Chapter 9   Angular Momentum

Quantum Mechanical Angular Momentum Operators

Classical angular momentum is a vector quantity denoted $\vec{L} = \vec{r} \text{ X } \vec{p}$. A common mnemonic to calculate the components is

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = (yp_z - zp_y)\hat{i} + (zp_x - xp_z)\hat{j} + (xp_y - yp_x)\hat{j}$$

$$= L_x\hat{i} + L_y\hat{j} + L_z\hat{j}.$$

Let's focus on one component of angular momentum, say $L_x = yp_z - zp_y$. On the right side of the equation are two components of position and two components of linear momentum. Quantum mechanically, all four quantities are operators. Since the product of two operators is an operator, and the difference of operators is another operator, we expect the components of angular momentum to be operators. In other words, quantum mechanically

$$\mathcal{L}_x = \mathcal{Y}\mathcal{P}_z - \mathcal{Z}\mathcal{P}_y, \qquad \mathcal{L}_y = \mathcal{Z}\mathcal{P}_x - \mathcal{X}\mathcal{P}_z, \qquad \mathcal{L}_z = \mathcal{X}\mathcal{P}_y - \mathcal{Y}\mathcal{P}_x.$$

These are the components. Angular momentum is the vector sum of the components. The sum of operators is another operator, so angular momentum is an operator. We have not encountered an operator like this one, however, this operator is comparable to a vector sum of operators; it is essentially a ket with operator components. We might write

$$|\mathcal{L}> = \begin{pmatrix} \mathcal{L}_x \\ \mathcal{L}_y \\ \mathcal{L}_z \end{pmatrix} = \begin{pmatrix} \mathcal{Y}\mathcal{P}_z - \mathcal{Z}\mathcal{P}_y \\ \mathcal{Z}\mathcal{P}_x - \mathcal{X}\mathcal{P}_z \\ \mathcal{X}\mathcal{P}_y - \mathcal{Y}\mathcal{P}_x \end{pmatrix}. \tag{9-1}$$

A word of caution concerning common notation—this is usually written just $\mathcal{L}$, and the ket/vector nature of quantum mechanical angular momentum is not explicitly written but implied.

Equation (9-1) is in abstract Hilbert space and is completely devoid of a representation. We will want to pick a basis to perform a calculation. In position space, for instance

$$\mathcal{X} \to x, \qquad \mathcal{Y} \to y, \qquad \text{and} \qquad \mathcal{Z} \to z,$$

and

$$\mathcal{P}_x \to -i\hbar\frac{\partial}{\partial x}, \qquad \mathcal{P}_y \to -i\hbar\frac{\partial}{\partial y}, \qquad \text{and} \qquad \mathcal{P}_z \to -i\hbar\frac{\partial}{\partial z}.$$

Equation (9–1) in position space would then be written

$$|\mathcal{L}> = \begin{pmatrix} -i\hbar y\frac{\partial}{\partial z} + i\hbar z\frac{\partial}{\partial y} \\ -i\hbar z\frac{\partial}{\partial x} + i\hbar x\frac{\partial}{\partial z} \\ -i\hbar x\frac{\partial}{\partial y} + i\hbar y\frac{\partial}{\partial x} \end{pmatrix}. \tag{9-2}$$

The operator nature of the components promise difficulty, because unlike their classical analogs which are scalars, the angular momentum operators do not commute.

Example 9–1:   Show the components of angular momentum in position space do not commute.

Let the commutator of any two components, say $\left[\mathcal{L}_x,\, \mathcal{L}_y\right]$, act on the function x.   This means

$$\left[\mathcal{L}_x,\, \mathcal{L}_y\right]x = \left(\mathcal{L}_x\,\mathcal{L}_y - \mathcal{L}_y\,\mathcal{L}_x\right)x$$

$$\rightarrow \left(-i\hbar y\frac{\partial}{\partial z} + i\hbar z\frac{\partial}{\partial y}\right)\left(-i\hbar z\frac{\partial}{\partial x} + i\hbar x\frac{\partial}{\partial z}\right)x - \left(-i\hbar z\frac{\partial}{\partial x} + i\hbar x\frac{\partial}{\partial z}\right)\left(-i\hbar y\frac{\partial}{\partial z} + i\hbar z\frac{\partial}{\partial y}\right)x$$

$$= \left(-i\hbar y\frac{\partial}{\partial z} + i\hbar z\frac{\partial}{\partial y}\right)\left(-i\hbar z\right) - \left(-i\hbar z\frac{\partial}{\partial x} + i\hbar x\frac{\partial}{\partial z}\right)(0)$$

$$= \left(\left(-i\hbar\right)^2 y\right) = -\hbar^2 y \neq 0,$$

therefore $\mathcal{L}_x$ and $\mathcal{L}_y$ do not commute. Using functions which are simply appropriate position space components, other components of angular momentum can be shown not to commute similarly.

Example 9–2:   What is equation (9–1) in the momentum basis?

In momentum space, the operators are

$$\mathcal{X} \rightarrow i\hbar\frac{\partial}{\partial p_x}, \qquad \mathcal{Y} \rightarrow i\hbar\frac{\partial}{\partial p_y}, \qquad \text{and} \qquad \mathcal{Z} \rightarrow i\hbar\frac{\partial}{\partial p_z},$$

and

$$\mathcal{P}_x \rightarrow p_x, \qquad \mathcal{P}_y \rightarrow p_y, \qquad \text{and} \qquad \mathcal{P}_z \rightarrow p_z.$$

Equation (9–1) in momentum space would be written

$$\big|\,\mathcal{L} > = \begin{pmatrix} i\hbar\frac{\partial}{\partial p_y}\,p_z - i\hbar\frac{\partial}{\partial p_z}\,p_y \\ i\hbar\frac{\partial}{\partial p_z}\,p_x - i\hbar\frac{\partial}{\partial p_x}\,p_z \\ i\hbar\frac{\partial}{\partial p_x}\,p_y - i\hbar\frac{\partial}{\partial p_y}\,p_x \end{pmatrix}.$$

Canonical Commutation Relations in Three Dimensions

We indicated in equation (9–3) the fundamental canonical commutator is

$$\left[\mathcal{X},\, \mathcal{P}\right] = i\hbar.$$

This is fine when working in one dimension, however, descriptions of angular momentum are generally three dimensional. The generalization to three dimensions[2,3] is

$$\left[\mathcal{X}_i,\, \mathcal{X}_j\right] = 0, \tag{9-3}$$

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[2] Cohen-Tannoudji, *Quantum Mechanics* (John Wiley & Sons, New York, 1977), pp 149 – 151.
[3] Sakurai, *Modern Quantum Mechanics* (Addison–Wesley Publishing Company, Reading, Massachusetts; 1994), pp 44 – 51.

which means any position component commutes with any other position component including itself,

$$[\mathcal{P}_i, \mathcal{P}_j] = 0, \qquad\qquad (9-4)$$

which means any linear momentum component commutes with any other linear momentum component including itself,

$$[\mathcal{X}_i, \mathcal{P}_j] = i\hbar\delta_{i,j}, \qquad\qquad (9-5)$$

and the meaning of this equation requires some discussion. This means a position component will commute with an unlike component of linear momentum,

$$[\mathcal{X}, \mathcal{P}_y] = [\mathcal{X}, \mathcal{P}_z] = [\mathcal{Y}, \mathcal{P}_x] = [\mathcal{Y}, \mathcal{P}_z] = [\mathcal{Z}, \mathcal{P}_x] = [\mathcal{Z}, \mathcal{P}_y] = 0,$$

but a position component and a like component of linear momentum are canonical commutators, *i.e.*,

$$[\mathcal{X}_x, \mathcal{P}_x] = [\mathcal{Y}, \mathcal{P}_y] = [\mathcal{Z}, \mathcal{P}_z] = i\hbar.$$

Commutator Algebra

In order to use the canonical commutators of equations (9–3) through (9–5), we need to develop some relations for commutators in excess of those discussed in chapter 3. For any operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, the relations below, some of which we have used previously, may be a useful list.

$$\begin{aligned}
[\mathcal{A}, \mathcal{A}] &= 0 \\
[\mathcal{A}, \mathcal{B}] &= -[\mathcal{B}, \mathcal{A}] \\
[\mathcal{A}, c] &= 0, \qquad \text{for any scalar } c, \\
[\mathcal{A}, c\mathcal{B}] &= c[\mathcal{A}, \mathcal{B}], \qquad \text{for any scalar } c, \\
[\mathcal{A} + \mathcal{B}, \mathcal{C}] &= [\mathcal{A}, \mathcal{C}] + [\mathcal{B}, \mathcal{C}] \\
[\mathcal{A}, \mathcal{B}\mathcal{C}] &= [\mathcal{A}, \mathcal{B}]\mathcal{C} + \mathcal{B}[\mathcal{A}, \mathcal{C}] \qquad\qquad (9-6)
\end{aligned}$$

$$\Big[\mathcal{A}, [\mathcal{B}, \mathcal{C}]\Big] + \Big[\mathcal{B}, [\mathcal{C}, \mathcal{A}]\Big] + \Big[\mathcal{C}, [\mathcal{A}, \mathcal{B}]\Big] = 0.$$

You may have encountered relations similar to these in classical mechanics where the brackets are Poisson brackets. In particular, the last relation is known as the Jacobi identity. We are interested in quantum mechanical commutators and there are two important differences. Classical mechanics is concerned with quantities which are intrinsically real and are of finite dimension. Quantum mechanics is concerned with quantitites which are intrinsically complex and are generally of infinite dimension. Equation (9–6) is a relation we want to develop further.

Example 9–3: Prove equation (9–6).

$$\begin{aligned}
[\mathcal{A}, \mathcal{B}\mathcal{C}] &= \mathcal{A}\mathcal{B}\mathcal{C} - \mathcal{B}\mathcal{C}\mathcal{A} \\
&= \mathcal{A}\mathcal{B}\mathcal{C} - \mathcal{B}\mathcal{A}\mathcal{C} + \mathcal{B}\mathcal{A}\mathcal{C} - \mathcal{B}\mathcal{C}\mathcal{A} \\
&= (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\mathcal{C} + \mathcal{B}(\mathcal{A}\mathcal{C} - \mathcal{C}\mathcal{A}) \\
&= [\mathcal{A}, \mathcal{B}]\mathcal{C} + \mathcal{B}[\mathcal{A}, \mathcal{C}],
\end{aligned}$$

where we have added zero, in the form $-\mathcal{B}\mathcal{A}\mathcal{C} + \mathcal{B}\mathcal{A}\mathcal{C}$, in the second line.

Example 9–4: Develop a relation for $\left[\mathcal{A}\mathcal{B},\, \mathcal{C}\right]$ in terms of commutators of individual operators.

$$\left[\mathcal{A}\mathcal{B},\, \mathcal{C}\right] = \mathcal{A}\mathcal{B}\mathcal{C} - \mathcal{C}\mathcal{A}\mathcal{B}$$
$$= \mathcal{A}\mathcal{B}\mathcal{C} - \mathcal{A}\mathcal{C}\mathcal{B} + \mathcal{A}\mathcal{C}\mathcal{B} - \mathcal{C}\mathcal{A}\mathcal{B}$$
$$= \mathcal{A}\left(\mathcal{B}\mathcal{C} - \mathcal{C}\mathcal{B}\right) + \left(\mathcal{A}\mathcal{C} - \mathcal{C}\mathcal{A}\right)\mathcal{B}$$
$$= \mathcal{A}\left[\mathcal{B},\, \mathcal{C}\right] + \left[\mathcal{A},\, \mathcal{C}\right]\mathcal{B}.$$

Example 9–5: Develop a relation for $\left[\mathcal{A}\mathcal{B},\, \mathcal{C}\mathcal{D}\right]$ in terms of commutators of individual operators.

Using the result of example 9–3,

$$\left[\mathcal{A}\mathcal{B},\, \mathcal{C}\mathcal{D}\right] = \left[\mathcal{A}\mathcal{B},\, \mathcal{C}\right]\mathcal{D} + \mathcal{C}\left[\mathcal{A}\mathcal{B},\, \mathcal{D}\right],$$

and using the result of example 9–4 on both of the commutators on the right,

$$\left[\mathcal{A}\mathcal{B},\, \mathcal{C}\mathcal{D}\right] = \left(\mathcal{A}\left[\mathcal{B},\, \mathcal{C}\right] + \left[\mathcal{A},\, \mathcal{C}\right]\mathcal{B}\right)\mathcal{D} + \mathcal{C}\left(\mathcal{A}\left[\mathcal{B},\, \mathcal{D}\right] + \left[\mathcal{A},\, \mathcal{D}\right]\mathcal{B}\right)$$
$$= \mathcal{A}\left[\mathcal{B},\, \mathcal{C}\right]\mathcal{D} + \left[\mathcal{A},\, \mathcal{C}\right]\mathcal{B}\mathcal{D} + \mathcal{C}\mathcal{A}\left[\mathcal{B},\, \mathcal{D}\right] + \mathcal{C}\left[\mathcal{A},\, \mathcal{D}\right]\mathcal{B},$$

which is the desired result.

