g_w(\frac{1}{2}) = z, g_w(1) = w, f_z(0) = \zeta, f_z(\frac{1}{2}) = z$. Thus, by defining the path:

   $$h_w(x) = \begin{cases} f_z & x \in [0, \frac{1}{2}] \\ g_w & x \in [\frac{1}{2}, 1] \end{cases}$$

   We define a path fully contained in Ω s.t. $h_w(0) = \zeta$ and $h_w(1) = w$ which implies $w \in \mathbb{C}_\zeta$. $\square$

2. *Proof.* From part 1, we know that Ω is the union of all its connected components i.e. $\Omega = \cup_{i=1}^{n} \mathbb{C}_i$. If $n = \infty$, let us look at the case of:

$$\Omega = \bigcup_{i=3}^{\infty} \left\{ z : \frac{\log(i+1)}{\log(i)} < |z| < \frac{\log(i)}{\log(i-1)} \right\}$$

Here, we see the connected components of Ω are just the sets making up the union by which Ω is defined. However, $\lim_{n\to\infty} \mathbb{C}_n = \{z : |z| = 1\}$, which is a closed set and hence violates our definition of a equivalence relation and thus implying Ω must only have countably many distinct connected components. □

3. *Proof.* Let S denote our compact set (i.e., $\Omega = S^c$). Since S is compact, and $S \subset \mathbb{C}$, we know (1), $\exists b \in \mathbb{C}$ s.t. $\forall s \in S, s < b$, and (2), $\sup(S) \in S$. Thus, we form $\mathbb{C}_j$ as follows:

$\mathbb{C}_j = \{z : |z| > \sup(S)\} \cup \{z : \exists f : [0,1] \to \mathbb{C} | f(0) = \sup(S), f(1) = z, \nexists x \in f([0,1]) \text{ s.t. } x \in S\}$

We note $\sup(S) = \{z : z = \max |w|, w \in S\}$ may not be a unique point, so we just take one of its elements to form $\mathbb{C}_j$).

It is now apparent after such construction that if $\Omega = S^c$ where S is compact, $\exists$ a unique set $\in \Omega$'s connected components which is unbounded. □

2c) Topological definitions applied in the complex plane

Show that the bounded regions determined by a closed curve are simply connected, while the unbounded region is doubly connected.

Proof. We prove this a little informally: From our discussion above (2a), we can see that both the bounded region and unbounded regions in question are path-connected. Furthermore, we note a more general idea from topology: If a space X is path-connected, and has n genus, then the space is $(n+1)$-connected. Therefore, since the bounded region has a 0 genus, it is simply (or 1-connected), and since the unbounded region has a 1 genus (the hole created by γ creates this), it is doubly connected (2-connected).

□

3 Holomorphic functions

3a) The Complex chain rule

Suppose U and V are open sets in the complex plane. Prove that if $f : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable (in the real sense, that is, as functions of the two real variables x and y), and $h(z) = g(f(z))$, then:

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z} \quad \text{and} \quad \frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}$$

Proof. For simplicity, we may write $f = f(z, \bar{z})$. Thus, $df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$ (and similary for $\bar{f}$). Thus, $dh = \frac{\partial g(f)}{\partial z}df + \frac{\partial g(f)}{\partial \bar{z}}d\bar{z} \implies$

$$dh = \frac{\partial g(f)}{\partial z}\left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}\right) + \frac{\partial g(f)}{\partial \bar{z}}\left(\frac{\partial \bar{f}}{\partial z}dz + \frac{\partial \bar{f}}{\partial \bar{z}}d\bar{z}\right)$$

Rearranging by the dz and $d\bar{z}$ terms:

$$dh = \left(\frac{\partial g(f)}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g(f)}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}\right)dz + \left(\frac{\partial g(f)}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g(f)}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}\right)d\bar{z}$$

And hence the terms in front of dz yield $\frac{\partial h}{\partial z}$ and in front of $d\bar{z}$ yield $\frac{\partial h}{\partial \bar{z}}$.

□

3b) Cauchy-Riemann Equations do not imply holomorphic at a point

Consider the function defined by:

$$f(x + iy) = \sqrt{|x||y|}, \text{ whenever } x, y \in \mathbb{R}$$

Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

Proof. We see that $u(x, y) = f(z)$ and $v(x, y) =$, thus:

$$\frac{\partial u}{\partial x}(0, 0) = \lim_{h\to 0, h\in\mathbb{R}} \frac{u(h, 0) - u(0, 0)}{h} = \frac{\sqrt{|h||0|} - 0}{h} = 0$$

And similar computation shows $\frac{\partial u}{\partial y}(0, 0) = 0$. Therefore, we may conclude the Cauchy-Riemann equations of $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ are satisfied around $(0, 0)$. However, when we consider:

$$\lim_{h(1+i)\to 0, h\in\mathbb{R}} \frac{f(h + ih) - f(0 + i0)}{h(1 + i)} = \frac{\sqrt{|h|^2} - 0}{h(1 + i)} = \frac{|h|}{h(1 + i)}$$

We find the limit does not exist and hence f is not holomorphic at $(0, 0)$. □

3c) Constant holomorphic functions

Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

1. *Re*(f) *is constant;*
2. *Im*(f) *is constant;*
3. $|f|$ *is constant;*

one can conclude that f is constant.

Proof. We recall the definition of holomorphic on Ω if $\forall z \in \Omega$, $f'(z)$ exists and is equal to: $\lim_{h\to 0, h\in\mathbb{C}} \frac{f(z+h)-f(z)}{h}$. Therefore, since $\text{Re}(f) = u(x,y)$, and $Re(f) = c\ \forall x, y$, we must have: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \implies \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. And hence $f = u + v$ is constant. The same holds for if $\text{Im}(f)$ is constant. If $|f| = c$ for some $c \in \mathbb{R}$, then $\lim_{|h|\to 0, h\in\mathbb{C}} \frac{f(z+|h|)-f(z)}{|h|} = 0\ \forall z$, and hence due to the existence of $f'(z)$ despite which path h takes to reach $h = 0$, we may conclude f is constant since we just showed $f'(z) = 0$.

$\square$

4 Power series

4a) Radii of convergence

Find the radius of convergence for the following series:

$$(i) \sum_{n=1}^{\infty} n^p z^n, \qquad (ii) \sum_{n=1}^{\infty} \frac{z^n}{n!}, \qquad (iii) \sum_{n=1}^{\infty} n! z^n, \qquad (iv) \sum_{n=1}^{\infty} z^{n!}$$

(i) We recall $\frac{1}{R} = \lim_{n\to\infty} |\frac{an+1}{a_n}| = \lim_{n\to\infty} |\frac{(n+1)^p}{n^p}| = 1$ and hence $R = 1$.

(ii) $\frac{1}{R} = \lim_{n\to\infty} |\frac{an+1}{a_n}| = \lim_{n\to\infty} |\frac{n!}{(n+1)!}| = 0$ and hence $R = \infty$.

(iii) Our limit is the inverse of (ii) and hence $R = 0$.

(iv) We notice $\sum_{n=1}^{\infty} z^{n!} = \sum_{n=1}^{\infty} z^n z^{(n-1)!}$. Therefore, we may think of $a_n = z^{(n-1)!}$. We compute:

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{z^{n!}}{z^{(n-1)!}}\right| = \lim_{n\to\infty} \left|z^{(n-1)!(n-1)}\right| = \begin{cases} 0 & \text{if } |z| < 1 \\ 1 & \text{if } |z| = 1 \\ \infty & \text{if } |z| > 1 \end{cases}$$

Thus, $R = \infty$ if $|z| < 1$, $R = 1$ if $|z| = 1$ and $R = 0$ if $|z| > 1$.

4b) Series

Show the following:

1. *The series* $f(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(n-z)^3}$ *converges absolutely* $\forall z \in \mathbb{C}\backslash\mathbb{Z}$.
2. *The partial sums* $\sum_{|n|<N} \frac{1}{(n-z)^3}$ *of the series converge normally to* f.
3. f *is meromorphic in* $\mathbb{C}$ *with poles at* $n \in \mathbb{Z}$ *having principal part* $\frac{1}{(n-z)^3}$.
4. $\int_\gamma f(z)dz = 0$ *for any toy contour in* $\mathbb{C}\backslash\mathbb{Z}$.
5. $f'(z) = \sum_{n\in\mathbb{Z}} \frac{3}{(n-z)^4}$ *(justify the term by term differentiation).*

Hint: Recall that normal convergence means uniform convergence on compact subsets and that this is equivalent to local uniform convergence.

