CHAPTER 7
MATRIX ALGEBRA

7.0 MATRIX ALGEBRA

Definition 7.1: Matrix

Matrix is a rectangular array of numbers which called elements arranged in rows and columns. A matrix with m rows and n columns is called of order $m \times n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = [a_{ij}]_{m\times n}$$

a_{ij} indicates the element in the i^{th} row and the j^{th} column.

7.1 ELEMENTARY ROW OPERATIONS (ERO)

- Important method to find the inverse of a matrix and to solve the system of linear equations.
- The following notations will be used while applying ERO

1. Interchange the i^{th} row with the j^{th} row of the matrix. This process is denoted as $\boldsymbol{B_i \leftrightarrow B_j}$.
2. Multiply the i^{th} row of the matrix with the scalar k where $k \neq 0$. This process is denoted as $\boldsymbol{kB_i}$.
3. Add the i^{th} row, that is multiplied by the scalar h to the j^{th} row that has been multiplied by the scalar k, where $h \neq 0$, and $k \neq 0$. This process can be denoted as $\boldsymbol{hB_i + kB_j}$. The purpose of this process is to change the elements in the i^{th} row.

Example 7.1:

Given the matrix $A = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 1 \\ -3 & 1 & 2 \end{pmatrix}$, perform the following operations consecutively: $B_1 \leftrightarrow B_2$, $B_2 + (-2)B_1$, $B_3 + 3B_1$, $B_3 + (-7)B_2$ and $-\frac{1}{2}B_3$.

Solution:

$$\begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 1 \\ -3 & 1 & 2 \end{pmatrix} \xrightarrow[B_1 \leftrightarrow B_2]{} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & 1 & 2 \end{pmatrix} \xrightarrow[B_2+(-2)B_1]{} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -3 & 1 & 2 \end{pmatrix} \xrightarrow[B_3+3B_1]{}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 7 & 5 \end{pmatrix} \xrightarrow[B_3+(-7)B_2]{} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow[-\frac{1}{2}B_3]{} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Notes:
If the matrix A is transformed to the matrix B by using ERO, then the matrix A is called *equivalent matrix* to the matrix B and can be denoted as $A \sim B$.

Definition 7.2: Rank of a Matrix

The rank of a matrix is the number of row that is non zero in that *echelon matrix* or *reduced echelon matrix*. The rank of matrix A is denoted as $p(A)$.

What is ***echelon matrix*** and ***reduced echelon matrix***?

$\begin{pmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow p(A) = 3$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow p(A) = 3$ $\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow p(A) = 2$	$\begin{pmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix} \Rightarrow p(A) = 3$ $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow p(A) = 3$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow p(A) = 2$
Example of Echelon Matrix and its rank of matrix	Example of Reduced Echelon Matrix and its rank of matrix

How can we get echelon matrix and reduced echelon matrix?

Using ERO of course! And the operation is not unique.

Example 7.2:

Given

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{pmatrix}$$

obtain

a) Echelon matrix

b) Reduced echelon matrix

c) Rank of matrix A

Solution:

a) $$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{pmatrix} \xrightarrow[B_3+(-3)B_1]{B_2+(-2)B_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & -5 & -10 \end{pmatrix} \xrightarrow[\left(-\frac{1}{5}\right)B_3]{\left(-\frac{1}{7}\right)B_2}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4/7 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[B_3+(-1)B_2]{} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4/7 \\ 0 & 0 & 10/7 \end{pmatrix} \xrightarrow[7/10 B_3]{} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4/7 \\ 0 & 0 & 1 \end{pmatrix}$$

b) $$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4/7 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[B_2+\left(-\frac{4}{7}\right)B_3]{B_1+(-2)B_2} \begin{pmatrix} 1 & 0 & 13/7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[B_1+\left(-\frac{13}{7}\right)B_3]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) $p(A) = 3$

7.2 DETERMINANT OF A MATRIX

- A scalar value that can be used to find the inverse of a matrix.
- The inverse of the matrix will be used to solve a system of linear equations.

Definition 7.3 : Determinant
The determinant of a matrix A is a scalar value and denoted by $|\boldsymbol{A}|$ or $\mathbf{det}\ (\boldsymbol{A})$.

I - The determinant of a 2x2 matrix is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

II - The determinant of a 3x3 matrix is defined by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

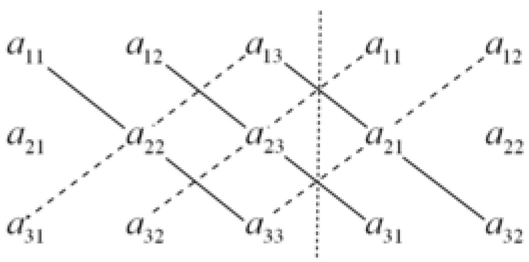


Figure 7.1: The determinant of a 3x3 matrix can be calculated by its diagonal

III - The determinant of a n x n matrix can be calculated by using **cofactor expansion**. (Note: *This involves minor and cofactor so we will see this method after reviewing minor and cofactor of a matrix)*

Definition 7.4: Minor

If

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \ddots & a_{ij} & \ddots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

then the **minor** of $\boldsymbol{a_{ij}}$, denoted by $\mathbf{D_{ij}}$ is the determinant of the submatrix that results from removing the i^{th} row and j^{th} column of **A**.