Angular Momentum Commutation Relations

Given the relations of equations (9–3) through (9–5), it follows that

$$\left[\mathcal{L}_x,\, \mathcal{L}_y\right] = i\hbar\,\mathcal{L}_z, \qquad \left[\mathcal{L}_y,\, \mathcal{L}_z\right] = i\hbar\,\mathcal{L}_x, \qquad \text{and} \qquad \left[\mathcal{L}_z,\, \mathcal{L}_x\right] = i\hbar\,\mathcal{L}_y. \qquad (9-7)$$

Example 9–6: Show $\left[\mathcal{L}_x,\, \mathcal{L}_y\right] = i\hbar\,\mathcal{L}_z$.

$$\left[\mathcal{L}_x,\, \mathcal{L}_y\right] = \left[\mathcal{Y}\mathcal{P}_z - \mathcal{Z}\mathcal{P}_y,\, \mathcal{Z}\mathcal{P}_x - \mathcal{X}\mathcal{P}_z\right]$$
$$= \left(\mathcal{Y}\mathcal{P}_z - \mathcal{Z}\mathcal{P}_y\right)\left(\mathcal{Z}\mathcal{P}_x - \mathcal{X}\mathcal{P}_z\right) - \left(\mathcal{Z}\mathcal{P}_x - \mathcal{X}\mathcal{P}_z\right)\left(\mathcal{Y}\mathcal{P}_z - \mathcal{Z}\mathcal{P}_y\right)$$

$$= \mathcal{Y}\mathcal{P}_z\mathcal{Z}\mathcal{P}_x - \mathcal{Y}\mathcal{P}_z\mathcal{X}\mathcal{P}_z - \mathcal{Z}\mathcal{P}_y\mathcal{Z}\mathcal{P}_x + \mathcal{Z}\mathcal{P}_y\mathcal{X}\mathcal{P}_z - \mathcal{Z}\mathcal{P}_x\mathcal{Y}\mathcal{P}_z + \mathcal{Z}\mathcal{P}_x\mathcal{Z}\mathcal{P}_y + \mathcal{X}\mathcal{P}_z\mathcal{Y}\mathcal{P}_z - \mathcal{X}\mathcal{P}_z\mathcal{Z}\mathcal{P}_y$$

$$= \left(\mathcal{Y}\mathcal{P}_z\mathcal{Z}\mathcal{P}_x - \mathcal{Z}\mathcal{P}_x\mathcal{Y}\mathcal{P}_z\right) + \left(\mathcal{Z}\mathcal{P}_y\mathcal{X}\mathcal{P}_z - \mathcal{X}\mathcal{P}_z\mathcal{Z}\mathcal{P}_y\right)$$

$$+ \left(\mathcal{Z}\mathcal{P}_x\mathcal{Z}\mathcal{P}_y - \mathcal{Z}\mathcal{P}_y\mathcal{Z}\mathcal{P}_x\right) + \left(\mathcal{X}\mathcal{P}_z\mathcal{Y}\mathcal{P}_z - \mathcal{Y}\mathcal{P}_z\mathcal{X}\mathcal{P}_z\right)$$

$$= \left[\mathcal{Y}\mathcal{P}_z,\, \mathcal{Z}\mathcal{P}_x\right] + \left[\mathcal{Z}\mathcal{P}_y,\, \mathcal{X}\mathcal{P}_z\right] + \left[\mathcal{Z}\mathcal{P}_x,\, \mathcal{Z}\mathcal{P}_y\right] + \left[\mathcal{X}\mathcal{P}_z,\, \mathcal{Y}\mathcal{P}_z\right].$$

Using the result of example 9–5, the plan is to express these commutators in terms of individual operators, and then evaluate those using the commutation relations of equations (9–3) through (9–5). In example 9–5, one commutator of the products of two operators turns into four commutators. Since we start with four commutators of the products of two operators, we are going to get 16

commutators in terms of individual operators. The good news is 14 of them are zero from equations (9–3), (9–4), and (9–5), so will be struck.

$$[\mathcal{L}_x, \mathcal{L}_y] = \mathcal{Y}[\mathcal{P}_z, \mathcal{Z}]\mathcal{P}_x + [\mathcal{Y}, \cancel{\mathcal{Z}}]\mathcal{P}_z\mathcal{P}_x + \mathcal{Z}\mathcal{Y}[\mathcal{P}_z, \cancel{\mathcal{P}_x}] + \mathcal{Z}[\mathcal{Y}, \cancel{\mathcal{P}_x}]\mathcal{P}_z$$

$$+ \mathcal{Z}[\mathcal{P}_y, \cancel{\mathcal{X}}]\mathcal{P}_z + [\mathcal{Z}, \cancel{\mathcal{X}}]\mathcal{P}_y\mathcal{P}_z + \mathcal{X}\mathcal{Z}[\mathcal{P}_y, \cancel{\mathcal{P}_z}] + \mathcal{X}[\mathcal{Z}, \mathcal{P}_z]\mathcal{P}_y$$

$$+ \mathcal{Z}[\mathcal{P}_x, \cancel{\mathcal{Z}}]\mathcal{P}_y + [\mathcal{Z}, \cancel{\mathcal{Z}}]\mathcal{P}_x\mathcal{P}_y + \mathcal{Z}\mathcal{Z}[\mathcal{P}_x, \cancel{\mathcal{P}_y}] + \mathcal{Z}[\mathcal{Z}, \cancel{\mathcal{P}_y}]\mathcal{P}_x$$

$$+ \mathcal{X}[\mathcal{P}_z, \cancel{\mathcal{Y}}]\mathcal{P}_z + [\mathcal{X}, \cancel{\mathcal{Y}}]\mathcal{P}_z\mathcal{P}_z + \mathcal{Y}\mathcal{X}[\mathcal{P}_z, \cancel{\mathcal{P}_z}] + \mathcal{Y}[\mathcal{X}, \cancel{\mathcal{P}_z}]\mathcal{P}_z$$

$$= \mathcal{Y}[\mathcal{P}_z, \mathcal{Z}]\mathcal{P}_x + \mathcal{X}[\mathcal{Z}, \mathcal{P}_z]\mathcal{P}_y$$

$$= \mathcal{Y}(-i\hbar)\mathcal{P}_x + \mathcal{X}(i\hbar)\mathcal{P}_y$$

$$= i\hbar(\mathcal{X}\mathcal{P}_y - \mathcal{Y}\mathcal{P}_x)$$

$$= i\hbar\mathcal{L}_z.$$

The other two relations, $[\mathcal{L}_y, \mathcal{L}_z] = i\hbar\,\mathcal{L}_x$ and $[\mathcal{L}_z, \mathcal{L}_x] = i\hbar\,\mathcal{L}_y$ can be calculated using similar procedures.

A Representation of Angular Momentum Operators

We would like to have matrix operators for the angular momentum operators $\mathcal{L}_x$, $\mathcal{L}_y$, and $\mathcal{L}_z$. In the form $\mathcal{L}_x$, $\mathcal{L}_y$, and $\mathcal{L}_z$, these are abstract operators in an infinite dimensional Hilbert space. Remember from chapter 2 that a subspace is a specific subset of a general complex linear vector space. In this case, we are going to find relations in a subspace C^3 of an infinite dimensional Hilbert space. The idea is to find three 3 X 3 matrix operators that satisfy relations (9–7), which are

$$[\mathcal{L}_x, \mathcal{L}_y] = i\hbar\,\mathcal{L}_z, \qquad [\mathcal{L}_y, \mathcal{L}_z] = i\hbar\,\mathcal{L}_x, \qquad \text{and} \qquad [\mathcal{L}_z, \mathcal{L}_x] = i\hbar\,\mathcal{L}_y.$$

One such group of objects is

$$\mathcal{L}_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar, \qquad \mathcal{L}_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar, \qquad \mathcal{L}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar. \quad (9-8)$$

You have seen these matrices in chapters 2 and 3. In addition to illustrating some of the mathematical operations of those chapters, they were used when appropriate there, so you may have a degree of familiarity with them here. There are other ways to express these matrices in C^3. Relations (9–8) are dominantly the most popular. Since the three operators do not commute, we arbitrarily have selected a basis for one of them, and then expressed the other two in that basis. Notice $\mathcal{L}_z$ is diagonal. That means the basis selected is natural for $\mathcal{L}_z$. The terminology usually used is the operators in equations (9–8) are **in the** $\underline{\mathcal{L}_z}$ **basis**.

We could have selected a basis which makes $\mathcal{L}_x$ or $\mathcal{L}_y$, and expressed the other two in terms of the natural basis for $\mathcal{L}_x$ or $\mathcal{L}_y$. If we had done that, the operators are different than

those seen in relations (9–8). The mathematics of this is not important at the moment, but it is important that you understand there are other self consistent ways to express these operators as 3 X 3 matrices.

Example 9–7:   Show $\left[\,\mathcal{L}_x,\,\mathcal{L}_y\,\right] = i\hbar\,\mathcal{L}_z$ using relations (9–8).

$$\left[\,\mathcal{L}_x,\,\mathcal{L}_y\,\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar$$

$$= \frac{\hbar^2}{2}\begin{pmatrix} i & 0 & -i \\ 0 & -i+i & 0 \\ i & 0 & -i \end{pmatrix} - \frac{\hbar^2}{2}\begin{pmatrix} -i & 0 & -i \\ 0 & i-i & 0 \\ i & 0 & i \end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix} i+i & 0 & -i+i \\ 0 & 0 & 0 \\ i-i & 0 & -i-i \end{pmatrix}$$

$$= \frac{\hbar^2}{2}\begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} = i\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar$$

$$= i\hbar\,\mathcal{L}_z.$$

Again, the other two relations can be calculated using similar procedures. In fact, the arithmetic for the other two relations is simpler. Why would this be so? ...Because $\mathcal{L}_z$ is a diagonal operator.

Remember $\mathcal{L}$ is comparable to a vector sum of the three component operators, so in vector/matrix notation would look like

$$|\mathcal{L}> = \begin{pmatrix} \mathcal{L}_x \\ \mathcal{L}_y \\ \mathcal{L}_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar \\ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar \end{pmatrix}.$$

Again, this operator will normally be denoted just $\mathcal{L}$. The $\mathcal{L}$ operator is a different sort of object than the component operators. It is a different object in a different space. Yet, we would like a way to address angular momentum with a 3 X 3 matrix which is in the same subspace as the components. We can do this if we use $\mathcal{L}^2$. This operator is

$$\mathcal{L}^2 = 2\hbar^2\mathcal{I} = 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (9-9)$$

Example 9–8: Show $\mathcal{L}^2 = 2\hbar^2\mathcal{I}$.

$$\mathcal{L}^2 = <\mathcal{L}\,\big|\,\mathcal{L}>$$

$$\rightarrow \Big\langle \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar, \; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar, \; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar \, \Bigg| \; \begin{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar \\ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar \end{pmatrix} \Big\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar$$

$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1+1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\hbar^2 + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1+1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2$$

$$= \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\hbar^2$$

$$= 2\hbar^2\mathcal{I}.$$

Complete Set of Commuting Observables ...A Discussion about Operators which do not Commute....

The intent of this section is to appreciate non–commutivity from a new perspective, and explain "what can be done about it" if the non–commuting operators represent physical quantities we want to measure. The following toy example is adapted from *Quantum Mechanics and Experience*[4].

We want two operators which do not commute. We are deliberately using simple operators in an effort to focus on principles. In a two dimensional linear vector space, the property of "hardness" is modelled

$$H_{ard} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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[4] Albert, *Quantum Mechanics and Experience* (Harvard University Press, Cambridge, Massachusetts, 1992), pp 30–33.

and has eigenvalues of ± 1 and eigenvectors

$$|1>_{\text{hard}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-1>_{\text{hard}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let's also consider the "color" operator,

$$C_{\text{olor}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with eigenvalues of ± 1 and eigenvectors

$$|1>_{\text{color}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-1>_{\text{color}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Note that a "hardness" or "color" eigenvector is a superposition of the eigenvectors of the other property, i.e.,

$$|1>_{\text{hard}} = \frac{1}{\sqrt{2}}|1>_{\text{color}} + \frac{1}{\sqrt{2}}|-1>_{\text{color}}$$

$$|-1>_{\text{hard}} = \frac{1}{\sqrt{2}}|1>_{\text{color}} - \frac{1}{\sqrt{2}}|-1>_{\text{color}}$$

$$|1>_{\text{color}} = \frac{1}{\sqrt{2}}|1>_{\text{hard}} + \frac{1}{\sqrt{2}}|-1>_{\text{hard}}$$

$$|-1>_{\text{color}} = \frac{1}{\sqrt{2}}|1>_{\text{hard}} - \frac{1}{\sqrt{2}}|-1>_{\text{hard}}$$

Hardness is a superposition of color states and color is a superposition of hardness states. That is the foundation of incompatibility, or non–commutivity. Each measurable state is a linear combination or superposition of the measurable states of the other property. To disturb one property is to disturb both properties.

Also in chapter 3, we indicated if two Hermitian operators commute, there exists a basis of common eigenvectors. Conversely, if they do not commute, there is no basis of common eigenvectors. We conclude there is no common eigenbasis for the "hardness" and "color" operators.

This is exactly the status of the three angular momentum component operators, except there are three vice two operators which do not commute with one another. None of the component operators commutes with any other. There is no common basis of eigenvectors between any two, so can be no common eigenbasis between all three.

