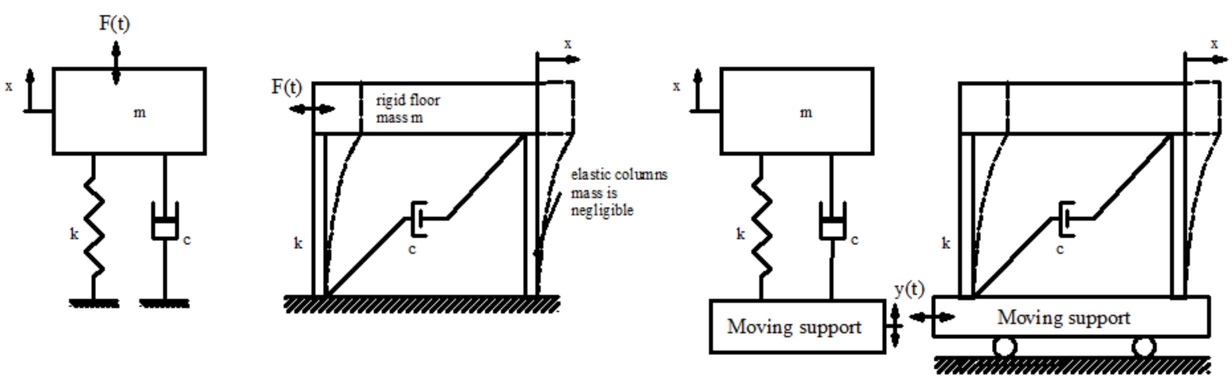
Chapter III

Harmonic Excitation of Single-Degree-of-Freedom systems “Forced Vibration”

There are many sources of excitations that cause machines and structures to vibrate. They include Unbalance rotating devices, Gusting winds, Vortex shedding, moving vehicles, Earthquakes, Rough road surfaces, and so on.

The forced vibrations of systems are usually caused by dynamic forces **F (t)** or support motions **y (t)** such as shown.



a) Forced Excitation

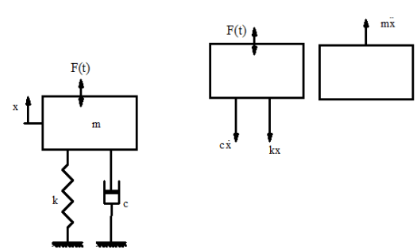
b) Support Excitation

I- Exciting Force F (t) = F_o sin ωt (or = F_o cos ωt)

$F \equiv Exciting\ force$

$F_o \equiv amplitude\ of\ exciting\ force$

$\omega \equiv Exciting\ frequency$



Equation of motion:

$$\sum F_x = -kx - c\dot{x} = m\ddot{x} + F_o \sin \omega t$$

$$m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t$$

$$\ddot{x} + \frac{2c\omega_n}{2m\omega_n}\dot{x} + \frac{k}{m}x = \frac{F_o}{m}\sin \omega t$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2 x = \frac{F_o}{m}\sin \omega t \qquad (1)$$

The last equation is the general equation of motion of single degree of freedom system.

Solution of equation of motion:

The complete solution of this equation is the sum of:

1- Homogeneous solution "x_h" (Free Response or natural response) which is dies out with time, it is often referred as a transient response, and
2- Particular solution "x_p" (Forced response) which is known as the steady state response.

The total response is

$$x = x_h + x_p$$

$$x_h = e^{-\zeta\omega_n t}[x_o \cos \omega_d t + \frac{v_o + \zeta\omega_n x_o}{\omega_d} \sin \omega_d t]$$

The particular solution or steady state response is best determined with the use of complex algebra,

Since $F = F_o \sin \omega t$

$$\therefore F = Imag.\left(F_o e^{i\omega t}\right) \quad i = \sqrt{-1} \qquad (2)$$

We can express the right-hand side of equation (1) as $\frac{F_o}{m} e^{i\omega t}$, with the provision that only the imaginary part of the term will be used in the solution process.

We assume the steady state response as,

$$x_p = Ae^{i\omega t}$$

$$\therefore \dot{x} = i\omega A e^{i\omega t} \qquad (3)$$

$$\ddot{x} = -\omega^2 A e^{i\omega t}$$

Substituting equations (2), (3) into (1) yields,

$$(-\omega^2 + i\omega.2\zeta\omega_n + {\omega_n}^2)Ae^{i\omega t} = \frac{F_o}{m}e^{i\omega t}$$

$$dividing\ by\ {\omega_n}^2 and\ noting\ that\ m{\omega_n}^2 = k$$

$$\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + i.2\zeta\frac{\omega}{\omega_n}\right]A = \frac{F_o}{k}$$

$$the\ bracted\ term\ can\ be\ written\ as$$

$$\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + i.2\zeta\frac{\omega}{\omega_n}\right] = \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}\, e^{i\varphi}$$

$$in\ which$$

$$\tan\varphi = \frac{\left[2\zeta\frac{\omega}{\omega_n}\right]}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}$$

$$A = \frac{\frac{F_o}{k}e^{-i\varphi}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}} = Xe^{-i\varphi}$$

$$X\ is\ the\ amplitude$$

$$Thus\ the\ steady\ state\ solution$$

$$x_p = Xe^{i(\omega t - \varphi)}$$

$$using\ the\ imaginary\ part\ of\quad e^{i(\omega t - \varphi)}$$

$$\therefore x_p = X\sin(\omega t - \varphi) \qquad (4)$$

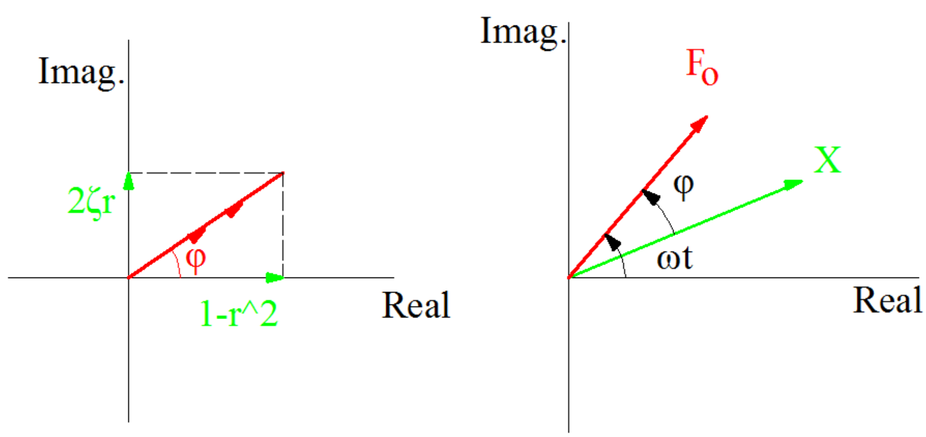
where

$$X = \frac{\frac{F_o}{k}}{\sqrt{[1 - r^2]^2 + [2\zeta r]^2}} \qquad (5)$$

and

$$\varphi = \tan^{-1}\frac{2\zeta r}{1 - r^2} \qquad (6)$$

$r = \frac{\omega}{\omega_n}$ *is the frequency ratio*



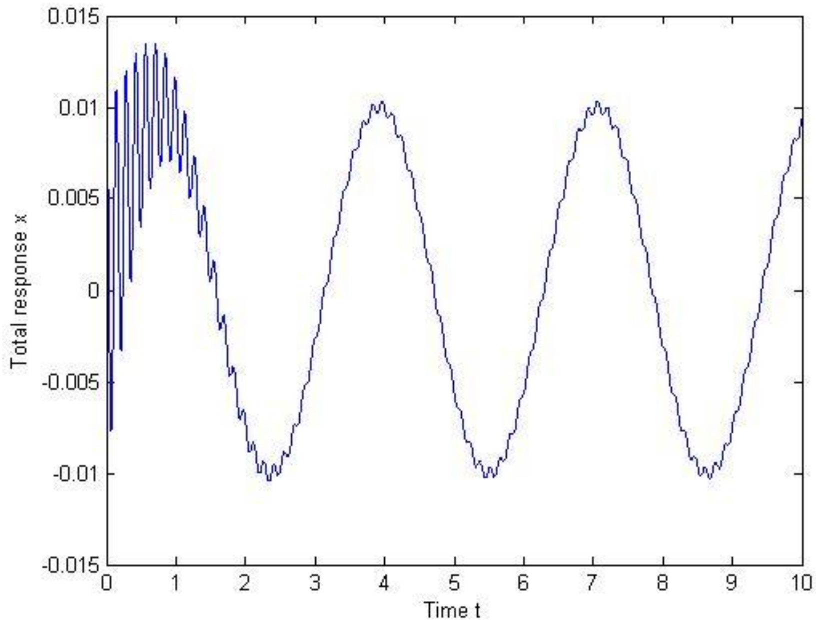
φ is called the phase angle, the angle by which the steady state response lags the exciting force as shown.

The complete solution,

$$x = e^{-\zeta\omega_n t}\left[x_o \cos\omega_d t + \frac{v_o + \zeta\omega_n x_o}{\omega_d}\sin\omega_d t\right] + X\sin(\omega t - \varphi) \qquad (7)$$

The vibratory motion described by equation (7) is a combination of two motions; one has a frequency ω_d and an exponentially decreasing amplitude, while the other has a frequency ω and constant amplitude of X.

As mentioned, the transient vibration disappears with time, leaving just the steady state motion.



For Undamped Systems:

For the undamped system "$\zeta = 0$". According to Eq. (6), "φ" is equal to zero or 180^o depending on the value of "r" whether it is less or more than one. This means that the displacement is in phase or out of phase with the force. The homogeneous part of the solution does not vanish. The general solution is written as

$$x = A\cos\omega_n t + B\sin\omega_n t + X\sin\omega t \qquad (8)$$

The constants "A" and "B" are determined from the initial conditions. Most probably, at the start of applying the external force, the initial displacement and velocity are zero. Thus, applying the conditions "x = 0" and "$\dot{x} = 0$" for "t = 0", we get

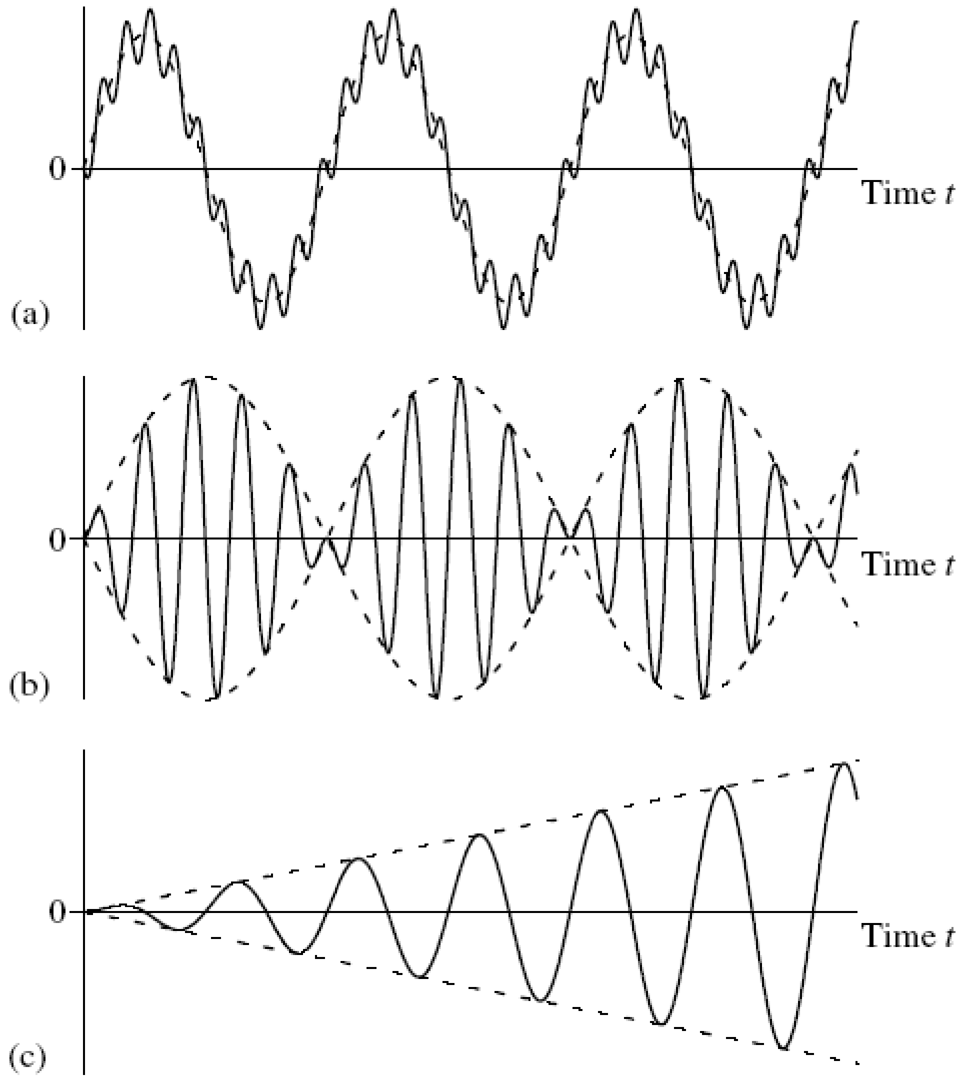
$$A = 0$$

$$B = -\frac{X_{st}.r}{1-r^2} \quad where, \qquad X_{st} = \frac{F_o}{k}$$

$$Therefore,$$

$$x = \frac{X_{st}}{1-r^2}(\sin\omega t - r\sin\omega_n t) \qquad (9)$$

The displacement "x" is formed of two frequencies.



Forced response of a harmonically excited undamped simple oscillator:

(a) for a large frequency difference;

(b) for a small frequency difference (beat phenomenon)

(c) response at resonance.

When ω is very close to ω_n "r ≈1" i.e. the exciting frequency is equal to the natural frequency, the amplitude, theoretically, is infinite. This situation is known as "resonance". Actually, the amplitude does not jump to infinity all of a sudden. It increases gradually. This is explained as follows.