1. *Proof.* Let us separate $\mathbb{Z}$ into 3 disjoint parts: $N_1 = \{n : |n| < |z|, |N - z| < 1\}$, $N_2 = \{n : |n-1| > 1\}$ (which will consist of 1 or 2 elements) and $n_3 = \{n : |n| > |z|, |n-z| < 1\}$. So, we have $\mathbb{Z} = n_1 \cup N_2 \cup N_3$ and $N_i \cap N_j = \emptyset, \forall j \neq i$. Therefore,

$$f(z) = \sum_{n \in N_1} \frac{1}{(n-z)^3} + \sum_{n \in N_2} \frac{1}{(n-z)^3} + \sum_{n \in N_3} \frac{1}{(n-z)^3}$$

From here, we recall if $|f(z)| = \sum_{n\in\mathbb{Z}} |\frac{1}{(n-z)^3}| \leq \sum_{n\in\mathbb{Z}} g(n)\ \forall n$, and $\sum_{n\in\mathbb{Z}} g(n)$ converges, then our series is absolutely convergent. If we denote $\hat{n} = \min_{n\in N_1^+}(|n-z|)$, then $\sum_{n\in N_1} |\frac{1}{(n-z)^3}| \leq \sum_{n\in N_1} |\frac{1}{(\hat{n}-z)^3}| = \frac{card(N_1)}{|(\hat{n}-z)^3|} < \infty$, and $\sum_{n\in N_2} |\frac{1}{(n-z)^3}| = 2\frac{1}{|(n-z)^3|} < \infty$ and $\sum_{n\in N_3} |\frac{1}{(n-z)^3}| = 2\sum_{n\in N_3^+} \frac{1}{(n-z)^3}$ and since $\sum_{n=k,k\in\mathbb{N}}^{\infty} (n^{-\rho})$ converges $\forall \rho > 1$ (see Riemann zeta function), we know this sum converges to some $c < \infty$. Therefore, we see by constructing g as follows, that f converges absolutely $\forall z \in \mathbb{C}\backslash\mathbb{Z}$.

$$g(z) = \begin{cases} \frac{1}{|(\hat{n}-z)^3|} & \text{if } n \in N_1 \\ f(z) & \text{if } n \in N_2 \cup N_3 \end{cases}$$

□

2. We will want to show that if $f_N = \sum_{|n|<N} \frac{1}{(n-z)^3}$, then $\lim_{N\to\infty} ||f_N - f||_\infty = 0$. Or, equivalently that $\forall \epsilon > 0\ \exists M \in \mathbb{N}$ such that $||f_N - f|| < \epsilon$ whenever $N \geq M$.

3. This is kind of trivial, no?

4. We first note due to the uniform convergence of f:

$$\int_\gamma \sum_{n\in\mathbb{N}} \frac{1}{(n-z)^3} dz = \sum_{n\in\mathbb{N}} \int_\gamma \frac{1}{(n-z)^3} dz$$

And since $z \neq n + iy$, f will be holomorphic on the toy contour in question and hence Cauchy's Theorem is applicable, so $\int_\gamma f(z)dz = 0$.

5 Integration along curves

5a) Integrating log's derivative

Assume that $f(z)$ is analytic and satisfies the inequality $|f(z) - a| < a$, $a \in \mathbb{R}$, in a region Ω. Show that:

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$

for every closed curve in Ω.

Proof. We recall the logarithm is analytic on $\mathbb{C}\backslash(-\infty, 0]$. Therefore,

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \int_\gamma \frac{d}{dz} \log(f(z)) dz = \int_a^b \frac{d}{dz} \log(\gamma(t))\gamma'(t)dt = \log(\gamma(b)) - \log(\gamma(a)) = 0$$

□

6 The Exponential and trigonometric functions

6a) Hyperbolic sine & cosine

The hyperbolic cosine and sine are defined by $\cosh(z) = \frac{1}{2}(e^z + e^{-z})$, $\sinh(z) = \frac{1}{2}(e^z - e^{-z})$. *Express them through* $\cos(iz)$, $\sin(iz)$. *Derive the addition formulas, and formulas for* $\cosh(2z), \sinh(2z)$. *Then, use the addition formulas to separate* $\cos(x+iy), \sin(x+iy)$ *in real and imaginary parts.*

Proof. We recall: $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}) \implies \cos(iz) = \frac{1}{2}(e^{-z} + e^z)$ and $\sin(iz) = \frac{1}{2i}(e^{-z} - e^z)$. Therefore:

$$\sin(iz) = i\sinh(z) \quad \text{and} \quad \cos(iz) = \cosh(z)$$

We have:

$$\cosh(z_1)\cosh(z_2) = \frac{1}{2}(e^{z_1} + e^{-z_1})\frac{1}{2}(e^{z_2} + e^{-z_2})$$

$$= \frac{1}{4}(e^{z_1+z_2} + e^{-(z_1+z_2)}) + \frac{1}{4}(e^{z_1-z_2} + e^{-(z_1-z_2)}) = \frac{1}{2}(\cosh(z_1+z_2) + \cosh(z_1-z_2))$$

And similarly for sinh:

$$\sinh(z_1)\sinh(z_2) = \frac{1}{2}(\cosh(z_1+z_2) - \cosh(z_1-z_2))$$

$$\implies \cosh(z_1+z_2) = \cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2)$$

By similar derivation, we find:

$$\sinh(z_1+z_2) = \sinh(z_1)\cosh(z_2) + \cosh(z_1)\sinh(z_2)$$

This now implies:

$$\cosh(2z) = \cosh^2(z) + \sinh^2(z) \quad \text{and} \quad \sinh(2z) = 2\cosh(z)\sinh(z)$$

□

7 Cauchy's Theorem

7a) Finitely many points with bounded neighbourhoods that lie on the interior of a rectifiable closed curve do not impact Cauchy's Theorem

Let Ω *be a simply connected open subset of* $\mathbb{C}$ *and let* $\gamma \subset \Omega$ *be a rectifiable closed path contained in* Ω. *Suppose that* f *is a function holomorphic in* Ω *except possibly at a finitely many points* $w_1, \ldots, w_n$ *inside* γ. *Prove that if* f *is bounded in a neighborhood around* $w_1, \ldots, w_n$, *then:*

$$\int_\gamma f(z)dz = 0$$

Proof. Let us define $\epsilon = \min(\epsilon_i)$ where ϵ_i is selected arbitrarily from the set $\{x : x \in \mathbb{R}, \forall y \in B_x(w_i)\ |f(y)| < |M_i|, x < \frac{\gamma_i - w_i}{2}\}$ where $\gamma_i :=$ the shortest path between w_i and any point on γ, but does not pass through the points $w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n$.
Let us define the circle of radius ϵ_i centered around w_i as C_{ϵ_i}. Thus, by Cauchy's Theorem, if $S = \gamma \cup \sum_{i=1}^n \gamma_i \cup \sum_{i=1}^n -C_{\epsilon_i} \cup \sum_{i=1}^n -\gamma_i$:

$$0 = \int_S f(z)dz = \int_\gamma f(z)dz + \sum_{i=1}^n \int_{\gamma_i} f(z)dz - \sum_{i=1}^n \int_{C_{\epsilon_i}} f(z)dz - \sum_{i=1}^n \int_{\gamma_i} f(z)dz$$

$$\implies \int_\gamma f(z)dz = \sum_{i=1}^n \int_{C_{\epsilon_i}} f(z)dz$$

Therefore, since $|f|$ is bounded by M_i at a neighborhood of $\epsilon_i \geq \epsilon$ radius around w_i, we have:

$$\int_\gamma f(z)dz = \sum_{i=1}^n \int_{C_{\epsilon_i}} f(z)dz \leq \sum_{i=1}^n \left| \int_{C_\epsilon} f(z)dz \right| \leq \sum_{i=1}^n \left(\sup_{z \in C_{\epsilon_i}} |f(z)| \cdot \text{length}(C_{\epsilon_i}) \right)$$
$$\leq n \cdot \max(\sup_{z \in C_{\epsilon_i}} |f(z)|) \cdot \max(\text{length}(C_{\epsilon_i})) = n \cdot |M| 2\pi \cdot \epsilon$$

Where $|M| = \max(|M_i|)$.
Since epsilon can be arbitrarily small, $n|M|2\pi\epsilon \to 0$ as $\epsilon \to 0$ and hence we are done.

□

8 Cauchy's Integral Formula

8a) Cauchy's Inequality

If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leq \frac{1}{1-|z|}$, find the best estimate of $|f^{(n)}(0)|$ that Cauchy's inequality will yield.

Proof. For $r < 1$, we will have Cauchy's Inequality yields:

$$|f^{(n)}(0)| \leq \frac{n!||f||_{C(0,r)}}{r^n} \leq \frac{n!}{r^n(1-r)} = h$$

To minimize h, we choose r s.t. $\frac{\partial h}{\partial r} = 0$. Computing:

$$\frac{\partial}{\partial r} \frac{n!}{r^n(1-r)} = n! \left(\frac{r - n(1-r)}{r^{n+1}(1-r)^2} \right) = 0 \iff r = \frac{n}{1+n}, \quad \text{Therefore:}$$

$$|f^{(n)}(0)| \leq \frac{n!}{(\frac{1}{n+1})(\frac{n}{1+n})^n} = (n+1)! \left(1 + \frac{1}{n} \right)^n$$

□

8b) Line integral computations

Compute:

$$1. \int_{|z|=2} z^n(1-z)^m dz, \quad 2. \int_{|z|=1} |z-a|^{-2}|dz|, \quad \text{and} \quad 3. \int_{|z|=\rho} |z-a|^{-4}|dz|, \text{ where } |a| \neq \rho$$

1. *Proof.* Case (1): If $m, n \geq 0$, then $z^n(1-z)^m$ is entire and hence by Cauchy's Theorem, $\int_{|z|=2} z^n(1-z)^m dz = 0$.

   Case (2): Assume $m < 0$ and $n \geq 0$. We now have:

$$\int_{|z|=2} z^n(1-z)^m dz = \mathbb{1}_{|m|-1<n} \frac{2\pi i}{(|m|-1)!} \frac{n!}{(n-|m|)!} = \mathbb{1}_{|m|-1<n} 2\pi i |m| \binom{n}{|m|}$$

   If $n < 0$ and $m \geq 0$, simply replace m and n in the solution above and change to $\pm$ as needed.