Back to the hardness and color operators. If we can find an operator with which both commute, say the two dimensional identity operator $\mathcal{I}$, we can ascertain the eigenstate of the system. If we measure an eigenvalue of 1 for color, the eigenstate is proportional to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, were we to operate on this with the identity operator, the eigenstate of system is either $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If we then measure with the hardness operator, the eigenvalue will be 1 if the state was $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, or -1 if the state was $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We have effectively removed the indeterminacy of the system by including

$\mathcal{I}$. If we measure either "hardness" or "color," and then operate with the identity, we attain a distinct, unique unit vector. There are two **complete sets of commuting operators** possible, $\mathcal{I}$ and H_{ard}, or $\mathcal{I}$ and C_{olor}.

The eigenvalues, indicated in the ket, and eigenvectors for the three angular momentum component operators are

$$|-\sqrt{2}> = \frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}, \qquad |0> = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \qquad |\sqrt{2}> = \frac{1}{2}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix},$$

for $\mathcal{L}_x$,

$$|-\sqrt{2}> = \frac{1}{2}\begin{pmatrix}1\\-i\sqrt{2}\\-1\end{pmatrix}, \qquad |0> = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad |\sqrt{2}> = \frac{1}{2}\begin{pmatrix}1\\i\sqrt{2}\\-1\end{pmatrix},$$

for $\mathcal{L}_y$, and

$$|-1> = \begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad |0> = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad |1> = \begin{pmatrix}1\\0\\0\end{pmatrix},$$

for $\mathcal{L}_z$. Notice like the nonsense operators hardness and color, none of the angular momentum component operators commute and none of the eigenvectors correspond. Also comparable, $\mathcal{L}^2$ is proportional to the identity operator, except in three dimensions. We can do something similar to the "hardness, color" case to remove the indeterminacy. It must be similar and not the same...because we need a fourth operator with which the three non–commuting component angular momentum operators all commute, and any one of the angular momentum components to form a complete set of commuting observables. We choose $\mathcal{L}^2$, which commutes with all three component operators, and $\mathcal{L}_z$, which is the conventional choice of components.

The requirement for a complete set of commuting observables is equivalent to **removing or lifting a degeneracy**. The idea is closely related to the discussion at the end of example 3–33. If you comprehend the idea behind that discussion, you have the basic principle of this discussion.

Also, "complete" here means all possiblities are clear, *i.e.*, that any degeneracy is removed. This is the same word but a different context than "span the space" as the word was used in chapter 2. Both uses are conventional and meaning is ascertained only by usage, so do not be confused by its use in both contexts.

Precurser to the Hydrogen Atom

The Hamiltonian for a spherically symmetric potential commutes with $\mathcal{L}^2$ and the three component angular momentum operators. So $\mathcal{H}$, $\mathcal{L}^2$, and one of the three component angular momentum operators, conventially $\mathcal{L}_z$, is a complete set of commuting observables for a spherically symmetric potential.

We will use a Hamiltonian with a Coulomb potential for the hydrogen atom. The Coulomb potential is rotationally invariant, or spherically symmetric. We have indicated $\mathcal{H}$, $\mathcal{L}^2$, and $\mathcal{L}_z$ form a complete set of commuting observables for such a system. You may be familiar with the principal quantum number n, the angular momentum quantum number l, and the magnetic quantum number m. We will find there is a correspondence between these two sets of three quantities, which is n comes from application of $\mathcal{H}$, l comes from application of $\mathcal{L}^2$, and m

comes from application of $\mathcal{L}_z$. A significant portion of the reason to address angular momentum and explain the concept of a complete set of commuting observables now is for use in the next chapter on the hydrogen atom.

Ladder Operators for Angular Momentum

We are going to address angular momentum, like the SHO, from both a linear algebra and a differential equation perspective. We are going to assume rotational invariance, or spherical symmetry, so we have $\mathcal{H}$, $\mathcal{L}^2$, and $\mathcal{L}_z$ as a complete set of commuting observables. We will address linear algebra arguments first. And we will work only with the components and $\mathcal{L}^2$, saving the Hamiltonian for the next chapter.

The four angular momentum operators are related as

$$\mathcal{L}^2 = \mathcal{L}_x^2 + \mathcal{L}_y^2 + \mathcal{L}_z^2 \quad \Rightarrow \quad \mathcal{L}^2 - \mathcal{L}_z^2 = \mathcal{L}_x^2 + \mathcal{L}_y^2.$$

The sum of the two components $\mathcal{L}_x^2 + \mathcal{L}_y^2$ would appear to factor

$$\left(\mathcal{L}_x + i\mathcal{L}_y\right)\left(\mathcal{L}_x - i\mathcal{L}_y\right),$$

and they would if the factors were scalars, but they are operators which do not commute, so this is not factoring. Just like the SHO, it is a good mnemonic, nevertheless.

Example 9–9:   Show $\mathcal{L}_x^2 + \mathcal{L}_y^2 \neq \left(\mathcal{L}_x + i\mathcal{L}_y\right)\left(\mathcal{L}_x - i\mathcal{L}_y\right)$.

$$\begin{aligned}
\left(\mathcal{L}_x + i\mathcal{L}_y\right)\left(\mathcal{L}_x - i\mathcal{L}_y\right) &= \mathcal{L}_x^2 - i\mathcal{L}_x\mathcal{L}_y + i\mathcal{L}_y\mathcal{L}_x + \mathcal{L}_y^2 \\
&= \mathcal{L}_x^2 + \mathcal{L}_y^2 - i\left(\mathcal{L}_x\mathcal{L}_y - \mathcal{L}_y\mathcal{L}_x\right) \\
&= \mathcal{L}_x^2 + \mathcal{L}_y^2 - i\left[\mathcal{L}_x,\, \mathcal{L}_y\right] \\
&= \mathcal{L}_x^2 + \mathcal{L}_y^2 - i\left(i\hbar\mathcal{L}_z\right) \\
&= \mathcal{L}_x^2 + \mathcal{L}_y^2 + \hbar\mathcal{L}_z \\
&\neq \mathcal{L}_x^2 + \mathcal{L}_y^2,
\end{aligned}$$

where the expression in the next to last line is a significant intermediate result, and we will have reason to refer to it.

Like the SHO, the idea is to take advantage of the commutation relations of equations (9–7). We will use the notation

$$\mathcal{L}_+ = \mathcal{L}_x + i\mathcal{L}_y, \qquad \text{and} \qquad \mathcal{L}_- = \mathcal{L}_x - i\mathcal{L}_y, \tag{9-12}$$

which together are often denoted $\mathcal{L}_\pm$. We need commutators for $\mathcal{L}_\pm$, which are

$$\left[\mathcal{L}^2,\, \mathcal{L}_\pm\right] = 0, \tag{9-13}$$

$$\left[\mathcal{L}_z,\, \mathcal{L}_\pm\right] = \pm\hbar\,\mathcal{L}_\pm. \tag{9-14}$$

Example 9–10:   Show $\left[\mathcal{L}^2,\, \mathcal{L}_+\right] = 0$.

$$\left[\mathcal{L}^2,\, \mathcal{L}_+\right] = \left[\mathcal{L}^2,\, \mathcal{L}_x + i\mathcal{L}_y\right] = \left[\mathcal{L}^2,\, \mathcal{L}_x\right] + i\left[\mathcal{L}^2,\, \mathcal{L}_y\right] = 0 + i(0) = 0.$$

Example 9–11:   Show $\left[\mathcal{L}_z,\, \mathcal{L}_+\right] = \hbar\,\mathcal{L}_+$.

$$\left[\mathcal{L}_z,\, \mathcal{L}_+\right] = \left[\mathcal{L}_z,\, \mathcal{L}_x + i\mathcal{L}_y\right] = \left[\mathcal{L}_z,\, \mathcal{L}_x\right] + i\left[\mathcal{L}_z,\, \mathcal{L}_y\right] = i\hbar\mathcal{L}_y + i\left(-i\hbar\mathcal{L}_x\right) = \hbar\left(\mathcal{L}_x + i\mathcal{L}_y\right) = \hbar\mathcal{L}_+.$$

We will proceed essentially as we did the the raising and lowering operators of the SHO. Since $\mathcal{L}^2$ and $\mathcal{L}_z$ commute, they share a common eigenbasis.

Example 9–12:   Show $\mathcal{L}^2$ and $\mathcal{L}_z$ commute.

$$\begin{aligned}
\left[\mathcal{L}^2,\, \mathcal{L}_z\right] &= \left[\mathcal{L}_x^2 + \mathcal{L}_y^2 + \mathcal{L}_z^2,\, \mathcal{L}_z\right] \\
&= \left[\mathcal{L}_x^2,\, \mathcal{L}_z\right] + \left[\mathcal{L}_y^2,\, \mathcal{L}_z\right] + \left[\mathcal{L}_z^2\!\!\!\!\Big/\mathcal{L}_z\right] \\
&= \left[\mathcal{L}_x\,\mathcal{L}_x,\, \mathcal{L}_z\right] + \left[\mathcal{L}_y\,\mathcal{L}_y,\, \mathcal{L}_z\right] \\
&= \mathcal{L}_x\left[\mathcal{L}_x,\, \mathcal{L}_z\right] + \left[\mathcal{L}_x,\, \mathcal{L}_z\right]\mathcal{L}_x + \mathcal{L}_y\left[\mathcal{L}_y,\, \mathcal{L}_z\right] + \left[\mathcal{L}_y,\, \mathcal{L}_z\right]\mathcal{L}_y \\
&= \mathcal{L}_x\left(-i\hbar\mathcal{L}_y\right) + \left(-i\hbar\mathcal{L}_y\right)\mathcal{L}_x + \mathcal{L}_y\left(i\hbar\mathcal{L}_x\right) + \left(i\hbar\mathcal{L}_x\right)\mathcal{L}_y \\
&= \left(-i\hbar\mathcal{L}_x\,\mathcal{L}_y + i\hbar\mathcal{L}_x\,\mathcal{L}_y\right) + \left(-i\hbar\mathcal{L}_y\,\mathcal{L}_x + i\hbar\mathcal{L}_y\,\mathcal{L}_x\right) \\
&= 0,
\end{aligned}$$

where we have used the results of example 9–4 and two of equations (9–7) in the reduction.

We assume $\mathcal{L}^2$ and $\mathcal{L}_z$ will have different eigenvalues when they operate on the same basis vector, so we need two indices for each basis vector. The first index is the eigenvalue for $\mathcal{L}^2$, we will use α for the eigenvalue, and the second index is the eigenvalue for $\mathcal{L}_z$, denoted by β.   If we had a third commuting operator, for instance $\mathcal{H}$ which we will add in the next chapter, we would need three eigenvalues to uniquely identify each ket. Here we are considering two commuting operators, so we need two indices representing the eigenvalues of the two commuting operators.

Considering just $\mathcal{L}^2$ and $\mathcal{L}_z$ here, the form of the eigenvalue equations must be

$$\mathcal{L}^2\big|\alpha,\, \beta\!> = \alpha\big|\alpha,\, \beta\!>, \tag{9-15}$$

$$\mathcal{L}_z\big|\alpha,\, \beta\!> = \beta\big|\alpha,\, \beta\!>, \tag{9-16}$$

where $\big|\alpha,\, \beta\!>$ is the eigenstate, α is the eigenvalue of $\mathcal{L}^2$, and β is the eigenvalue of $\mathcal{L}_z$. Equation (9–14)/example 9–11 give us

$$\left[\mathcal{L}_z,\, \mathcal{L}_+\right] = \mathcal{L}_z\,\mathcal{L}_+ - \mathcal{L}_+\,\mathcal{L}_z = \hbar\,\mathcal{L}_+$$

$$\Rightarrow \quad \mathcal{L}_z\,\mathcal{L}_+ = \mathcal{L}_+\,\mathcal{L}_z + \hbar\,\mathcal{L}_+.$$

Using this in equation (9–16),

$$\begin{aligned}
\mathcal{L}_z\,\mathcal{L}_+\big|\alpha,\, \beta\!> &= \left(\mathcal{L}_+\,\mathcal{L}_z + \hbar\,\mathcal{L}_+\right)\big|\alpha,\, \beta\!> \\
&= \mathcal{L}_+\,\mathcal{L}_z\big|\alpha,\, \beta\!> + \hbar\,\mathcal{L}_+\big|\alpha,\, \beta\!> \\
&= \mathcal{L}_+\,\beta\big|\alpha,\, \beta\!> + \hbar\,\mathcal{L}_+\big|\alpha,\, \beta\!> \\
&= \left(\beta + \hbar\right)\mathcal{L}_+\big|\alpha,\, \beta\!> .
\end{aligned} \tag{9-17}$$

Summarizing,

$$\mathcal{L}_z\big(\mathcal{L}_+|\alpha,\,\beta\!>\,\big) = \big(\beta+\hbar\big)\big(\mathcal{L}_+|\alpha,\,\beta\!>\,\big),$$

which means $\mathcal{L}_+|\alpha,\,\beta\!>$ is itself an eigenvector of $\mathcal{L}_z$ with eigenvalue $\big(\beta+\hbar\big)$. The effect of $\mathcal{L}_+$ is to increase the eigenvalue of $\mathcal{L}_z$ by the amount $\hbar$, so it is called the **raising operator**. Note that it raises only the eigenvalue of $\mathcal{L}_z$. A better name would be the raising operator for $\mathcal{L}_z$, but the convention is when angular momentum is being discussed is to refer simply to the raising operator, and you need to know it applies only to $\mathcal{L}_z$.

