

# CHAPTER 8: MATRICES and DETERMINANTS

The material in this chapter will be covered in your Linear Algebra class ([Math 254 at Mesa](#)).

## SECTION 8.1: MATRICES and SYSTEMS OF EQUATIONS

### PART A: MATRICES

A matrix is basically an organized box (or “array”) of numbers (or other expressions). In this chapter, we will typically assume that our matrices contain only numbers.

#### Example

Here is a matrix of size  $2 \times 3$  (“2 by 3”), because it has 2 rows and 3 columns:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

The matrix consists of 6 entries or elements.

In general, an  $m \times n$  matrix has  $m$  rows and  $n$  columns and has  $mn$  entries.

#### Example

Here is a matrix of size  $2 \times 2$  (an order 2 square matrix):

$$\begin{bmatrix} \mathbf{4} & -1 \\ 3 & \mathbf{2} \end{bmatrix}$$

The **boldfaced** entries lie on the main diagonal of the matrix.  
(The other diagonal is the skew diagonal.)

**PART B: THE AUGMENTED MATRIX FOR A SYSTEM OF LINEAR EQUATIONS**Example

Write the augmented matrix for the system: 
$$\begin{cases} 3x + 2y + z = 0 \\ -2x - z = 3 \end{cases}$$

Solution

Preliminaries:

Make sure that the equations are in (what we refer to now as) standard form, meaning that ...

- All of the variable terms are on the left side (with  $x$ ,  $y$ , and  $z$  ordered alphabetically), and
- There is only one constant term, and it is on the right side.

Line up like terms vertically.

Here, we will rewrite the system as follows:

$$\begin{cases} 3x + 2y + z = 0 \\ -2x \quad - z = 3 \end{cases}$$

(Optional) Insert “1”s and “0”s to clarify coefficients.

$$\begin{cases} 3x + 2y + 1z = 0 \\ -2x + 0y - 1z = 3 \end{cases}$$

**Warning:** Although this step is not necessary, people often mistake the coefficients on the  $z$  terms for “0”s.


Write the augmented matrix:

Coefficients of			Right
$x$	$y$	$z$	sides
3	2	1	0
-2	0	-1	3

<u>Coefficient matrix</u>	<u>Right-hand side (RHS)</u>
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Augmented matrix

We may refer to the first three columns as the  $x$ -column, the  $y$ -column, and the  $z$ -column of the coefficient matrix.

**Warning:** If you do not insert “1”s and “0”s, you may want to read the equations and fill out the matrix row by row in order to minimize the chance of errors. Otherwise, it may be faster to fill it out column by column.

The augmented matrix is an efficient representation of a system of linear equations, although the names of the variables are hidden.

**PART C: ELEMENTARY ROW OPERATIONS (EROs)**

Recall from Algebra I that equivalent equations have the same solution set.

**Example**

Solve:  $2x - 1 = 5$

$$2x - 1 = 5$$

$$2x = 6$$

$$x = 3 \Rightarrow \text{Solution set is } \{3\}.$$

To solve the first equation, we write a sequence of equivalent equations until we arrive at an equation whose solution set is obvious.

The steps of adding 1 to both sides of the first equation and of dividing both sides of the second equation by 2 are like “legal chess moves” that allowed us to maintain equivalence (i.e., to preserve the solution set).

Similarly, equivalent systems have the same solution set.

Elementary Row Operations (EROs) represent the legal moves that allow us to write a sequence of row-equivalent matrices (corresponding to equivalent systems) until we obtain one whose corresponding solution set is easy to find. There are three types of EROs:

1) **Row Reordering**Example

Consider the system: 
$$\begin{cases} 3x - y = 1 \\ x + y = 4 \end{cases}$$

If we switch (i.e., interchange) the two equations, then the solution set is not disturbed:

$$\begin{cases} x + y = 4 \\ 3x - y = 1 \end{cases}$$

This suggests that, when we solve a system using augmented matrices,  
...

**We can switch any two rows.**

Before:

$$\begin{array}{l} R_1 \left[ \begin{array}{cc|c} 3 & -1 & 1 \end{array} \right] \\ R_2 \left[ \begin{array}{cc|c} 1 & 1 & 4 \end{array} \right] \end{array}$$

Here, we switch rows  $R_1$  and  $R_2$ , which we denote by:  $R_1 \leftrightarrow R_2$

After:

$$\begin{array}{l} \text{new } R_1 \left[ \begin{array}{cc|c} 1 & 1 & 4 \end{array} \right] \\ \text{new } R_2 \left[ \begin{array}{cc|c} 3 & -1 & 1 \end{array} \right] \end{array}$$

**In general, we can reorder the rows of an augmented matrix in any order.**

Warning: Do **not** reorder columns; in the coefficient matrix, that will change the order of the corresponding variables.

2) **Row Rescaling**Example

Consider the system: 
$$\begin{cases} \frac{1}{2}x + \frac{1}{2}y = 3 \\ y = 4 \end{cases}$$

If we multiply “through” both sides of the first equation by 2, then we obtain an equivalent equation and, overall, an equivalent system:

$$\begin{cases} x + y = 6 \\ y = 4 \end{cases}$$

This suggests that, when we solve a system using augmented matrices,  
...

**We can multiply (or divide) “through” a row by any nonzero constant.**

Before:

$$\begin{array}{l} R_1 \left[ \begin{array}{cc|c} 1/2 & 1/2 & 3 \end{array} \right] \\ R_2 \left[ \begin{array}{cc|c} 0 & 1 & 4 \end{array} \right] \end{array}$$

Here, we multiply through  $R_1$  by 2, which we denote by:  $R_1 \leftarrow 2 \cdot R_1$ , or  $(\text{new } R_1) \leftarrow 2 \cdot (\text{old } R_1)$

After:

$$\begin{array}{l} \text{new } R_1 \left[ \begin{array}{cc|c} 1 & 1 & 6 \end{array} \right] \\ R_2 \left[ \begin{array}{cc|c} 0 & 1 & 4 \end{array} \right] \end{array}$$

### 3) Row Replacement

(This is perhaps poorly named, since ERO types 1 and 2 may also be viewed as “row replacements” in a literal sense.)

When we solve a system using augmented matrices, ...

**We can add a multiple of one row to another row.**

Technical Note: This combines ideas from the Row Rescaling ERO and the Addition Method from Chapter 7.

#### Example

Consider the system: 
$$\begin{cases} x + 3y = 3 \\ -2x + 5y = 16 \end{cases}$$

Before:

$$\begin{array}{l} R_1 \left[ \begin{array}{cc|c} 1 & 3 & 3 \end{array} \right] \\ R_2 \left[ \begin{array}{cc|c} -2 & 5 & 16 \end{array} \right] \end{array}$$

Note: We will sometimes boldface items for purposes of clarity.

It turns out that we want to add twice the first row to the second row, because we want to replace the “ $-2$ ” with a “0.”

We denote this by:

$$R_2 \leftarrow R_2 + 2 \cdot R_1, \text{ or } (\text{new } R_2) \leftarrow (\text{old } R_2) + 2 \cdot R_1$$

old $R_2$	$-2$	$5$		$16$
$+2 \cdot R_1$	$2$	$6$		$6$
new $R_2$	<b>0</b>	<b>11</b>		<b>22</b>

Warning: It is **highly** advised that you write out the table!  
People often rush through this step and make mechanical errors.

Warning: Although we can also **subtract** a multiple of one row from another row, we generally prefer to **add**, instead, even if that means that we multiply “through” a row by a negative number. Errors are common when people subtract.

After:

$$\begin{array}{l} \text{old } R_1 \\ \text{new } R_2 \end{array} \left[ \begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 11 & 22 \end{array} \right]$$

Note: In principle, you could replace the old  $R_1$  with the rescaled version, but it turns out that we like having that “1” in the upper left hand corner!

If matrix  $B$  is obtained from matrix  $A$  after applying one or more EROs, then we call  $A$  and  $B$  row-equivalent matrices, and we write  $A \sim B$ .

Example

$$\left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 7 & 8 & 9 \end{array} \right] \sim \left[ \begin{array}{cc|c} 7 & 8 & 9 \\ 1 & 2 & 3 \end{array} \right]$$

Row-equivalent augmented matrices correspond to equivalent systems, assuming that the underlying variables (corresponding to the columns of the coefficient matrix) stay the same and are in the same order.



**PART D: GAUSSIAN ELIMINATION (WITH BACK-SUBSTITUTION)**

This is a method for solving systems of linear equations.

**Historical Note:** This method was popularized by the great mathematician Carl Gauss, but the Chinese were using it as early as 200 BC.

**Steps**

Given a square system (i.e., a system of  $n$  linear equations in  $n$  unknowns for some  $n \in \mathbf{Z}^+$ ; we will consider other cases later) ...

- 1) Write the **augmented matrix**.
- 2) Use **EROs** to write a sequence of row-equivalent matrices until you get one in the form:

$$\left[ \begin{array}{cccc|c} 1 & & & & ? \\ & 1 & & & ? \\ & & \ddots & & ? \\ 0 & & & \ddots & ? \\ & & & & 1 \end{array} \right]$$

If we begin with a square system, then all of the coefficient matrices will be square.

**We want “1”s along the main diagonal and “0”s all below.**

The other entries are “wild cards” that can potentially be any real numbers.

This is the form that we are aiming for. Think of this as “checkmate” or “the top of the jigsaw puzzle box” or “the TARGET” (like in a trig ID).

**Warning:** As you perform EROs and this form crystallizes and emerges, you usually want to avoid “undoing” the good work you have already done. For example, if you get a “1” in the upper left corner, you usually want to preserve it. For this reason, it is often a good strategy to **“correct” the columns from left to right** (that is, from the leftmost column to the rightmost column) in the coefficient matrix. Different strategies may work better under different circumstances.