   Case (3): If $m, n < 0$, then:

$$\int_{|z|=2} z^n(1-z)^m dz = 2\pi i \left[\frac{1}{(n-1)!} \frac{\partial^{(n-1)}}{\partial^{(n-1)} z} \Big((1-z)^m \Big) + \frac{1}{(m-1)!} \frac{\partial^{(m-1)}}{\partial^{(m-1)} z} \Big((z)^n \Big) \right]$$

$$= 2\pi i \left[\binom{|m|+|n|-2}{|n|-1} - \binom{|m|+|n|-2}{|n|-1} \right] = 0$$

   $\square$

2. *Proof.* We first note if $z = \rho e^{it}$, $t \in [0, 2\pi)$, (which is equivalent to $\gamma := \{z : |z| = \rho\}$, then $dz = i\rho e^{it} dt$. Thus, $|dz| = |\gamma'(t)| dt = |\rho^2(\sin^2(t) + \cos^2(t))| dt = \rho \frac{1}{ie^{it}} dz = -\rho i z^{-1}$.

   We next note: $\bar{z} = \overline{\rho(\cos(t) + i\sin(t))} = \rho(\cos(t) - i\sin(t)) = \rho(\cos(-t) + i\sin(-t)) = \rho e^{-it} = z^{-1}$.

   First, assume $a \in \text{int}(\Gamma)$, where $\partial\Gamma = \gamma$. We see:

$$\int_\gamma \frac{1}{|z-a|^2} |dz| = \int_\gamma \frac{-i}{(z-a)\overline{(z-a)}z} dz - \int_\gamma \frac{-i}{(z-a)(1-\bar{a}z)} dz$$

   If $f(\xi) = \frac{-i}{(1-\xi\bar{a})}$, then by Cauchy's Formula we have:

$$\int_\gamma \frac{1}{|z-a|^2} |dz| = 2\pi i f(a) = \frac{2\pi}{1-|a|^2}$$

   If $a \notin \text{int}(\Gamma)$, then $\bar{a}^{-1} \in \text{int}(\Gamma)$ and therefore:

$$\int_\gamma \frac{1}{|z-a|^2} |dz| = \int_\gamma \frac{-i}{(z-a)(1-\hat{a}z)} dz = \int_\gamma \frac{i\bar{a}^{-1}}{(z-a)(z-\bar{a}^{-1})} dz$$

and if $g(\xi) = \frac{i\bar{a}^{-1}}{(\xi - a)}$, then by Cauchy's Formula we have:

$$\int_\gamma \frac{1}{|z-a|^2}|dz| = 2\pi i g(\hat{a}^{-1}) = \frac{2\pi}{|a|^2 - 1}$$

□

3. *Proof.* This integral follows quite nicely as an augmented generalization of the previous example. We recall the residue formula, and note if $\gamma = \{z : |z| = \rho\}$, then if $a \in$ or $\notin \gamma$, then so is $\bar{a}$ and both a and $\bar{a}$ are $\notin$ and $\in \gamma$ respectively. Therefore:

$$\int_{|z|=\rho} \frac{1}{|z-a|^4}|dz| = \int_{|z|=\rho} \frac{i\rho\bar{a}^{-1}}{(z-a)^2(z-\bar{a})(z-\bar{a}^{-1})}dz$$

So, if $|a| < \rho$:

$$\int_{|z|=\rho} \frac{1}{|z-a|^4}|dz| = 2\pi i \left[\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{i\rho\bar{a}^{-1}}{(z-\bar{a})(z-\bar{a}^{-1})} \right)\Bigg|_{z=a} + \frac{i\rho\bar{a}^{-1}}{(\bar{a}-a)^2(\bar{a}-\bar{a}^{-1})} \right]$$

And if $|a| > \rho$:

$$\int_{|z|=\rho} \frac{1}{|z-a|^4}|dz| = 2\pi i \frac{i\rho\bar{a}^{-1}}{(\bar{a}^{-1}-a)^2(\bar{a}^{-1}-\bar{a})}$$

□

8c) A More General Version of Liouville's Theorem

Show that if f is entire and if there exists a constant $C > 0$ and a positive integer n such that $|f(z)| \leq C|z|^n$ for all sufficiently large $|z|$, then f is a polynomial.

Proof. Let us first make explicit the condition of "sufficiently large $|z|$" (which will refere to as "The Large $|z|$ Condition") . We will say that if $|z| \geq R, R \in \mathbb{R}^+$, then $\exists\ C > 0$ and n s.t. $|f(z)| \leq C|z|^n$.

Now, let us choose m s.t. $m > n, m \in \mathbb{N}$ and $r \geq R$, Then, we will have (and with the parametrization of $z = \gamma(\theta) := z_0 + re^{i\theta}, \theta \in [0, 2\pi])$:

$$
\begin{aligned}
|f^{(m)}(z_0)| &= \left|\frac{m!}{2\pi i}\int_{|z|=r}\frac{f(z)}{(z-z_0)^{m+1}}dz\right| && \text{By Cauchy's Integral Formula} \\
&\leq \frac{m!}{2\pi}\int_{|z|=r}\frac{|f(z)|}{|z-z_0|^{m+1}}|dz| \\
&\leq \frac{m!}{2\pi}\int_{|z|=r}\frac{C|z|^n}{|z-z_0|^{m+1}}|dz| && \text{By The Large } |z| \text{ Condition} \\
&= \frac{m!C}{2\pi}\int_0^{2\pi}\frac{|re^{i\theta}|^n}{|re^{i\theta}|^{m+1}}|ire^{i\theta}|d\theta \\
&= \frac{m!C}{2\pi}\int_0^{2\pi}\frac{1}{r^{m-n}}d\theta && \text{Since } |dz| = |\gamma'(\theta)|d\theta \text{ and } |ire^{i\theta}| = r \\
&= \frac{m!C}{r^{m-n}}
\end{aligned}
$$

Now, since f is entire and since $m-n \geq 1$, we may let $r \to \infty$, which $\implies$ $\forall m > n,\ |f^{(m)}(z_0)| = 0 \equiv f^{(m)}(z_0) = 0$. Next, we recall that if f is entire, we may write f as: $f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j$. By Cacuhy's Integral Formula, we have the formula for each a_j as follows:

$$a_j = \frac{f^{(j)}(z_0)}{j!}$$

And since it was shown that $\forall m > n$, $f^{(m)}(z_0) = 0$, we know $\forall a_j$ s.t. $j > n$, $a_j = 0$. Therefore, f is in the polynomial form (of max degree $= n$):

$$f(z) = \sum_{j=0}^{n} a_j(z-z_0)^j$$

□

8d) An Application of Parseval's and Cauchy's Integral Formulae

Show that if f is an entire function satisfying $|f(z)| \leq C\sqrt{|z|}|\cos(z)|$ for some constant C, then f is identically zero. (Hint: compare f to the function $z\cos(z)$.)

Proof. We first prove the following Theorem:

Theorem. 8.1: Can't Think of A Nice Name for This One... Yet.

$$\text{If} \quad f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j, \quad z \in B_R(a)$$

and if $0 < r < R$, then:

$$\sum_{j=0}^{\infty}|a_j|^2 r^{2j} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(z_0 + re^{i\theta})|^2 d\theta$$

Proof. We know that since $f(z)$ has a series form as given above, then naturally:

$$f(z_0 + re^{i\theta}) = \sum_{n=0}^{\infty} a_j r^j e^{i\theta j}$$

And since $r < R$, this series will be converging uniformly on $[-\pi, \pi]$. Thus,

$$a_j r^j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(z_0 + re^{i\theta})}{e^{i\theta j}} d\theta$$

And by Parseval's Formula, we get:

$$\sum_{j=0}^{\infty} |a_j|^2 r^{2j} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(z_0 + re^{i\theta})|^2 d\theta$$

$\square$

Coming back to our question, since f is entire, we may write:

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

And by applying the above theorem, and we will have:

$$\begin{aligned}
\sum_{j=0}^{\infty} |a_j|^2 r^{2j} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(z_0 + re^{i\theta})|^2 d\theta \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sqrt{|re^{i\theta}|} |\cos\left(re^{i\theta}\right)| \right)^2 d\theta \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sqrt{|re^{i\theta}|} \right)^2 d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} r d\theta \\
&= r
\end{aligned}$$

Hence, we have $|a_0|^2 + |a_1|^2 r^2 + |a_2|^2 r^4 + \cdots \leq r$, which is true $\iff$ $a_j = 0 \ \forall j \geq 1$, i.e. f is constant and equal to a_0. Next, we simply note that since $|f(0)| \leq \sqrt{|0|} |\cos(0)| = 0$, then it must be that $a_0 = 0$, which $\implies$ f is identically zero.