Were we to calculate similarly, we would find $\mathcal{L}_-|\alpha,\,\beta\!>$ is itself an eigenvector of $\mathcal{L}_z$ with eigenvalue $\big(\beta-\hbar\big)$. The effect of $\mathcal{L}_-$ is to decrease the eigenvalue by the amount $\hbar$, so it is called the **lowering operator**. Again, the convention when angular momentum is being discussed is to refer to the lowering operator without reference to $\mathcal{L}_z$.

Example 9–13: Show $|\alpha,\,\beta\!>$ is an eigenvector of $\mathcal{L}^2$. Equation (9–13) yields

$$\big[\mathcal{L}^2,\,\mathcal{L}_+\big] = \mathcal{L}^2\,\mathcal{L}_+ - \mathcal{L}_+\,\mathcal{L}^2 = 0$$

$$\Rightarrow \quad \mathcal{L}^2\,\mathcal{L}_+ = \mathcal{L}_+\,\mathcal{L}^2.$$

Then

$$\mathcal{L}^2\,\mathcal{L}_+\big|\alpha,\,\beta\!> = \mathcal{L}_+\,\mathcal{L}^2\big|\alpha,\,\beta\!> = \mathcal{L}_+\,\alpha\big|\alpha,\,\beta\!> = \alpha\,\mathcal{L}_+\big|\alpha,\,\beta\!>,$$

or summarizing

$$\mathcal{L}^2\big(\mathcal{L}_+\big|\alpha,\,\beta\!>\,\big) = \alpha\big(\mathcal{L}_+\big|\alpha,\,\beta\!>\,\big),$$

so $\mathcal{L}_+|\alpha,\,\beta\!>$ is itself an eigenvector of $\mathcal{L}^2$ with eigenvalue α. Similarly, $\mathcal{L}_-|\alpha,\,\beta\!>$ is itself an eigenvector of $\mathcal{L}^2$ with eigenvalue α.

It is important that $\mathcal{L}_+|\alpha,\,\beta\!>$ is itself an eigenvector of $\mathcal{L}^2$, but be sure to notice that the raising/lowering operator has no effect on the eigenvalue of $\mathcal{L}^2$. The eigenvalue of $\mathcal{L}^2$ acting on an eigenstate is α. The eigenvalue of $\mathcal{L}^2$ acting on a combination of the raising/lowering operator and an eigenstate is still α.

Eigenvalue Solution for the Square of Orbital Angular Momentum

Recalling the relation between the four angular momentum operators,

$$\mathcal{L}^2 - \mathcal{L}_z^2 = \mathcal{L}_x^2 + \mathcal{L}_y^2,$$

we are going use the eigenvalue equations and apply these operators to the generic eigenstate, *i.e.*,

$$\begin{aligned}
\big(\mathcal{L}^2 - \mathcal{L}_z^2\big)\big|\alpha,\,\beta\!> &= \mathcal{L}^2\big|\alpha,\,\beta\!> -\mathcal{L}_z^2\big|\alpha,\,\beta\!> \\
&= \alpha\big|\alpha,\,\beta\!> -\mathcal{L}_z\beta\big|\alpha,\,\beta\!> \\
&= \alpha\big|\alpha,\,\beta\!> -\beta^2\big|\alpha,\,\beta\!> \\
&= \big(\alpha-\beta^2\big)\big|\alpha,\,\beta\!> .
\end{aligned}$$

Forming an adjoint eigenstate and a braket,

$$\begin{aligned}
<\alpha,\beta\big|\mathcal{L}^2 - \mathcal{L}_z^2\big|\alpha,\,\beta\!> &= \,<\alpha,\beta\big|\mathcal{L}_x^2 + \mathcal{L}_y^2\big|\alpha,\,\beta\!> && (9-18) \\
&= \,<\alpha,\beta\big|\alpha - \beta^2\big|\alpha,\,\beta\!> \\
&= \,\big(\alpha-\beta^2\big) <\alpha,\beta\big|\alpha,\,\beta\!> && (9-19) \\
&= \alpha - \beta^2 \geq 0, && (9-20)
\end{aligned}$$

where we have assumed orthonormality of eigenstates in equation (9–19). The condition that the difference in equation (9–20) is non–negative is from the fact the braket is expressible in terms of a sum of $\mathcal{L}_x^2$ and $\mathcal{L}_y^2$, as seen in equation (9–18). Both $\mathcal{L}_x$ and $\mathcal{L}_y$ are Hermitian, so their eigenvalues are real. The sum of the squares of the eigenvalues, corresponding to operations by $\mathcal{L}_x^2$ and $\mathcal{L}_y^2$ in equation (9–18), must be non–negative. In mathematical vernacular, $\mathcal{L}_x^2$ and $\mathcal{L}_y^2$ are positive definite.

Equation (9–20) is equivalent to $\alpha \geq \beta^2$, which means β is bounded for a given value of α. Therefore there is an eigenstate $|\alpha, \beta_{\max}>$ which cannot be raised, and another eigenstate $|\alpha, \beta_{\min}>$ which cannot be lowered. In other words, we have a ladder which has a bottom, like the SHO, and a top, unlike the SHO. In a calculation similar to example 9–9,

$$\mathcal{L}_-\mathcal{L}_+ = \mathcal{L}_x^2 + \mathcal{L}_y^2 - \hbar = \mathcal{L}^2 - \mathcal{L}_z^2 - \hbar\mathcal{L}_z,$$

so

$$\mathcal{L}_-\mathcal{L}_+|\alpha, \beta_{\max}> = \vec{0}$$
$$\Rightarrow \quad \left(\mathcal{L}^2 - \mathcal{L}_z^2 - \hbar\mathcal{L}_z\right)|\alpha, \beta_{\max}> = 0 \qquad\qquad (9-21)$$
$$\Rightarrow \quad \mathcal{L}^2|\alpha, \beta_{\max}> - \mathcal{L}_z^2|\alpha, \beta_{\max}> - \hbar\mathcal{L}_z|\alpha, \beta_{\max}> = 0$$
$$\Rightarrow \quad \alpha|\alpha, \beta_{\max}> - \beta_{\max}^2|\alpha, \beta_{\max}> - \hbar\,\beta_{\max}|\alpha, \beta_{\max}> = 0$$
$$\Rightarrow \quad \left(\alpha - \beta_{\max}^2 - \hbar\,\beta_{\max}\right)|\alpha, \beta_{\max}> = 0$$
$$\Rightarrow \quad \alpha - \beta_{\max}^2 - \hbar\,\beta_{\max} = 0$$
$$\Rightarrow \quad \alpha = \beta_{\max}^2 + \hbar\,\beta_{\max}. \qquad\qquad (9-22)$$

Similarly,

$$\mathcal{L}_+\mathcal{L}_-|\alpha, \beta_{\min}> = \vec{0}$$
$$\Rightarrow \quad \alpha = \beta_{\max}^2 - \hbar\,\beta_{\max}. \qquad\qquad (9-23)$$

Equating equations (9–22) and (9–23), we get

$$\beta_{\max}^2 + \hbar\beta_{\max} - \beta_{\min}^2 + \hbar\beta_{\min} = 0.$$

This is quadratic in both $\beta_{\max}$ and $\beta_{\min}$, and to solve the equation, we will use the quadratic formula to solve for $\beta_{\max}$, or

$$\beta_{\max} = -\frac{1}{2}\hbar \pm \frac{1}{2}\sqrt{\hbar^2 - 4(-\beta_{\min}^2 + \hbar\,\beta_{\min})}$$
$$= -\frac{1}{2}\hbar \pm \frac{1}{2}\sqrt{4\beta_{\min}^2 - 4\hbar\,\beta_{\min} + \hbar^2}$$
$$= -\frac{1}{2}\hbar \pm \frac{1}{2}\sqrt{(2\beta_{\min} - \hbar)^2}$$
$$= -\frac{1}{2}\hbar \pm \frac{1}{2}(2\beta_{\min} - \hbar)$$
$$\Rightarrow \quad \beta_{\max} = -\beta_{\min}, \quad \beta_{\min} - \hbar. \qquad\qquad (9-24)$$

The case $\beta_{\max} = -\beta_{\min}$ is the maximum separation case. It gives us the top and bottom of the ladder. We assume the rungs of the ladder are

separated by $\hbar$, because that is the amount of change indicated by the raising and lowering operators. The picture corresponds to figure 11–1. If there is other than minimum separation, say there are n steps between the bottom and top rungs of the ladder, there is a total separation of $n\hbar$ between the bottom and the top. From figure 11–1 we expect

$$2\beta_{\max} = n\hbar \quad \Rightarrow \quad \beta_{\max} = \frac{n\hbar}{2}.$$

Using this in equation (11–22),

$$\alpha = \beta_{\max}\left(\beta_{\max} + \hbar\right)$$

$$= \frac{n\hbar}{2}\left(\frac{n\hbar}{2} + \hbar\right)$$

$$= \hbar^2\left(\frac{n}{2}\right)\left(\frac{n}{2} + 1\right).$$

We are going to re-label, letting $j = n/2$, so

$$\alpha = \hbar^2 j(j+1). \tag{11-25}$$

FIGURE 11-1. n STEPS WITH EACH STEP OF DISTANCE $\hbar$.

Wait a minute.... The fact $j = \hbar/2$ vice just $\hbar$ does not appear consistent with the assumption that the rungs of the ladder are separated by $\hbar$...and it isn't. It appears the rungs of the ladder are separated by $\hbar/2$ vice $\hbar$.

What has occurred is that we have actually solved a more general problem than intended. Because of symmetry, the linear algebra arguments have given us the solution for **total angular momentum**. Total angular momentum is

$$\vec{J} = \vec{L} + \vec{S}, \tag{11-26}$$

where $\vec{L}$ is **orbital angular momentum**, $\vec{S}$ is **spin angular momentum** or just **spin**. We posed the problem for orbital angular momentum, but because total angular momentum and spin obey analogous commutation relations to orbital angular momentum, we arrive at the solution for total angular momentum. Equations (11–7) indicated components of orbital angular momentum do not commute,

$$\left[\mathcal{L}_x, \mathcal{L}_y\right] = i\hbar\mathcal{L}_z, \qquad \left[\mathcal{L}_y, \mathcal{L}_z\right] = i\hbar\mathcal{L}_x, \qquad \text{and} \qquad \left[\mathcal{L}_z, \mathcal{L}_x\right] = i\hbar\mathcal{L}_y,$$

and for the ladder operator solution, we formed $\mathcal{L}_\pm = \mathcal{L}_x \pm i\mathcal{L}_y$. The commutation relations among the components of total angular momentum and spin angular momentum are exactly the same, *i.e.*,

$$\left[\mathcal{J}_x, \mathcal{J}_y\right] = i\hbar\mathcal{J}_z, \qquad \left[\mathcal{J}_y, \mathcal{J}_z\right] = i\hbar\mathcal{J}_x, \qquad \text{and} \qquad \left[\mathcal{J}_z, \mathcal{J}_x\right] = i\hbar\mathcal{J}_y,$$

and

$$\left[\mathcal{S}_x, \mathcal{S}_y\right] = i\hbar\mathcal{S}_z, \qquad \left[\mathcal{S}_y, \mathcal{S}_z\right] = i\hbar\mathcal{S}_x, \qquad \text{and} \qquad \left[\mathcal{S}_z, \mathcal{S}_x\right] = i\hbar\mathcal{S}_y.$$
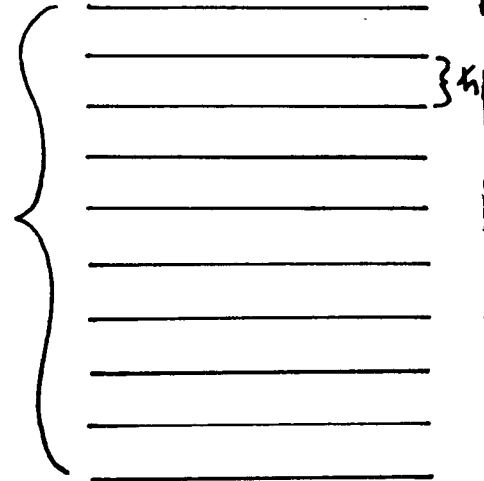
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separated by $\hbar$, because that is the amount of change indicated by the raising and lowering operators. The picture corresponds to figure 9–1. If there is other than minimum separation, say there are n steps between the bottom and top rungs of the ladder, there is a total separation of $n\hbar$ between the bottom and the top. From figure 9–1 we expect

$$2\beta_{\text{max}} = n\hbar \quad \Rightarrow \quad \beta_{\text{max}} = \frac{n\hbar}{2}.$$

Using this in equation (9–22),

$$\alpha = \beta_{\text{max}}\big(\beta_{\text{max}} + \hbar\big)$$

$$= \frac{n\hbar}{2}\left(\frac{n\hbar}{2} + \hbar\right)$$

$$= \hbar^2\left(\frac{n}{2}\right)\left(\frac{n}{2} + 1\right).$$

We are going to re–label, letting $j = n/2$, so

$$\alpha = \hbar^2\, j\big(j + 1\big). \tag{9 - 25}$$

Wait a minute.... The fact $j = \hbar/2$ vice just $\hbar$ does not appear consistent with the assumption that the rungs of the ladder are separated by $\hbar$...and it isn't. It appears the rungs of the ladder are separated by $\hbar/2$ vice $\hbar$.