According to Eq. (9), take the limit as "ω" tends to "ω" by differentiating the nominator and the denominator with respect to "ω" and substitute "$\omega = \omega_n$", then

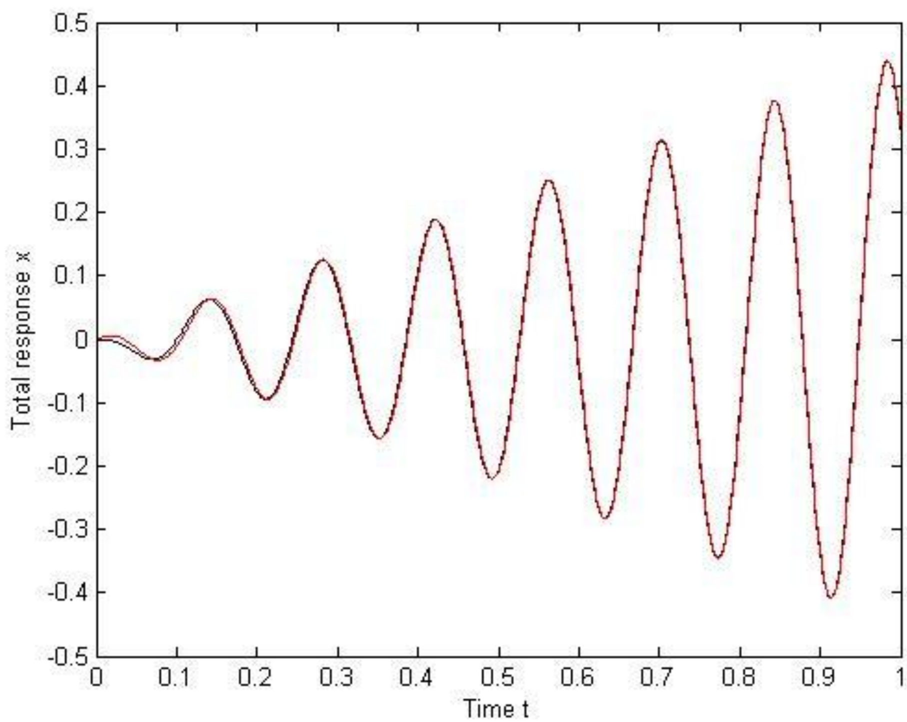
$$x = \frac{X_{st}}{2}(\sin \omega_n t - \omega_n t \cos \omega_n t) \qquad (10)$$

$$x = -\frac{X_{st}}{2}\omega_n t \cos \omega_n t \qquad (11)$$

where,

$\sin \omega_n t \quad$ *is very small*

The plot of Eqs. (10) and (11) is shown in Fig.



Steady-State Response:

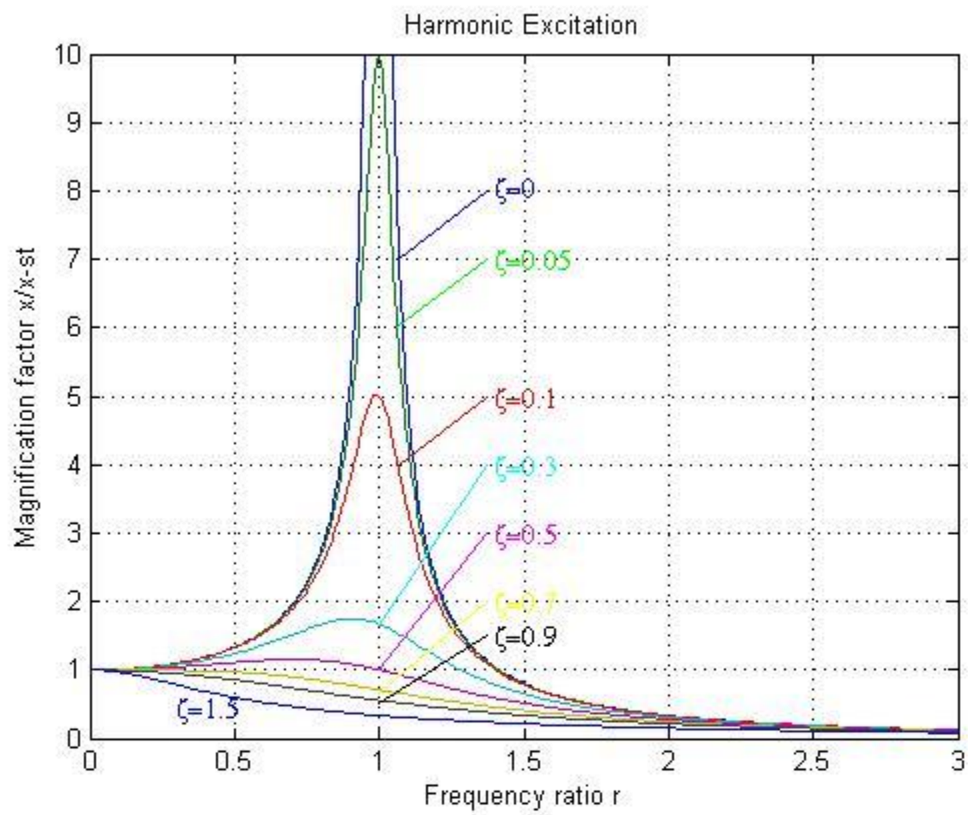
Equation (7) can be written as:

$$x = X \sin(\omega t - \varphi) \qquad (12)$$

Equation (5) in dimensionless form,

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{[1-r^2]^2+[2\zeta r]^2}} \tag{13}$$

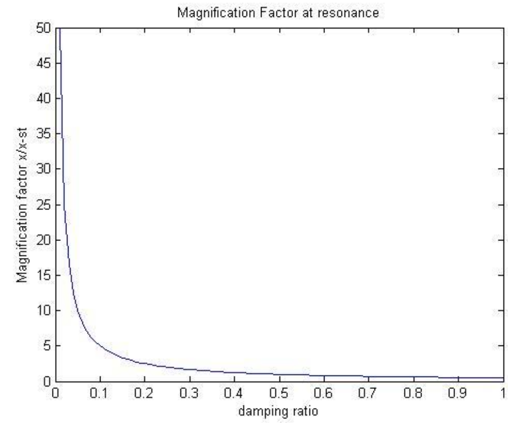
$$\frac{X}{X_{st}} = M.F \ \ manification\ factor$$



A plot of equation (12) for various magnitudes of damping is shown. These curves reveal some important characteristics of steady-state vibration of a system subjected to a harmonic excitation:

1- $r \ll 1$, the M.F. is nearly 1 approaching the static loading condition,
2- $r \approx 1$, and ζ is small, the M.F. becomes very large,
3- $r \gg 1$, the system approaches a motionless state,

4- ζ has a negligible effect on the M.F. when r <<1 and r>>1, but has a very significant effect in the region of r ≈ 1,



5- setting the derivative of the right-hand side of equation (13) w.r.t r equal to zero yields $r_{peak} = \sqrt{1-2\zeta^2} < 1.0$,which shows that the M.F. is maximum just short of r = 1.0 depending upon the magnitude of ζ, this condition is referred as ***resonance***,

6- The M.F. at resonance is given by $\frac{X}{X_{st}} = \frac{1}{2\zeta}$,

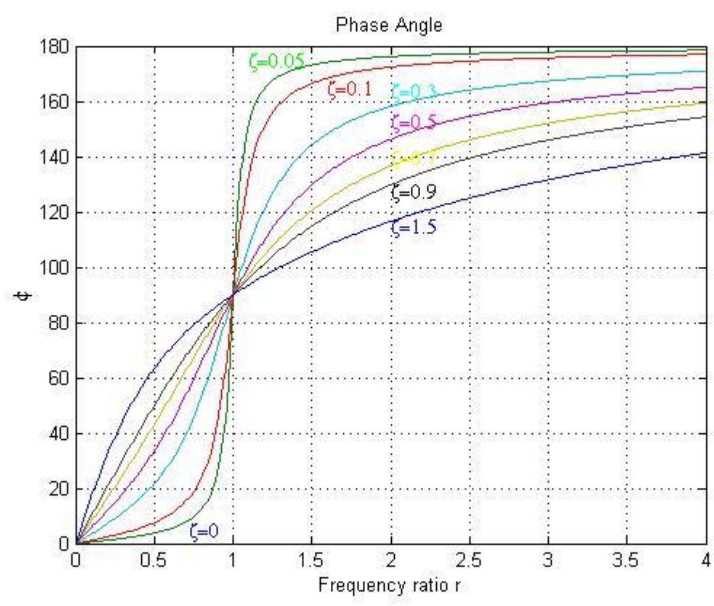
7- M.F. increases as the damping drops below 4% , at 0.01 the M.F. is 50 times greater than the static displacement X_{st} caused by applying F_o statically,

8- How to reduce the M.F. (or the amplitude of vibration X)?

r < 1.0	r = 1.0	r >1.0
ζ↑	ζ↑	ζ↑
m↓	m↑	m↑
k↑	k↓	k↓

The Phase Angle "φ"

A family of curves of equation (7) is shown,



1- For values of "r << 1", φ is small, this means that the excitation F is nearly in phase with the displacement x.

2- For values of "r < 1", "0 < φ < 90°". This means that the displacement is lagging behind the force.

3- For "r = 1", the phase angle is equal to 90° for all values of the damping factor, F is in phase with the velocity $\dot{x}$.

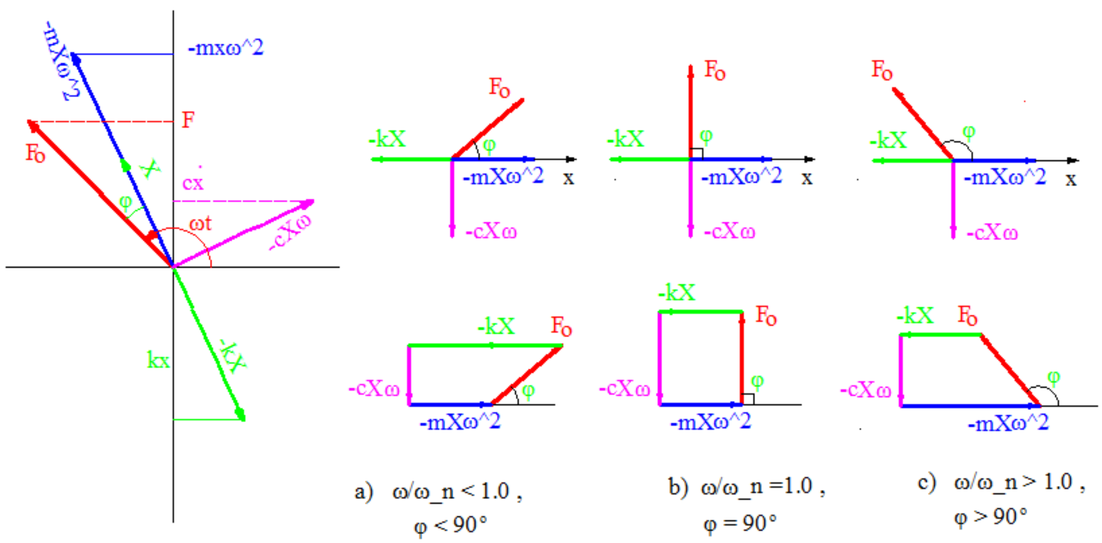
4- For "r > 1", "90° < φ < 180°.
5- For large values of "r>> 1.0", the phase angle approaches "180°". The force and the displacement are out of phase.
6- For no damping (ζ = 0), φ = 0, when r < 1.0 and φ = 180, when r > 1.0.
7- The excitation force F and the steady-state response x do not attain their maximum values at the same time, φ is a measure of this time difference.

Graphical Analysis:

$$m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t$$

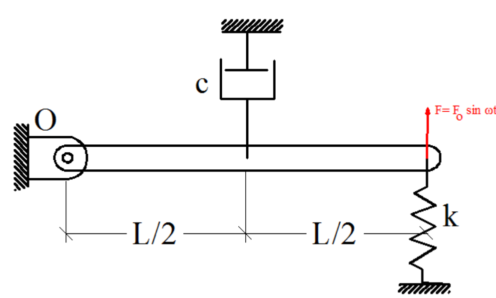
$$\varphi = \tan^{-1} \frac{c\omega}{k - m\omega^2}$$

$$F_0{}^2 = (c\omega x_o)^2 + (kx_o - m\omega^2 x_o)^2$$



Example 1:

For the system shown determine:



(a) the differential equation of motion of the uniform slender rod if the damping is sufficient to keep the oscillation small for all values of the exciting frequency ω,

(b) the damped natural frequency in terms of the system parameters,

(c) the of the damping coefficient c for critical damping, and

(d) The amplitude of steady-state response.

If the rod was steel and had a magnification factor of 2.5 at resonance. Then replace the steel rod with aluminum one of identical length and cross section. Assuming that c and k are the same for both systems, find the magnification factor with the aluminum rod.