8e) The image of non-constant entire function is dense

Show that if f is a non-constant entire function, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

This statement is actually a Corollary of Liouville's theorem:

Proof. Assume $f(\mathbb{C})$ is not dense, which $\implies$ $\exists\ z_0 \in \mathbb{C}$ and $r > 0, r \in \mathbb{R}$ s.t. $B_r(z_0) \cap f(\mathbb{C}) = \emptyset$ ($\iff \forall z \in \mathbb{C}, |f(z) - z_0| \geq r$). Thus, if we define g as: $g(z) = \frac{1}{f(z) - z_0}$. Then:

$$|g(z)| = \frac{1}{|f(z) - z_0|} \leq \frac{1}{r}$$

Since $|g(z)| \leq \frac{1}{r} \in \mathbb{R}^+$, and is entire since $g^{-1}(z) \neq 0 \; \forall z \in \mathbb{C}$, we may use Liouville's theorem to say g is constant. Thus, f must also be constant and hence we have just contradicted our assumption. $\square$

$\square$

9 Residues

9a) A Classic residue computation question

Find the residue at i of $\frac{1}{(1+z^2)^n}$.
Suggestion: Expand $\frac{1}{(1+z^2)^n}$ in powers of $z-i$ by using the expansion of $\frac{1}{(1-w)^n}$ derived by differentiating the geometric series for $\frac{1}{(1-w)}$ $n-1$ times.

Proof. We first note:

$$f(z) = \frac{1}{(1+z^2)^n} = \frac{1}{(z-i)^n(z+i)^n}$$

Therefore,

$$\mathrm{Res}(f;i) = \int_\gamma \frac{1}{(z-i)^n(z+i)^n} = \frac{2\pi i}{(n-1)!}\left[\frac{\partial^{n-1}}{\partial^{n-1}z}\left(\frac{1}{(z+i)^n}\right)\bigg|_{z=i}\right] = (-1)^{n+1}\binom{2n-1}{n-1}\left(\frac{1}{2i}\right)^{2n-1}$$

$\square$

9b) Sine's residues

Using Euler's formula: $\sin(\pi z) = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$, show that the complex zeros of $\sin(\pi z)$ are exactly at the integers, and that they are each of order 1. Then, calculate the residue of $\frac{1}{\sin(\pi z)}$ at $z = n \in \mathbb{Z}$.

Proof. Let $w = e^{i\pi z}$, so $\sin(\pi z) = \frac{1}{2i}(w - \frac{1}{w}) = 0 \iff w - \frac{1}{w} = 0 \iff w^2 = 1 \iff w = \pm 1$. Therefore, $\sin(\pi z) = 0 \iff e^{i\pi z} = \pm 1$ which happens $\iff z \in \mathbb{Z}$.

We note if $w = e^{i\pi z}$, then $dz = \frac{1}{i\pi w}dw$. Therefore,

$$\mathrm{Res}\left(\frac{1}{\sin(\pi z)}, n\right) = \int_\gamma \frac{1}{i\pi w(w - \frac{1}{w})} = 2\int_\gamma \frac{1}{(w-1)(w+1)} = 2\frac{1}{(1+1)} = 1$$

$\square$

9c) A nice trigonometric integral

Prove that:

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos(\theta)} = \frac{2\pi}{\sqrt{a^2-b^2}} \text{ if } a > |b| \text{ and } a, b \in \mathbb{R}$$

Proof. If we let $z = e^{i\theta}$, then $dz = ie^{i\theta}d\theta$ and hence $d\theta = \frac{1}{iz}dz$. Thus,

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos(\theta)} = \int_{|z|=1} \frac{dz}{iz(a + b\frac{1}{2}(z + \frac{1}{z}))} = \int_{|z|=1} \frac{-2idz}{bz^2 + 2az + b}$$

We simply via the handy formula of $az^2 + bz + c = \big(z - \big[-\frac{b}{2} + \frac{\sqrt{b^2-4(a)(c)}}{2}\big]\big)\big(z - \big[-\frac{b}{2} - \frac{\sqrt{b^2-4(a)(c)}}{2}\big]\big)$:

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos(\theta)} = \int_{|z|=1} \frac{-2idz}{(z - [-2a + i\sqrt{b^2 - a^2}])(z - [-2a - i\sqrt{b^2 - a^2}])}$$

And hence since $a > |b|$, we have two poles of degree 1 within $|z| = 1$, and hence:

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos(\theta)} = 2\pi i\Bigg[-2i\bigg(\frac{1}{2\sqrt{b^2 - a^2}} + \frac{1}{2\sqrt{b^2 - a^2}}\bigg)\Bigg] = \frac{2\pi}{\sqrt{b^2 - a^2}}$$

□

9d) An application of Rouché's Theorem

Determine the number of zeros of $f(z) = z^3 - 3z + 4$ in the closed ball $\{|z - 1| \leq 1\}$ and show that they are simple.

Proof. We note: $f(z) = z^3 - 3z + 4 = (z - 1)^3 + 3(z - 1)^2 + 2$. Therefore, if we let $w = z + 1$, we now have the equivalent problem of analyzing the zeros of $f(w) = w^3 + 3w^2 + 2$ inside $|w| \leq 1$. Thus, we see that when $|w| = 1$, $|f(w) - (3w^2 + 1)| = |w^3 + 1| \leq 2 \leq |3w^2 + 1|$ with $|3w^2 + 1| = 2 \iff w = i, -i$, but at $i, -i$, $|w^3 + 1| = \sqrt{2} < 2$, and hence $|f(w) - g(w)| < |g(w)|$, $g(w) = 3w^2 + 1$. Since $g(w)$ has two roots within $|w| \leq 1$, so too does $f(w)$ by Rouché's Theorem. Furthermore, since $f'(w) = 3w(w + 3)$ has roots at 0 and -3, both of which are not roots of $f(w)$, we may conclude that the two roots of $f(w)$ are simple.

□

9e) Contour integration part I

Evaluate

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1 + e^x} dx$$

(Hint: Use a contour integral around the rectangle with vertices $\pm R, \pm R + 2\pi i$)

Proof. Let us define Γ as a path equal to $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. We define γ_1 as the path along the real axis from $-R$ to R, γ_2 as the path from R to $R + 2\pi i$, γ_3 as the path from $R + 2\pi i$ to $-R + 2\pi i$, and finally γ_4 as the path from $-R + 2\pi i$ to $-R$. Also for simplicity of notation, $f = \frac{e^{\alpha z}}{1+e^z}$.

We now note that inside Γ $\exists$ only one pole, at $z = \pi i$ (since $e^{i\pi} + 1 = 0 :=$ Euler's Identity). We thus compute:

$$\text{Res}(f; i\pi) = \frac{e^{\alpha z}}{\frac{\partial}{\partial z}(1 + e^z)}\bigg|_{z=i\pi} = -e^{\alpha i\pi}$$

Thus, by recalling the residual theorem, we have:

$$\int_{\Gamma} f = \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f = -2\pi i e^{\alpha i\pi}$$

We now look at γ_2 in that:

$$\left|\int_{\gamma_2} f\right| = \left|\int_0^{2\pi} \frac{ie^{\alpha(R+it)}}{1+e^{R+it}}dt\right| \leq \int_0^{2\pi} \left|\frac{e^{\alpha(R+it)}}{1+e^{R+it}}\right|dt \leq \int_0^{2\pi} \frac{e^{\alpha R}}{e^R-1}dt$$

And hence $\lim_{R\to\infty}\int_{\gamma_2} f = 0$ since $\lim_{R\to\infty}\frac{e^{\alpha R}}{e^R-1} = 0$ when $0 < \alpha < 1$.

By using the same reasoning, we can also show that $\lim_{R\to\infty}\int_{\gamma_4} f = 0$ since:

$$\left|\int_{\gamma_4} f\right| = \left|-\int_0^{2\pi} \frac{ie^{\alpha(-R+it)}}{1+e^{-R+it}}dt\right| \leq \int_0^{2\pi} \frac{e^{-\alpha R}}{1-e^{-R}}dt$$

And noting the limit of the rightmost term above goes to 0 like for γ_2.

We now look at γ_3 in that:

$$-\int_{\gamma_3} f = \int_{-R}^{R} \frac{e^{\alpha(t+2\pi i)}}{1+e^t} = e^{\alpha 2\pi i}\int_{\gamma_1} f$$

Therefore, taking $R \to \infty$, we see:

$$\left(1-e^{\alpha 2\pi i}\right)\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x} = -2\pi i e^{\alpha i\pi}$$

And by rearanging this, we may conclude that:

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x} = \frac{\pi}{\sin(\alpha\pi)}$$

□

9f) Contour integration part II

Evaluate with residues:

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}dx$$

Proof. Let us consider $f(z) = \frac{\sqrt{1-z^2}}{1+z^2}$ defined by a branch cut from -1 to 1 and let $f(0) = +\frac{1}{2}$ on the top side of the cut. If we define $\Gamma := re^{i\theta}, \theta \in [0, 2\pi], r > 1$, then by the Residual Theorem (and since $1 + x^2 = (x + i)(x - i)$), we have:

$$\int_{\Gamma} f(z)dz = 2\pi i\text{Res}(f; \pm i) - \int_{-1}^{1} f(x)dx - \int_{1}^{-1} -f(x)dx$$

Since we must deform our contour, Γ, around the branch cuts and around the poles. Next, if we have defined Γ as we did, then by letting $r \to \infty$, we have:

$$\begin{aligned}\lim_{r\to\infty}\int_\Gamma f(z)dz &= \lim_{r\to\infty}\int_0^{2\pi}\frac{\sqrt{1-r^2e^{i2\theta}}}{1+r^2e^{i\theta}}ire^{i\theta}d\theta\\ &= \int_0^{2\pi} d\theta\\ &= 2\pi\end{aligned}$$

(We also note that we could have also computed the integral by considering the Laurent series of $-iz\frac{\sqrt{1-z^{-2}}}{1+z^2} = \frac{\sqrt{1-z^2}}{1+z^2}$, and only one term in its expansion evaluates to a non-zero number under integration, specifically to 2π).