What has occurred is that we have actually solved a more general problem than intended. Because of symmetry, the linear algebra arguments have given us the solution for **total angular momentum**. Total angular momentum is

$$\vec{J} = \vec{L} + \vec{S}, \tag{9 - 26}$$

where $\vec{L}$ is **orbital angular momentum**, $\vec{S}$ is **spin angular momentum** or just **spin**. We posed the problem for orbital angular momentum, but because total angular momentum and spin obey analogous commutation relations to orbital angular momentum, we arrive at the solution for total angular momentum. Equations (9–7) indicated components of orbital angular momentum do not commute,

$$\big[\mathcal{L}_x,\,\mathcal{L}_y\big] = i\hbar\,\mathcal{L}_z, \qquad \big[\mathcal{L}_y,\,\mathcal{L}_z\big] = i\hbar\,\mathcal{L}_x, \qquad \text{and} \qquad \big[\mathcal{L}_z,\,\mathcal{L}_x\big] = i\hbar\,\mathcal{L}_y,$$

and for the ladder operator solution, we formed $\mathcal{L}_\pm = \mathcal{L}_x \pm i\mathcal{L}_y$. The commutation relations among the components of total angular momentum and spin angular momentum are exactly the same, *i.e.*,

$$\big[\mathcal{J}_x,\,\mathcal{J}_y\big] = i\hbar\,\mathcal{J}_z, \qquad \big[\mathcal{J}_y,\,\mathcal{J}_z\big] = i\hbar\,\mathcal{J}_x, \qquad \text{and} \qquad \big[\mathcal{J}_z,\,\mathcal{J}_x\big] = i\hbar\,\mathcal{J}_y,$$

and

$$\big[\mathcal{S}_x,\,\mathcal{S}_y\big] = i\hbar\,\mathcal{S}_z, \qquad \big[\mathcal{S}_y,\,\mathcal{S}_z\big] = i\hbar\,\mathcal{S}_x, \qquad \text{and} \qquad \big[\mathcal{S}_z,\,\mathcal{S}_x\big] = i\hbar\,\mathcal{S}_y.$$

If we had started out with $\mathcal{J}_\pm = \mathcal{J}_x \pm i\mathcal{J}_y$, or $\mathcal{S}_\pm = \mathcal{S}_x \pm i\mathcal{S}_y$, we would have come out with exactly the same result. In fact, this is the problem we solved, except using the symbol $\mathcal{L}$ vice $\mathcal{J}$ or $\mathcal{S}$.

We will reinforce in chapter 13 that spin can have half integral values, or values of multiples of $\hbar/2$. Since spin can be half integral, values of total angular momentum can be half integral. When we use symbols such as $\mathcal{L}^2$ and $\mathcal{L}_i$, we get the information contained in the commutation relations, independent of whatever symbols we choose. Had we used explicit representations, such as equations (9–8) and (9–9), we would get the same information, however, limited by the representation. In that case, only integral values would be possible, though the form of the result analogous to equation (9–25) would remain the same. Using l as the quantum number for orbital angular momentum, the eigenvalue for orbital angular momentum squared is

$$\alpha = \hbar^2 \, l(l+1). \qquad (9-27)$$

A comment about the picture and notation is appropriate. The first impression is that this is similar to classical mechanics. The earth orbits the sun and has orbital angular momentum in that regard, and also spins on its axis so has spin angular momentum as well. It is tempting to apply this picture to a quantum mechanical system, say an electron in a hydrogen atom. It simply does not apply. The electron is not a small ball spinning on its axis as it orbits the proton. Per the first chapter, an electron is not a particle, it is not a wave, it is an electron. There is no classical analogy for an electron, and many of the manifestations of quantum mechanical angular momentum are similarly not classical analogs.

Equation (9–26) says the total is the sum of the parts, but it is an operator equation which in Dirac notation is $|\mathcal{J}> = |\mathcal{L}> + |\mathcal{S}>$. Since earlier development was in this form, it may be useful to assist you to realize that each of these three operators has three components which are also each operators. Equation (9-26) is standard notation nevertheless.

Eigenvalue Solution for the Z Component of Orbital Angular Momentum

We have calculated the eigenvalue of $\mathcal{L}^2$, but still need to find the eigenvalue of $\mathcal{L}_z$. We know one of the possible eigenvalues of $\mathcal{L}_z$ is zero from the last of equations (9–8), the explicit representations, regardless of the eigenstate. We have also calculated

$$\mathcal{L}_z\big(\mathcal{L}_+\big|\alpha, \, \beta> \big) = \big(\beta + \hbar\big)\big(\mathcal{L}_+\big|\alpha, \, \beta> \big).$$

If we start with an eigenstate that has the z component of angular momentum equal to zero,

$$\mathcal{L}_z\big(\mathcal{L}_+\big|\alpha, \, 0> \big) = \big(0 + \hbar\big)\big(\mathcal{L}_+\big|\alpha, \, 0> \big) = \hbar\big(\mathcal{L}_+\big|\alpha, \, 0> \big),$$

so $\hbar$ is the next eigenvalue. Using $\hbar$ as the eigenvalue,

$$\mathcal{L}_z\big(\mathcal{L}_+\big|\alpha, \, \hbar> \big) = \big(\hbar + \hbar\big)\big(\mathcal{L}_+\big|\alpha, \, \hbar> \big) = 2\hbar\big(\mathcal{L}_+\big|\alpha, \, \hbar> \big),$$

so $2\hbar$ is the next eigenvalue. If we use this as an eigenvalue,

$$\mathcal{L}_z\big(\mathcal{L}_+\big|\alpha, \, 2\hbar> \big) = \big(\hbar + 2\hbar\big)\big(\mathcal{L}_+\big|\alpha, \, 2\hbar> \big) = 3\hbar\big(\mathcal{L}_+\big|\alpha, \, 2\hbar> \big),$$

and $3\hbar$ is the next eigenvalue up the ladder. We can continue, and will attain integral values of $\hbar$. But we cannot continue forever, because we determined β is bounded by the eigenvalue of $\mathcal{L}^2$. What is the maximum value? We go back to figure 9–1 and the result from this figure is

$$\beta_{\text{max}} = \frac{n\hbar}{2},$$

where we want only integral values for the orbital angular momentum, so this becomes

$$\beta_{\text{max}} = l\hbar.$$

Were we to do the same calculation with the lowering operator, that is

$$\mathcal{L}_z\big(\mathcal{L}_-\big|\alpha,\,0{>}\,\big) = -\hbar\big(\mathcal{L}_-\big|\alpha,\,0{>}\,\big),$$

we step down the ladder in increments of $-\hbar$ until we get to β_{min}. Remember β_{min} also has a minimum, which is of the same magnitude but negative or

$$\beta_{\text{min}} = -l\hbar.$$

So we have eigenvalues which climb to $l\hbar$ and drop to $-l\hbar$ in integral increments of $\hbar$. The eigenvalue of the z component of angular momentum is just an integer times $\hbar$, from minimum to maximum values. The symbol conventionally used to denote this integer is m, so

$$\mathcal{L}_z\big|\alpha,\,\beta{>} = m\hbar\big|\alpha,\,\beta{>}, \quad -l < m < l$$

is the eigenvalue/eigenvector equation for the z component of angular momentum. The quantum number m, occasionally denoted m_l, is known as the **magnetic quantum number**.

Eigenvalue/Eigenvector Equations for Orbital Angular Momentum

If we use l vice α to denote the state of total angular momentum, realizing l itself is not an eigenvalue of $\mathcal{L}^2$, and m to denote the state of the z component of angular momentum, realizing the eigenvalue of $\mathcal{L}_z$ is actually $m\hbar$, the eigenvalue/eigenvector equations for $\mathcal{L}^2$ and $\mathcal{L}_z$ are

$$\mathcal{L}^2\big|l,\,m{>} = \hbar^2\,l(l+1)\big|l,\,m{>}, \tag{9-28}$$

$$\mathcal{L}_z\big|l,\,m{>} = m\hbar\big|l,\,m{>}, \quad -l < m < l \tag{9-29}$$

which is the conventional form of the two eigenvalue/eigenvector equations for $\mathcal{L}^2$ and $\mathcal{L}_z$.

Angular Momentum Eigenvalue Picture for Eigenstates

What is $|l, m>$? It is an eigenstate of the commuting operators $\mathcal{L}^2$ and $\mathcal{L}_z$. The quantum numbers l and m are not eigenvalues. The corresponding eigenvalues are $\hbar^2 l(l+1)$ and $m\hbar$. Were we to use eigenvalues in the ket, the eigenstate would look like $|\hbar^2 l(l+1), m\hbar>$. But just l and m uniquely identify the state, and that is more economical, so only the quantum numbers are conventionally used. This is essentially the same sort of convenient shorthand used to denote an eigenstate of a SHO $|n>$, vice using the eigenvalue $|(n+\frac{1}{2})\hbar\omega>$.

Only one quantum number is needed to uniquely identify an eigenstate of a SHO, but two are needed to uniquely identify an eigenstate of angular momentum. Because the angular momentum component operators do not commute, a complete set of commuting observables are needed. Each of the component operators commutes with $\mathcal{L}^2$, so we use it and one other, which is $\mathcal{L}_z$ chosen by convention. One quantum number is needed for each operator in the complete set. Multiple quantum numbers used to identify a ket denote a complete set of commuting observables is needed.

Remember that a system is assumed to exist in a linear combination of all possible eigenstates until we measure. If we measure, what are the possible outcomes? Possible outcomes are the eigenvalues. For a given value of the orbital angular momentum quantum number, the magnetic quantum number can assume integer values ranging from $-l$ to l. The simplest case is $l = 0 \Rightarrow m = 0$ is the only possible value of the magnetic quantum number. The possible outcomes of a measurement of such a system are eigenvalues of $\hbar^2(0)(0+1) = 0\hbar^2$ or just 0 for $\mathcal{L}^2$, and $m\hbar = (0)\hbar$ or just 0 for $\mathcal{L}_z$ as well, corresponding to figure 11–2.a.
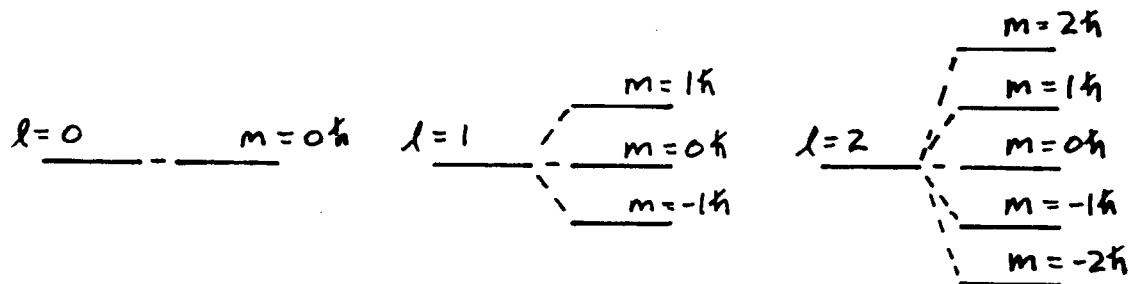

Figure $11 - 2$.a. $l = 0$.   Figure $11 - 2$.b. $l = 1$.   Figure $11 - 2$.c. $l = 2$.

If we somehow knew $l = 1$, which could be ascertained by a measurement of $\hbar^2(1)(1+1) = 2\hbar^2$ for $\mathcal{L}^2$, the possible values of the magnetic quantum number are $m = -1$, 0, or 1, so the eigenvalues which could be measured are $-\hbar$, 0, or $\hbar$ for $\mathcal{L}_z$, per figure 11–2.b. If we measured $\hbar^2(2)(2+1) = 6\hbar^2$ for $\mathcal{L}^2$, we would know we had $l = 2$, and the possible values of the magnetic quantum number are $m = -2$, -1, 0, 1, or 2, so the eigenvalues which could be measured are $-2\hbar$, $-\hbar$, 0, $\hbar$, or $2\hbar$ for $\mathcal{L}_z$, per figure 11-2.c. Though the magnetic quantum number is bounded by the orbital angular momentum quantum number, the orbital angular momentum quantum number is not bounded, so we can continue indefinitely. Notice there are $2l+1$ possible values of m for every value of l.