(sp. wt. of alum. = 27.04 KN/m^3, sp. wt. of steel = 78.4 KN/m^3)

Solution:

a) $\sum M_o = I_o\ddot{\theta}$

$\therefore I_o\ddot{\theta} + c\left(\frac{l}{2}\right)^2\dot{\theta} + kl^2\theta = F_o l \sin\omega t$ (Equation of motion)

b) $\omega_n = \sqrt{\frac{kl^2}{I_o}}$ (natural frequency)

c) $c_c = 2I_o\omega_n = 2l\sqrt{kI_o}$ $\zeta = \frac{c\left(\frac{l}{2}\right)^2}{c_c} = \frac{cl}{8}\sqrt{\frac{1}{kI_o}}$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

d) Steady-state response: $\theta = \frac{\theta_{st}}{\sqrt{[1-r^2]^2+[2\zeta r]^2}}$, $\theta_{st} = \frac{F_o}{kl}$

$*\ For\ Aluminum\ I_o = \frac{1}{3}m_{al}l^2$, $For\ Steel\ I_o = \frac{1}{3}m_{st}l^2$

$\frac{(I_o)_{st}}{(I_o)_{al}} = \frac{\rho_{st}}{\rho_{al}},$

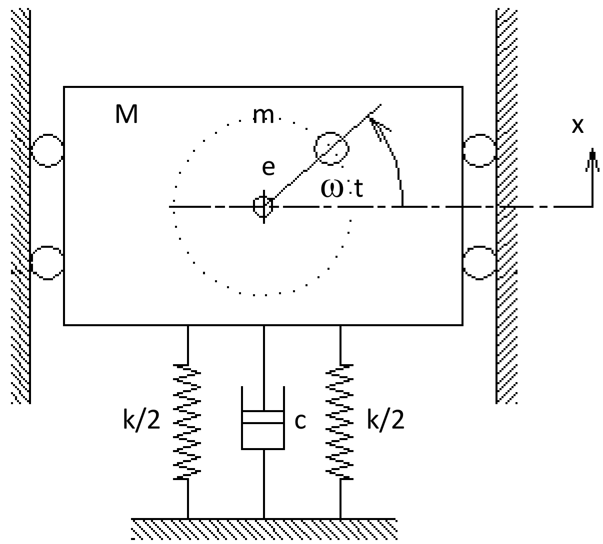
$\frac{\zeta_{al}}{\zeta_{st}} = \sqrt{\frac{\rho_{st}}{\rho_{al}}}$

$M.F.)_{\omega=\omega_n} = \frac{1}{2\zeta}$, $\frac{M.F_{al}}{M.F._{st}})_{resonance} = \frac{\zeta_{st}}{\zeta_{al}} = \sqrt{\frac{\rho_{al}}{\rho_{st}}}$

$M.F.)_{al} = 2.5\sqrt{\frac{27.04}{78.4}} = 1.47$

II- Impressed Force Due to Rotating Unbalance:

Rotating unbalance is one of the major causes of vibration in machines. Even with the best balancing process there still exists, even with small amount, an unbalance which causes vibration especially when the operating speed is near resonance. Consider the case of a machine of a total mass "M" supported by springs of total stiffness "k" and a damper with damping coefficient "c". The unbalance is represented by a mass "m" with eccentricity "e" rotating with an angular speed "γ". The machine is constrained to move in the vertical direction only.



The vertical displacement of the machine is "x" from the equilibrium position.

The equation of motion is given by:

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t$$

$$\ddot{x} + \frac{2c\omega_n}{2M\omega_n}\dot{x} + \frac{k}{M}x = \frac{me\omega^2}{M}\sin \omega t$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = \frac{me\,\omega^2}{M}\sin \omega t \qquad (1)$$

The steady-state solution of equation (1):

$$\therefore x = X\sin(\omega t - \varphi) \qquad (2)$$

where

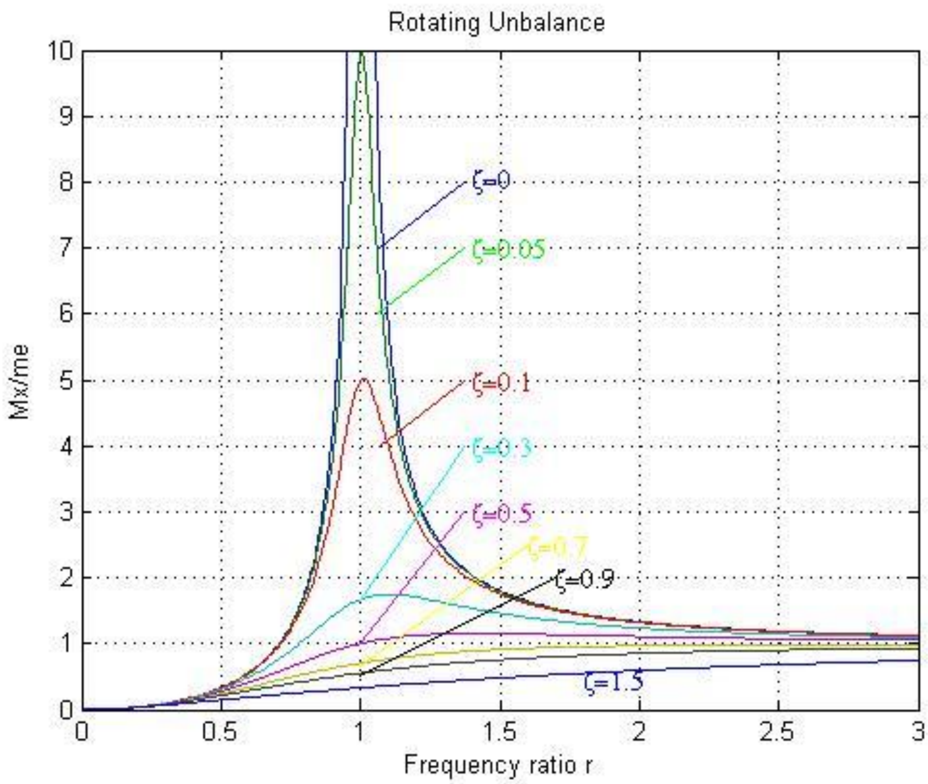
$$X = \frac{\frac{me\omega^2}{k}}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}}$$

Or, in dimensionless form,

$$\frac{MX}{me} = \frac{r^2}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}} \qquad (3)$$

and

$$\varphi = \tan^{-1}\frac{2\zeta r}{1-r^2} \qquad (4)$$



A plot of equation (3) for various magnitudes of damping is shown. These curves reveal some important characteristics of steady-state vibration of a system subjected to rotating unbalance:

1- r << 1, the $\frac{MX}{me}$ is nearly 0, 2- r ≈ 1, and ζ is small, the $\frac{MX}{me}$ becomes very large,

3- r >>1, the value of $\frac{MX}{me}$ tends to one,

4- ζ has a negligible effect on the $\frac{MX}{me}$ when r <<1 and r>>1, but has a very significant effect in the region of r ≈ 1,

5- setting the derivative of the right-hand side of equation (3) w.r.t r equal to zero yields $r_{peak} = \frac{1}{\sqrt{1-2\zeta^2}} > 1.0$,which shows that the $\frac{MX}{me}$ is maximum just short of r = 1.0 depending upon the magnitude of ζ, this condition is referred as ***resonance***,

6- The $\frac{MX}{me}$ at resonance is given by $\frac{MX}{me} = \frac{1}{2\zeta}$,

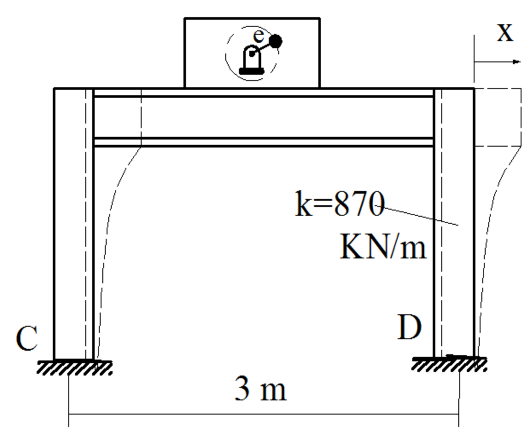
7- How to reduce the amplitude of vibration X?

r < 1.0	r = 1.0	r >1.0
ζ↑	ζ↑	ζ↑
m↓	m↑	m↑
k↑	k↓	k↓

Example 2:

The frame shown consists of a steel beam welded rigidly to two vertical channels. An eccentric exciter weighing 250 N is attached to the beam, which weighs 10 KN and is used to excite the frame. The unbalance weight of the exciter is 25 N and it has an eccentricity of 5 cm. By varying the rotational speed of the exciter until resonance occurs, the maximum horizontal amplitude was found to be 3.75 mm. Assuming no bending on the beam and considering the channels to be completely fixed at C and D, determine,

(a) The natural frequency in Hz.,
(b) The damping factor, and
(c) The magnification factor at resonance.



Solution:

M = (10000+250)/9.81

= 1045 Kg

$$a)\ \omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{870000}{1045}} = 28.85\frac{rad}{s} \qquad f = 4.6\ Hz$$

$$b)\ at\ resonance:\ \ \omega = \omega_n$$

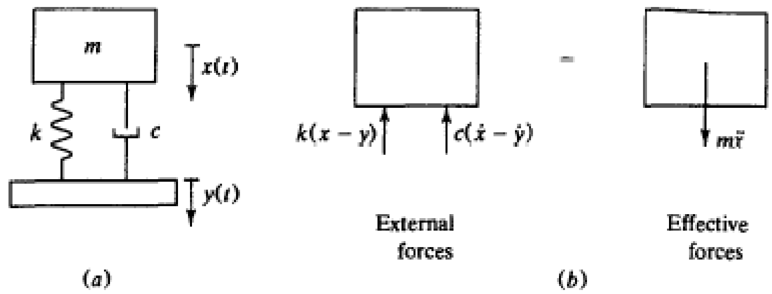
$$\frac{MX}{me} = \frac{1}{2\zeta}$$

$$\zeta = 0.0162$$

$$c)\ M.F. = \frac{MX}{me} = 30.75$$

III- Support Excitation:

In many applications dynamical systems are subjected to base excitations. A machine operating in a factory is affected by the vibration of other machines. Another example of base excitation is the earthquake which affects greatly the buildings. To study the effect of base excitation, consider the spring-mass-damper system shown.



The base moves with a harmonic motion "y" which is given by:

y = Y sin ωt

The vibratory motion of a system subjected to support excitation may be analyzed in terms of:

i) The absolute motion: "motion w.r.t. a coordinate system attached to the earth"

$$\sum F_x = M\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \qquad (I)$$

$$\ddot{x} + \frac{2c\omega_n}{2M\omega_n}\dot{x} + \frac{k}{M}x = \frac{c}{M}\dot{y} + \frac{k}{M}y$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = 2\zeta\omega_n\dot{y} + {\omega_n}^2 y \qquad (1)$$

which is the differential equation of absolute motion

We assume the steady state response as,

$$x = Ae^{i\omega t}$$

$$\therefore \dot{x} = i\omega Ae^{i\omega t} \qquad (2)$$

$$\ddot{x} = -\omega^2 Ae^{i\omega t}$$

Using the imaginary part of $Ye^{i\omega t}$ for Y= sin ωt

$$y = Ye^{i\omega t}$$

$$\therefore \dot{y} = i\omega Ye^{i\omega t} \qquad (3)$$

Substituting the terms in equations (2) and (3) into equation (1), the results may be arranged as:

$$\left[1-\left(\frac{\omega}{\omega_n}\right)^2+i.2\zeta\frac{\omega}{\omega_n}\right]A=\left[1+i.2\zeta\frac{\omega}{\omega_n}\right]Y \qquad (4)$$

Equation (4) can be written more simply as:

(a + i b) A = (1 + i b) Y

The ratio,

$$\frac{A}{Y}=\frac{1\ +\ \mathrm{i\,b}}{\mathrm{a}\ +\ \mathrm{i\,b}}=\frac{\sqrt{1+b^2}e^{i\varphi_1}}{\sqrt{a^2+b^2}e^{i\varphi_2}} \qquad (5)$$

$$or\ \frac{A}{Y}=\frac{\sqrt{1+b^2}}{\sqrt{a^2+b^2}}e^{-i\varphi}$$

$$where\ \varphi=\varphi_2-\varphi_1$$

$$\varphi_1=\tan^{-1}b$$

$$\varphi_2=\tan^{-1}\frac{b}{a}$$

$$\therefore x=X\sin(\omega t-\varphi)$$

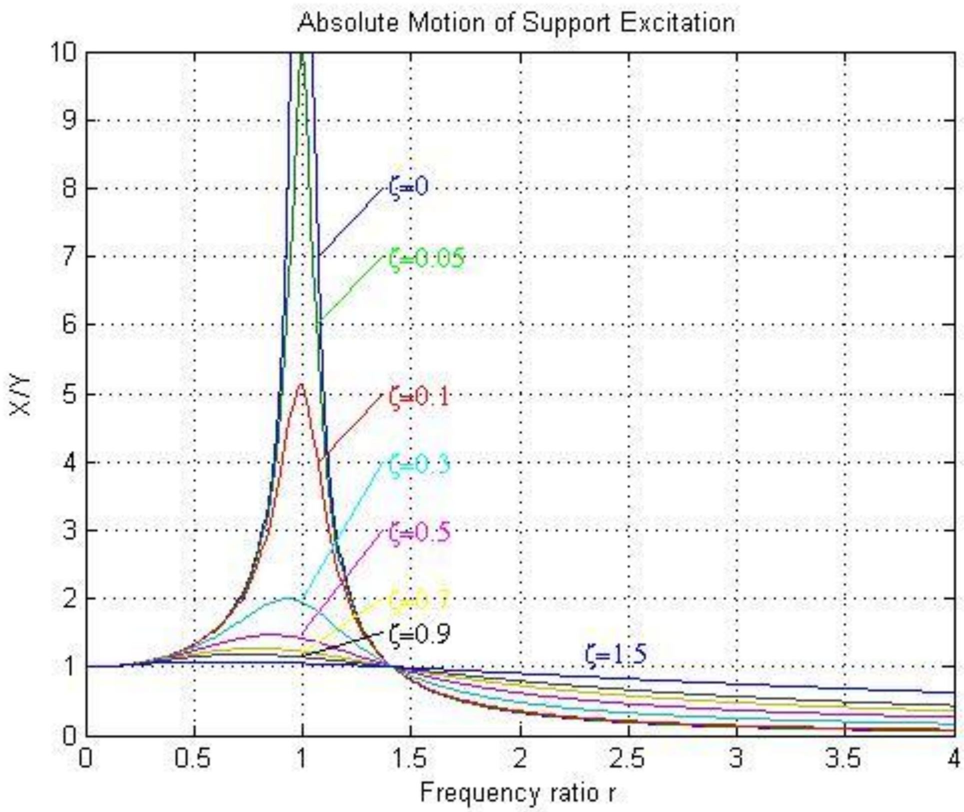
$$where\ \frac{X}{Y}=\frac{\sqrt{1+[2\zeta r]^2}}{\sqrt{[1-r^2]^2+[2\zeta r]^2}} \qquad (6)$$