Next, if we solve for $2\pi i\mathrm{Res}(f;\pm i)$:

$$2\pi i\big(\mathrm{Res}(f;\pm i)\big) = 2\pi i\left(\frac{\sqrt{2}}{2i}+\frac{\sqrt{2}}{2i}\right) = 2\pi\sqrt{2}$$

And hence:

$$\int_{-1}^{1}\frac{\sqrt{1-x^2}}{1+x^2}dx = \frac{1}{2}\big(2\pi\sqrt{2}-2\pi\big) = \pi(\sqrt{2}-1)$$

□

9g) Rational and entire polynomial functions

1. *Show that an entire function is a polynomial if and only if it has a pole at infinity.*
2. *Show that a meromorphic function on $P^1(\mathbb{C})$ is rational.*

1. *Proof.* Let us first state a Lemma (Corollary from the Laurent Series Development found in Conway, pg. 105):

Lemma. 9.1: Conditions for Poles

Let z be an isolated singularity of f and let $f(z) = \sum_{-\infty}^{\infty} a_j(z-z_0)^j$ be its Laurent Expansion in $A(a,0,R)$. Then,

(a) $z = z_0$ is a removable singularity $\iff$ $a_n = 0$ for $n \le -1$.

(b) $z = z_0$ is a pole of order n $\iff$ $a_{-n} \neq 0$ and $a_m = 0$ for $m \le -(n+1)$.

The proof for (b) (in assuming (a)) is as follows (also from Conway):

Proof. Suppose $a_m = 0$ for $m \le -(n+1)$, $(z-z_0)^n f(z)$ has a Laurent Expansion which has no negative powers of $(z-z_0)$. Thus, by (a), $(z-z_0)^n f(z)$ has a removable singularity at $z-z_0$. The converse argument retraces the steps made for the forward argument.

□

Now, in coming back to our question; assume $f : \mathbb{C} \to \mathbb{C}$ is an entire function with a pole, say of order n at ∞. Since f is entire, we may write:

$$f(z) = \sum_{i=0}^{\infty} a_i (z)^i$$

Next, by having a pole at ∞, naturally $f\left(\frac{1}{z}\right)$ will have a pole (also of order n) at $z = 0$. Next, by our construction of $f(z)$ in series form, we thus see that $f\left(\frac{1}{z}\right)$ will have the form:

$$f\left(\frac{1}{z}\right) = \sum_{i=0}^{\infty} a_i \left(\frac{1}{z}\right)^i = \sum_{-\infty}^{i=0} a_{-i} (z)^i$$

As such, we may now invoke part (b) of our Lemma as follows: Since $f\left(\frac{1}{z}\right)$ has a pole of order n at $z = 0$, we know that $\forall - i \leq -(n+1)$, $a_{-i} = 0$, which is equivalent to saying: $\forall i \geq (n+1), a_i = 0$. As such, we know that:

$$\begin{aligned} f\left(\frac{1}{z}\right) &= \sum_{-\infty}^{i=0} a_{-i}(z)^i = \sum_{i=-n}^{i=0} a_{-i}(z)^i \\ \implies f(z) &= \sum_{i=0}^{n} a_i (z)^i \end{aligned}$$

Conversely, assume that $f : \mathbb{C} \to \mathbb{C}$ is an entire function with polynomial form:

$$f(z) = \sum_{i=0}^{n} a_i (z)^i$$

We now once again consider the function $f\left(\frac{1}{z}\right)$, which must have series expansion:

$$f\left(\frac{1}{z}\right) = \sum_{i=-n}^{0} a_{-i}(z)^i$$

We now have the reverse criterion required by part (b) of or Lemma to conclude that $f\left(\frac{1}{z}\right)$ must have a pole of order n at $z = 0$, and naturally by the exact same fashion of the preceding argumentation, $f(z)$ must have a pole of order n at ∞. $\square$

2. *Proof.* Let $f : P^1(\mathbb{C}) \to P^1(\mathbb{C})$ be a meromorphic function in the extended complex plane. First, let us recall that since:

   (a) $P^1(\mathbb{C})$ is compact.

   (b) Any discrete and closed subset of a compact set is discrete and compact, aka finite.

   (c) Any set of poles under a meromorphic function must be closed and discrete.

The set of poles under a meromorphic function $f : P^1(\mathbb{C}) \to P^1(\mathbb{C})$ must be finite. Let us call this set of poles: $z_1, z_2, \ldots, z_k$, each with a multiplicity of $m_1, m_2, \ldots, m_k$ respectively. If we then define F by taking out these poles, i.e.,

$$F(z) = f(z) \prod_{j=1}^{k} (z - z_j)^{m_j}$$

Then it must be that F has at most a pole at ∞ of order say $m_\infty \in \mathbb{N}$. As such, $|F(z)| \leq C|z|^n \ \forall z \in \mathbb{C}$. Thus, as we showed in Question 1, F must be a polynomial of order at most m_∞, and by definition, $\prod_{j=1}^{k} (z - z_j)^{m_j}$ is a polynomial of order $\sum_{j=1}^{k} m_j$. As such, we find that:

$$f(z) = \frac{F(z)}{\prod_{j=1}^{k} (z - z_j)^{m_j}} \in \{\text{Rational Functions}\}$$

□

10 Infinite Sums and Products

10a) Proof of Wallis's Product

Prove Wallis's product formula:

$$\prod_{m=1}^{\infty} \frac{(2m)(2m)}{(2m-1)(2m+1)} = \frac{\pi}{2}$$

(Hint: Use the product formula for $\sin(z)$ *at* $z = \frac{\pi}{2}$*.)*

Proof. We recall (from Ahlfors 5.2.3, Equation 24) that:

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right)$$

Thus, if we take $z = \frac{1}{2}$:

$$1 = \sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(\frac{4n^2 - 1}{(2n)(2n)}\right) = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(\frac{(2n+1)(2n-1)}{(2n)(2n)}\right)$$

$$\implies \frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)(2n)}{(2n-1)(2n+1)}$$

□

10b) Properties of the Fibonacci Numbers

The Fibonacci numbers are defined by $c_0 = 0, c_1 = 1$,

$$c_n = c_{n-1} + c_{n-2}, \quad n \geq 2$$

Show that the c_n *are the Taylor coefficients for a rational function, and determine a closed expression for* c_n.

Proof. Let us begin by letting $f(z) := \sum_{k=0}^{\infty} c_k z^k$ where c_k are the Fibonacci numbers. Next, we see that:

$$\begin{aligned}
f(z) &= c_0 + c_1 z + \sum_{k=0}^{\infty} z^{k+2} c_{k+2} \\
&= c_0 + c_1 z + z \sum_{k=2}^{\infty} z^{k+1}(c_{k+1} + c_k) && \text{Since } c_n = c_{n-1} + c_{n-2} \\
&= c_0 + c_1 z + z \sum_{k=2}^{\infty} c_{k+1} z^{k+1} + z^2 \sum_{k=2}^{\infty} c_{k+2} z^k \\
&= c_0 + c_1 z + z\left(\sum_{k=0}^{\infty} c_k z^k - c_0\right) + z^2 \sum_{k=0}^{\infty} c_k z^k \\
&= (0) + (1)z + z(f(z) - 0) + z^2(f(z)) \\
\implies f(z) &= \frac{z}{1 - z - z^2}
\end{aligned}$$

Now, if we let $w_1 = \frac{1+\sqrt{5}}{2}$ and $w_2 = \frac{1-\sqrt{5}}{2}$, we make the quick realization that since $w_1 \cdot w_2 = -1 \implies w_1 = \frac{-1}{w_2}$, and hence $(1 - z - z^2) = -(z + w_1)(z + w_2) = (1 - w_1 z)(1 - w_2 z)$. Hence we may now look for A and B s.t. $\frac{z}{1-z-z^2} = \frac{A}{z-w_1} + \frac{B}{z-w_2}$. We see that:

$$\begin{aligned}
\frac{z}{1 - z - z^2} &= \frac{A}{1 - w_1 z} + \frac{B}{1 - w_2 z} \\
\iff z &= A(1 - w_2 z) + B(1 - w_1 z) \\
\iff A &= -B \text{ and } A w_2 + B w_1 = -1 \\
\iff B\left(1 - \frac{w_2}{w_1}\right) &= \frac{-1}{w_1} \text{ and } A = -B \\
\iff B &= \frac{-1}{w_1 - w_2} = \frac{-1}{\sqrt{5}} \text{ and } A = \frac{1}{\sqrt{5}} \\
\implies f(z) &= \frac{z}{1 - z - z^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1 - w_1 z} - \frac{1}{1 - w_2 z}\right) \\
&= \frac{1}{\sqrt{5}}\left(\sum_{k=0}^{\infty} \left((w_1)^k - (w_2)^k\right) z^k\right)
\end{aligned}$$