**For now, assume that we have succeeded in obtaining this form; this means that the system has exactly one solution.**

What if it is impossible for us to obtain this form? We shall discuss this matter later (starting with [Notes 8.21](#)).

3) Write the **new system**, complete with variables.

This system will be equivalent to the given system, meaning that they share the same solution set. The new system should be **easy to solve** if you ...

4) Use **back-substitution** to find the values of the unknowns.

We will discuss this later.

5) Write the solution as an ordered  $n$ -tuple (pair, triple, etc.).

6) **Check** the solution in the given system. (Optional)

**Warning: This check will not capture other solutions if there are, in fact, infinitely many solutions.**

**Technical Note: This method actually works with complex numbers in general.**

**Warning: You may want to quickly check each of your steps before proceeding. A single mistake can have massive consequences that are difficult to correct.**

Example

$$\text{Solve the system: } \begin{cases} 4x - y = 13 \\ x - 2y = 5 \end{cases}$$

Solution**Step 1) Write the augmented matrix.**

You may first want to insert “1”s and “0”s where appropriate.

$$\begin{cases} 4x - 1y = 13 \\ 1x - 2y = 5 \end{cases}$$

$$\begin{array}{l} R_1 \left[ \begin{array}{cc|c} 4 & -1 & 13 \end{array} \right] \\ R_2 \left[ \begin{array}{cc|c} 1 & -2 & 5 \end{array} \right] \end{array}$$

Note: It’s up to you if you want to write the “ $R_1$ ” and the “ $R_2$ .”

**Step 2) Use EROs until we obtain the desired form:**  $\left[ \begin{array}{cc|c} 1 & ? & ? \\ 0 & 1 & ? \end{array} \right]$

Note: There may be different “good” ways to achieve our goal.

We want a “1” to replace the “4” in the upper left.

Dividing through  $R_1$  by 4 will do it, but we will then end up with fractions. Sometimes, we can’t avoid fractions. Here, we can.

Instead, let’s switch the rows.

$$R_1 \leftrightarrow R_2$$

**Warning: You should keep a record of your EROs. This will reduce eyestrain and frustration if you want to check your work!**

$$\begin{array}{l} R_1 \left[ \begin{array}{cc|c} 1 & -2 & 5 \end{array} \right] \\ R_2 \left[ \begin{array}{cc|c} 4 & -1 & 13 \end{array} \right] \end{array}$$

We now want a “0” to replace the “4” in the bottom left. Remember, we generally want to “correct” columns from left to right, so we will attack the position containing the  $-1$  later.

We cannot multiply through a row by 0.

Instead, we will use a row replacement ERO that exploits the “1” in the upper left to “kill off” the “4.” This really represents the elimination of the  $x$  term in what is now the second equation in our system.

$$(\text{new } R_2) \leftarrow (\text{old } R_2) + (-4) \cdot R_1$$

The notation above is really unnecessary if you show the work below:

old $R_2$	4	-1		13
$+(-4) \cdot R_1$	-4	8		-20
new $R_2$	<b>0</b>	<b>7</b>		<b>-7</b>

$$\begin{array}{l} R_1 \left[ \begin{array}{cc|c} 1 & -2 & 5 \end{array} \right] \\ R_2 \left[ \begin{array}{cc|c} 0 & 7 & -7 \end{array} \right] \end{array}$$

We want a “1” to replace the “7.”

We will divide through  $R_2$  by 7, or, equivalently, we will multiply

through  $R_2$  by  $\frac{1}{7}$ :

$$R_2 \leftarrow \frac{1}{7} \cdot R_2, \text{ or}$$

$$\begin{array}{l} R_1 \left[ \begin{array}{cc|c} 1 & -2 & 5 \end{array} \right] \\ R_2 \left[ \begin{array}{cc|c} 0 & 7 & -7 \end{array} \right] \leftarrow \div 7 \end{array}$$

$$\begin{array}{l} R_1 \left[ \begin{array}{cc|c} 1 & -2 & 5 \end{array} \right] \\ R_2 \left[ \begin{array}{cc|c} 0 & 1 & -1 \end{array} \right] \end{array}$$

We now have our desired form.

Technical Note: What's best for computation by hand may not be best for computer algorithms that attempt to maximize precision and accuracy. For example, the strategy of partial pivoting would have kept the "4" in the upper left position of the original matrix and would have used it to eliminate the "1" below.

Note: Some books remove the requirement that the entries along the main diagonal all have to be "1"s. However, when we refer to Gaussian Elimination, we will require that they all be "1"s.

### Step 3) Write the new system.

You may want to write down the variables on top of their corresponding columns.

$$\begin{array}{cc} x & y \\ \left[ \begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -1 \end{array} \right] \end{array}$$

$$\begin{cases} x - 2y = 5 \\ y = -1 \end{cases} \uparrow$$

This is called an upper triangular system, which is very easy to solve if we ...

**Step 4) Use back-substitution.**

We start at the bottom, where we immediately find that  $y = -1$ .

We then work our way up the system, plugging in values for unknowns along the way whenever we know them.

$$\begin{aligned}x - 2y &= 5 \\x - 2(-1) &= 5 \\x + 2 &= 5 \\x &= 3\end{aligned}$$

**Step 5) Write the solution.**

The solution set is:  $\{(3, -1)\}$ .

Books are often content with omitting the  $\{ \}$  brace symbols. Ask your instructor, though.

**Warning:** Observe that the order of the coordinates is the **reverse** of the order in which we found them in the back-substitution procedure.

**Step 6) Check. (Optional)**

$$\text{Given system: } \begin{cases} 4x - y = 13 \\ x - 2y = 5 \end{cases}$$

$$\begin{cases} 4(3) - (-1) = 13 \\ (3) - 2(-1) = 5 \end{cases}$$

$$\begin{cases} 13 = 13 \\ 5 = 5 \end{cases}$$

Our solution checks out.

Example (#62 on p.556)

$$\text{Solve the system: } \begin{cases} 2x + 2y - z = 2 \\ x - 3y + z = -28 \\ -x + y = 14 \end{cases}$$

Solution

**Step 1) Write the augmented matrix.**

You may first want to insert “1”s and “0”s where appropriate.

$$\begin{cases} 2x + 2y - 1z = 2 \\ 1x - 3y + 1z = -28 \\ -1x + 1y + 0z = 14 \end{cases}$$

$$\begin{array}{l} R_1 \left[ \begin{array}{ccc|c} 2 & 2 & -1 & 2 \end{array} \right] \\ R_2 \left[ \begin{array}{ccc|c} 1 & -3 & 1 & -28 \end{array} \right] \\ R_3 \left[ \begin{array}{ccc|c} -1 & 1 & 0 & 14 \end{array} \right] \end{array}$$

**Step 2) Use EROs until we obtain the desired form:**  $\left[ \begin{array}{ccc|c} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{array} \right]$

We want a “1” to replace the “2” in the upper left corner.

Dividing through  $R_1$  by 2 would do it, but we would then end up with a fraction.

Instead, let’s switch the first two rows.

$$R_1 \leftrightarrow R_2$$

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ \mathbf{2} & 2 & -1 & 2 \\ -\mathbf{1} & 1 & 0 & 14 \end{array} \right]$$

We now want to “eliminate down” the first column by using the “1” in the upper left corner to “kill off” the boldfaced entries and turn them into “0”s.

**Warning:** Performing more than one ERO before writing down a new matrix often risks mechanical errors. However, when eliminating down a column, we can usually perform several row replacement EROs without confusion before writing a new matrix. (The same is true of multiple row rescalings and of row reorderings, which can represent multiple row interchanges.) Mixing ERO types before writing a new matrix is probably a bad idea, though!

old $R_2$	2	2	-1		2
$+(-2) \cdot R_1$	-2	6	-2		56
new $R_2$	<b>0</b>	<b>8</b>	<b>-3</b>		<b>58</b>

old $R_3$	-1	1	0		14
$+R_1$	1	-3	1		-28
new $R_3$	<b>0</b>	<b>-2</b>	<b>1</b>		<b>-14</b>

Now, write down the new matrix:

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ \mathbf{0} & 8 & -3 & 58 \\ \mathbf{0} & -2 & 1 & -14 \end{array} \right]$$

The first column has been “corrected.” From a strategic perspective, we may now think of the first row and the first column (in blue) as “locked in.” (EROs that change the entries therein are not necessarily “wrong,” but you may be in danger of being taken further away from the desired form.)



We will now focus on the second column. We want:

$$\left[ \begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{1} & ? & ? \\ 0 & \mathbf{0} & ? & ? \end{array} \right]$$

Here is our current matrix:

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{8} & -3 & 58 \\ 0 & -2 & 1 & -14 \end{array} \right]$$

If we use the “ $-2$ ” to kill off the “ $\mathbf{8}$ ,” we can avoid fractions for the time being. Let’s first switch  $R_2$  and  $R_3$  so that we don’t get confused when we do this. (We’re used to eliminating **down** a column.)

Technical Note: The computer-based strategy of partial pivoting would use the “ $\mathbf{8}$ ” to kill off the “ $-2$ ,” since the “ $\mathbf{8}$ ” is larger in absolute value.

$$R_2 \leftrightarrow R_3$$

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & -2 & 1 & -14 \\ 0 & \mathbf{8} & -3 & 58 \end{array} \right]$$

Now, we will use a row replacement ERO to eliminate the “ $\mathbf{8}$ .”

old $R_3$	0	8	-3		58
$+4 \cdot R_2$	0	-8	4		-56
new $R_3$	<b>0</b>	<b>0</b>	<b>1</b>		<b>2</b>

Warning: Don’t ignore the “0”s on the left; otherwise, you may get confused.

Now, write down the new matrix:

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & -2 & 1 & -14 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Once we get a “1” where the “-2” is, we’ll have our desired form. We are fortunate that we already have a “1” at the bottom of the third column, so we won’t have to “correct” it.