To determine the phase angle φ between X and Y, multiply the numerator and denominator of equation (5) by a – i b (the conjugate of a + ib),

$$\frac{(1\ +\ \mathrm{i\,b})(\mathrm{a}-\mathrm{ib})}{(\mathrm{a}\ +\ \mathrm{i\,b})(\mathrm{a}-\mathrm{ib})}=\frac{\mathrm{a}\ +\ \mathrm{b}^2}{\mathrm{a}^2\ +\ \mathrm{b}^2}+i\frac{\mathrm{b}(\mathrm{a}-1)}{\mathrm{a}^2\ +\ \mathrm{b}^2}$$

Dividing the imaginary part by its real part gives:

$$\varphi=\tan^{-1}\frac{2\zeta r^3}{1-r^2+(2\zeta r)^2} \qquad (7)$$



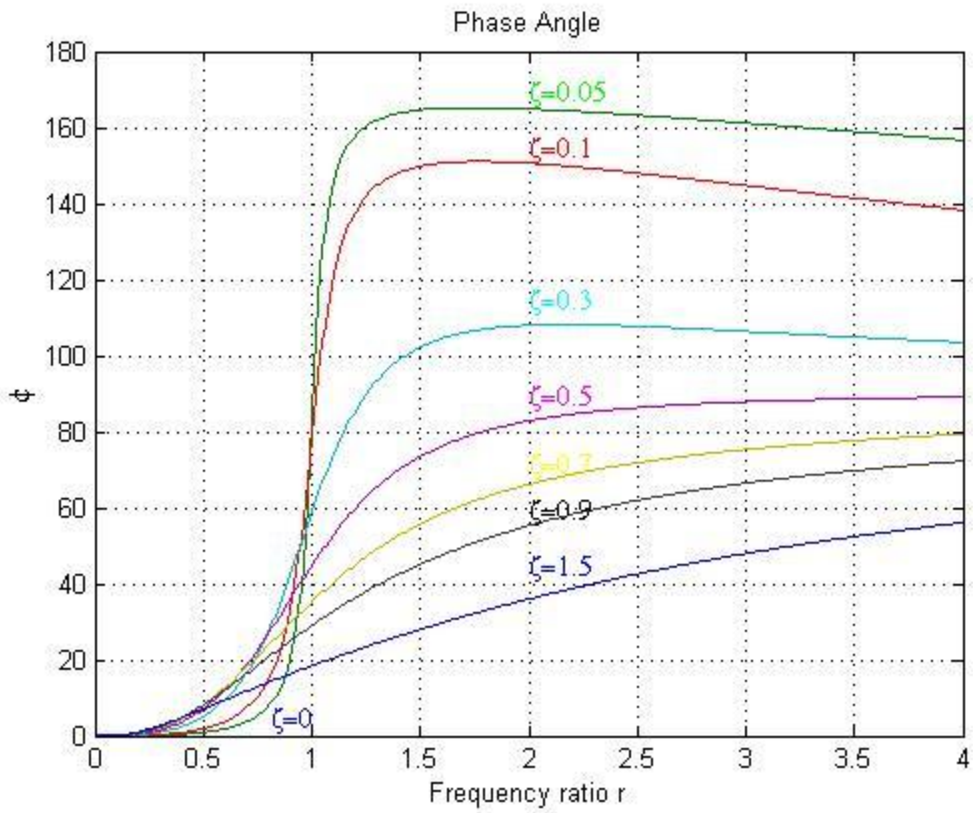
A plot of equation (6) for various magnitudes of damping is shown. These curves reveal some important characteristics of steady-state vibration of a system subjected to support excitation:

1- The amplitude ratio $\frac{X}{Y}$ =1 for all values of the damping when r=$\sqrt{2}$.
2- $\frac{X}{Y}$ is less than 1 when, r > $\sqrt{2}$ so r=$\sqrt{2}$ is the beginning of the region of ***vibration isolation***.
3- r >>$\sqrt{2}$, the value of $\frac{X}{Y}$ is quite small, which mean that the mass is essentially stationary.
4- The amplitude ratio $\frac{X}{Y}$ attains a maximum for $0 < \zeta < 1$ at the frequency ratio $r = r_{peak} < 1$ given by:

$$r_{peak} = \frac{1}{2\zeta}[\sqrt{1+8\zeta^2} - 1]^{\frac{1}{2}}$$

The Phase Angle "φ"

A family of curves of equation (7) is shown,



1- For values of "r << 1", φ is small,
2- For large values of "r>> 1.0", the phase angle approaches "180^{o}".
3- For no damping (ζ = 0), φ = 0, when r < 1.0 and φ = 180, when r > 1.0.
4- The excitation displacement y and the steady-state response x do not attain their maximum values at the same time, φ is a measure of this time difference.

ii) The relative motion: "the displacement z of the mass M relative to the support motion y = y (t)"

$z = x - y$

$\dot{z} = \dot{x} - \dot{y}$

$\ddot{z} = \ddot{x} - \ddot{y}$

Substituting into equation (I)

$$M(\ddot{z} + \ddot{y}) + c\dot{z} + kz = 0$$

$$\ddot{z} + 2\zeta\omega_n\dot{z} + {\omega_n}^2 z = -\ddot{y}$$

$$z = Ae^{i\omega t}$$

$\therefore \dot{z} = i\omega A e^{i\omega t}$

$\ddot{z} = -\omega^2 A e^{i\omega t}$

Substituting, and solving for the ratio $\frac{Z}{Y}$, we obtain:

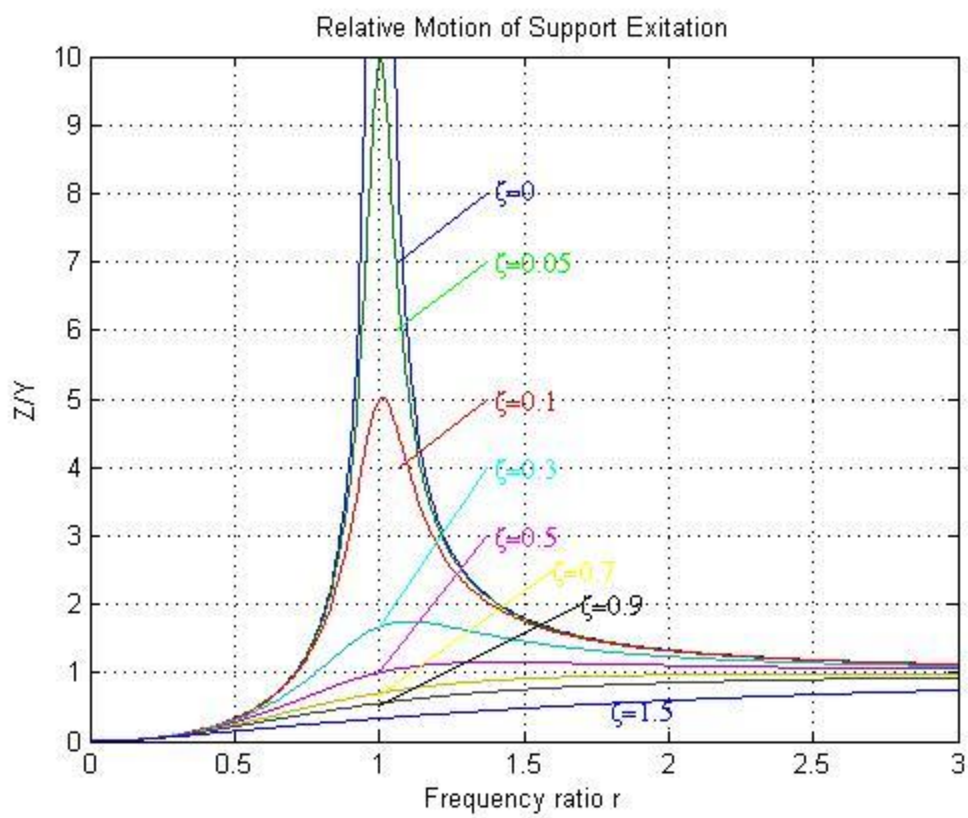
$\therefore z = Z \sin(\omega t - \varphi)$

Where,

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{[1 - r^2]^2 + [2\zeta r]^2}} \qquad (8)$$

and

$$\varphi = \tan^{-1} \frac{2\zeta r}{1 - r^2} \qquad (9)$$

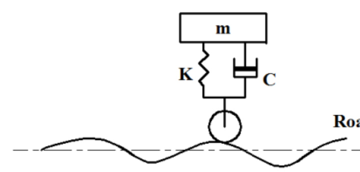


A plot of equation (8) for various magnitudes of damping is shown. These curves reveal some important characteristics of steady-state relative vibration of a system subjected to support excitation:

1- When the amplitude ratio $\frac{Z}{Y} \approx 1$, which corresponds to the absolute ratio $\frac{X}{Y} = 0$ the mass is essentially stationary.
2- When r > 3, $\frac{Z}{Y} \approx 1$ which indicate that the relative amplitude Z is the same as the amplitude of the moving support Y. This is the principle in ***measuring vibratory motion***.

Example 3:

The trailer shown is being pulled over an undulating road at a velocity v. The contour of the road is such that it can be approximated by a sine wave having a wavelength of 3 m. and amplitude of 1.5 mm. The total static deflection of the springs and tires of the trailer due to its weight has been measured as 38 mm. Assuming that damping is viscous of magnitude 0.05, determine:



a) The speed v at which the amplitude of the trailer will be maximum,
b) The maximum amplitude, and
c) The amplitude when the speed 90 Km/hr.

Solution:

The contour of the road: $y = Y \sin \omega t$

The distance traveled: $S = v.t$

$$l = v.\tau = \frac{2\pi v}{\omega}$$

$$\omega = \frac{2\pi v}{l}$$

$$y = Y\sin\left(\frac{2\pi vt}{l}\right)$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}} = \sqrt{\frac{9.81}{38 * 10^{-3}}} = 16.07\,\frac{rad}{s}$$

The maximum amplitude X is when $\omega = \omega_n$

$$\omega = \frac{2\pi v}{l} = 16.07$$

$$\therefore\ v = 7.67\,\frac{m}{s} = 27.62\,\frac{Km}{hr}$$

$b)\ \zeta = 0.05\ \ r = 1\ \ Y = 1.5 * 10^{-3}\ m$

$$\frac{X}{Y} = \frac{\sqrt{1 + [2\zeta r]^2}}{\sqrt{[1 - r^2]^2 + [2\zeta r]^2}}$$

$$X_{max} = 150.75\ mm$$

$$c)\ \ at\ 90\,\frac{Km}{hr}\quad \omega = \frac{2\pi * 90 * 10^3}{3 * 3600} = 52.36\,\frac{rad}{s}$$

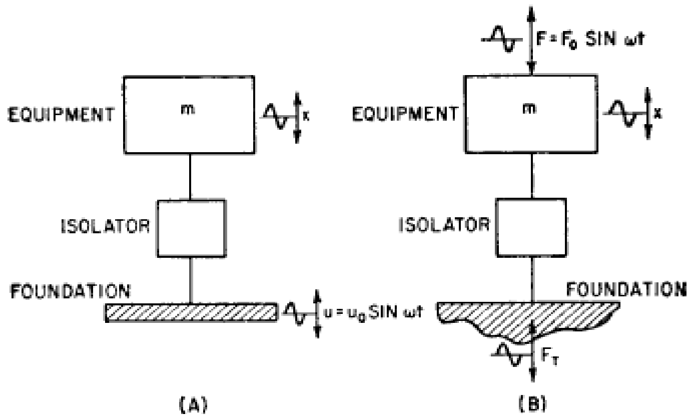
$$r = 3.26$$

$$\frac{X}{Y} = 0.11$$

$$X = 1.64\ mm$$

IV- Vibration Isolation:

Vibration isolation analysis is concerned with reducing the magnitude of force transmitted from moving components of machines to supporting foundation or with reducing the support motion transmitted to instruments or equipments.



The transmitted forces can be reduced by mounting the machine on isolation mounts, which are pads of rubber or some type of elastomer such as neoprene. The pads are modeled by a spring and a dashpot.

- Motor-compressor units in refrigerators are supported on isolation mounts to minimize the force transmitted to the refrigerator frame, and in turn to the floor upon which refrigerator sits.
- Instruments and equipments can malfunction or even suffer serious damage if not isolated from vibrating supports upon which they were mounted. For example, an electron microscope housed in a building that is close to street carrying heavy traffic would need to be isolated from the floor of the building.