And since $f(z) = \sum_{k=0}^{\infty} c_k z^k$, by equating the coefficients here, we see that:

$$c_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right)$$

$\square$

10c) Evaluating Positive even Integer Values of the Riemann Zeta Function

Comparing the coefficients in the Laurent series for $\cot(\pi z)$ *and its expression as a sum of partial fractions, find the values of: (Give a complete justification of the steps that are needed.)*

$$\sum_1^\infty \frac{1}{n^2}, \quad \sum_1^\infty \frac{1}{n^4}$$

Proof. Let us begin by performing a little algebra on $\cot(\pi z)$:

$$\begin{aligned}
\cot(\pi z) &= \frac{\cos(\pi z)}{\sin(\pi z)} = i\left(\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}\right)\left(\frac{e^{i\pi z}}{e^{i\pi z}}\right) = i\left(\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1} + \frac{1}{e^{2\pi i z}-1}\right) \\
&= i\left(\frac{e^{2\pi i z}-1}{e^{2\pi i z}-1} + \frac{2}{e^{2\pi i z}-1}\right) = i + \frac{2i}{e^{2\pi i z}-1} \\
\implies \pi z\cot(\pi z) &= i\pi z + \frac{2\pi i z}{e^{2\pi i z}-1}
\end{aligned}$$

Next, we define what are called the "Bernoulli Numbers", denoted B_m. They are defined as:

$$\frac{y}{e^y-1} = \sum_{m=0}^\infty \frac{B_m}{m!}y^m$$

$$\text{If we let } y = 2\pi i z, \text{ we obtain: } \quad \frac{2\pi i z}{e^{2\pi i z}-1} = \sum_{m=0}^\infty \frac{B_m}{m!}(2\pi i z)^m$$

$$\implies \pi z\cot(\pi z) = i\pi z + \sum_{m=0}^\infty \frac{B_m}{m!}(2\pi i z)^m$$

Now, we turn our attention to the series (derived in Ahlfors, 5.2.1 Equation 11), which states: $\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^\infty \frac{2z}{z^2-n^2}$, therefore for $|z| < 1$:

$$
\begin{aligned}
z\pi\cot(\pi z) &= 1+\sum_{n=1}^{\infty}\frac{2z^2}{z^2-n^2} \\
&= 1-\sum_{n=1}^{\infty}\frac{2z^2}{n^2\left(1-\left(\frac{z}{n}\right)^2\right)} \\
&= 1-2\sum_{n=1}^{\infty}\left(\left(\frac{z}{n}\right)^2\left(\sum_{k=0}^{\infty}\left(\frac{z}{n}\right)^{2k}\right)\right) && \text{Since if } |w|<1,\ \sum_{k=0}^{\infty}w^k=\frac{1}{1-w} \\
&= 1-2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{z}{n}\right)^{2k} \\
&= 1-2\sum_{k=1}^{\infty}z^{2k}\sum_{n=1}^{\infty}n^{-2k} && \text{Since the series are absolutely convergent}
\end{aligned}
$$

Thus, we can now equate our two formulas for $\pi z\cot(\pi z)$:

$$
\begin{aligned}
&1-2\sum_{k=1}^{\infty}z^{2k}\sum_{n=1}^{\infty}n^{-2k}=i\pi z+\sum_{m=0}^{\infty}\frac{B_m}{m!}(2\pi iz)^m \iff \\
&z^2\sum_{n=1}^{\infty}\frac{1}{n^2}+z^4\sum_{n=1}^{\infty}\frac{1}{n^4}\cdots=\frac{1-i\pi z}{2}-\frac{B_0}{2\cdot 0!}-\frac{B_1(2\pi iz)}{2\cdot 1!}+\frac{B_2(2\pi z)^2}{2\cdot 2!}-\frac{B_3(2i\pi z)^3}{2\cdot 3!}-\frac{B_4(2\pi z)^3}{2\cdot 4!}\cdots \\
&\implies \sum_{n=1}^{\infty}\frac{1}{n^{2k}}=(-1)^{k+1}\cdot\left(\frac{2^{2k-1}\pi^{2k}B_{2k}}{(2k)!}\right)\quad \forall k\in\mathbb{N}
\end{aligned}
$$

Now, since $B_i=1,\frac{1}{2},\frac{1}{6},0,\frac{-1}{30},0,\frac{1}{42},0,\frac{-1}{30},\ldots \implies$

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2}&=\frac{\pi^2}{6} \\
\sum_{n=1}^{\infty}\frac{1}{n^4}&=\frac{\pi^4}{90} \\
\sum_{n=1}^{\infty}\frac{1}{n^6}&=\frac{\pi^6}{945} \\
&\vdots
\end{aligned}
$$

□

10d) The cotangent function's properties with regard to infinite sums

Express:

$$\sum_{-\infty}^{\infty} \frac{1}{z^3 - n^3}$$

in closed form.

Proof. First, let us recall the formulae: $a^3 - b^3 = (a-b)(a^2+ab-b^2)$ and $ax^2+bx+c = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$, thus we will have:

$$\begin{aligned} z^3 - n^3 &= (z-n)(z^2+zn+n^2) \\ &= (z-n)\left(z + n \cdot \frac{1+i\sqrt{3}}{2}\right)\left(z + n \cdot \frac{1-i\sqrt{3}}{2}\right) \\ &= (z-n)\big(z + ne^{i\frac{\pi}{3}}\big)\big(z + ne^{-i\frac{\pi}{3}}\big) \end{aligned}$$

We now look for a way to write: $\frac{1}{z^3-n^3}$ in the form of: $\frac{A}{(z-n)} + \frac{B}{\left(z+ne^{i\frac{\pi}{3}}\right)} + \frac{C}{\left(z+ne^{-i\frac{\pi}{3}}\right)}$. However, before doing this, let us make simplify $\frac{3z^2}{z^3-n^3}$ first since this will tremendously simplify our derivations. Furthermore, for the remainder, we denote $w = ne^{i\frac{\pi}{3}}$ (and hence $\bar{w} = ne^{-i\frac{\pi}{3}}$). Our first simplification will be that if we consider $\frac{3z^2}{z^3-n^3}$ in the form of $\frac{A}{(z-n)} + \frac{D}{(z^2+zn+n^2)}$:

$$\begin{aligned} \frac{3z^2}{z^3-n^3} &= \frac{A}{(z-n)} + \frac{D}{(z^2+zn+n^2)} \\ \iff A(z^2+zn+n^2) &+ D(z-n) = 3z^2 \end{aligned}$$

Setting $A = 1$ and $D = 2z+n$ nicely solves this equation. Now, we look for B, C as specified before:

$$\begin{aligned} &\frac{2z+n}{z^2+zn+n^2} = \frac{A}{z+w} + \frac{B}{z+\bar{w}} \\ \iff &2z+n = A(z+\bar{w}) + B(z+w) \\ \iff &A+B = 2, \text{ and } A\bar{w} + Bw = n \\ \iff &B = \frac{n(2w-1)}{w-\bar{w}} = \frac{(2(\frac{1}{2}(e^{i\frac{\pi}{3}})) - 1)}{2i\mathrm{Im}(w)} = \frac{n(\sqrt{3})i}{2ni\frac{\sqrt{3}}{2}} = 1, \text{ and } A = 1 \end{aligned}$$

Therefore, we now have that:

$$\begin{aligned} \frac{3z^2}{z^3-n^3} &= \frac{1}{z-n} + \frac{1}{z+w} + \frac{1}{z+\bar{w}} \\ &= \frac{1}{z-n} + \frac{\bar{w}}{z\bar{w}+n} + \frac{w}{zw+n} \\ &= \frac{1}{z-n} + \frac{-\bar{w}}{z(-\bar{w})-n} + \frac{-w}{z(-w)-n} \end{aligned}$$

Next, we recall (from Ahlfors 5.2.1 Equation 10) that if $|z| < 1$, that: $\lim_{m\to\infty} \sum_{n-m}^{m} \frac{1}{z-n}$, and since if $|z| < 1 \implies |-wz| < 1$ and $|wz| < 1$ ($|w| = |\bar{w}| = 1$), and as such:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{z^3 - n^3} &= \frac{1}{3z^2} \lim_{m\to\infty} \frac{1}{z^3 - n^3} \\
&= \frac{1}{3z^2} \lim_{m\to\infty} \left(\frac{1}{z(-\bar{w}) - n} - \frac{\bar{w}}{z\bar{w} - n} - \frac{w}{z(-w) - n} \right. \\
&= \frac{1}{3z^2} \Big(\pi \cot(\pi z) - \bar{w}\pi \cot(-\bar{w}\pi z) - w\pi \cot(-w\pi z) \Big) \\
&= \frac{1}{3z^2} \Big(\pi \cot(\pi z) - \pi e^{-i\frac{\pi}{3}} \cot(-e^{-i\frac{\pi}{3}}\pi z) - e^{i\frac{\pi}{3}} \cot(-e^{i\frac{\pi}{3}}\pi z) \Big)
\end{aligned}$$

□

10e) The Blaschke Product

Suppose $f(z)$ is a holomorphic function defined on the unit disk with a zero at 0 of order s and the other zeros a_k satisfying $\sum_k (1 - |a_k|) < \infty$ (or equivalently $\sum_k \log |a_k| > -\infty$). Show that it admits a factorization $f = BG$ where B is a product of:

$$B(z) = z^s \prod_k \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a}_k z}$$

and $G(z)$ is a holomorphic functions without zeroes. Show that $B(z)$ is holomorphic.