We will divide through  $R_2$  by  $-2$ , or, equivalently, we will multiply through  $R_2$  by  $-\frac{1}{2}$ .

$$R_2 \leftarrow \left( -\frac{1}{2} \right) \cdot R_2, \text{ or}$$

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & -2 & 1 & -14 \\ 0 & 0 & 1 & 2 \end{array} \right] \leftarrow \div(-2)$$

We finally obtain a matrix in our desired form:

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & 1 & -1/2 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

**Step 3) Write the new system.**

$$\begin{array}{ccc} x & y & z \\ \left[ \begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & 1 & -1/2 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{array}$$

$$\begin{cases} x - 3y + z = -28 \\ y - \frac{1}{2}z = 7 \quad \uparrow \\ z = 2 \quad \uparrow \end{cases}$$

**Step 4) Use back-substitution.**

We immediately have:  $z = 2$

Use  $z = 2$  in the second equation:

$$\begin{aligned} y - \frac{1}{2}z &= 7 \\ y - \frac{1}{2}(2) &= 7 \\ y - 1 &= 7 \\ \mathbf{y} &= \mathbf{8} \end{aligned}$$

Use  $y = 8$  and  $z = 2$  in the first equation:

$$\begin{aligned} x - 3y + z &= -28 \\ x - 3(8) + (2) &= -28 \\ x - 24 + 2 &= -28 \\ x - 22 &= -28 \\ \mathbf{x} &= \mathbf{-6} \end{aligned}$$

**Step 5) Write the solution.**

The solution set is:  $\{(-6, 8, 2)\}$ .

**Warning:** Remember that the order of the coordinates is the **reverse** of the order in which we found them in the back-substitution procedure.

**Step 6) Check. (Optional)**

$$\text{Given system: } \begin{cases} 2x + 2y - z = 2 \\ x - 3y + z = -28 \\ -x + y = 14 \end{cases}$$

$$\begin{cases} 2(-6) + 2(8) - (2) = 2 \\ (-6) - 3(8) + (2) = -28 \\ -(-6) + (8) = 14 \end{cases}$$

$$\begin{cases} 2 = 2 \\ -28 = -28 \\ 14 = 14 \end{cases}$$

Our solution checks out.

**PART E: WHEN DOES A SYSTEM HAVE NO SOLUTION?**

If we **ever** get a row of the form:

$$0 \quad 0 \quad \cdots \quad 0 \quad | \quad (\text{non-0 constant}),$$

then STOP! We know at this point that the solution set is  $\emptyset$ .

Example

Solve the system: 
$$\begin{cases} x + y = 1 \\ x + y = 4 \end{cases}$$

Solution

The augmented matrix is:

$$\begin{array}{l} R_1 \left[ \begin{array}{cc|c} 1 & 1 & 1 \end{array} \right] \\ R_2 \left[ \begin{array}{cc|c} 1 & 1 & 4 \end{array} \right] \end{array}$$

We can quickly subtract  $R_1$  from  $R_2$ . We then obtain:

$$\begin{array}{l} R_1 \left[ \begin{array}{cc|c} 1 & 1 & 1 \end{array} \right] \\ R_2 \left[ \begin{array}{cc|c} 0 & 0 & 3 \end{array} \right] \end{array}$$

The new  $R_2$  implies that the solution set is  $\emptyset$ .

Comments: This is because  $R_2$  corresponds to the equation  $0 = 3$ , which cannot hold true for any pair  $(x, y)$ .

If we get a row of all “0”s, such as:

$$0 \ 0 \ \dots \ 0 \ | \ 0,$$

then what does that imply? The story is more complicated here.

### Example

$$\text{Solve the system: } \begin{cases} x + y = 4 \\ x + y = 4 \end{cases}$$

### Solution

The augmented matrix is:

$$\begin{array}{l} R_1 [1 \ 1 \ | \ 4] \\ R_2 [1 \ 1 \ | \ 4] \end{array}$$

We can quickly subtract  $R_1$  from  $R_2$ . We then obtain:

$$\begin{array}{l} R_1 [1 \ 1 \ | \ 4] \\ R_2 [0 \ 0 \ | \ 0] \end{array}$$

The corresponding system is then:

$$\begin{cases} x + y = 4 \\ 0 = 0 \end{cases}$$

The equation  $0 = 0$  is pretty easy to satisfy. All ordered pairs  $(x, y)$  satisfy it. In principle, we could delete this equation from the system. However, we tend not to delete rows in an augmented matrix, even if they consist of nothing but “0”s. The idea of changing the size of a matrix creeps us out.

The solution set is:

$$\{(x, y) \mid x + y = 4\}$$

The system has infinitely many solutions; they correspond to all of the points on the line  $x + y = 4$ .

However, a row of all “0”'s does **not** automatically imply that the corresponding system has infinitely many solutions.

### Example

Consider the augmented matrix:

$$\begin{array}{l} R_1 \\ R_2 \end{array} \left[ \begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

Because of  $R_1$ , the corresponding system actually has no solution.

See [Notes 7.12](#) for a similar example.

The augmented matrices we have seen [in this Part](#) are **not** row equivalent to any matrix of the form

$$\begin{array}{l} R_1 \\ R_2 \end{array} \left[ \begin{array}{cc|c} 1 & ? & ? \\ 0 & 1 & ? \end{array} \right].$$

There was no way to get that desired form using EROs.

What form do we aim for, then?

**PART F: ROW-ECHELON FORM FOR A MATRIX**

If it is impossible for us to obtain the form

$$\left[ \begin{array}{cccc|c} 1 & & & & \\ & 1 & & ? & \\ & & \cdot & \cdot & \\ 0 & & \cdot & \cdot & \\ & & & & 1 & ? \end{array} \right]$$

(maybe because our coefficient matrix isn't even square), then what do we aim for? We aim for row-echelon form; in fact, the above form is a special case of row-echelon form.

Properties of a Matrix in Row-Echelon Form

1) If there are any “all-0” rows, then they must be at the bottom of the matrix.

Aside from these “all-0” rows,

2) Every row must have a “1” (called a “leading 1”) as its leftmost non-0 entry.

3) The “leading 1”s must “flow down and to the right.”

More precisely: The “leading 1” of a row must be in a column to the right of the “leading 1”s of all higher rows.

Example

The matrix below is in Row-Echelon Form:

$$\left[ \begin{array}{ccccc|c} \mathbf{1} & 3 & 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 9 & 2 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The “leading 1”s are boldfaced.

The “1” in the upper right corner is **not** a “leading 1.”



**PART G: REDUCED ROW-ECHELON (RRE) FORM FOR A MATRIX**

This is a special case of Row-Echelon Form.

Properties of a Matrix in Reduced Row-Echelon (RRE) Form

- 1-3) It is in Row-Echelon form. (See [Part F](#).)
- 4) Each “leading 1” has all “0”s elsewhere in its column.

Property 4) leads us to eliminate **up** from the “leading 1”s.

Recall the matrix in Row-Echelon Form that we just saw:

$$\left[ \begin{array}{ccccc|c} \mathbf{1} & 3 & 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 9 & 2 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

In order to obtain RRE Form, we must use row replacement EROs to kill off the three entries in purple (the “7,” the “4,” and the “9”); we need “0”s in those positions.

**PART H: GAUSS-JORDAN ELIMINATION**

This is a matrix-heavy alternative to Gaussian Elimination in which we use EROs to go all the way to RRE Form.

A matrix of numbers can have infinitely many Row-Echelon Forms [that the matrix is row-equivalent to], but it has only one **unique** RRE Form.

Technical Note: The popular MATLAB (“Matrix Laboratory”) software has an “rref” command that gives this unique RRE Form for a given matrix.

In fact, we can efficiently use Gauss-Jordan Elimination to help us describe the solution set of a system of linear equations with infinitely many solutions.

**Example**

Let’s say we have a system that we begin to solve using Gaussian Elimination. Let’s say we obtain the following matrix in Row-Echelon Form:

$$\left[ \begin{array}{ccc|c} \mathbf{1} & -2 & 3 & 9 \\ 0 & \mathbf{1} & 3 & 5 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Before [this Part](#), we would stop with the matrices and write out the corresponding system.

In Gauss-Jordan Elimination, however, we’re not satisfied with just any Row-Echelon Form for our final augmented matrix. We demand RRE Form.

To obtain RRE Form, we must eliminate **up** from two of the “leading **1**”s and kill off the three purple entries: the “**-2**” and the two “**3**”s. We need “**0**”s in those positions.

In Gaussian Elimination, we “corrected” the columns from left to right in order to preserve our good works. At this stage, however, when we eliminate **up**, we prefer to correct the columns from **right to left** so that we can take advantage of the “**0**”s we create along the way.

(Reminder:)

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \left[ \begin{array}{ccc|c} \mathbf{1} & -2 & 3 & 9 \\ 0 & \mathbf{1} & 3 & 5 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Use row replacement EROs to eliminate the two “3”s in the third column. Observe that we use a “leading 1” from a lower row to kill off an entry from a higher row.

old $R_2$	0	1	3		5
$+(-3) \cdot R_3$	0	0	-3		-6
new $R_2$	<b>0</b>	<b>1</b>	<b>0</b>		<b>-1</b>

old $R_1$	1	-2	3		9
$+(-3) \cdot R_3$	0	0	-3		-6
new $R_1$	<b>1</b>	<b>-2</b>	<b>0</b>		<b>3</b>

New matrix:

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \left[ \begin{array}{ccc|c} \mathbf{1} & -2 & 0 & 3 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Now, use a row replacement ERO to eliminate the “-2” in the second column.

old $R_1$	1	-2	0		3
$+2 \cdot R_2$	0	2	0		-2
new $R_1$	<b>1</b>	<b>0</b>	<b>0</b>		<b>1</b>

Observe that our “right to left” strategy has allowed us to use “0”s to our advantage.