Transmissibility of Forces:

The force transmitted to the foundation through the isolation system is:

$$f_{TR} = kx + c\dot{x}$$

$$X = \frac{\frac{F_o}{k}}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}}$$

$$x = Xe^{i(\omega t - \varphi)}$$

$$\dot{x} = i\omega Xe^{i(\omega t - \varphi)}$$

$$\therefore f_{TR} = (k + ic\omega)Xe^{i(\omega t - \varphi)}$$

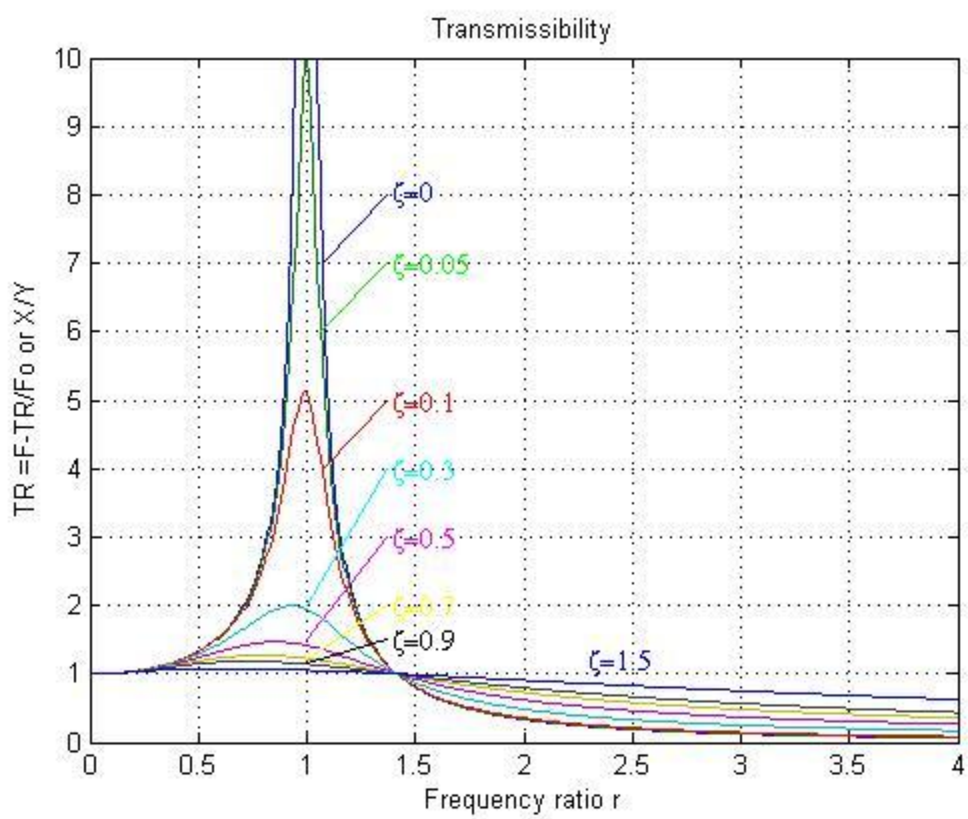
$$= kX\sqrt{1 + (2\zeta r)^2}\ \ e^{i(\omega t - \varphi)}$$

The magnitude of force transmitted:

$$F_{TR} = kX\sqrt{1 + (2\zeta r)^2} \qquad (10)$$

$$TR = \frac{F_{TR}}{F_o} = \frac{\sqrt{1 + [2\zeta r]^2}}{\sqrt{[1 - r^2]^2 + [2\zeta r]^2}} = \frac{X}{Y} \qquad (11)$$

TR is referred to as ***<u>transmissibility</u>***; the ratio $\frac{X}{Y}$ is used to determine what portion of the support motion amplitude Y is being transmitted to the system being excited by the support motion.

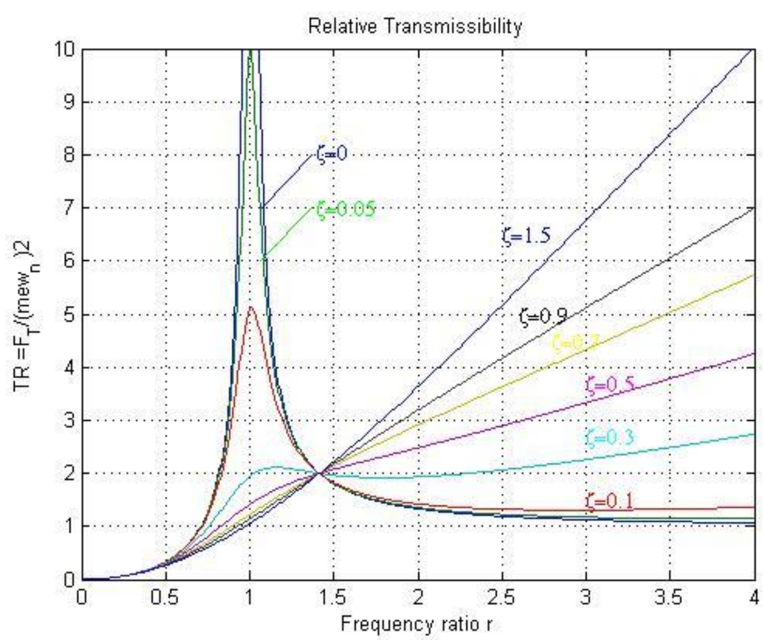


1- The region of vibration isolation begins at r > $\sqrt{2}$ since either ratio of equation (11) must be less than 1 for vibration isolation. Thus, for a given excitation frequency ω, the isolation mounts must be selected so that the natural frequency ω_n of the resulting system is less than $\omega\sqrt{2}$. Since $\omega_n = \sqrt{\frac{k}{m}}$ and the mass of the mounts is generally much less than the mass of the system, appropriate isolation mounts are usually selected on the basis of their stiffness. However, there are certain systems for which isolation is accomplished by adding mass to the system when the exciting frequency ω is very low.

2- Since the transmissibility of an exciting force or support motion decreases as r increases in the isolation region, the less stiff the isolation mounts the greater the efficiency of the isolation system, some damping must be present to minimize the peak response when the system passes through resonance during start-up or shut down.

3- When r > 3 the response curves are about the same for different of damping below 20 percent ($\zeta < 0.2$). This shows that in this region the transmissibility of a force or support motion is relatively unaffected by changing the damping. This is a fortunate feature of vibration isolation, since accurate values of ζ are generally not known.

4- Relative transmissibility is the ratio:



$$TR_R = \frac{F_{TR}}{me\omega_n^2} = \frac{r^2\sqrt{1 + [2\zeta r]^2}}{\sqrt{[1 - r^2]^2 + [2\zeta r]^2}}$$

Since the transmissibility is relatively unaffected by the damping in the isolation region, it is common practice to neglect damping in equation (11) when isolating a system.

$$TR = \frac{1}{r^2 - 1} \qquad (12)$$

In which negative root has been used so that equation (12) will yield positive transmissibility.

The reduction R in transmissibility is given by

R = 1 – TR (13)

and is used to indicate the efficiency of an isolation system.

From equations (12), (13)

$$1 - R = \frac{1}{r^2 - 1}$$

$from\ which$

$$r = \sqrt{\frac{2 - R}{1 - R}} \qquad (14)$$

Equation (14) can be used to determine the required stiffness k of an isolation system to accomplish a desired reduction R.

Expressing ω as

$$\omega = \frac{2\pi N}{60}$$

$and\ \omega_n\ as$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}}$$

we obtain

$$N = \frac{30}{\pi}\sqrt{\frac{kg(2-R)}{W(1-R)}} \quad (rpm\ or\ cpm)$$

substituting

$$W = k\delta_{st},$$

$$N = \frac{30}{\pi}\sqrt{\frac{g(2-R)}{\delta_{st}(1-R)}} \quad (rpm\ or\ cpm) \qquad (15)$$

In these equations:

k = stiffness of isolation system (N/m)

g = acceleration of gravity (9.81 m/s^2)

δ_{st} = static deflection (m)

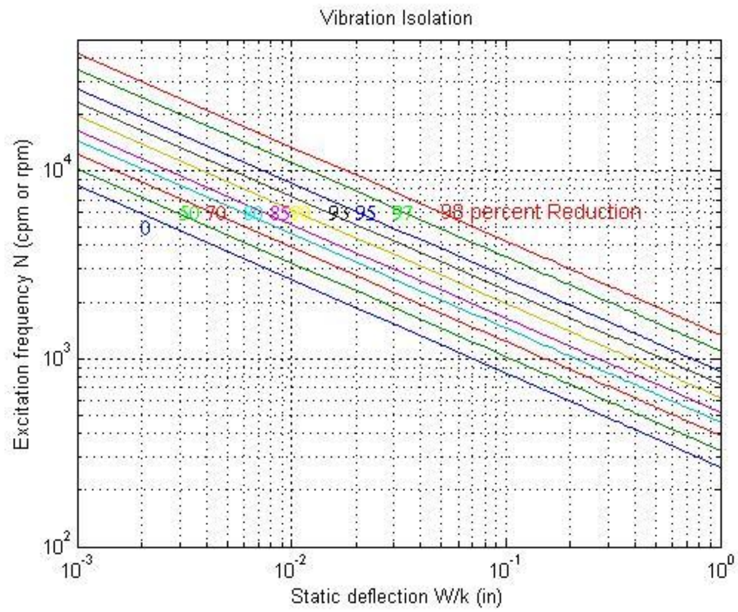
W = weight of machine or structure (N)

Equation (15) can be plotted on log-log paper to facilitate the design of isolation systems by providing a graph such as the one shown. Taking the logarithm of both sides of equation (15) gives

$$\log N = -\frac{1}{2}\log\delta_{st} + \log\frac{30}{\pi}\sqrt{\frac{g(2-R)}{(1-R)}} \qquad (16)$$

which has the form of equation of a straight line, $y = mx + b$,

A plot of equation (16) is shown:



These curves can be used to determine the stiffness k that a system must have for a specified reduction in transmissibility. It is usually difficult to provide isolation at very low excitation frequencies. At those frequencies, static deflection δ_{st}, can become so large that isolation becomes impractical.

When it becomes necessary to provide a highly efficient isolation system ($R \geq 90\%$) at fairly low excitation frequencies, the machine or instrument to be isolated is sometimes attached to, or rested upon, a rather large mass M (such as a block of concrete).

It is often necessary to consider isolating a system for more than one excitation frequency, in such instances, it should be apparent that the lowest excitation frequency is the one of primary importance, as the reduction R for an excitation frequency ω_2 would be even greater than that for ω_1 when $\omega_1 < \omega_2$.

Example 4: The machine shown has an armature with a small imbalance that is causing a force F_{TR} to be transmitted to the foundation upon which the machine rests. The machine weighs 18 KN and has an operating speed of 2000 rpm. It is desired to reduce the amplitude of the transmitted force by 80 percent, using isolation pads represented by the springs shown.

Solution:

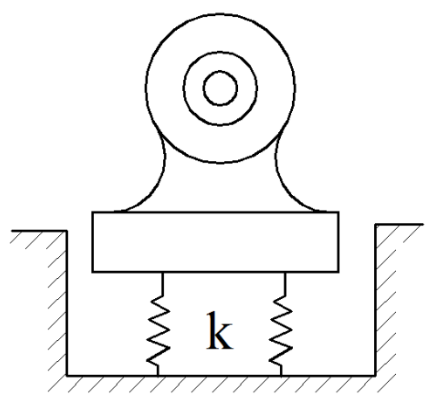
$R = 0.8$

$$N = \frac{30}{\pi}\sqrt{\frac{kg(2-R)}{W(1-R)}}$$

$k = 1.34 * 10^7 \quad N/m$

Example 5:

A large machine that weighs 135 KN is found to be transmitting a force of 2250 N to its foundation when running at 1200 rpm. The total (equivalent) spring stiffness is $2.66*10^7$ N/m. Determine the magnitude of the unbalance force F_o developed by the machine. What is the amplitude of vibration of the machine?



Solution:

$\omega = 2\pi f = 125.66 \quad rad/s$

$$\omega_n = \sqrt{\frac{k}{m}} = 43.97 \quad rad/s$$

$r = 2.86$

$$TR = \frac{F_{TR}}{F_o} = \frac{1}{r^2 - 1}$$

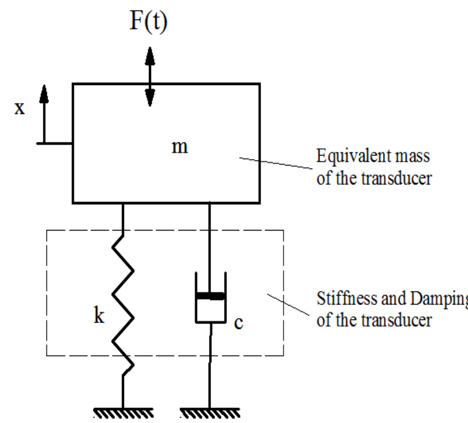
$F_o = 16.154 \; N \quad F_{TR} = kx$

$\therefore x = 0.0085 \, mm$

Transducers

[Force and Vibratory Motion]

Transducers are devices transfer energy from one form to another, and have different configurations. Our discussion is concerned with transducers that are used to measure dynamic forces and vibratory motion. Such transducers are usually modeled as simple spring-and-mass systems with viscous damping.



"Flat response" frequency region:

It is only in this region that the response (the output) of the transducer is essentially independent of the frequency components present in the dynamic phenomenon being measured. This flat region depends upon the ratios formed by the circular frequency ω present in the dynamic phenomenon being measured and the natural frequency ω_n of the transducer.

Force Transducers:

Load cells or pressure transducers that are used to measure forces or pressures frequently utilize resistance strain gages bonded to the elastic elements of the transducer to sense the strains resulting from the forces the transducer experiences.

The output of the transducer is proportional to the strain of the elastic element of the transducer, and corresponds to the displacement x of the mass shown.

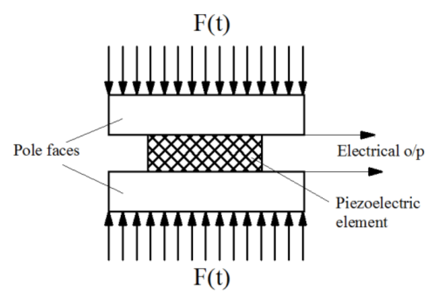
$$\frac{X}{X_{st}} = \frac{1}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}}$$

- $\frac{X}{X_{st}}$ is very nearly unity for small values of r. This means that the amplitude X of the response is the same as the static displacement X_{st} resulting from a statically applied force of magnitude F_o in this "flat response" frequency region.