A semi–classical diagram is often used. A simple interpretation of $|l, m>$ is that it is a vector quantized in length of

$$|\mathcal{L}| - |\mathcal{L}| = \sqrt{\mathcal{L}^2} = \hbar\sqrt{l(l+1)}.$$

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What is $|l, m>$? It is an eigenstate of the commuting operators $\mathcal{L}^2$ and $\mathcal{L}_z$. The quantum numbers l and m are not eigenvalues. The corresponding eigenvalues are $\hbar^2 l(l+1)$ and $m\hbar$. Were we to use eigenvalues in the ket, the eigenstate would look like $|\hbar^2 l(l+1), m\hbar>$. But just l and m uniquely identify the state, and that is more economical, so only the quantum numbers are conventionally used. This is essentially the same sort of convenient shorthand used to denote an eigenstate of a SHO $|n>$, vice using the eigenvalue $\left|\left(n + \frac{1}{2}\right)\hbar\omega>\right.$

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A semi–classical diagram is often used. A simple interpretation of $|l, m>$ is that it is a vector quantized in length of

$$|\mathcal{L}| \rightarrow |\mathcal{L}| = \sqrt{\mathcal{L}^2} = \hbar\sqrt{l(l+1)}.$$

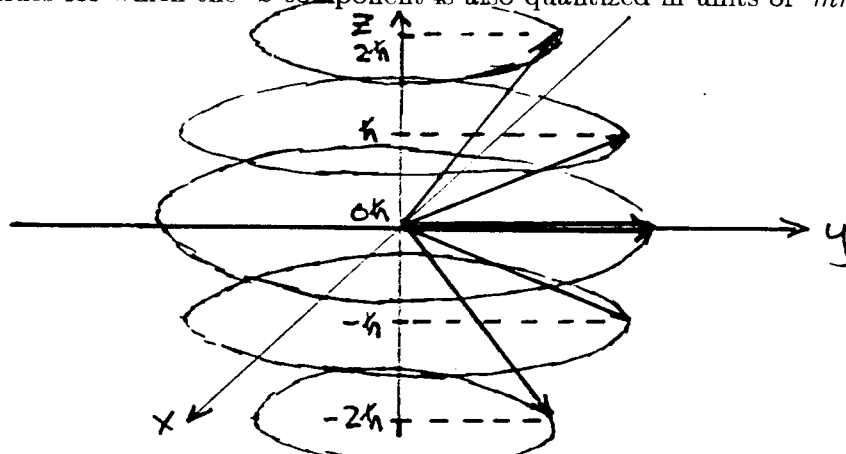This vector has values for which the z component is also quantized in units of $m\hbar$. These

Figure 11 − 3. Semi − Classical Picture for $l = 2$.

features are illustrated in figure 11–3 for $l = 2$. The vectors are free to rotate around the z axis at any azimuthal angle ϕ, but are fixed at polar angles θ determined by the fact the projection on the z axis must be $-2\hbar$, $-\hbar$, 0, $\hbar$, or $2\hbar$. Notice there is no information concerning the x or y components other than the square of their sum is fixed. We could express this for $|\psi(t)> = |l, m>$ by stating the projection on the xy plane will be $\cos(\omega t)$ or $\sin(\omega t)$. In such a case we can determine x and y component expectation values from symmetry alone, $i.e.$,

$$<\mathcal{L}_x> = 0, \qquad <\mathcal{L}_y> = 0.$$

Finally, what fixes any axis in space? And how do we know which axis is the z axis? The answer is we must introduce some asymmetry. Without an asymmetry of some sort, the axes and their labels are arbitrary. The practical assymmetry to introduce is a magnetic field, and that will establish a component quantization axis which will be the z axis.

Eigenvalue/Eigenvector Equations for the Raising and Lowering Operators

Using quantum number notation, the fact $\mathcal{L}_+|\alpha, \beta>$ is and eigenstate of $\mathcal{L}_z$ would be written

$$\begin{aligned} L_z\mathcal{L}_+|l, m> &= (m\hbar + \hbar)\mathcal{L}_+|l, m> \\ &= (m+1)\hbar\mathcal{L}_+|l, m> \\ &= \gamma\mathcal{L}_z|l, m+1>, \end{aligned}$$

where γ is a proportionality constant. Then

$$\begin{aligned} \mathcal{L}_z\mathcal{L}_+|l, m> &= \mathcal{L}_z\gamma|l, m+1> \\ \Rightarrow \mathcal{L}_+|l, m> &= \gamma|l, m+1>, \end{aligned}$$

is the eigenvalue/eigenvector equation for the raising operator, where γ is evidently the eigenvalue, and the eigenvector is raised by one element of quantization in the z component. This means if the z component of the state on which the raising operator acts is $m\hbar$, the new state has a z component of $m\hbar + \hbar = (m+1)\hbar$, and thus the index $m+1$ is used in the new eigenket.

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Figure $9-3$. Semi $-$ Classical Picture for $l = 2$.

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want to solve for γ and have an equation analogous to the forms of equations (9–28) and (9–29). Forming the adjoint equation,

$$<l,\, m|\mathcal{L}_+^\dagger = <l,\, m+1|\gamma^* \;\Rightarrow\; <l,\, m|\mathcal{L}_- = <l,\, m+1|\gamma^*,$$

because $\mathcal{L}_+^\dagger = \mathcal{L}_-$.   Forming a braket with the original equation

$$<l,\, m|\mathcal{L}_-\mathcal{L}_+|l,\, m> = <l,\, m+1|\gamma^*\gamma|l,\, m+1>\,.$$

Though we did it for $\beta_{\max}$, the maximum eigenvalue of $\mathcal{L}_z$, the algebra leading to equation (9–22) remains the same for any β, any eigenvalue of $\mathcal{L}_z$, so we know

$$\mathcal{L}_-\mathcal{L}_+ = \alpha - \beta^2 - \hbar\beta = \hbar^2\, l(l+1) - m^2\hbar^2 - m\hbar^2.$$

Using this in the braket,

$$<l,\, m|\hbar^2\Big(l(l+1) - m^2 - m\Big)|l,\, m> \;=\; <l,\, m+1|\gamma^*\gamma|l,\, m+1>$$

$$\Rightarrow\; \hbar^2\Big(l(l+1) - m(m+1)\Big) <l,\, m|l,\, m> \;=\; |\gamma^*\gamma|\, <l,\, m+1|l,\, m+1>$$

$$\Rightarrow\; \hbar^2\Big(l(l+1) - m(m+1)\Big) = |\gamma|^2 \tag{9 – 30}$$

$$\Rightarrow\; \gamma = \sqrt{l(l+1) - m(m+1)}\,\hbar,$$

where we used the orthonormality of eigenstates to arrive at equation (9–30). The eigenvalue/ eigenvector equation is then

$$\mathcal{L}_+|l,\, m> = \sqrt{l(l+1) - m(m+1)}\,\hbar|l,\, m+1>\,.$$

Were we to do the similar calculation for $\mathcal{L}_-$, we find

$$\mathcal{L}_-|l,\, m> = \sqrt{l(l+1) - m(m-1)}\,\hbar|l,\, m-1>\,.$$

These are most often expressed as one relation,

$$\mathcal{L}_\pm|l,\, m> = \sqrt{l(l+1) - m(m\pm 1)}\,\hbar|l,\, m\pm 1>\,. \tag{9 – 31}$$

Example 9–14:   For the eigenstate $|l,\, m> = |3,\, m>$, what measurements are possible for $\mathcal{L}^2$ and $\mathcal{L}_z$?

The only measurements that are possible are the eigenvalues. From equation (9–28), the eigenvalue of $\mathcal{L}^2$ is $\hbar^2\, l(l+1) = \hbar^2\, 3(3+1) = 12\hbar^2$.

For $l = 3$, the possible eigenvalues of $\mathcal{L}_z$ can range from $-3\hbar$ to $3\hbar$ in increments of $\hbar$.   Explicitly, the measurements that are possible for $\mathcal{L}_z$ for the eigenstate $|3,\, m>$ are $-3\hbar,\ -2\hbar,\ -\hbar,\ 0,\ \hbar,\ 2\hbar,$ or $3\hbar$.

Example 9–15:   What are $\mathcal{L}_+$ and $\mathcal{L}_-$ operating on the eigenstate $|2,\, -1>$?

Using equation (9–31),

$$\mathcal{L}_+|2, -1> = \sqrt{2(2+1) - (-1)((-1)+1)}\,\hbar|2, -1+1>$$
$$= \sqrt{2(3) - (-1)(0)}\,\hbar|2, 0>$$
$$= \sqrt{6}\,\hbar|2, 0> .$$

$$\mathcal{L}_-|2, -1> = \sqrt{2(2+1) - (-1)((-1)-1)}\,\hbar|2, -1-1>$$
$$= \sqrt{2(3) - (-1)(-2)}\,\hbar|2, -2> = \sqrt{6-2}\,\hbar|2, -2> = \sqrt{4}\,\hbar|2, -2>$$
$$= 2\hbar|2, -2> .$$

Possibilities, Probabilities, Expectation Value, Uncertainty, and Time Dependence

Examples 9–16 through 9–21 are intended to interface, apply, and extend calculations developed previously to eigenstates of angular momentum. As indicated earlier, a state vector will be a linear combination of eigenstates, which this development should reinforce. Examples 9–16 through 9–21 all refer to the $t = 0$ state vector

$$|\psi(t=0)> = A\big(|2, 1> +3|1, -1>\big) \qquad (9-32)$$

is is a linear combination of two eigenstates.

Example 9–16: Normalize the state vector of equation (9–32).

There are two eigenstates, so we can work in a two dimensional subspace. We can model the first eigenstate $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the second $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then the state vector can be written

$$|\psi(0)> = A\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = A\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Another way to look at it is the state vector is two dimensional with one part the first eigenstate and three parts the second eigenstate. This technique makes the normalization calculation, and a number of others, particularly simple.

$$(1, 3)A^*A\begin{pmatrix} 1 \\ 3 \end{pmatrix} = |A|^2(1+9) = 10|A|^2 = 1$$

$$\Rightarrow \quad A = \frac{1}{\sqrt{10}} \quad \Rightarrow \quad |\psi(0)> = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{10}}\big(|2, 1> +3|1, -1>\big).$$

Example 9–17: What are the possibilities and probabilities of a measurement of $\mathcal{L}^2$?

The possibilities are the eigenvalues. There are two eigenstates, each with its own eigenvalue. If we measure and put the system into the first eigenstate, we measure the state corresponding to the quantum number $l = 2$, which has the eigenvalue $\hbar^2\,l(l+1) = \hbar^2\,2(2+1) = 6\hbar^2$. If

we measure and place the state vector into the second eigenstate corresponding to the quantum number $l = 1$, the eigenvalue measured is $\hbar^2\, l(l+1) = \hbar^2\, 1(1+1) = 2\hbar^2$.

Since the state function is normalized,

$$P(\mathcal{L}^2 = 6\hbar^2) = \big|<\psi|\psi_i>\big|^2 = \left|\frac{1}{\sqrt{10}}(1,\ 3)\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{10}\,|1+0|^2 = \frac{1}{10}.$$

$$P(\mathcal{L}^2 = 2\hbar^2) = \big|<\psi|\psi_i>\big|^2 = \left|\frac{1}{\sqrt{10}}(1,\ 3)\begin{pmatrix}0\\1\end{pmatrix}\right|^2 = \frac{1}{10}\,|0+3|^2 = \frac{9}{10}.$$

Example 9–18: What are the possibilities and probabilities of a measurement of $\mathcal{L}_z$?

For exactly the same reasons, the possible results of a measurement are $m = 1 \ \Rightarrow \ \hbar$ is the first eigenvalue and $m = -1 \ \Rightarrow \ -\hbar$ is the second possible eigenvalue. Using exactly the same math,

$$P(\mathcal{L}_z = \hbar) = \frac{1}{10}, \qquad P(\mathcal{L}_z = -\hbar) = \frac{9}{10}.$$

Example 9–19: What is the expectation value of $\mathcal{L}^2$?

$$<\mathcal{L}^2> = \sum P(\alpha_i)\alpha_i = \frac{1}{10}6\hbar^2 + \frac{9}{10}2\hbar^2 = \frac{6}{10}\hbar^2 + \frac{18}{10}\hbar^2 = \frac{24}{10}\hbar^2 = 2.4\hbar^2.$$

Example 9–20: What is the uncertainty of $\mathcal{L}^2$?

$$\triangle\mathcal{L}^2 = \sqrt{\sum P(\alpha_i)\big(\alpha_i - <\mathcal{L}^2>\big)^2} = \left[\frac{1}{10}\big(6\hbar^2 - 2.4\hbar^2\big)^2 + \frac{9}{10}\big(2\hbar^2 - 2.4\hbar^2\big)^2\right]^{1/2}$$

$$= \hbar^2\left[\frac{1}{10}(3.6)^2 + \frac{9}{10}\big(-0.4\big)^2\right]^{1/2} = \hbar^2\big[1.296 + 0.144\big]^{1/2} = \hbar^2\sqrt{1.44}$$

$$= 1.2\hbar^2.$$

Example 9–21: What is the time dependent state vector?

$$|\psi(t)> \ = \sum |j><j|\psi(0)> e^{-iE_jt/\hbar}$$

$$= \begin{pmatrix}1\\0\end{pmatrix}(1,\ 0)\frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix}e^{-iE_1t/\hbar} + \begin{pmatrix}0\\1\end{pmatrix}(0,\ 1)\frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix}e^{-iE_2t/\hbar}$$

$$= \frac{1}{\sqrt{10}}\begin{pmatrix}1\\0\end{pmatrix}e^{-iE_1t/\hbar} + \frac{3}{\sqrt{10}}\begin{pmatrix}0\\1\end{pmatrix}e^{-iE_2t/\hbar}$$

$$= \frac{1}{\sqrt{10}}|2,\ 1> e^{-iE_1t/\hbar} + \frac{3}{\sqrt{10}}|1,\ -1> e^{-iE_2t/\hbar}$$

which is as far as we can go with the given information. We need a specific system and an energy operator, a Hamiltonian, to attain specific E_i.