Here is the final RRE Form:

$$\begin{array}{c} x \quad y \quad z \\ R_1 \left[ \begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 1 \end{array} \right] \\ R_2 \left[ \begin{array}{ccc|c} 0 & \mathbf{1} & 0 & -1 \end{array} \right] \\ R_3 \left[ \begin{array}{ccc|c} 0 & 0 & \mathbf{1} & 2 \end{array} \right] \\ R_4 \left[ \begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

We can read off our solution now!

$$\begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$$

Solution set:  $\{(1, -1, 2)\}$ .

As you can see, some work has been moved from the back-substitution stage (which is now deleted) to the ERO stage.

**PART I: SYSTEMS WITH INFINITELY MANY SOLUTIONS (OPTIONAL?)**Example

$$\text{Solve the system: } \begin{cases} x - 2y + z + 5w = 3 \\ 2x - 4y + z + 7w = 5 \end{cases}$$

Warning: In fact,  $w$  is often considered to be the fourth coordinate of ordered 4-tuples of the form  $(x, y, z, w)$ .

Solution

The augmented matrix is:

$$\begin{array}{l} R_1 \\ R_2 \end{array} \left[ \begin{array}{cccc|c} 1 & -2 & 1 & 5 & 3 \\ 2 & -4 & 1 & 7 & 5 \end{array} \right]$$

Let's first go to Row-Echelon Form, which is required in both Gaussian Elimination and Gauss-Jordan Elimination – that is, unless it is clear at some point that there is no solution.

We will use a row replacement ERO and use the “1” in the upper left corner to kill off the “2” in the lower left corner and get a “0” in there.

old $R_2$	2	-4	1	7		5
$+(-2) \cdot R_1$	-2	4	-2	-10		-6
new $R_2$	<b>0</b>	<b>0</b>	<b>-1</b>	<b>-3</b>		<b>-1</b>

New matrix:

$$\begin{array}{l} R_1 \\ R_2 \end{array} \left[ \begin{array}{cccc|c} 1 & -2 & 1 & 5 & 3 \\ 0 & 0 & \mathbf{-1} & \mathbf{-3} & \mathbf{-1} \end{array} \right]$$

We now need a “1” where the boldfaced “-1” is.

To obtain Row-Echelon Form, we multiply through  $R_2$  by  $(-1)$ :

$$(\text{new } R_2) \leftarrow (-1) \cdot (\text{old } R_2)$$

$$\begin{array}{cccc|c} & x & y & z & w & \\ R_1 & \mathbf{1} & -2 & 1 & 5 & 3 \\ R_2 & 0 & 0 & \mathbf{1} & 3 & 1 \end{array}$$

RHS

The “leading **1**”s are boldfaced.

We first observe that the system is consistent, because of the following rule:

An augmented matrix in Row-Echelon Form corresponds to an **inconsistent** system (i.e., a system with no solution)  $\Leftrightarrow$  (if and only if) there is a “leading **1**” in the RHS.

In other words, it corresponds to a **consistent** system  $\Leftrightarrow$  there are **no** “leading **1**”s in the RHS.

**Warning:** There is a “1” in our RHS here in our Example, but it is **not** a “leading **1**.”

Each of the variables that correspond to the columns of the coefficient matrix (here,  $x$ ,  $y$ ,  $z$ , and  $w$ ) is either a basic variable or a free variable.

A variable is called a basic variable  $\Leftrightarrow$   
It corresponds to a column that has a “leading **1**.”

A variable is called a free variable  $\Leftrightarrow$   
It corresponds to a column that does **not** have a “leading **1**.”

In this Example,  $x$  and  $z$  are basic variables, and  $y$  and  $w$  are free variables.

Let's say our system of linear equations is consistent.

If there are no free variables, then the system has only one solution.

Otherwise, if there is at least one free variable, then the system has infinitely many solutions.

At this point, we know that the system in our Example has infinitely many solutions.

If we want to completely describe the solution set of a system with infinitely many solutions, then we should use Gauss-Jordan Elimination and take our matrix to RRE Form. We must kill off the "1" in purple below.

$$\begin{array}{cccc}
 & x & y & z & w \\
 R_1 & \left[ \begin{array}{cccc|c} \mathbf{1} & -2 & \mathbf{1} & 5 & 3 \end{array} \right] \\
 R_2 & \left[ \begin{array}{cccc|c} 0 & 0 & \mathbf{1} & 3 & 1 \end{array} \right] \\
 & & & & \text{RHS}
 \end{array}$$

old $R_1$	1	-2	<b>1</b>	5		3
$+(-1) \cdot R_2$	0	0	-1	-3		-1
new $R_1$	<b>1</b>	<b>-2</b>	<b>0</b>	<b>2</b>		<b>2</b>

Our RRE Form:

$$\begin{array}{cccc}
 & x & y & z & w \\
 R_1 & \left[ \begin{array}{cccc|c} \mathbf{1} & -2 & 0 & 2 & 2 \end{array} \right] \\
 R_2 & \left[ \begin{array}{cccc|c} 0 & 0 & \mathbf{1} & 3 & 1 \end{array} \right] \\
 & & & & \text{RHS}
 \end{array}$$

The corresponding system:

$$\begin{cases} x - 2y + 2w = 2 \\ z + 3w = 1 \end{cases}$$

Now for some steps we haven't seen before.

We will parameterize (or parametrize) the free variables:

$$\text{Let } y = a,$$

$$w = b,$$

where the parameters  $a$  and  $b$  represent any pair of real numbers.

Both of the parameters are allowed to “roam freely” over the reals.

Let's rewrite our system using these parameters:

$$\begin{cases} x - 2a + 2b = 2 \\ z + 3b = 1 \end{cases}$$

This is a system consisting of two variables and two parameters.

We then solve the equations for the basic variables,  $x$  and  $z$ :

$$\begin{cases} x = 2 + 2a - 2b \\ z = 1 - 3b \end{cases}$$

Remember that  $y = a$  and  $w = b$ , so we have:

$$\begin{cases} x = 2 + 2a - 2b \\ y = a \\ z = 1 - 3b \\ w = b \end{cases}$$

Note: In your Linear Algebra class ([Math 254 at Mesa](#)), you may want to line up like terms.



We can now write the solution set.

$$\left\{ (2 + 2a - 2b, a, 1 - 3b, b) \mid a \text{ and } b \text{ are real numbers} \right\}$$

### Comments

This set consists of infinitely many solutions, each corresponding to a different pair of choices for  $a$  and  $b$ .

Some solutions:

$a$	$b$	$\Rightarrow$	(	$x,$	$y,$	$z,$	$w$	)
0	0	$\Rightarrow$	(	2,	0,	1,	0	)
-4	7	$\Rightarrow$	(	-20,	-4,	-20,	7	)

Because we have two parameters, the graph of the solution set is a 2-dimensional plane existing in 4-dimensional space. Unfortunately, we can't see this graph! Nevertheless, this is the kind of thinking you will engage in in your Linear Algebra class ([Math 254 at Mesa](#))!

## **SECTION 8.2: OPERATIONS WITH MATRICES**

We will not discuss augmented matrices until [Part G](#).

For now, we will simply think of a matrix as a box of numbers.

### **PART A: NOTATION**

The matrix  $A = [a_{ij}]$ , meaning that  $A$  consists of entries labeled  $a_{ij}$ , where  $i$  is the row number, and  $j$  is the column number.

#### Example

$$\text{If } A \text{ is } 2 \times 2, \text{ then } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Note:  $a_{12}$  and  $a_{21}$  are not necessarily equal. If they are, then we have a symmetric matrix, which is a square matrix that is symmetric about its main diagonal. An example of a symmetric matrix is:  $\begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}$

**PART B: WHEN DOES  $A = B$ ?**

Two matrices (say  $A$  and  $B$ ) are equal  $\Leftrightarrow$   
They have the same size, and they have the same numbers (or expressions) in the same positions.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

If the matrix on the left is  $A$ , then the matrix on the right is  $A^T$  (“ $A$  transpose”). For the two matrices, the rows of one are the columns of the other.

Example

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The two matrices have different sizes.

The matrix on the left is  $1 \times 2$ .

It may be seen as a row vector, since it consists of only 1 row.

The matrix on the right is  $2 \times 1$ .

It may be seen as a column vector, since it consists of only 1 column.

Observe that the matrices are transposes of each other.

Think About It: What kind of matrix is, in fact, equal to its transpose?

**PART C: BASIC OPERATIONS**

Matrix addition: If two or more matrices have the same size, then you add them by adding corresponding entries. If the matrices do not have the same size, then the sum is undefined.

Matrix subtraction problems can be rewritten as matrix addition problems.

Scalar multiplication: To multiply a matrix by a scalar (i.e., a real number [in this class](#)), you multiply each entry of the matrix by the scalar.

Example

If

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$B$  is the  $2 \times 3$  zero matrix, denoted by “0” or “ $0_{2 \times 3}$ ” – it is the additive identity for the set of  $2 \times 3$  real matrices. However, when we refer to “identity matrices,” we typically refer to multiplicative identities, which we will discuss later.

$$C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

then ...

1) Find  $A + B + 2C$ 

$$\begin{aligned}
 A + B + 2C &= \underbrace{\begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Perform matrix addition.}} + \underbrace{2 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix}}_{\text{Perform scalar multiplication}} \\
 &= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & -6 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 4 & 1 \\ -1 & 5 & -4 \end{bmatrix}
 \end{aligned}$$

2) Find  $A - 5C$ 

$$\begin{aligned}
 A - 5C &= A + (-5)C \\
 &= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix} + (-5) \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix} \\
 &= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} -5 & -10 & 0 \\ 0 & -5 & 15 \end{bmatrix} \\
 &= \begin{bmatrix} -3 & -10 & 1 \\ -1 & -2 & 17 \end{bmatrix}
 \end{aligned}$$

**PART D: (A ROW VECTOR) TIMES (A COLUMN VECTOR)**

We will deal with this basic multiplication problem before we go on to matrix multiplication in general.