Proof. We would first like to prove that for any holomorphic function, f, which is defined on the unit disk (plus has the noted constraints in the question), it may be written as $f = g\frac{|\beta|}{\beta} \cdot \big(\frac{(\beta - z)}{(1-\overline{\beta}z)}\big)^{s_\beta} := g(\varphi_\beta(z))^{s_\beta}$, where β is a zero of f of order s_β, $\beta \neq 0$, and g has all the zeros as f except β. If this is true, then it'll follow immediately that we can write $f = BG$ as defined in the question above. To prove this, we first note that: $\big|\frac{|\beta|}{\beta}\big| = 1$. Next, by the Schwarz Lemma, if $|f(z)| = |z|$ for some non-zero z or $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$. If we say $h_1 := f \circ \big((\varphi_\beta(z))^{s_\beta}\big)^{-1}$, then since $f(\beta) = 0 \implies h_1(\beta) = |\beta| \implies f = g\varphi_\beta$ as we wanted to show.

Now, let us define $\varphi_{a_k}(z) = \frac{|a_k|a_k - |a_k|z}{a_k - |a_k|^2 z}$, which is holomorphic since the quotient of holomorphic functions on Ω, ($f = \frac{h}{g}$), is holomorphic on $\Omega\backslash\{S\}$ where $g(s) = 0\ \forall s \in S$. Thus, since $a_k - |a_k|^2 z \neq 0 \forall |z| < 1 \implies \varphi_k(z)$ is holomorphic $\forall k \in \mathbb{N}$ inside the unit disc.

Let us now recall that if $\varphi_n(z) = 1 + \phi_k(z)$ and $\sum_{n=1}^{\infty} |\phi_k(z)|$ converges uniformly on a compact subsets, then $\prod_{n=1}^{\infty} \varphi_n(z)$ also converges uniformly on a compact subsets. Thus, we take a look at $\phi_{a_k}(z) = \varphi_{a_k}(z) - 1$:

$$\begin{aligned}
\frac{|a_k|}{a_k} \cdot \frac{a_k - z}{1 - \overline{a}_k z} - 1 &= \frac{|a_k|(a_n - z) - a_k(1 - \overline{a}_k z)}{a_k(1 - \overline{a}_k z)} \\
&= \frac{|a_k|a_k - a_k + |a_k|^2 z - |a_k|z}{a_k(1 - \overline{a}_k z)} \\
&= \frac{(|a_k| - 1)a_k + |a_k|(|a_k| - 1)z}{a_k(1 - \overline{a}_k z)} \\
&= \frac{(|a_k| - 1)(a_k + |a_k|z)}{a_k(1 - \overline{a}_k z)} \\
&= \frac{(|a_k| - 1)\big(1 + \frac{|a_k|}{a_k} z\big)}{1 - \overline{a}_k z}
\end{aligned}$$

Now, since we are operating within the context of a unit disk, we may assume $|z| < r, r \leq 1$; thus:

$$\begin{aligned}
\left| \frac{|a_k|}{a_k} \cdot \frac{a_k - z}{1 - \overline{a}_k z} - 1 \right| &= \left| \frac{-(1 - |a_k|)\big(1 + \frac{|a_k|}{a_k} z\big)}{1 - \overline{a}_k z} \right| \\
&\leq \frac{(1 + |z|)(1 - |a_k|)}{1 - |z|} \\
&\leq \frac{1 + r}{1 - r} \cdot (1 - |a_k|)
\end{aligned}$$

Now, we recall that the Blaschke Condition of: $\sum_k (1 - |a_k|) < \infty$, and hence:

$$\sum_k \left| \frac{|a_k|}{a_k} \cdot \frac{a_k - z}{1 - \overline{a}_k z} - 1 \right| \leq \frac{1 + r}{1 - r} \sum_k (1 - |a_k|) < \infty$$

And hence we have proven that $B(z)$ converges uniformly on compact subsets, and hence is holomorphic by Weirstrauss' Theorem.

□

10f) Existence of a function mapping from an arbitrary sequence of complex numbers to another

Let $a_1, a_2, a_3, \ldots$ be a sequence of distinct complex numbers with $a_n \to \infty$. Let $b_1, b_2, b_3, \ldots$ be an arbitrary sequence of complex numbers. Show that there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f(a_n) = b_n \ \forall n$.

Proof. Let us define the function f as:

$$f(z) = \sum_{k=0}^{\infty} f_k b_k \quad \text{where } f_n = \begin{cases} \mathbb{1} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

However, we might run into problems unless we first alter the series $\{a_i\}$ and $\{b_i\}$, we first remove all duplicate entries since we will want to have the property of if $a_j = a_k$, then $b_j = b_k$. Next, if $\exists j \in \mathbb{N}$ s.t. $a_j = 0$, let us remove this entry and it's corresponding b_j. If we solve the problem with a function g, which maps a_i to, say, $b_i^{'} = \frac{b_i - b_1}{a_n}$, then by making $f(z) = zb_1 + b_1$, we see that $f(a_n) = b_n \forall n$, when $a_1 = 0$.

Now, if we let

$$g(z) := \prod_{i=1}^{\infty}(1 - \frac{z}{a_i})e^{\sum_{i=1}^{\infty} \frac{z}{i \cdot a_i}}$$

And f_i be defined as:

$$f_i(z) := \frac{1}{g'(a_i)} \cdot \frac{g(z)}{z - a_i} e^{\gamma_n(z - a_i)}$$

Then, naturally we see that $g(a_i) = 0$ and $\lim_{z \to a_i} f_n(a_i) = 1$, and hence a_i is a removable singularity of f_n and hence f_n is entire. Thus, if we can find γ_i s.t. f as defined as a sum of f_k's and b_k's which converges uniformly on all compact sets, by Weirstrass' Theorem, f will be an entire function.

Let $S := \{z : |z| < r\}$. On S, g is bounded and hence we will have that $|g(z)| \leq C_r$. Also, since $a_i \to \infty$, $\exists n_r \in \mathbb{N}$ s.t. $\forall i > n_r |a_i| > 2r + 1$. Therefore, $|z - a_i| \geq 2r + 1 - r = r + 1 > 1$. Thus,

$$\begin{aligned}
|f_i(z)| &\leq \left|\frac{b_i}{g'(a_i)}\right| \cdot \frac{|g(z)|}{|z - a_i|}\left|e^{\gamma_i(z - a_i)}\right| \\
&\leq \left|\frac{b_i}{g'(a_i)}\right| C_r \left|e^{\gamma_i(z - a_i)}\right| \\
&= \left|\frac{b_i}{g'(a_i)}\right| C_r \left|e^{\xi_i(\frac{z}{a_i} - 1)}\right| && \text{where } \xi_i = \frac{\gamma_i}{a_i} \\
&= \left|\frac{b_i}{g'(a_i)}\right| C_r e^{\xi_i(\mathrm{Re}(\frac{z}{a_i}) - 1)} \\
&\leq \left|\frac{b_i}{g'(a_i)}\right| C_r e^{\xi_i(\frac{r}{|a_i|} - 1)} && \text{if } \xi_i \geq 0 \\
&\leq \left|\frac{b_i}{g'(a_i)}\right| C_r e^{\frac{-\xi_i}{2}}
\end{aligned}$$

Now, if we choose $\xi_i = 2\left(\left|\frac{b_i}{g'(a_i)}\right| + i - 1\right) \implies e^{\frac{-\xi_i}{2}} = e^{1 - \left|\frac{b_i}{g'(a_i)}\right|} e^{-i}$. And as such, $\left|\frac{b_i}{g'(a_i)}\right| e^{\frac{-\xi_i}{2}} \leq e^{-i}$, and hence we can conclude that: not only have we constructed a function with the proprieties specified, but also that this function is entire since:

$$f(z) \leq \sup(C_r) \sum_{i=1}^{\infty} \left|\frac{b_i}{g'(a_i)}\right| e^{\frac{-\xi_i}{2}} \leq \sup(C_r) \sum_{i=1}^{\infty} e^{-i} < \infty$$

□

10g) Riemann Zeta Function Convergence

Show that the Riemann zeta function:

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

converges for Re$(z) > 1$, and represent its derivative in series form.

Proof.

$$\frac{\partial^m \zeta(z)}{\partial^m z} = (-1)^m \sum_{n=2}^{\infty} \log^m(n) n^{-z} \dots$$

The rest is forthcoming... □

11 Normal Families and Automorphisms

11a) z^n is a Normal Family except on the unit circle

Show that the functions z^n, n a non-negative integer, form a normal family in the extended sense on $|z| < 1$, also on $|z| > 1$, but not in any region that contains a point of the unit circle.

(Outline of Proof)

Proof. If $|z| < 1$, we claim that any infinite sequence of functions, say, z^{n_k} always has a converging subsequence. This is because if $n \to \infty$, then naturally the functions go to zero uniformly on compact subset. If $n \not\to \infty$, then there must be some index which occurs infinitely many times and hence there is the convergence is obvious.

For the region outsides $|z| > 1$, either z^n goes to infinity uniformly on compact subsets (if n_k diverges) or goes to some function z^k if some exponents occurs infinitely many times.

However, for $|z| = 1$, then consider the family z^n, where n goes from 1 to ∞. Then it convergence to a function which is not continuous which contradicts the defintion of a normal family. □

11b) Derivatives of normal families are normal families

Let $\{f_\alpha\}$ be a normal family in the extended sense of holomorphic functions on a domain Ω. Show that $\{f'_\alpha\}$ is a normal family.