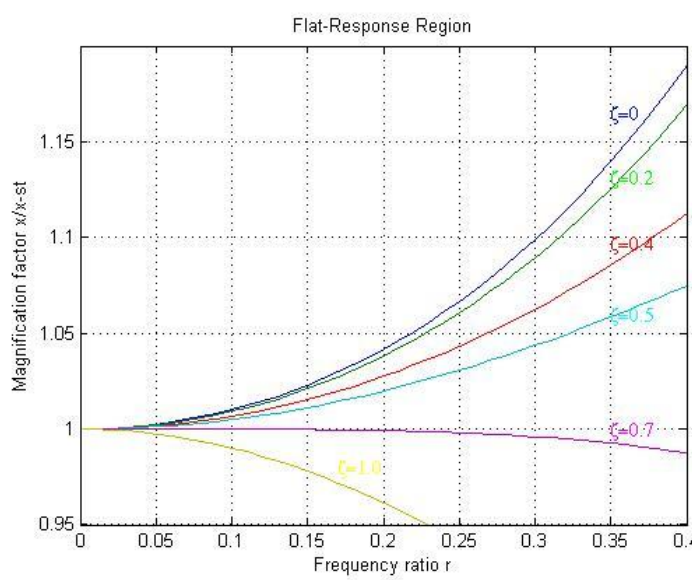
1- If $r \leq 0.2$ the response of the transducer is independent of the excitation frequency ω and $\frac{X}{X_{st}} \approx 1$

2- The maximum error in the deviation of $\frac{X}{X_{st}}$ from unity in the flat response region is less than 5%, regardless of the magnitude of damping $0 \leq \xi \leq 1.0$

Conclusion: accurate force measurements can be obtained from a force transducer when it is used to measure forces having frequencies up to 20% of the natural frequency of the transducer. For example, a transducer having natural frequency of 1000 Hz. would yield accurate measurements of the dynamic force if the frequency of that force were no greater than 200 Hz.



Dynamic forces generally contain more than one frequency component, the highest frequency component should be less than 20% of the natural frequency of the transducer.

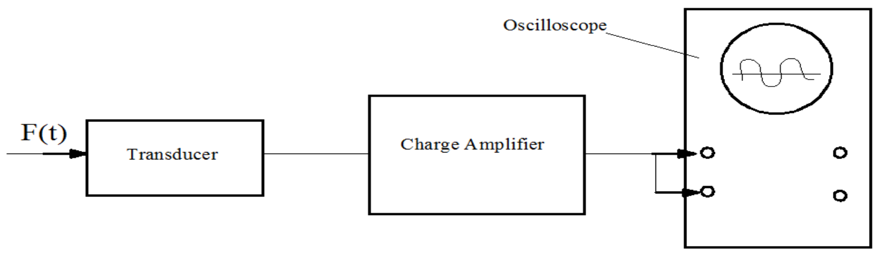


Force transducers that utilize a piezoelectric material (usually a polarized ferroelectric ceramic) as a sensing element can have fundamental natural frequency above 100,000 Hz.

The deformation of the piezoelectric element produces a charge q on the pole faces that is proportional to the force F (t)

Since the voltage V (volts), charge q (coulombs) and capacitance C (farads) are related by: V=q/C, the sensitivity of piezoelectric force transducers can be measured in terms of Pico-coulombs (pc = 10^{-12} c) per unit force (N) or mille-volts per unit of force.

Since the capacitance C indicates the capacitance of both the piezoelectric element and the cable connecting the transducer to oscillograph, the voltage V will be reduced by the capacitance of the cable. This loss can be eliminated by adding a charge amplifier as shown.



Phase Distortion:

Phase distortion causes changes in the shape of a wave in the time domain.

$$\varphi_i = \tan^{-1} \frac{2\zeta \frac{\omega_i}{\omega_n}}{1 - (\frac{\omega_i}{\omega_n})^2}$$

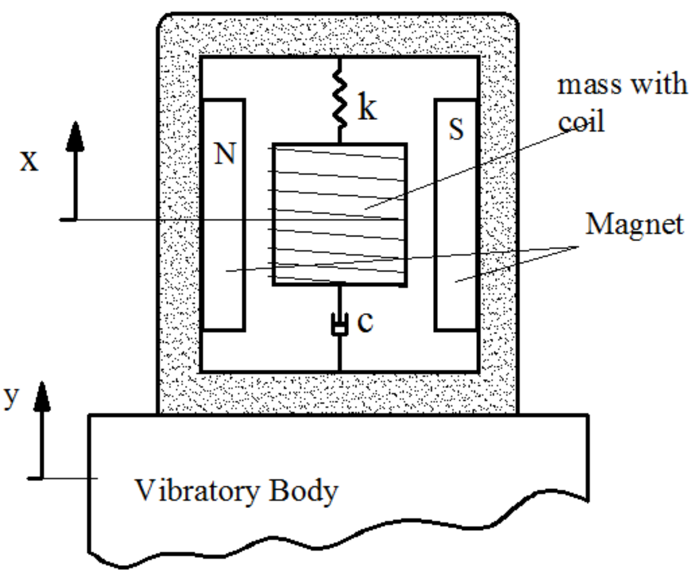
$\varphi_i = 0$ if there is no damping.

Since phase distortion is negligible in most force transducers because of their low inherent damping, there is no practical reason to make $\xi = 0.707$ in designing force transducers.

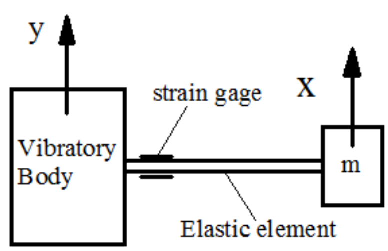
Vibration Measuring Transducers "Seismic Transducers"

Two general types of seismic transducers are used for vibration measurements. One is the ***vibrometer*** (seismometer), which is designed with a low natural frequency. The other is the ***accelerometer***, which is designed with a high natural frequency.

Most vibrometers are electromagnetic transducers, which consists of a moving mass m within a coil and a permanent magnet fixed to the case as shown. In some electromagnetic transducers, the moving mass is the permanent magnet and the coil is fixed to the transducer case. In either case, the voltage output from the coil is proportional to the rate at which the magnetic flux lines are cut (proportional to the *relative velocity* between the mass m and the vibrating body).



Vibrometer



Accelerometer

One type of accelerometer consists of a mass m attached to some type of elastic element such as the small cantilever beam shown. When it is mounted on a vibrating body, its output is proportional to the *absolute acceleration* of the mass m, which is equal to the acceleration of the vibrating body (the support acceleration). Electrical-resistance strain gages are sometimes used to sense this acceleration since the strain in the elastic element caused by the inertia of the mass m is proportional to the absolute acceleration of the mass m.

At present, the most widely used accelerometer for measuring shock and vibration is the piezoelectric accelerometer, which is similar in many ways to the piezoelectric force transducer.

1-Vibrometer (Low Frequency Transducer)

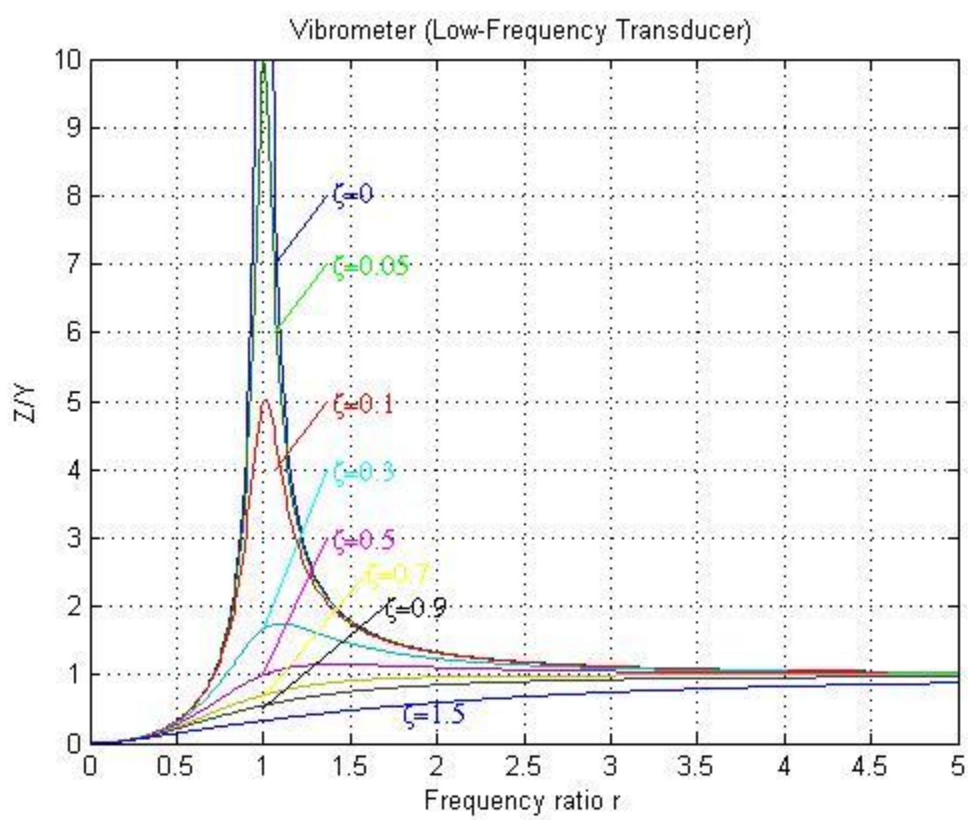
To determine the characteristics of this type transducer and its natural frequency range, we refer to equation (8) which is:

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}} \qquad (8)$$

A plot of this equation is shown, when r > 3, the steady-state amplitude ratio $\frac{Z}{Y} \approx 1$ for a wide range of damping factors. Therefore, in this frequency range,

$Z \approx Y$

and since $Z = X - Y$



$\therefore X = 0$

This means the mass m is remains stationary as the case moves with the vibrating body.

A vibration meter used with this type of transducer containing integrating and differentiating networks will yield direct readings of the displacement y, the velocity$\dot{y}$, and the acceleration $\ddot{y}$ of the vibrating body to which the transducer is attached.

The usable frequency range of a vibrometer depends upon its natural frequency ω_n and damping present.

Increasing damping, extend the lower end of the flat-response range (increase the speed range of this instrument), also, increasing the accuracy of the instrument (decreasing the percentage error of instrument reading).

As $\zeta \uparrow$, the speed range ↑, the error ↓,

As k↓, $\omega_n \downarrow$, the speed range ↑, the error ↓,

As m↑, $\omega_n \downarrow$, the speed range ↑, the error ↓,

2- Accelerometer (High Frequency Transducer)

Rewriting equation (8) in the form:

$$\frac{Z\omega_n^2}{Y\omega^2} = \frac{1}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}}$$

If the natural frequency ω_n is greater than the frequency of the vibrating body ω, the ratio r is small,

$$\frac{Z\omega_n^2}{Y\omega^2} \approx 1$$

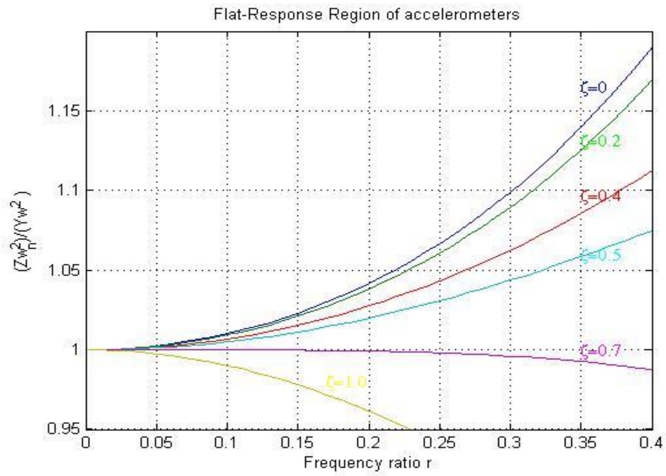
Or, $Z = \frac{1}{\omega_n^2} Y\omega^2 = \textit{the acceleration amplitude of the vibrating body}$

Increasing damping, extend the upper end of the flat-response range (increase the speed range of this instrument), also, increasing the accuracy of the instrument (decreasing the percentage error of instrument reading).

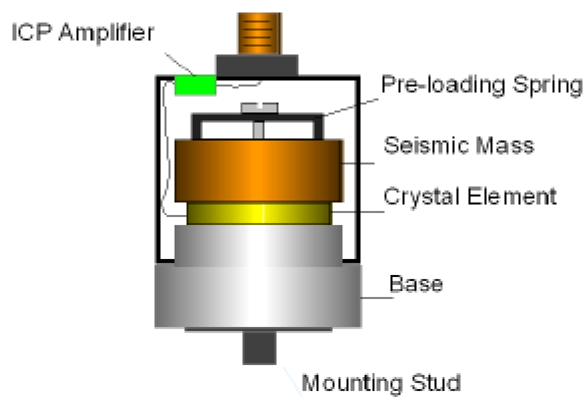
As $\zeta \uparrow$, the speed range $\uparrow$, the error $\downarrow$,

As $k\uparrow$, $\omega_n\uparrow$, the speed range $\uparrow$, error $\downarrow$,

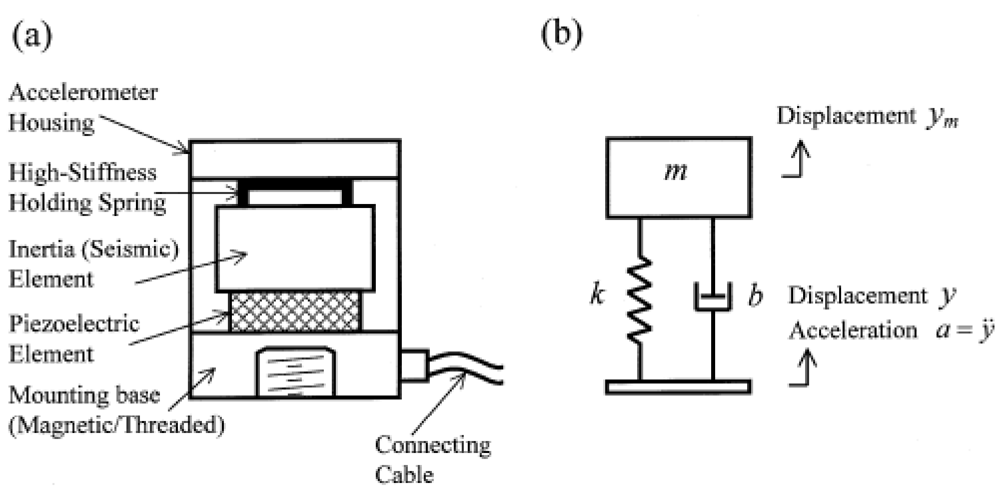
As $m\downarrow$, $\omega_n\uparrow$, the speed range $\uparrow$, error $\downarrow$,