Let's say we have a row vector  $[a_1 \ a_2 \ \cdots \ a_n]$  and a column vector  $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ .

Observe that they have the same number of entries; otherwise, our product will be undefined. This is how we multiply the row vector and the column vector (in that order); the resulting product may be viewed as either a scalar or a  $1 \times 1$  matrix, depending on the context of the problem:

$$[a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{cases} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n & \text{(as a scalar)} \\ \text{or} \\ [a_1 b_1 + a_2 b_2 + \cdots + a_n b_n] & \text{(as a } 1 \times 1 \text{ matrix)} \end{cases}$$

In words, we add the products of corresponding entries.

This should remind you of the dot product of two vectors, which we saw in [Section 6.4: Notes 6.28](#).

**Warning:** A column vector times a row vector (in that order) gives you something very different, namely an  $n \times n$  matrix. We will see why in the next Part.

**Example**

$$\begin{aligned} [1 \ 0 \ 3] \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix} &= (1)(4) + (0)(5) + (3)(-1) \\ &= 4 + 0 - 3 \\ &= 1 \end{aligned}$$

The product may also be written as  $[1]$ .

**PART E: MATRIX MULTIPLICATION ( $AB$ )**

When multiplying matrices, we do **not** simply multiply corresponding entries, although MATLAB does have an operation for that.

Technical Definition (Optional?)

(Bear in mind that the “tricks” that we will discuss later will make all of this easier to swallow.)

Given two matrices  $A$  and  $B$ , the matrix product  $AB$  is defined  $\Leftrightarrow$

The rows of  $A$  and the columns of  $B$  have the same “length” (i.e., number of entries).

That is: (the number of columns of  $A$ ) = (the number of rows of  $B$ )

If  $AB$  is defined, then the entry in its  $i^{\text{th}}$  row and  $j^{\text{th}}$  column equals:

(the  $i^{\text{th}}$  row of  $A$ ) times (the  $j^{\text{th}}$  column of  $B$ )

for appropriate values of  $i$  and  $j$ .

Another way of looking at this:

If we let  $C = AB$ , where  $C = [c_{ij}]$ , then:

$c_{ij} =$  (the  $i^{\text{th}}$  row of  $A$ ) times (the  $j^{\text{th}}$  column of  $B$ )

for appropriate values of  $i$  and  $j$ .

Example and Tricks

Consider the matrix product

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 4 & -1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}}_B.$$

Is  $AB$  defined? If so, what is its size?

Let's consider the sizes of  $A$  and  $B$ :

$$\begin{array}{cc} A & B \\ 2 \times \boxed{3} & \boxed{3} \times 2 \end{array}$$

If the two boxed “inner numbers” are equal, then  $AB$  is defined, because:

$$\left( \text{the number of columns of } A \right) = \left( \text{the number of rows of } B \right),$$

as specified in our [Technical Definition](#).

**Warning:** The two “outer numbers” (the “2”s here) need not be equal. The fact that they are means that the matrix will be square.

The size of  $AB$  is given by the two “outer numbers” in order.

Here,  $AB$  will be  $2 \times 2$ .

In general, if  $A$  is  $m \times \boxed{n}$ , and  $B$  is  $\boxed{n} \times p$ , then  $AB$  will be  $m \times p$ .

The trick coming up will help explain why.



Find  $AB$ .

We will use a “traffic intersection” model.

To begin our trick, we will write  $B$  to the “northeast” (i.e., entirely above and to the right) of  $A$ .

Draw thin lines (in blue below) through the rows of  $A$  and thin lines (in red below) through the columns of  $B$  so that all intersection points are shown. These intersection points correspond to the entries of  $AB$ . We can see immediately that the size of  $AB$  will be  $2 \times 2$ .

**Warning:** Make sure your lines are thin and are placed so that, for example, no “-” signs or “1”s are written over.

$$\begin{array}{c}
 \begin{array}{ccc}
 & & B \\
 & & \left[ \begin{array}{cc} 4 & -1 \\ 3 & 0 \\ 1 & 2 \end{array} \right] \\
 A & & \\
 \left[ \begin{array}{ccc} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array} \right] & \left[ \begin{array}{cc} c_{11} & c_{12} \\ c_{21} & c_{22} \end{array} \right] \\
 & & AB
 \end{array}
 \end{array}$$

At each intersection point, we take the corresponding row of  $A$  and the corresponding column of  $B$ , and we multiply them as we did in [Part D](#). (We are essentially taking the dot product of the two vectors whose lines intersect at that point.)

$$c_{11} = (1)(4) + (-1)(3) + (0)(1) = 4 - 3 + 0 = \mathbf{1}$$

$$c_{12} = (1)(-1) + (-1)(0) + (0)(2) = -1 + 0 + 0 = \mathbf{-1}$$

$$c_{21} = (2)(4) + (1)(3) + (3)(1) = 8 + 3 + 3 = \mathbf{14}$$

$$c_{22} = (2)(-1) + (1)(0) + (3)(2) = -2 + 0 + 6 = \mathbf{4}$$

$$\text{Therefore, } AB = \begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{14} & \mathbf{4} \end{bmatrix}.$$

If the rows of  $A$  do not have the same length as the columns of  $B$  (so that dot products cannot be taken), then the matrix product  $AB$  is undefined.

Observe that this is all consistent with the [Technical Definition](#).

Even though you may not have to use the  $c_{ij}$  notation, it may be a good idea to show some work for partial credit purposes.

**Warning: Matrix multiplication is **not commutative**. It is often the case that  $AB \neq BA$ . In fact, one product may be defined, while the other is not.**

Think About It: When are  $AB$  and  $BA$  both defined?

Why was matrix multiplication defined in this way? The answer lies in your Linear Algebra course ([Math 254 at Mesa](#)). The idea of “compositions of linear transformations” is key. You’ll see.



**PART G: MATRIX NOTATION and SYSTEMS OF LINEAR EQUATIONS**

Example (#56 on p.570)

$$\text{Consider the system: } \begin{cases} x_1 + x_2 - 3x_3 = 9 \\ -x_1 + 2x_2 = 6 \\ x_1 - x_2 + x_3 = -5 \end{cases}$$

Observe that this is a **square** system of linear equations; the number of equations (3) equals the number of unknowns (3).

We can write this system as a matrix (or matrix-vector) equation,  $AX = B$ :

$$\underbrace{\begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 9 \\ 6 \\ -5 \end{bmatrix}}_B$$

$A$  is the coefficient matrix (and it is square), and  $B$  is the RHS.  
 $X$  may be thought of as a vector of variables or as a solution vector.

We will use Gauss-Jordan Elimination on the augmented matrix  $[A|B]$  to solve for  $X$ . This is what we basically did in [Section 8.1](#).

In order to save time, we will skip the steps that take us to Row-Echelon Form. Don't do this in your own work, though!

The  $\sim$  symbol indicates row-equivalence, not equality.

$$\begin{aligned} [A|B] &= \left[ \begin{array}{ccc|c} 1 & 1 & -3 & 9 \\ -1 & 2 & 0 & 6 \\ 1 & -1 & 1 & -5 \end{array} \right] \\ &\quad \vdots \\ &\sim \left[ \begin{array}{ccc|c} 1 & 1 & -3 & 9 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & -2 \end{array} \right] \end{aligned}$$

There are actually infinitely many possible Row-Echelon Forms for  $[A|B]$ ; the last matrix is just one of them. However, there is only one RRE Form. Let's find it. We proceed with Gauss-Jordan Elimination by eliminating up from the "leading 1"s.

$$\left[ \begin{array}{ccc|c} \mathbf{1} & 1 & -3 & 9 \\ 0 & \mathbf{1} & -1 & 5 \\ 0 & 0 & \mathbf{1} & -2 \end{array} \right]$$

We eliminate up the third column:

old $R_2$	0	<b>1</b>	<b>-1</b>		5
$+R_3$	0	0	1		-2
new $R_2$	<b>0</b>	<b>1</b>	<b>0</b>		<b>3</b>

old $R_1$	1	1	<b>-3</b>		9
$+3 \cdot R_3$	0	0	3		-6
new $R_1$	<b>1</b>	<b>1</b>	<b>0</b>		<b>3</b>

New matrix:

$$\left[ \begin{array}{ccc|c} \mathbf{1} & \mathbf{1} & 0 & 3 \\ 0 & \mathbf{1} & 0 & 3 \\ 0 & 0 & \mathbf{1} & -2 \end{array} \right]$$

We eliminate up the second column:

old $R_1$	1	<b>1</b>	0		3
$+(-1) \cdot R_2$	0	-1	0		-3
new $R_1$	<b>1</b>	<b>0</b>	<b>0</b>		<b>0</b>

RRE Form:

$$\left[ \begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 3 \\ 0 & 0 & \mathbf{1} & -2 \end{array} \right]$$

Observe that the coefficient matrix is  $I_3$ , the  $3 \times 3$  identity.

The corresponding system is pretty nifty:

$$\begin{cases} x_1 = 0 \\ x_2 = 3 \\ x_3 = -2 \end{cases}$$

This immediately gives us our solution vector,  $X$ :

$$X = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

In general, if  $AX = B$  represents a square system of linear equations that has exactly one solution, then the RRE Form of  $[A|B]$  will be  $[I|X]$ , where  $I$  is the identity matrix that is the same size as  $A$ . We simply grab our solution from the new RHS.

## SECTION 8.3: THE INVERSE OF A SQUARE MATRIX

### PART A: (REVIEW) THE INVERSE OF A REAL NUMBER

If  $a$  is a nonzero real number, then  $aa^{-1} = a\left(\frac{1}{a}\right) = 1$ .