Angular Momentum Operators in Spherical Coordinates

The conservation of angular momentum, or rotational invariance, implies circular or spherical symmetry. We want to examine spherical symmetry, because spherical symmetry is often a reasonable assumption for simple physical systems. We will assume a hydrogen atom is spherically symmetric, for instance. Remember in spherical coordinates,

$$
\begin{aligned}
x &= r\sin\theta\cos\phi, & r &= \left(x^2 + y^2 + z^2\right)^{1/2} \\
y &= r\sin\theta\sin\phi, & \theta &= \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right) \\
z &= r\cos\theta, & \phi &= \tan^{-1}\left(y/x\right).
\end{aligned}
$$

From these it follows that position space representations in spherical coordinates are

$$
\mathcal{L}_x = i\hbar\left(\sin\phi\,\frac{\partial}{\partial\theta} + \cos\phi\,\cot\theta\,\frac{\partial}{\partial\phi}\right),
$$

$$
\mathcal{L}_y = i\hbar\left(-\cos\phi\,\frac{\partial}{\partial\theta} + \sin\phi\,\cot\theta\,\frac{\partial}{\partial\phi}\right),
$$

$$
\mathcal{L}_z = -i\hbar\frac{\partial}{\partial\phi}, \tag{9-32}
$$

$$
\mathcal{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right), \tag{9-33}
$$

$$
\mathcal{L}_\pm = \pm\hbar\,e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\,\frac{\partial}{\partial\phi}\right). \tag{9-34}
$$

Example 9–22: Derive equation (9–32).

From equation (9–2),

$$
\mathcal{L}_z = i\hbar\left(-x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right).
$$

We can develop the desired partial differentials from the relation between azimuthal angle and position coordinates, or

$$
\phi = \tan^{-1}\left(y/x\right) \quad\Rightarrow\quad y = x\tan\phi
$$

$$
\Rightarrow \quad \frac{\partial y}{\partial\phi} = x\,\partial\left(\tan\phi\right) = x\sec^2\phi = \frac{x}{\cos^2\phi}
$$

$$
\Rightarrow \quad \partial y = \frac{x\,\partial\phi}{\cos^2\phi}.
$$

The same relation gives us

$$
x = \frac{y}{\tan\phi} = y\frac{\cos\phi}{\sin\phi} = y\cos\phi\,\sin^{-1}\phi
$$

$$
\Rightarrow \quad \frac{\partial x}{\partial\phi} = y\left(-\sin\phi\,\sin^{-1}\phi + \cos\phi\,(-1)\sin^{-2}\phi\cos\phi\right)
$$

$$= -y\left(1 + \frac{\cos^2\phi}{\sin^2\phi}\right) = -y\left(\frac{\sin^2\phi + \cos^2\phi}{\sin^2\phi}\right) = -\frac{y}{\sin^2\phi}$$

$$\Rightarrow \quad \partial x = -\frac{y\,\partial\phi}{\sin^2\phi}.$$

Using the partial differentials in the Cartesian formulation for the z component of angular momentum,

$$\mathcal{L}_z = i\hbar\left(-x\,\cos^2\frac{\partial}{x\,\partial\phi} + y\left(-\sin^2\phi\frac{\partial x}{y\partial\phi}\right)\right)$$

$$= -i\hbar\left(\cos^2\phi + \sin^2\phi\right)\frac{\partial}{\partial\phi}$$

$$= -i\hbar\frac{\partial}{\partial\phi}.$$

Example 9–23: Given the spherical coordinate representations of $\mathcal{L}_x$ and $\mathcal{L}_y$, show equation (9–34) is true for $\mathcal{L}_+$.

$$\mathcal{L}_+ = \mathcal{L}_x + i\mathcal{L}_y$$

$$= i\hbar\left(\sin\phi\,\frac{\partial}{\partial\theta} + \cos\phi\,\cot\theta\,\frac{\partial}{\partial\phi}\right) + i\left[i\hbar\left(-\cos\phi\,\frac{\partial}{\partial\theta} + \sin\phi\,\cot\theta\,\frac{\partial}{\partial\phi}\right)\right]$$

$$= \hbar\left[i\sin\phi\,\frac{\partial}{\partial\theta} + i\cos\phi\,\cot\theta\,\frac{\partial}{\partial\phi} + \cos\phi\,\frac{\partial}{\partial\theta} - \sin\phi\,\cot\theta\,\frac{\partial}{\partial\phi}\right]$$

$$= \hbar\left[(\cos\phi + i\sin\phi)\,\frac{\partial}{\partial\theta} + (i\cos\phi - \sin\phi)\cot\theta\,\frac{\partial}{\partial\phi}\right]$$

$$= \hbar\left[(\cos\phi + i\sin\phi)\,\frac{\partial}{\partial\theta} + i\,(\cos\phi + i\sin\phi)\cot\theta\,\frac{\partial}{\partial\phi}\right]$$

$$= \hbar\left[\left(e^{i\phi}\right)\frac{\partial}{\partial\theta} + i\left(e^{i\phi}\right)\cot\theta\,\frac{\partial}{\partial\phi}\right]$$

$$= \hbar^2\,e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right).$$

An outline of the derivations of the all components and square of angular momentum in spherical coordinates is included in Ziock[5]. These calculations can be "messy" by practical standards.

Special Functions Used for the Hydrogen Atom

Two special functions are particularly useful in describing a hydrogen atom assumed to have spherical symmetry. These are **spherical harmonics** and **Associated Laguerre functions**. The plan will be to separate the Schrodinger equation into radial and angular equations. The solutions to the radial equation can be expressed in terms of associated Laguerre polynomials, which we will examine in the next chapter. The solutions to the angular equation can be expressed in terms of spherical harmonic functions, which we will examine in the next section. Spherical harmonics are closely related to a third special function, **Legendre functions**. They are so closely related, the spherical harmonics can be expressed in terms of **associated Legendre polynomials**.

[5] Ziock *Basic Quantum Mechanics* (John Wiley & Sons, New York, 1969), pp. 91–94

The name spherical harmonic comes from the geometry the functions naturally describe, spherical, and the fact any solution of Laplace's equation is known as harmonic. Picture a ball. The surface may be smooth, which is likely the first picture you form. Put a rubber band around the center, and you get a minima at the center and bulges, or maxima, in the top and bottom half. Put rubber bands on the circumference, like lines of longitude, and you get a different pattern of maxima and minima. We could imagine other, more complex patterns of maxima and minima. When these maxima and minima are symmetric with respect to an origin, the center of the ball, Legendre functions, associated Legendre functions, and spherical harmonics provide useful descriptions.
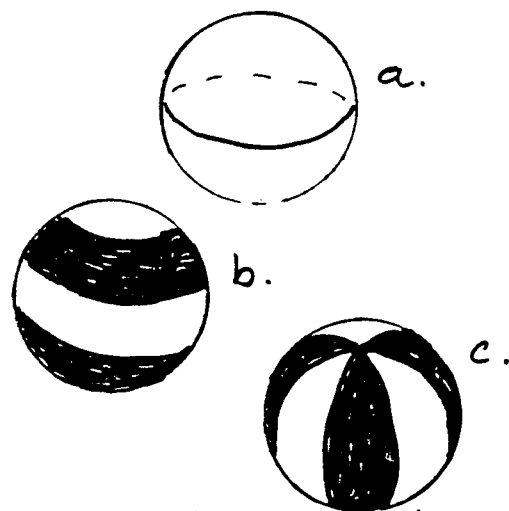

FIGURE 11-4.a) SPHERE, b) ZONAL HARMONIC, c) SECTORAL HARMONIC.

Properties that makes these special functions particularly useful is they are orthogonal and complete. Any set that is orthogonal can be made orthonormal. We have used orthonormality in a number of calculations, and the property of orthonormality continues to be a practical necessity. They are also complete in the sense any phenomenon can be described by an appropriate linear combination. Other complete sets of orthonormal functions we have encountered are sines and cosines for the square well, and Hermite polynomials for the SHO. A set of complete, orthonormal functions is equivalent to a linear vector space; these special functions are different manifestations of a complex linear vector space.

Spherical Harmonics

The ket $|l, m>$ is an eigenstate of the commuting operators $\mathcal{L}^2$ and $\mathcal{L}_z$, but it is an abstract eigenstate. That $|l, m>$ is abstract is irrelevant for the eigenvalues, since eigenvalues are properties of the operators. We would, however, like a representation useful for description for the eigenvectors. Per chapter 4, we can form an inner product with an abstract vector to attain a representation. Using a guided choice, the angles of spherical coordinate system will yield an appropriate representation. Just as $<x|g> = g(x)$, we will write

$$<\theta, \phi|l, m> = Y_{l,m}(\theta, \phi).$$

The functions of polar and azimuthal angles, $Y_{l,m}(\theta, \phi)$, are the spherical harmonics.

The spherical harmonics are related so strongly to the geometry of the current problem, they can be derived from the spherical coordinate form of the eigenvalue/eigenvector equation (11–29), $\mathcal{L}_z|l, m> = m\hbar|l, m>$, and use of the raising/lowering operator equation (11–31).

Using the spherical coordinate system form of the operator and the functional forms of the eigenstates, equation (11–29) is

$$-i\hbar\frac{\partial}{\partial\phi}Y_{l,m}(\theta, \phi) = m\hbar Y_{l,m}(\theta, \phi).$$

We are going to assume the spherical harmonics are separable, that they can be expressed as a product of a function of θ and a second function of ϕ, or

$$Y_{l,m}(\theta, \phi) = f_{l,m}(\theta)\, g_{l,m}(\phi).$$

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$$Y_{l,m}(\theta, \phi) = f_{l,m}(\theta) \, g_{l,m}(\phi).$$

Using this in the differential equation,

$$-i\hbar \frac{\partial}{\partial \phi} f_{l,m}(\theta) \, g_{l,m}(\phi) = m\hbar \, f_{l,m}(\theta) \, g_{l,m}(\phi)$$

$$\Rightarrow \quad -i \, f_{l,m}(\theta) \frac{\partial}{\partial \phi} g_{l,m}(\phi) = m f_{l,m}(\theta) \, g_{l,m}(\phi)$$

$$\Rightarrow \quad -i \frac{\partial}{\partial \phi} g_{l,m}(\phi) = m \, g_{l,m}(\phi)$$

$$\Rightarrow \quad \frac{\partial g_{l,m}(\phi)}{g_{l,m}(\phi)} = im \, \partial\phi$$

$$\Rightarrow \quad \ln g_{l,m}(\phi) = im\phi$$

$$\Rightarrow \quad g_{l,m}(\phi) = e^{im\phi}.$$

Notice the exponential has no dependence on l, so we can write

$$g_m(\phi) = e^{im\phi}, \tag{9-35}$$

which is the azimuthal dependence.

Remember that there is a top and bottom to the ladder for a given l. The top of the ladder is at $m = l$. If we act on an eigenstate on the top of the ladder, we get zero, meaning

$$\mathcal{L}_+ |l, \, l> = 0,$$

Using the spherical coordinate forms of the raising operator and separated eigenstate including equation (9–35), this is

$$\hbar \, e^{i\phi} \left[\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right] f_{l,l}(\theta) \, e^{il\phi} = 0$$

$$\Rightarrow \quad e^{il\phi} \frac{\partial}{\partial \theta} f_{l,l}(\theta) + i \, f_{l,l}(\theta) \cot \theta \, (il) e^{il\phi} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial \theta} f_{l,l}(\theta) - l \, f_{l,l}(\theta) \cot \theta = 0.$$

The solution to this is $f_{l,l}(\theta) = A(\sin \theta)^l$. To see that it is a solution,

$$\frac{\partial}{\partial \theta} f_{l,l}(\theta) = \frac{\partial}{\partial \theta} A(\sin \theta)^l = A l (\sin \theta)^{l-1} \cos \theta,$$

and substituting this in the differential equation,

$$A l (\sin \theta)^{l-1} \cos \theta - l \left[A(\sin \theta)^l \right] \frac{\cos \theta}{\sin \theta} = A l (\sin \theta)^{l-1} \left[\cos \theta - \sin \theta \frac{\cos \theta}{\sin \theta} \right] = 0.$$

So the unnormalized form of the $m = l$ spherical harmonics is

$$Y_{l,m}(\theta, \phi) = A(\sin \theta)^l e^{im\phi}. \tag{9-36}$$

Example 9–24 derives $Y_{1,1}(\theta, \phi)$ starting with equation (9–36).

So how do we get the spherical harmonics for which $m \neq l$? The answer is to attain a $Y_{l,l}(\theta, \phi)$ and operate on it with the lowering operator. Example 9–25 derives $Y_{1,0}(\theta, \phi)$ in this manner.

One comment before we proceed. The spherical harmonics of equation (9–36) can be made orthonormal, so we need to calculate the normalization constants, A for each $Y_{l,m}(\theta, \phi)$. Having selected a representation, this is most easily approached by the appropriate form of integration. The appropriate form of integration for spherical angles is with respect to solid angle, $d\Omega = \sin\theta d\theta d\phi$, or

$$\int Y_{l,m}^*(\theta, \phi) \, Y_{l,m}(\theta, \phi) \, d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \big| Y_{l,m}(\theta, \phi)\big|^2 = 1,$$

which will also be illustrated in examples 9–24 and 9–25. These and other special functions are addressed in most mathematical physics texts including Arken[6] and Mathews and Walker[7].