Piezoelectric Accelerometer

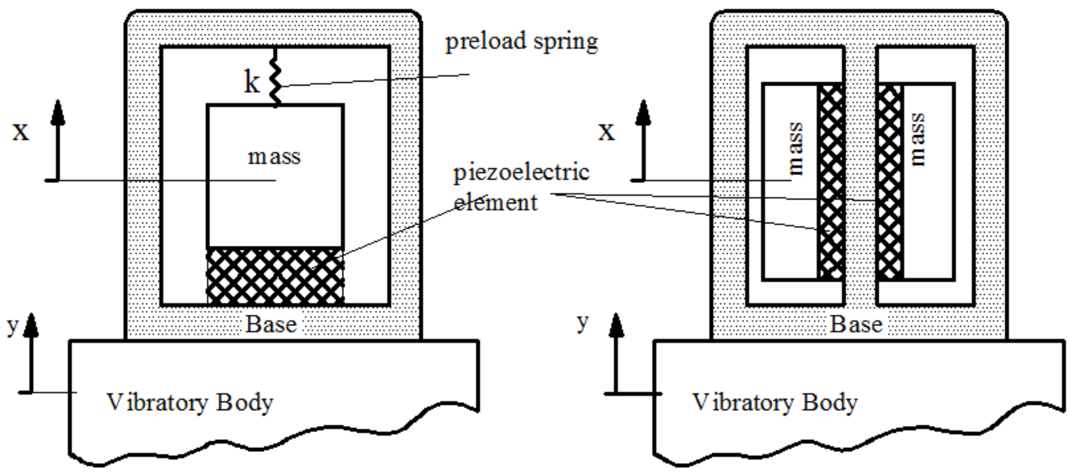


The compression-type Piezoelectric Accelerometer



(a) Schematic diagram of a piezoelectric accelerometer, and (b) A simplified model

The most widely used accelerometer for measuring shock and vibration is the piezoelectric accelerometer.



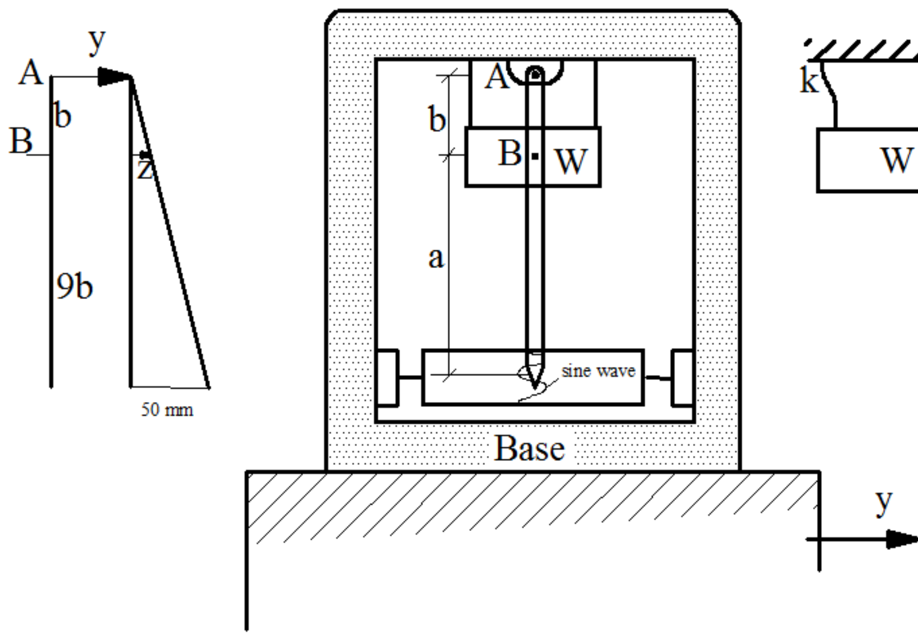
Compression type   Shear type

There are two types of this accelerometer, as shown; the *shear* type is less affected by airborne vibration (sound) than is the *compression* type. The *iso-shear* type (not shown) which contain multiple piezoelectric elements in shear, include high sensitivity and high signal to noise ratio. Both produce an electrical charge q proportional to the base acceleration $\ddot{y}$

- These accelerometers are small, rugged, and reliable transducers that have quite stable characteristics over long periods of time.
- Their small size (typically 0.25 to 0.75 in, diameter) facilitates their use in small confined areas.
- Their light weight (typically 0.2 to 20 g) permits their use on lightweight test objects without affecting the vibration characteristics being measured.
- For most acceleration measurements (10 g's or less), they are easily mounted using a wax-type material such as beeswax, when the acceleration is above 10g's, other means should be employed, such as an epoxy cement or threaded mounting studs.
- They are self-generating and as such require no external power supply.

- Small ones with small mass elements can have natural frequencies over 100,000 Hz, and will produce accurate measurements for frequency components of up to around 10,000 Hz, with negligible phase distortion.
- Natural frequencies of 30,000 to 50,000 Hz are typical for general-purpose piezoelectric accelerometers, depending upon the design.
- Charge amplifiers are used to reduce the sensitivity loss due to the capacitance in the cable connecting the transducer to such instruments as oscilloscopes and frequency analyzers. The sensitivity is usually given as Pico-coulombs per g of acceleration (pC/g), and the output of the charge amplifier is volts/g or millivolts/g.

Example 6:



The instrument shown is attached to the 85th floor of the Empire state building to measure the lateral oscillations of the building caused by strong gusting winds. The weight W is rigidily attached to the two vertical elastic elements each have stiffness k. The vertical penholder is pinned to the frame of the instrument at A and to the weight at B. Previous observations have shown that the lateral oscillations of the 85th floor can have amplitudes as large as 60 cm when it is vibrating at 0.2 Hz Should the instrument be designed as an accelerometer or as a vibrometer? Give a brief explanation of the reason for your answer.

Solution:

If designed as a vibrometer;

$Z \approx Y$

1- the drum being over 120 cm. long to record an amplitude of 60 cm

2- $\omega = 2 * \pi * 0.2$, then ω_n must be very small so that $r = (\omega / \omega_n) \geq 3$

If r = 4, W = 5 N

$\omega_n = \omega / 4 = 0.1 * \pi$

so that $2k = k_e = \omega_n^2 * m = 0.05$ N/m which is impractical

So, instrument should be designed as an accelerometer.

Example 7:

In example 6, assume that the instrument was designed as an accelerometer for which W = 0.4 N, and a = 9 b.

Determine:

a) the value of the spring constant k for frequency ratio of 0.1
b) the amplitude Y of the oscillation of the 85th floor when the chart amplitude reading on the drum is 50 mm with a period of 5 seconds.

Solution:

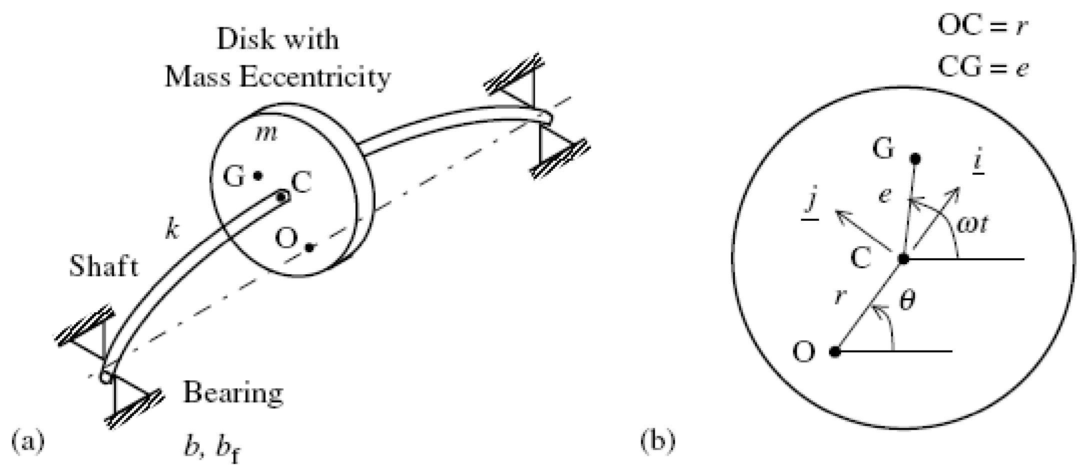
$r = 0.1$, $\omega_n = 10\ \omega = 4\ \pi$, $k_e = \omega_n^2 * m = 2k$, $k = 3.22$ N/m

$Z = Y r^2 = Y(0.01)$, $Y = 100\ Z$, $Z/b = 0.05/10b$

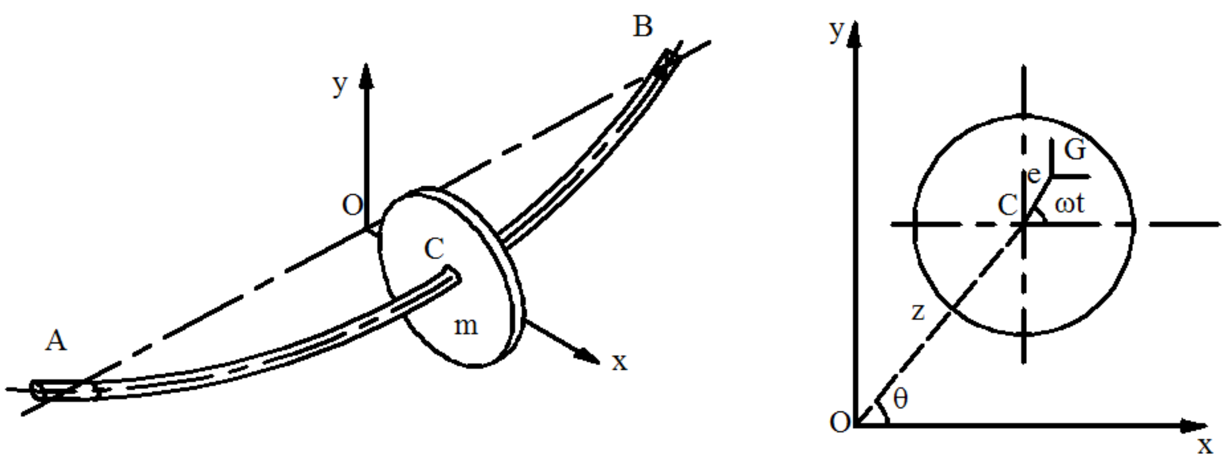
$Z = 0.005$ m

$Y = 0.5$ m

Critical Speed of Rotating Shafts



(a) A whirling shaft carrying a disk with mass eccentricity; (b) end view of the disk and whirling shaft.



When a shaft is rotating about its longitudinal axis bends about that axis (line AB), the bent shaft will whirl about its original axis of rotation as well as continuing to rotate about its longitudinal axis. Unbalanced disks, loose or worn bearing, and gyroscopic effects are examples of things that can cause shafts to whirl.

G ≡ mass center

e ≡ eccentricity

C ≡ geometric center

O ≡ intersection of the bearing centerline with the plane of the disk

z ≡ is the lateral deflection of the shaft z = OC

ω ≡ the angular velocity of the shaft-and-disk system w.r.t the longitudinal axis of the shaft

$\dot{\theta}$ ≡ the angular velocity of the rotating plane formed the bent shaft and the line AB

φ ≡ the angle between the position vector $\vec{z}$ and the position of vector $\vec{e}$

The mass of the shaft is small compared with that of attached disk (m)

$$x_G = x + e \cos \omega t$$
$$y_G = y + e \sin \omega t$$
$$\therefore \ddot{x}_G = \ddot{x} - e\omega^2 \cos \omega t$$
$$, \ddot{y}_G = \ddot{y} - e\omega^2 \sin \omega t$$

For circular shaft, the stiffness and damping of the shaft are the same

$k = k_x = k_y$

and, $c = c_x = c_y$

$$-kx - c\dot{x} = m(\ddot{x} - e\omega^2 \cos \omega t)$$
$$and \quad -ky - c\dot{y} = m(\ddot{y} - e\omega^2 \sin \omega t)$$

which can be reduced to:

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = e\omega^2 \cos \omega t$$
$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = e\omega^2 \sin \omega t$$

Assuming steady-state solution

$$x = Ae^{i\omega t}\ , \qquad y = Be^{i\omega t}$$

$\therefore \dot{x} = i\omega A e^{i\omega t}$, $\dot{y} = i\omega B e^{i\omega t}$

$\ddot{x} = -\omega^2 A e^{i\omega t}$, $\ddot{y} = -\omega^2 B e^{i\omega t}$

$$A = B = \frac{e.r^2}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}} e^{-i\varphi}$$

$$\varphi = \tan^{-1}\frac{2\zeta r}{1-r^2}$$

The rectangular components of the disk center C are:

$x = X\cos(\omega t - \varphi)$,

$y = Y\sin(\omega t - \varphi)$

Squaring the above components

$$z = \sqrt{x^2 + y^2}$$

$$\therefore z = X = Y = \frac{e.r^2}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}}$$

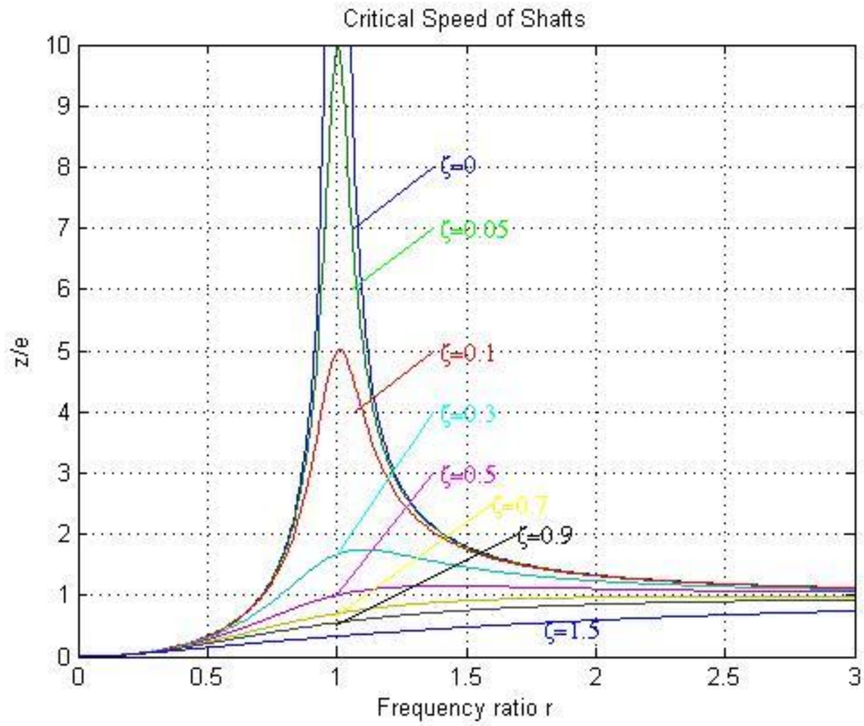
$$\theta = \tan^{-1}\frac{y}{x} = \omega t - \varphi$$

$and\ \dot{\theta} = \omega$

The above relationships show that the plane formed by the bent shaft and line AB whirls about line AB with angular velocity $\dot{\theta}$ that is equal to the angular velocity ω of the shaft-and-disk system.