$a^{-1}$ , or  $\frac{1}{a}$ , is the multiplicative inverse of  $a$ , because its product with  $a$  is 1, the multiplicative identity.

Example

$3\left(\frac{1}{3}\right) = 1$ , so 3 and  $\frac{1}{3}$  are multiplicative inverses of each other.

### PART B: THE INVERSE OF A SQUARE MATRIX

If  $A$  is a square  $n \times n$  matrix, sometimes there exists a matrix  $A^{-1}$  (“A inverse”) such that

$$AA^{-1} = I_n \text{ and } A^{-1}A = I_n.$$

An invertible matrix and its inverse commute with respect to matrix multiplication.

Then,  $A$  is invertible (or nonsingular), and  $A^{-1}$  is unique.

In this course, an invertible matrix is assumed to be square.

Technical Note: A nonsquare matrix may have a left inverse matrix or a right inverse matrix that “works” on one side of the product and produces an identity matrix. They cannot be the same matrix, however.

**PART C: FINDING  $A^{-1}$** 

We will discuss a shortcut for  $2 \times 2$  matrices in [Part F](#).

Assume that  $A$  is a given  $n \times n$  (square) matrix.

$A$  is invertible  $\Leftrightarrow$  Its RRE Form is the identity matrix  $I_n$  (or simply  $I$ ).

It turns out that a sequence of EROs that takes you from an invertible matrix  $A$  down to  $I$  will also take you from  $I$  down to  $A^{-1}$ . (A good Linear Algebra book will have a proof for this.) We can use this fact to efficiently find  $A^{-1}$ .

We construct  $[A \mid I]$ . We say that  $A$  is in the “left square” of this matrix, and  $I$  is in the “right square.”

We apply EROs to  $[A \mid I]$  until we obtain the RRE Form  $[I \mid A^{-1}]$ .

That is, as soon as you obtain  $I$  in the left square, you grab the matrix in the right square as your  $A^{-1}$ .

If you **ever** get a row of “0”s in the left square, then it will be impossible to obtain  $[I \mid A^{-1}]$ , and  $A$  is noninvertible (or singular).

**Example**

Let’s go back to our  $A$  matrix from [Section 8.2: Notes 8.44](#).

$$A = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Find  $A^{-1}$ .



Solution

We construct  $[A \mid I]$ :

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array} \right]$$

We perform Gauss-Jordan Elimination to take the left square down to  $I$ . The right square will be affected in the process, because we perform EROs on **entire** rows “all the way across.”

We will show a couple of row replacement EROs, and then we will leave the remaining steps to you.

We will kill off the purple entries and put “0”s in their places.

old $R_2$	-1	2	0		0	1	0
$+R_1$	1	1	-3		1	0	0
new $R_2$	<b>0</b>	<b>3</b>	<b>-3</b>		<b>1</b>	<b>1</b>	<b>0</b>

old $R_3$	1	-1	1		0	0	1
$+(-1) \cdot R_1$	-1	-1	3		-1	0	0
new $R_3$	<b>0</b>	<b>-2</b>	<b>4</b>		<b>-1</b>	<b>0</b>	<b>1</b>

New matrix:

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 3 & -3 & 1 & 1 & 0 \\ 0 & -2 & 4 & -1 & 0 & 1 \end{array} \right]$$

(Your turn! Keep going ....)

RRE Form:

(Remember, this form is unique.)

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & 1 \\ 0 & 1 & 0 & 1/6 & 2/3 & 1/2 \\ 0 & 0 & 1 & -1/6 & 1/3 & 1/2 \end{array} \right]$$

$\underbrace{\hspace{10em}}_I \quad \underbrace{\hspace{10em}}_{A^{-1}}$

Check. (Optional)

You can check that  $AA^{-1} = I$ . If that holds, then it is automatically true that  $A^{-1}A = I$ . (The right inverse and the left inverse of an invertible matrix must be the same. An invertible matrix must commute with its inverse.)

**PART D: THE INVERSE MATRIX METHOD FOR SOLVING SYSTEMS**

In [Section 8.2: Notes 8.44](#), we expressed a system in the matrix form  $AX = B$ :

$$\underbrace{\begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 9 \\ 6 \\ -5 \end{bmatrix}}_B$$

$A$  should look familiar. In [Part C](#), we found its inverse,  $A^{-1}$ .

How can we express  $X$  directly in terms of  $A$  and  $B$ ?

Review from Algebra I (Optional)

Let's say we want to solve  $ax = b$ , where  $a \neq 0$  and  $a$  and  $b$  are real constants.

$$\begin{aligned} ax &= b \\ \underbrace{\left(\frac{1}{a}\right)a}_=1 x &= \left(\frac{1}{a}\right)b \\ x &= \frac{b}{a} \end{aligned}$$

Because  $a^{-1}$  represents the multiplicative inverse of  $a$ , we can say that  $a^{-1} = \frac{1}{a}$ , and the steps can be rewritten as follows:

$$\begin{aligned} ax &= b \\ \underbrace{a^{-1}a}_=1 x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

Solving the Matrix Equation  $AX = B$ 

Assume that  $A$  is invertible.

Note: Even though 0 is the only real number that is noninvertible (in a multiplicative sense), there are many matrices other than zero matrices that are noninvertible.

It is assumed that  $A$ ,  $X$ , and  $B$  have “compatible” sizes. That is,  $AX$  is defined, and  $AX$  and  $B$  have the same size.

The steps should look familiar:

$$\begin{aligned}AX &= B \\ \underbrace{A^{-1}A}_{=I} X &= A^{-1}B \\ X &= A^{-1}B\end{aligned}$$

The Inverse Matrix Method for Solving a System of Linear Equations

If  $A$  is invertible, then the system  $AX = B$  has a unique solution given by  $X = A^{-1}B$ .

Comments

- We must left multiply both sides of  $AX = B$  by  $A^{-1}$ . If we were to right multiply, then we would obtain  $AXA^{-1} = BA^{-1}$ ; both sides of that equation are undefined, unless  $A$  is  $1 \times 1$ . Remember that matrix multiplication is not commutative. Although  $x = ba^{-1}$  would have been acceptable in our Algebra I discussion (because multiplication of real numbers is commutative),  $X = BA^{-1}$  would be inappropriate here.
- $I$  is the identity matrix that is the same size as  $A$ . It plays the role that “1” did in our Algebra I discussion, because 1 was the multiplicative identity for the set of real numbers.
- This result is of more theoretical significance than practical significance. The Gaussian Elimination (with Back-Substitution) method we discussed earlier is often more efficient than this inverse-based process. However,  $X = A^{-1}B$  is good to know if you’re using software you’re not familiar with. Also, there’s an important category of matrices called orthogonal matrices, for which  $A^{-1} = A^T$ ; this makes matters a whole lot easier, since  $A^T$  is trivial to find.

Back to Our Example

We will solve the system from [Section 8.2: Notes 8.44](#) using this Inverse Matrix Method.

$$\underbrace{\begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 9 \\ 6 \\ -5 \end{bmatrix}}_B$$

Solution

It helps a lot that we've already found  $A^{-1}$  in [Part C](#); that's the bulk of the work.

$$\begin{aligned} X &= A^{-1}B \\ &= \begin{bmatrix} 1/3 & 1/3 & 1 \\ 1/6 & 2/3 & 1/2 \\ -1/6 & 1/3 & 1/2 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \end{aligned}$$

This agrees with our result from the Gauss-Jordan Elimination method we used in [Section 8.2: Notes 8.44-8.46](#).

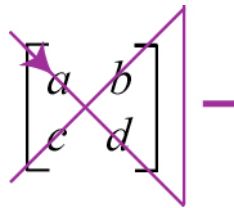
**PART E : THE DETERMINANT OF A  $2 \times 2$  MATRIX ("BUTTERFLY RULE")**

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then the determinant of  $A$ , denoted by  $\det(A)$  or  $|A|$ , is given by:

$$\det(A) = ad - bc$$

i.e.,  $\det(A) = (\text{product along main diagonal}) - (\text{product along skew diagonal})$

The following “butterfly” image may help you recall this formula.



**Warning:**  $|A|$  should not be confused with absolute value notation.

See [Section 8.4](#).

We will further discuss determinants in [Section 8.4](#).

**PART F : SHORTCUT FORMULA FOR THE INVERSE OF A  $2 \times 2$  MATRIX**

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If  $\det(A) = 0$ , then  $A^{-1}$  does not exist.

Remember that we:

**Switch** the entries along the **main diagonal**.

**Flip the signs** on (i.e., take the opposite of) the entries along the **skew diagonal**.

This formula is consistent with the method from **Part C**.

Example

If  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , find  $A^{-1}$ .

Solution

First off:

$$\begin{aligned} \det(A) &= (1)(4) - (2)(3) \\ &= -2 \end{aligned}$$

Now:

$$\begin{aligned} A^{-1} &= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \end{aligned}$$



## SECTION 8.4: THE DETERMINANT OF A SQUARE MATRIX

### PART A: INTRO

Every square matrix consisting of scalars (for example, real numbers) has a determinant, denoted by  $\det(A)$  or  $|A|$ , which is also a scalar.

### PART B: SHORTCUTS FOR COMPUTING DETERMINANTS

(We will discuss a general method in [Part C](#). The shortcuts described here for small matrices may be derived from that method.)

#### 1 × 1 Matrices

$$\text{If } A = [c], \text{ then } \det(A) = c.$$

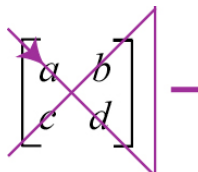
**Warning:** It may be confusing to write  $|A| = c$ . Don't confuse determinants (which **can** be negative in value) with absolute values (which **cannot**).