Proof. Let us begin by stating the following Lemma (cpt := "compact set" and open := "open set"):

Lemma. 11.1: Uniform Convergence of Derivatives

Let $f_j : \Omega^{open} \to \mathbb{C}, j \in \mathbb{N}$ be a sequence of holomorphic functions and $f : \Omega^{open} \to \mathbb{C}$ s.t. $\forall K^{cpt} \subset \Omega^{open}$, $f_j(K^{cpt}) \to f(K^{cpt})$. Then $\forall k \in \mathbb{N} \cup \{0\}$, we will have:

$$\left(\frac{\partial}{\partial z}\right)^k f_j(z) \to \left(\frac{\partial}{\partial z}\right)^k f(z)$$

Proof. Let $K^{cpt} \subset \Omega$. Therefore, $\exists r > 0$ s.t. $\forall z \in K$, $\overline{\Delta(z, 2r)} \subset \Omega$. If we fix such an r, then $K_r^{cpt} = \cup_{z \in \Omega} \overline{\Delta(z, r)} \subset \Omega$ and is compact. Next, $\forall z \in K_r$, we have from Cauchy's Inequality, that:

$$\begin{aligned} \left| \left(\frac{\partial}{\partial z}\right)^k (f_{j_1}(z) - f_{j_2}(z)) \right| &\leq \frac{k!}{r^k} \sup_{|\zeta - z| \leq r} \left| f_{j_1}(z) - f_{j_2}(z) \right| \\ &\leq \frac{k!}{r^k} \sup_{\zeta \in K_r^{cpt}} \left| f_{j_1}(z) - f_{j_2}(z) \right| \end{aligned}$$

Thus, the right hand side above goes to zero as $j_1, j_2 \to \infty$, and since K_r, $\{f_j\}$ must converge uniformly on K_r. And as such, $\left\{\left(\frac{\partial}{\partial z}\right)^k f_j\right\}$ is uniformly Cauchy on K. □

Now, we return to the problem at hand. Let us pick any sequence s.t. $\{f'_n\} \subseteq \{f'_\alpha\}$; therefore, we know that $\exists$ a subsequence, say n_k which converges uniformly on compact sets: $K \subset \Omega$. Now, by our above Lemma, we know that $\{f_{n_k}\}$ converges uniformly on K. Hence, for any sequence of $\{f'_n\}$, we can find a subsequence which converges uniformly on compact subsets, which implies that $\{f'_\alpha\}$ is a normal family by definition.

□

11c) An automorphic injective hol. mapping on a bounded domain given one point being the identity implies the whole mapping is too

Let Ω be a bounded domain and let ϕ be an injective holomorphic mapping of Ω to itself. Let $a \in \Omega$ and suppose that $\phi(a) = a$ and $\phi'(a) = 1$. Prove that ϕ must be the identity.

(Hint: Write the power series for ϕ centred at a

$$\phi(z) = a + (z - a) + \text{ higher order terms}$$

and consider $\phi, \phi \circ \phi, \phi \circ \phi \circ \phi, \ldots$. Estimate the coefficient of the first nonzero term in the power series after the linear term, assuming that it exists, and show that it must in fact be zero.)

Proof. First and foremost, we recall Montel's (Simpler) Theorem:

Lemma. 11.2: Montel's (Simpler) Theorem

Any uniformly bounded family of holomorphic functions defined on an open subset of the complex numbers is normal.

Thus, we can say that the family $\Phi = \{\phi^n | n \in \mathbb{N}\}$ ($\phi^n = \phi \circ \cdots \circ \phi$ n-times) is normal. Let us now assume that $a = 0$ since this will not affect any of our derivations. Let us now choose $r > 0$ s.t. $\overline{\Delta(a = 0, r)} \subset \Omega$. Now since Φ is a normal holomorphic family, hence we may write ϕ as:

$$\phi(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

If we now assume for sake of contradiction that $a_k = 0\ \forall 1 < k < n$, but not for $a_k, k \geq n$ then (ht:= higher [order] terms):

$$\phi(z) = z + a_n z^n + \text{ht}$$

We would like to prove by induction that $\phi^n = z + n a_n z^n + \text{ ht}$, for $m = 2$, we have:

$$\begin{aligned}
\phi \circ (\phi(z)) &= \phi(z) + a_n \phi(z)^n + \text{ ht} \\
&= z + a_m z^n + \text{ ht} + a_m (z + a_m z^n + \text{ ht})^n + \text{ ht} \\
&= z + 2a_m z^n + \text{ ht}
\end{aligned}$$

And if we assume a standard inductive hypothesis of truth for $n - 1$, we have that:

$$\begin{aligned}\phi^m(z) = \phi \circ (\phi^{m-1}(z)) &= \phi^{m-1}(z) + a_n(\phi^{m-1}(z))^n + \text{ ht} \\ &= z + (n-1)a_n z^n + \text{ ht} + a_n(z + (m-1)a_n z^n + \text{ ht})^n + \text{ ht} \\ &= z + m a_n z^n + \text{ ht}\end{aligned}$$

Now, we may make use of Cauchy's Inequality: I.e., since ϕ is holomorphic in $\Delta(0, r)$,

$$\left|\frac{\partial^k}{\partial z^k}\Big(\phi^m(z)\Big)\right| \leq \frac{k! \sup_{z \in \Delta(0,r)} |\phi^m(z)|}{r^k}$$

However, by our previous work for ϕ^m, we see that if $k \leq n$:

$$\frac{\partial^k}{\partial z^k}\phi^m(z) = (k!)(m)(a_n)z^{n-k}$$

Therefore combining these two findings, and setting $k = n$, we can see that:

$$\begin{aligned}|a_m| &= \left|\frac{1}{k!m}\frac{\partial^n}{\partial z^n}\phi^m(z)\right| \\ &\leq \left|\frac{1}{k!m}\right|\left|\frac{\partial^n}{\partial z^n}\phi^m(z)\right| \\ &\leq \left(\frac{1}{k!m}\right)\left(\frac{k! \sup_{z \in \Delta(0,r)} |\phi^m(z)|}{r^k}\right) \\ &= \frac{\sup_{z \in \Delta(0,r)} |\phi^m(z)|}{mr^n}\end{aligned}$$

Now, we proceed as follows: Since n is fixed, a_n and r^n are constant. Next, since $\phi : \Omega \to \Omega$, and Ω is a bounded domain, $\sup_\Delta |\phi^m|$ is uniformly bounded by some constant, say C, independent of m. This corresponds to the norm of the point farthest from the origin in the domain.

Thus, we have: $|a_n| \leq \frac{K}{m}$ where $K \in \mathbb{N}$ independent of m. Since m can be chosen arbitrarily large, we have $a_n = 0$, and hence have reached a contradiction.

Thus, we have shown that $\forall n > 1, a_n = 0 \implies$ since ϕ is injective, we cannot have $\phi(z) = a \forall z$, and hence must therefore be the identity.

□

11d) Bijective automorphisms properties

Let Ω be a simply connected domain in $\mathbb{C}$ and let p, q be distinct points of Ω. Let $f_1, f_2 \in \mathcal{A}ut(\Omega)$. Show that if $f_1(p) = f_2(p)$ and $f_1(q) = f_2(q)$, then $f_1 = f_2$.

Proof. Since $\Omega \subset \mathbb{C}$ and is simply connected, by the Riemann Mapping Theorem, Ω is either conformally equivalent to the unit disk, Δ, or to the entire complex plane $\mathbb{C}$. We now state the following Lemma from Greene and Krantz's Function Theory of One Complex Variable:

Lemma. 11.3: Uniqueness of Biholomorphic Automorphisms

1. If Ω as defined in the problem is conformally equivalent to the unit disk, Δ, then if $f \in \mathcal{A}(\Omega)$, f must take the form: $f(z) = e^{i\theta}\left(\frac{z+a}{1+\overline{a}z}\right)$.
2. If Ω as defined in the problem is conformally equivalent to $\mathbb{C}$, then if $f \in \mathcal{A}(\Omega)$, f must take the form: $f(z) = a + bz, a \neq 0$.

We first assume that Ω is in the second case (Ω conformally equivalent to $\mathbb{C}$). Thus, we see that if $p \neq q$, then:

$$
\begin{aligned}
& f_1(p) = a_1 + b_1 p = f_2(p) = a_2 + b_2 p, \text{ and } f_1(q) = a_1 + b_1 q = f_2(q) = a_2 + b_2 q \\
\iff & f_1(p) - f_2(p) = f_1(q) - f_2(q) = 0 \\
\iff & f_1(p) - f_1(q) = f_2(p) - f_2(q) \\
\iff & b_1(p-q) = b_2(p-q) \\
\iff & b_1 = b_2 \qquad \text{since } p \neq q \\
& \implies a_1 = a_2 \qquad \text{else } f_1(p) \neq f_2(p)
\end{aligned}
$$

As such, assume Ω is in the first case (Ω conformally equivalent to the unit disk). Thus, we have:

$$
\begin{aligned}
& f_i(z) = e^{i\theta}\left(\frac{z+a}{1+\overline{a}z}\right) \\
\iff & f_i^{-1}(f_i(z)) = e^{-i\theta}\left(\frac{f_i(z)-a}{1-\overline{a}f_i(z)}\right)
\end{aligned}
$$

Therefore following this reasoning and noting $f_i(z) = z \iff z^2 + z(1-e^{i\phi})/\overline{a} - e^{i\phi}a/\overline{a} = 0$, solving for creates like above a system of linear equations and hence implies that $a_1 = a_2$.
And hence we have shown that $f_1 = f_2$ in both cases.

$\square$