A list of the first few spherical harmonics is

$$Y_{0,0}(\theta, \phi) = \frac{1}{4\pi}$$

$$Y_{2,0}(\theta, \phi) = \sqrt{\frac{5}{16\pi}} \left(3\cos^2\theta - 1\right)$$

$$Y_{1,\pm 1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}$$

$$Y_{3,\pm 3}(\theta, \phi) = \sqrt{\frac{35}{64\pi}} \sin^3\theta \, e^{\pm 3i\phi}$$

$$Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{3,\pm 2}(\theta, \phi) = \sqrt{\frac{105}{32\pi}} \sin^2\theta \, \cos\theta \, e^{\pm 2i\phi}$$

$$Y_{2,\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{\pm 2i\phi}$$

$$Y_{3,\pm 1}(\theta, \phi) = \sqrt{\frac{21}{64\pi}} \sin\theta \left(5\cos^2\theta - 1\right) e^{\pm i\phi}$$

$$Y_{2,\pm 1}(\theta, \phi) = \sqrt{\frac{15}{8\pi}} \sin\theta \, \cos\theta \, e^{\pm i\phi}$$

$$Y_{3,0}(\theta, \phi) = \sqrt{\frac{7}{16\pi}} \left(5\cos^3\theta - 3\cos\theta\right)$$

Table 9 − 1. The First Sixteen Spherical Harmonic Functions.

A few comments about the list are appropriate. First, notice the symmetry about $m = 0$. For example, $Y_{2,1}$ and $Y_{2,-1}$ are exactly the same except for the sign of the argument of the exponential. Second, notice the $Y_{l,0}$ are independent of ϕ. When $m = 0$, the spherical harmonic functions are constant with respect to azimuthal angle. Next, per the previous sentences, it is common to refer to spherical harmonic functions without explicitly indicating that the arguments are polar and azimuthal angles. Finally, and most significantly, some texts will use a negative sign leading the spherical harmonic functions for which $m < 0$. This is a different choice of phase. We will use the convention denoted in table 9–1, where all spherical harmonics are positive. Used consistently, either choice is reasonable and both choices have advantages and disadvantages.

Figure 9–2 illustrates the functional form of the first 16 spherical harmonic functions. Note that the radial coordinate has not yet been addressed. Angular distribution is all that is being

[6] Arfken *Mathematical Methods for Physicists* (Academic Press, New York, 1970), 2nd ed., chapters 9–13.

[7] Mathews and Walker *Mathematical Methods of Physics* (The Benjamin/Cummings Publishing Co., Menlo Park, California, 1970), 2nd ed., chapter 7.
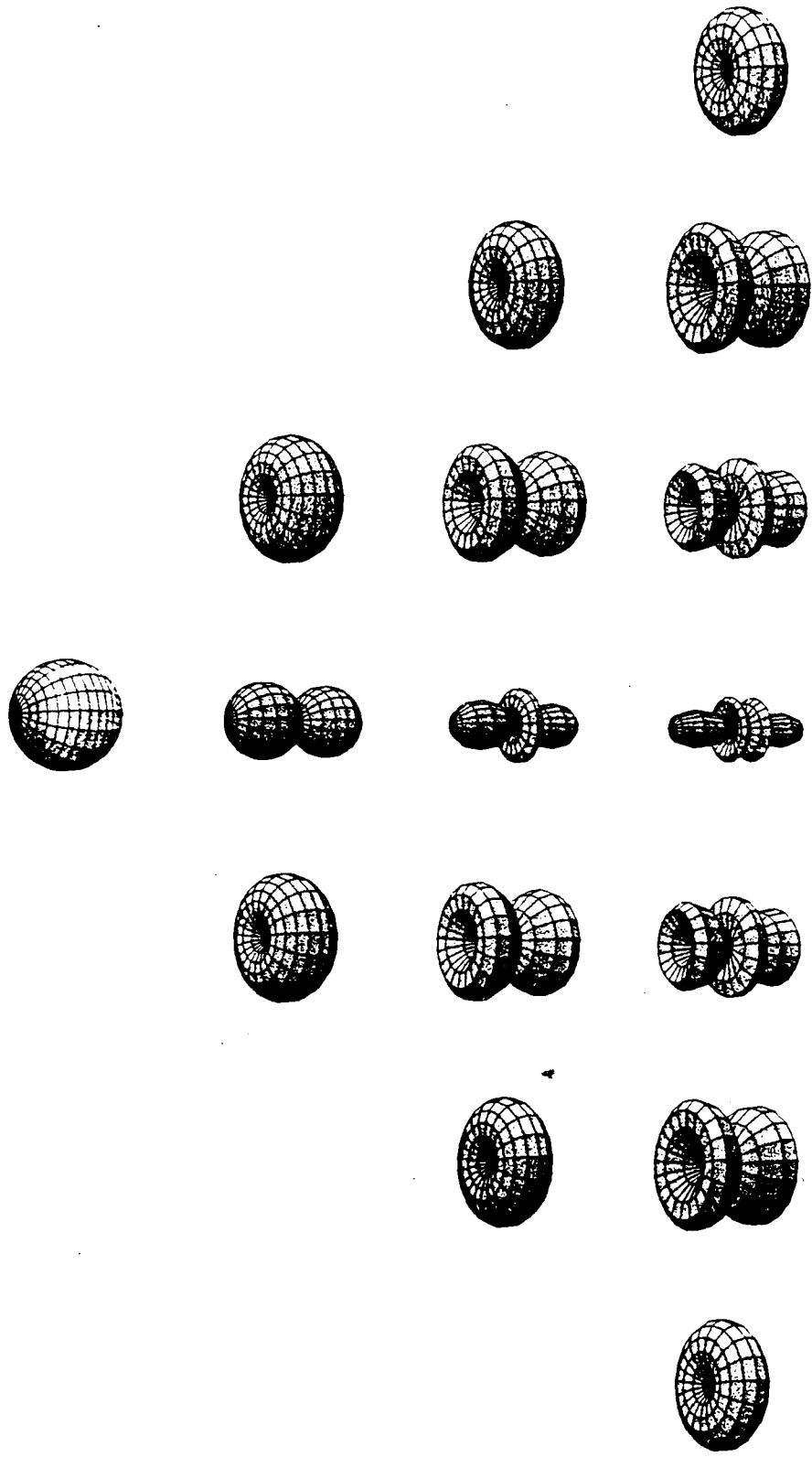
illustrated. The radial coordinate will be examined in the next chapter. The size of any of the individual pictures in figure 9–2 is arbitrary; they could be very large or very small. We assume a radius of one unit to draw the sketches. In other words, you can look at the smooth sphere of $Y_{0,0}$ as having radius one unit, and the relative sizes of other spherical harmonic functions are comparable on the same radial scale.

Figure $9 - 2$. Illustrations of the First Sixteen Spherical Harmonic Functions.

There is a technique here we want to exploit when we address radial functions. The spherical harmonics are orthonormal so are normalized. The figures represent spherical harmonics of magnitude one, multiplied by one, so remain orthonormal. We want the radial functions to be orthonormal, or individually to have magnitude one. Just as we have assumed a one unit radius to draw the figures here, if we multiply two quantities of magnitude one, we attain a product of magnitude one. If the angular function and radial function are individually normalized, the product function will be normalized as well.

Example 9–24: Show $Y_{l,l} = A\big(\sin\theta\big)^l e^{im\phi}$ yields the normalized $Y_{1,1}$ of table 9–1.

$$Y_{1,1} = A\big(\sin\theta\big)^1 e^{i(1)\phi} = A\sin\theta\, e^{i\phi}.$$

To normalize this,

$$1 = \int (Y_{1,1})^* \, Y_{1,1} \, d\Omega = \int A^* \sin\theta\, e^{-i\phi} \, A\sin\theta\, e^{i\phi} \, d\Omega$$

$$= |A|^2 \int \sin^2\theta\, e^0 \, d\Omega = |A|^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin^2\theta \, \sin\theta$$

$$= |A|^2 \int_0^{\pi} d\theta \, \sin^3\theta \int_0^{2\pi} d\phi = 2\pi |A|^2 \int_0^{\pi} d\theta \, \sin^3\theta$$

$$= 2\pi |A|^2 \left[-\frac{1}{3}\cos\theta\big(\sin^2\theta + 2\big) \right]_0^{\pi} = \frac{2\pi}{3}|A|^2 \left[\cos\theta\big(\sin^2\theta + 2\big) \right]_{\pi}^{0}$$

$$= \frac{2\pi}{3}|A|^2 \left[\cos(0)\big(\sin^2(0) + 2\big) - \cos(\pi)\big(\sin^2(\pi) + 2\big) \right]$$

$$= \frac{2\pi}{3}|A|^2 \left[(1)(2) - (-1)(2) \right] = \frac{2\pi}{3}|A|^2 [4]$$

$$\Rightarrow \quad \frac{8\pi}{3}|A|^2 = 1 \quad \Rightarrow \quad A = \sqrt{\frac{3}{8\pi}}$$

$$\Rightarrow \quad Y_{1,1} = \sqrt{\frac{3}{8\pi}} \, \sin\theta\, e^{i\phi},$$

which is identical to $Y_{1,1}$ in table 9–1.

Example 9–25: Derive $Y_{1,0}$ from the result of the previous example using the lowering operator.

A lowering operator acting on an abstract eigenstate is $\mathcal{L}_- |l,\, m> \, = B\,|l,\, m-1>,$ where B is a proportionality constant. Using the spherical angle representation on the eigenstate $Y_{1,1}$, this eigenvalue/eigenvector equation is

$$-\hbar\, e^{-i\phi} \left(\frac{\partial}{\partial\theta} - i\cot\theta \frac{\partial}{\partial\phi} \right) Y_{1,1} = B\, Y_{1,0},$$

where B is the eigenvalue. Using the unnormalized form of $Y_{1,1}$, we have

$$B\, Y_{1,0} = -\hbar\, e^{-i\phi} \left(\frac{\partial}{\partial\theta} - i\cot\theta \frac{\partial}{\partial\phi} \right) A\sin\theta\, e^{i\phi}$$

$$= -A\hbar\, e^{-i\phi} \left(e^{i\phi} \frac{\partial}{\partial\theta}\sin\theta - i\cot\theta \, \sin\theta \frac{\partial}{\partial\phi} e^{i\phi} \right)$$

$$= -A\hbar\, e^{-i\phi} \left(e^{i\phi}\cos\theta - i\frac{\cos\theta}{\sin\theta}\sin\theta (i)e^{i\phi} \right)$$

$$= -A\hbar\big(\cos\theta + \cos\theta\big) = -2A\hbar\big(\cos\theta\big)$$

$$\Rightarrow \quad Y_{1,0} = C \cos \theta,$$

where all constants have been combined to form C, which becomes simply a normalization constant. We normalize this using the same procedure as the previous example,

$$1 = \int C^* \cos \theta \, C \cos \theta \, d\Omega = |C|^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \cos^2 \theta \, \sin \theta$$

$$= 2\pi |C|^2 \int_0^{\pi} d\theta \, \cos^2 \theta \, \sin \theta = 2\pi |C|^2 \left[-\frac{\cos^3 \theta}{3} \right]_0^{\pi} = \frac{2\pi}{3} |C|^2 \left[\cos^3 \theta \right]_{\pi}^0$$

$$= \frac{2\pi}{3} |C|^2 \left[\cos^3(0) - \cos^3(\pi) \right] = \frac{2\pi}{3} |C|^2 \left[1 - (-1) \right] = \frac{2\pi}{3} |C|^2 [2]$$

$$\Rightarrow \quad \frac{4\pi}{3} |C|^2 = 1 \quad \Rightarrow \quad C = \sqrt{\frac{3}{4\pi}}$$

$$\Rightarrow \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos \theta,$$

which is identical to $Y_{1,0}$ as listed in table 9–1.

Generating Function for Spherical Harmonics

A generating functions for higher index spherical harmonics is

$$Y_{l,m}(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_{l,m}(\cos \theta) \, e^{im\phi}, \qquad m \geq 0,$$

and

$$Y_{l,-m}(\theta, \phi) = Y_{l,m}^*(\theta, \phi), \qquad m < 0,$$

where the $P_{l,m}(\cos \theta)$ are associated Legendre polynomials. Associated Legendre polynomials can be generated from Legendre polynomials using

$$P_{l,m}(u) = (-1)^m \sqrt{(1-u^2)^m} \frac{d^m}{du^m} P_l(u),$$

where the $P_l(u)$ are Legendre polynomials. Legendre polynomials can be generated using

$$P_l(u) = \frac{(-1)^l}{2^l l!} \frac{d^l}{du^l} (1-u^2)^l.$$

Notice the generating function for spherical harmonics contains the restriction $m \geq 0$. Our strategy to attain spherical harmonics with $m < 0$ will be to form them from the adjoint of the corresponding spherical harmonic with $m > 0$ as indicated. The advantage of this strategy is we do not need to consider associated Legendre polynomials with $m < 0$, though those also have meaning and can be attained using

$$P_{l,-m}(u) = \frac{(l-m)!}{(l+m)!} P_{l,m}(u),$$

in our phase scheme.