#### 2 × 2 Matrices (“Butterfly Rule”)

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } \det(A) = ad - bc.$$

$$\text{i.e., } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \text{ (Brackets are typically left out.)}$$

We discussed this case in [Section 8.3: Notes 8.55](#).



3 × 3 Matrices (“Sarrus’s Rule,” named after George Sarrus)

If  $A$  is  $3 \times 3$ , then, to find  $\det(A)$ :

- 1) Rewrite the 1<sup>st</sup> and 2<sup>nd</sup> columns on the right (as “Columns 4 and 5”).
- 2) **Add** the products along the three full diagonals that extend from upper left to lower right.
- 3) **Subtract** the products along the three full diagonals that extend from lower left to upper right.

The wording above is admittedly awkward. Look at this Example:

Example

$$\text{Let } A = \begin{bmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix}. \text{ Find } \det(A).$$

$$\text{i.e., Find } \begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}.$$

Solution

We begin by rewriting the 1<sup>st</sup> and 2<sup>nd</sup> columns on the right.

$$\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} \begin{array}{cc} -1 & 1 \\ 3 & 2 \\ 0 & -1 \end{array}$$



**PART C: “EXPANSION BY COFACTORS” METHOD FOR COMPUTING DETERMINANTS**

This is hard to explain without an Example to lean on!

This method works for square matrices of **any** size.

Example

$$\text{Find } \begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}.$$

(In Part B, we already found out this equals 10.)

Solution

Choose a “magic row or column” to expand along, preferably one with “0”s. We will call its entries our magic entries.

In principle, you could choose any row or any column. Here, let’s choose the 1<sup>st</sup> column, in part because of the “0” in the lower right corner.

$$\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}$$

Because we are dealing with a  $3 \times 3$  matrix, we will set up the  $3 \times 3$  sign matrix. This is always a “checkerboard” matrix that begins with a “+” sign in the upper left corner and then alternates signs along rows and columns.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

We really only need the signs corresponding to our magic row or column.

Note: The sign matrix for a  $4 \times 4$  matrix is given below.

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

Technical Note: The sign of the  $(i, j)$  entry of the sign matrix is the sign of  $(-1)^{i+j}$ , where  $i$  is the row number of the entry, and  $j$  is the column number.

Back to our Example:

$$\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} \text{ with sign matrix } \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

The following may be confusing until you see it “in action” on [the next page](#).

Our cofactor expansion for the determinant will consist of three terms that correspond to our three magic entries. Each term will have the form:

(Sign from sign matrix) (Magic entry) (Corresponding minor),

where the “corresponding minor” is the determinant of the submatrix that is obtained when the row and the column containing the magic entry are deleted from the original matrix.

Note: The “corresponding cofactor” is the same as the corresponding minor, except that you incorporate the corresponding sign from the sign matrix. In particular, if the corresponding sign is a “-” sign, then the cofactor is the opposite of the minor. Then, the determinant is given by the sum of the products of the magic entries with their corresponding cofactors.

Here, we have:

$$\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} = +(-1) \begin{vmatrix} \textcircled{-1} & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} \\ - (3) \begin{vmatrix} -1 & 1 & -2 \\ \textcircled{3} & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} \\ + (0) \begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ \textcircled{0} & -1 & -1 \end{vmatrix} \begin{array}{l} \text{Who} \\ \text{cares?} \end{array}$$

Observe that the third minor is irrelevant, because we know that the third term will be 0, anyway. This is why we like choosing magic rows and columns that have “0”s in them!

There are various ways to write out the cofactor expansion quickly and accurately. With practice, you should find the one that works best for you. Some people may need to write out the step above.

We now have:

$$\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} = +(-1) \underbrace{\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix}}_{\substack{=-2-(-1) \\ =-1}} \\ - (3) \underbrace{\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix}}_{\substack{=-1-(2) \\ =-3}} \\ = (-1)(-1) - 3(-3) \\ = 10$$

Note 1: It is a coincidence that the magic entries  $-1$  and  $3$  are equal to their corresponding cofactors here.

Note 2: Observe that we got the same answer when we used Sarrus's Rule back in [Part B](#). We better have!

Note 3: Observe that we expand the determinant of a  $3 \times 3$  matrix in terms of the determinants of up to three  $2 \times 2$  matrices. Likewise, we expand the determinant of a  $4 \times 4$  matrix in terms of the determinants of up to four  $3 \times 3$  matrices. This is why we like exploiting "0"s along a magic row or column – and why it is often painful to compute determinants of large matrices using this cofactor expansion method.

Note 4: An efficient alternative method employs the EROs we discussed back in [Section 8.1](#) on Gaussian Elimination:

- Row Replacement EROs preserve determinants.

For example,

$$\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

- A single Row Interchange (Switch) ERO flips the sign of the determinant.

For example,

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = - \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

- When computing determinants, a nonzero scalar may be "factored out" of an entire row or an entire column.

For example,

$$\begin{vmatrix} 4 & 8 \\ 7 & 9 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 \\ 7 & 9 \end{vmatrix}$$

Note 5: The following basic determinant properties are useful, particularly in the Gaussian Elimination method for computing determinants:

- If a square matrix has a row or a column consisting of all “0”s, then its determinant is 0.

For example,

$$\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

- If a square matrix is in triangular form (i.e., has all “0”s above or below the main diagonal), then its determinant equals the product of the entries along the main diagonal.

For example,

$$\begin{vmatrix} 2 & 70 & 30 \\ 0 & 3 & 50 \\ 0 & 0 & 4 \end{vmatrix} = (2)(3)(4) = 24$$

Can you see how the above properties are derived from the Cofactor Expansion method for computing determinants?



Example

$$\text{Find } \begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}.$$

Solution

Remember that there is no nice analog to Sarrus's Rule here, because we are dealing with a  $4 \times 4$  matrix.

Let's expand along the 2<sup>nd</sup> row so that we can exploit its "0"s.

We have:

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} \text{ with partial sign matrix } \begin{vmatrix} + & - & + \\ & & - \\ & & \\ & & \end{vmatrix}$$

Observe that, as far as the sign matrix goes, we only need to know that the "-" sign corresponds to the magic "3." To find this out, you could either start with the "+" in the upper left corner and snake your way to that position (see above), or you could observe that the "3" is in

Row 2, Column 3, and  $(-1)^{2+3} = (-1)^5 = -1$ .

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = - (3) \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$

Use Sarrus's Rule or Cofactor Expansion. It turns out this equals 2. You show work!

$$\begin{aligned} &= -3(2) \\ &= -6 \end{aligned}$$

**PART D : THE CROSS PRODUCT OF TWO VECTORS IN  $\mathbf{R}^3$** 

In [Section 6.4](#), we discussed the dot product of two vectors in  $\mathbf{R}^n$  ( $n$ -dimensional real space).

There is another common way to multiply two vectors in  $\mathbf{R}^3$  (3-dimensional real space), specifically.

Given two vectors  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$  in  $\mathbf{R}^3$ , the cross product  $\mathbf{a} \times \mathbf{b}$  is given by:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

where  $\mathbf{i} = \langle 1, 0, 0 \rangle$ ,  $\mathbf{j} = \langle 0, 1, 0 \rangle$ , and  $\mathbf{k} = \langle 0, 0, 1 \rangle$  are the standard unit vectors in  $\mathbf{R}^3$ .

This notation is informal, because the determinant is only “officially” defined if our matrix consists only of scalars.

Note: Although the dot product operation is commutative (i.e.,  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$  for two vectors  $\mathbf{a}$  and  $\mathbf{b}$  in the same space), the cross product operation is not. In fact, the cross product operation is anticommutative, meaning that  $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ . Recall from [Notes 8.63](#) that:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Geometrically,  $\mathbf{a} \times \mathbf{b}$  is a vector that is perpendicular (or orthogonal) to both  $\mathbf{a}$  and  $\mathbf{b}$ .

## SECTION 8.5: APPLICATIONS OF DETERMINANTS

### PART A: CRAMER'S RULE FOR SOLVING SYSTEMS

A square system of linear equations is a system of  $n$  linear equations in  $n$  unknowns, where  $n \in \mathbf{Z}^+$ . Cramer's Rule uses determinants to solve such a system. For now, we assume that the unknowns are  $x, y$ , etc. and that they make up  $X$ , the vector of unknowns.

#### Cramer's Rule

Write the augmented matrix for the system  $AX = B$ :

$$\left[ A \mid B \right]$$

- $A$  is the coefficient matrix.

If the system is square,  $A$  will be a square matrix.

- $B$  is the right-hand side (*RHS*); you could use *RHS*, instead.

Compute the following determinants:

- Let  $D = |A|$ , or  $\det(A)$ .
- Let  $D_x = |A_x|$ , or  $\det(A_x)$ .

where  $A_x$  is identical to  $A$ , except that the  $x$ -column of  $A$  is replaced by  $B$ , the *RHS*.

(continued on next page)

Cramer's Rule (cont.)

- Let  $D_y = |A_y|$ , or  $\det(A_y)$ ,

where  $A_y$  is identical to  $A$ , except that the  $y$ -column of  $A$  is replaced by  $B$ , the *RHS*.

- $D_z$ ,  $A_z$ , etc. are defined analogously as necessary.

If  $D \neq 0$ , **there is a unique solution** given by:

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D} \quad (\text{if applicable}), \text{ etc.}$$

If  $D = 0$ , there is **not** a unique solution. Then:

- If all of the other determinants,  $D_x$ ,  $D_y$ , etc. are also 0, then the system has infinitely many solutions.
- Otherwise, the system has no solution. The solution set is  $\emptyset$ , the empty set.

Note: Observe that the formulas for  $x$ ,  $y$ , etc. fall apart if  $D = 0$ .

Note: In fact, if  $A$  is square, then its determinant  $D \neq 0$  if and only if  $A$  is invertible, which is true if and only if  $AX = B$  has a unique solution (given by  $X = A^{-1}B$ ). See the Inverse Matrix Method for solving systems in [Section 8.3, Part D](#).