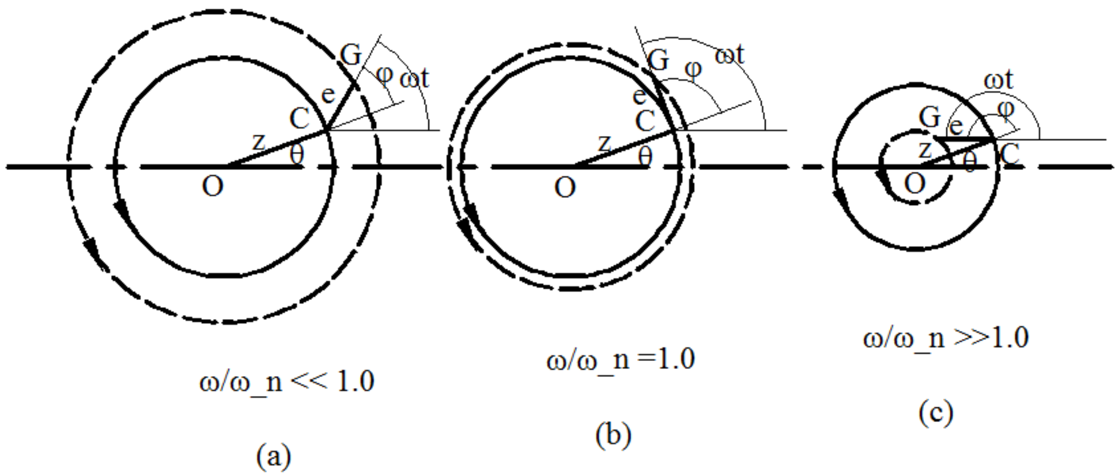
Prove that O,C and G at the same straight-line if there is no damping?

- For no damping $\theta = \omega t$
- $\therefore O, C\ and\ G\ at\ the\ same\ straight\ line$



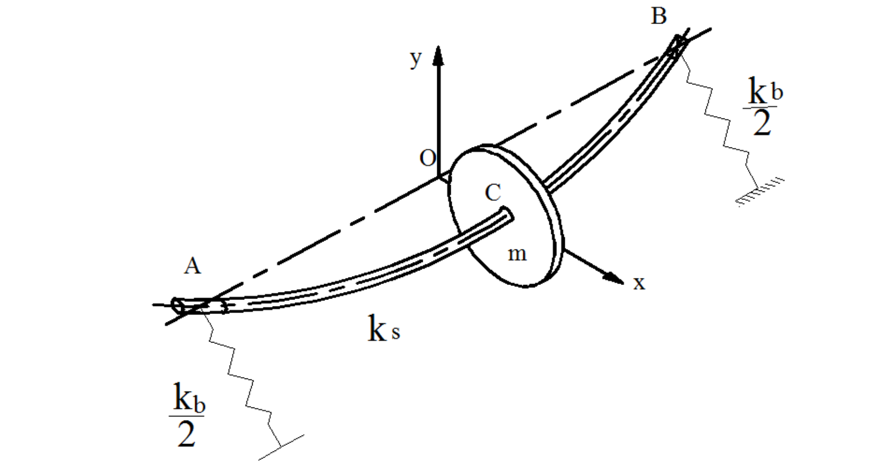
A plot of the dimensionless ratio z/e versus the frequency ratio ω/ω_n for various magnitudes of damping is shown:

- When r << 1, z/e ≈ 0 , φ = 0º, which means that the center of gravity G of the disk rotates on the outside of the path of the center C of the disk with a radius of rotation OG ≈ z + e [Fig. (a)]
- When r = 1, (the critical speed) z=e/2ξ, φ = 90º, that the magnitude of the whirling motion can be quite large if the damping factor is small, G rotates in a circular path of radius OG = $\sqrt{z^2 + e^2}$ [Fig. (b)]
- When r >> 1, z/e ≈ 1, φ = 180º, Thus, G rotates on the inside of the path of C with a radius of OG ≈ z-e [Fig. (c)]



- The deflection of the shaft must not exceed a given value, from the above curve, according to this value:
  a) When r <1, we can obtain the maximum value of ω at which the shaft must not run above it,
  b) When r >1, we can obtain the minimum value of ω at which the shaft must not run below it.
- The effective spring constant k of the shaft depends not only upon the size of the shaft but also upon the degree of bending resistance provided by the bearing. For example, if the bearings are attached to a rigid supports that prevent rotation of the bearing about any axis perpendicular to line AB, the stiffness k used would be the stiffness for a fixed-fixed beam. If the bearing were free to rotate, the stiffness k used would be that for a pinned-pinned beam.
- In starting and stopping rotating machines such as turbine that operates at speeds above their natural frequencies (critical speeds), large amplitudes of vibration can build up as the machine passes through the critical speed. These can be minimized by passing through the critical speed as quickly as possible since the amplitude does occur over some finite length of time.

Elastic Support



z_b ≡ bearing deflection, z_s ≡ shaft deflection, z ≡ total deflection,

$$z = z_s + z_b$$

k_b ≡ bearing stiffness, k_s ≡ shaft stiffness, k ≡ equivalent stiffness,

$$k = \frac{k_b k_s}{k_b + k_s}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$k_b z_b = k_s z_s = kz$$

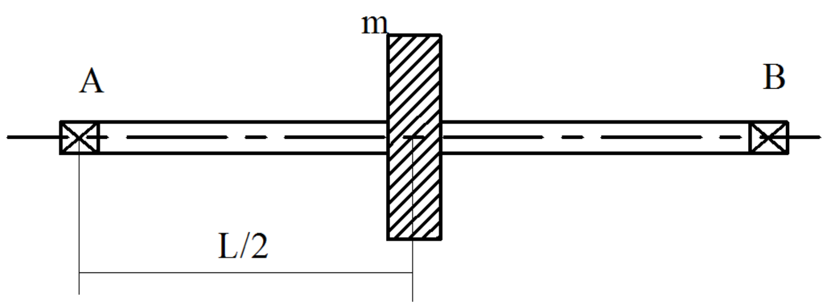
Example 6:

The shaft-and-disk system shown is supported by self-aligning bearings so that the steel shaft can be considered as a simply supported beam for purposes of choosing a spring constant k. the rotating disk is fixed to the shaft midway between bearings A and B, the data for the system are as follows:

m = 12 Kg, l = 0.5 m, d = 25.4 mm and E = 206.8 GPa.

Since the shaft is to operate over a range of speeds varying from 2400 to 3600 rpm. there is some concern that the bearing forces R_A and R_B could become large at the disks, the eccentricity of the disk is 0.05 mm and damping assumed 0.02. Determine:

a) the critical speed,
b) the maximum bearing forces that could be anticipated in operating over a range of 2400 to 3500 rpm.



Solution:

$$k = \frac{48EI}{l^3},$$

$$I = \frac{\pi d^4}{64} = 2.043 * 10^{-8}\ m^4$$

$$k = 1.622 * 10^6\ N/m$$

$$\omega_n = 367.65\quad rad./s \qquad\qquad or\ N = 3511\ rpm$$

$$*\ \omega_n\ is\ within\ the\ range\ of\ 2400\ to\ 3600\ rpm,$$

$$Max\ displacement\ when\ r = 1$$

$$\therefore z = \frac{e}{2\zeta} = 1.25\ mm$$

$$The\ bearing\ forces:$$

$$R_A = R_B = \frac{kz}{2} = 1013.8\ N$$

$$\frac{R_A}{R_{A_{static}}} = \frac{1013.8}{9.81 * 12} = 17.224$$

Equivalent Viscous Damping

A viscous damping model was used in which the damping force $c\dot{x}$ was assumed to be proportional to the velocity and was represented by a dashpot. Such model provides ordinary linear differential equations that are solved quite easily mathematically.

Energy-dissipating mechanisms in materials such as molecular friction, fluid resistance, and so on are very complicated phenomena and difficult to model mathematically. Such modeling is further complicated by the fact that energy dissipation or damping can result from combinations of different types of damping mechanisms.

An equivalent-viscous damping factor ζ_e can be determined for a non-viscous damping mechanism by equating the energy dissipated per cycle by viscous damping mechanism to that dissipated by the non- viscous damping mechanism.

Since the major effect of damping on a forced vibration occurs at or near resonance , equivalent viscous damping factors are usually determined at resonance.

Coulomb damping is an exception, since the amplitude of vibration goes to infinity when $\omega/\omega_n = 1$ for this type of damping.

Energy Dissipated by Viscous Damping:

The steady state response of a viscously damped system is:

$$x = X \sin(\omega t - \varphi)$$

$$x = X\omega \cos(\omega t - \varphi)$$

$$X = \frac{F_o/_k}{\sqrt{[1-r^2]^2+[2\zeta r]^2}}$$

The energy dissipated by $F_d = c\dot{x}$ for one cycle:

$$E_d = \oint c\dot{x}\,dx = c\int_0^{\tau} \dot{x}\frac{dx}{dt}dt = c\int_0^{\tau}\dot{x}^2 dt \qquad \tau = \frac{2\pi}{\omega} \quad E_d$$

$$= cX^2\omega^2\int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t-\varphi)dt$$

$$= \frac{cX^2\omega^2}{2}\int_0^{\frac{2\pi}{\omega}} [1+\cos 2(\omega t-\varphi)]dt$$

$$= \pi c X^2\omega \qquad \text{(I)}$$

$$\frac{c}{m} = 2\zeta\omega_n \qquad and\ \omega_n = \omega = \sqrt{\frac{k}{m}} \quad at\ resonance,$$

$$\therefore E_d = \pi 2\zeta k X^2$$

which gives the energy dissipated per cycle by viscous damping at resonance.

Velocity-Squared Damping: (Fluid Damping)

Velocity-squared damping is commonly used to describe the damping mechanism of a system vibrating in a fluid medium. The damping force is assumed to be proportional to the square of the velocity and can be approximated by

$$F_d = \pm(\frac{C\rho A}{2})\dot{x}^2 = -\alpha|\dot{x}|\dot{x}$$

where $\dot{x}$= velocity of vibrating body relative to fluid medium m/s

$|\dot{x}|$ = absolute value of $\dot{x}$

C = drag coefficient (dimensionless)

A = projected area of body perpendicular to $\dot{x}$ (m^2)

ρ = mass density of fluid (Kg/m^3)

$\alpha = \frac{C\rho A}{2}$

The equivalent viscous damping factor:

$$\zeta_e = \frac{2}{3}\frac{C\rho AX}{\pi m}$$

Coulomb Damping (Dry friction):

The friction damping force $F_d = \pm \mu W$

The energy dissipated in 1 cycle is $E_d = 4XF_d$

The equivalent-viscous-damping coefficient $c_e = 4F_d / \pi X\omega$ (from equation (I))

$$\zeta_e = \frac{2F_d}{\omega_n m\pi X\omega}$$

To investigate the response of a Coulomb-damped system at resonance

$$2\zeta_e\frac{\omega}{\omega_n} = \frac{4F_d}{\pi kX}$$

$$X = \frac{F_o/k}{\sqrt{[1-r^2]^2 + \left[\frac{4F_d}{\pi kX}\right]^2}}$$

Solving for the amplitude X yields:

$$X = \frac{F_o}{k}\frac{\sqrt{1-(\frac{4F_d}{\pi F_o})^2}}{1-r^2}$$

The last equation shows that the amplitude X is theoretically infinite at resonance for Coulomb-damped system. It should be noted that for X to be real, the quantity $\frac{4F_d}{\pi F_o}$ must be less than 1.

Structural Damping (complex Stiffness):

The energy dissipation caused by cyclic stress and strain within a structural material is often referred as structural damping (hysteretic damping, solid damping, and displacement damping).

The energy dissipated in 1 cycle is $E_d = \beta X^2$ (in which β is a constant having the units of force /displacement like a spring constant. The magnitude of β can vary with the size, shape, material, and temperature of the structural system.

Equating the last equation by equation (I), the equivalent-viscous-damping coefficient c_e for structural damping is:

$$c_e = \frac{\beta}{\pi\omega},$$

$$\zeta_e = \frac{\eta}{2}, \ in\ which\ \eta = \frac{\beta}{\pi k}\ is\ the\ structral\ damping\ factor$$