Note: One advantage that this method has over Gaussian Elimination with Back-Substitution is that the value of one unknown can be found without having to find the values of any others.

Technical Note: For large systems, the Expansion by Cofactors Method for computing determinants (found in [Section 8.4, Part C](#)) may be impractical. See Notes 3, 4, and 5 in [Notes 8.63 and 8.64](#).

Example (Two linear equations in two unknowns)

$$\text{Solve the system: } \begin{cases} 2x - 9y = 5 \\ 3x - 3y = 11 \end{cases}$$

Solution

The augmented matrix is:

$$\begin{array}{cc} & x & y & & \\ & & & & \\ \left[ \begin{array}{cc|c} 2 & -9 & 5 \\ 3 & -3 & 11 \end{array} \right] \\ & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & & \\ & A & B & & \end{array}$$

Compute the necessary determinants:

Note: Your instructor may want you to show more work here.

$$D = |A| = \begin{vmatrix} x & y \\ 2 & -9 \\ 3 & -3 \end{vmatrix} = 21$$

Warning: When constructing the  $A_x$  and  $A_y$  matrices, which are “mutated” versions of the  $A$  matrix, remember to replace the correct column with  $B$ , the *RHS*. You replace the column corresponding to the subscript, which is the variable that the matrix helps solve for. See the Warning in the next Example.

$$D_x = |A_x| = \begin{vmatrix} B & y \\ 5 & -9 \\ 11 & -3 \end{vmatrix} = 84$$

$$D_y = |A_y| = \begin{vmatrix} x & B \\ 2 & 5 \\ 3 & 11 \end{vmatrix} = 7$$

Since  $D \neq 0$ , the system has a unique solution, which is given by:

$$x = \frac{D_x}{D} = \frac{84}{21} = 4$$

$$y = \frac{D_y}{D} = \frac{7}{21} = \frac{1}{3}$$

Warning: You may have been tempted to write down the fraction  $\frac{21}{7}$ . Remember that non-integers may appear in your solutions.

The solution set is then:  $\left\{ \left( 4, \frac{1}{3} \right) \right\}$ .

Note: Our solution may be checked in the original system.

Note: Observe that we can solve for  $x$  without solving for  $y$ .

Example (Three linear equations in three unknowns)

$$\text{Solve the system: } \begin{cases} x + z = 0 \\ x - 3y = 1 \\ 4y - 3z = 3 \end{cases}$$

Solution

The augmented matrix is:

$$\begin{array}{ccc|c} x & y & z & \\ \hline 1 & 0 & 1 & 0 \\ 1 & -3 & 0 & 1 \\ 0 & 4 & -3 & 3 \end{array}$$

$\underbrace{\hspace{10em}}_A \quad \underbrace{\hspace{2em}}_B$

Compute the necessary determinants:

Note: Your instructor may want you to show more work here.

$$D = |A| = \begin{vmatrix} x & y & z \\ 1 & 0 & 1 \\ 1 & -3 & 0 \\ 0 & 4 & -3 \end{vmatrix} = 13$$

**Warning:** When constructing the  $A_x$  matrix, remember to replace the  $x$ -column with  $B$ , the  $RHS$ , and leave the  $y$ - and  $z$ -columns intact. (If you remember this, then the two variable case may be less confusing.) The  $A_y$  and  $A_z$  matrices are constructed analogously.

$$D_x = |A_x| = \begin{vmatrix} B & y & z \\ 0 & 0 & 1 \\ 1 & -3 & 0 \\ 3 & 4 & -3 \end{vmatrix} = 13$$

$$D_y = |A_y| = \begin{array}{c} x \quad B \quad z \\ \left| \begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & -3 \end{array} \right| = 0 \end{array}$$

$$D_z = |A_z| = \begin{array}{c} x \quad y \quad B \\ \left| \begin{array}{ccc} 1 & 0 & 0 \\ 1 & -3 & 1 \\ 0 & 4 & 3 \end{array} \right| = -13 \end{array}$$

Since  $D \neq 0$ , the system has a unique solution, which is given by:

$$x = \frac{D_x}{D} = \frac{13}{13} = 1$$

$$y = \frac{D_y}{D} = \frac{0}{13} = 0$$

$$z = \frac{D_z}{D} = \frac{-13}{13} = -1$$

The solution set is then:  $\{(1, 0, -1)\}$ .

Note: Our solution may be checked in the original system.



**PART B: AREA AND VOLUME**

In Calculus: In Multivariable Calculus (Calculus III: Math 252 at Mesa), you may study triple scalar products (when dealing with three-dimensional vectors) and Jacobians, which employ the following ideas.

**Determinants and Area**

Assume that  $A$  is a  $2 \times 2$  matrix of real numbers. Consider the position vectors corresponding to either the rows or the columns of  $A$ .

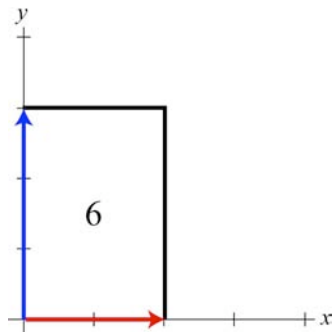
The area of the parallelogram determined by those vectors is given by  $|\det(A)|$ , or  $||A||$ , the absolute value of the determinant of  $A$ .

(If it is 0, the vectors are collinear – they lie on the same line, and the parallelogram is degenerate.)

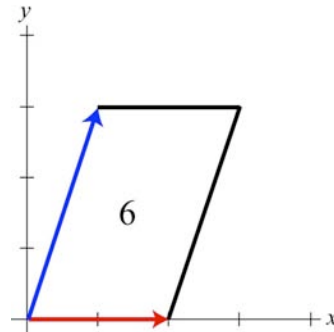
In these Examples, we will consider the position vectors corresponding to the **columns** of the matrices.

**Example**

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

**Example**

$$\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6$$

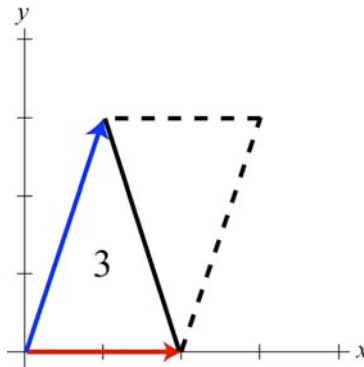


**Think About It:** Can you give other reasons why these parallelograms have the same area?

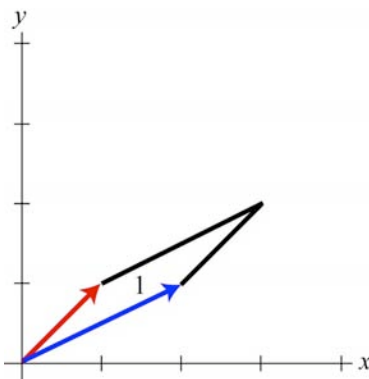
Follow-Up Example

The area of the **triangle** determined by the position vectors of interest equals **half** the area of the **parallelogram** determined by them.

$$\frac{1}{2} \left| \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \right| = \frac{1}{2} (6) = 3$$

Example

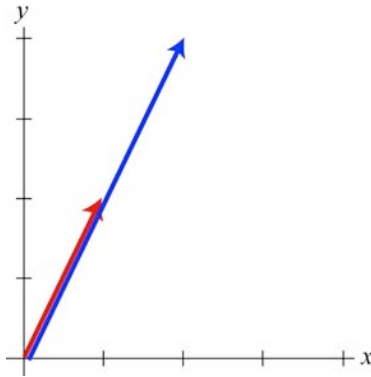
$$\left| \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \right| = |-1| = 1$$



Think About It: Why do you think the determinant is negative in this Example? Test your guess by trying out some examples of your own.

Example

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$$



The position vectors here are collinear.

Technical Note: We may analyze the row vectors or the column vectors of the matrix for the purposes of finding area or volume, because a square matrix and its transpose (see [Notes 8.35](#)) have the same determinant. i.e., If  $A$  is a square matrix, then  $\det(A) = \det(A^T)$ .

Technical Note: If the rows or the columns of a square matrix are reordered, then the determinant will change by at most a sign, and its absolute value stays the same. Therefore, the row or column vectors may be written in the matrix in any order for the purposes of finding area or volume.

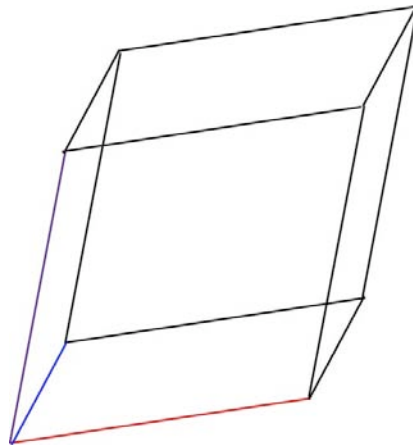
### Determinants and Volume

Assume that  $A$  is a  $3 \times 3$  matrix of real numbers. Consider the position vectors corresponding to either the rows or the columns of  $A$ .

The volume of the parallelepiped determined by those vectors is given by  $|\det(A)|$ , or  $\|A\|$ , the absolute value of the determinant of  $A$ .

(If it is 0, the vectors are coplanar – they lie on the same plane, and the parallelepiped is degenerate.)

A parallelepiped:



In Calculus: In Multivariable Calculus ([Calculus III: Math 252 at Mesa](#)), you may study the triple scalar product (“TSP”) of the row or column vectors ( $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ , say) in the  $3 \times 3$  matrix  $A$ . The TSP equals  $\det(A)$ . The TSP can also be written, and is usually defined, in terms of dot and cross products as:  $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ , or  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ . For more information, see [my Math 252 notes on Section 14.4 in the Swokowski Calculus text](#).