<u>Example 7.3:</u>

Find the minor $\mathbf{D_{12}}$ for matrix **A**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

<u>Solution:</u>

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \mathbf{D}_{\mathbf{12}} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

<u>Example 7.4:</u>

Given

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 2 & 4 & -5 \end{pmatrix}$$

Calculate the minor of a_{11} and a_{32}

<u>Solution:</u>

$$\mathrm{D}_{11} = \begin{vmatrix} -1 & 3 \\ 4 & -5 \end{vmatrix} = (-1)(-5) - (4)(3) = -7$$

$$\mathrm{D}_{32} = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = (1)(3) - (0)(2) = 3$$

Definition 7.5: Cofactor
If **A** is a square matrix $n \times n$, then the cofactor of $\boldsymbol{a_{ij}}$ is given by

$$\mathbf{A_{ij}} = (-1)^{i+j}\mathbf{D_{ij}}$$

Example 7.5:

Find the cofactor A_{23} from the given matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix}$$

Solution:

$$A_{23} = (-1)^{2+3}D_{23}$$
$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 4 \\ -1 & 9 \end{vmatrix} = (-1)\big(9-(-4)\big) = -13$$

Example 7.6:

From Example 7.4, find the cofactor of a_{11} and a_{32}

Solution:

$$A_{11} = (-1)^{1+1}D_{11} = (-1)^2\begin{vmatrix} -1 & 3 \\ 4 & -5 \end{vmatrix} = (1)(-7) = -7$$
$$A_{32} = (-1)^{3+2}D_{32} = (-1)^5\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = (-1)(3) = -3$$

Theorem 7.1: Cofactor Expansion

If A is an n x n matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The determinant of A ($\det(A)$) can be written as the sum of its cofactors multiplied by the entries that generated them.

a) Cofactor expansion along the j^{th} column

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}$$

b) Cofactor expansion along the i^{th} row

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum^{n} a_{ij}C_{ij}$$

Example 7.7:

Compute the determinant of the following matrix

a) $A = \begin{pmatrix} 4 & 2 & 1 \\ -2 & -6 & 3 \\ -7 & 5 & 0 \end{pmatrix}$ b) $B = \begin{pmatrix} 5 & -2 & 2 & 7 \\ 1 & 0 & 0 & 3 \\ -3 & 1 & 5 & 0 \\ 3 & -1 & -9 & 4 \end{pmatrix}$

Solution:

a) Expanding along the third row

$$|A| = (-7)(-1)^{3+1}\begin{vmatrix} 2 & 1 \\ -6 & 3 \end{vmatrix} + (5)(-1)^{3+2}\begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix}$$

$$+(0)(-1)^{3+3}\begin{vmatrix} 4 & 2 \\ -2 & -6 \end{vmatrix}$$

$$|A| = (-7)(1)(12) + (5)(-1)(14) + 0 = -154$$

b) Expanding along the second row

$$|B| = (1)(-1)^{2+1}\begin{vmatrix} -2 & 2 & 7 \\ 1 & 5 & 0 \\ -1 & -9 & 4 \end{vmatrix} + (0)(-1)^{2+2}\begin{vmatrix} 5 & 2 & 7 \\ -3 & 5 & 0 \\ 3 & -9 & 4 \end{vmatrix}$$
$$+(0)(-1)^{2+3}\begin{vmatrix} 5 & -2 & 7 \\ -3 & 1 & 0 \\ 3 & -1 & 4 \end{vmatrix} + (3)(-1)^{2+4}\begin{vmatrix} 5 & -2 & 2 \\ -3 & 1 & 5 \\ 3 & -1 & -9 \end{vmatrix}$$

$$|B| = (1)(-1)(-76) + 0 + 0 + (3)(1)(4) = 88$$

Example 7.8:

Given

$$B = \begin{pmatrix} 1 & 5 & 7 \\ -3 & 0 & 4 \\ 1 & 0 & -3 \end{pmatrix},$$

calculate the determinant of B.

Solution:

Since the second column has two zero elements, cofactor expansion can be done along the second column.

$$|B| = (5)(-1)^{1+2}\begin{vmatrix} -3 & 4 \\ 1 & -3 \end{vmatrix} + (0)(-1)^{2+2}\begin{vmatrix} 1 & 7 \\ 1 & -3 \end{vmatrix}$$
$$+ (0)(-1)^{3+2}\begin{vmatrix} 1 & 7 \\ -3 & 4 \end{vmatrix}$$
$$= (5)(-1)^3(5) + 0 + 0 = -25$$

PROPERTIES OF THE DETERMINANT

PROPERTY 1: If A is a square matrix, then $|A| = |A^T|$. For example,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}.$$

PROPERTY 2: If the matrix B is obtained by interchanging with any two rows or two columns of the matrix A, then $|A| = -|B|$. For example,

$$\begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{vmatrix} = -\begin{vmatrix} \boldsymbol{c} & \boldsymbol{d} \\ \boldsymbol{a} & \boldsymbol{b} \end{vmatrix}.$$

PROPERTY 3: If any two rows (or columns) of the matrix A are identical, then $|A| = 0$. For example,

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0.$$

PROPERTY 4: If the matrix B is obtained by multiplying every element in the row or the column of the matrix A with a scalar k, then $|B| = k|A|$. For example,

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

PROPERTY 5: If the matrix B is obtained by multiplying a scalar k of one row of the matrix A is added to another row of A, then $|B| = |A|$. This operation is denoted as $B_1 \to B_1 + kB_2$. For example,

$$\begin{vmatrix} a + kc & b + kd \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

PROPERTY 6: If the matrix A has a zero row, then $|A| = 0$. For example,

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0.$$

By using the right properties, we can also find the determinant.

Example 7.9:

Evaluate $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 2 & 6 & 4 & 8 \\ 3 & 1 & 1 & 2 \end{vmatrix}$

Solution:

1. From Property 4, we can factorize 2 from row 3.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 2 & 6 & 4 & 8 \\ 3 & 1 & 1 & 2 \end{vmatrix} = \mathbf{2}\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 1 & 3 & 2 & 4 \\ 3 & 1 & 1 & 2 \end{vmatrix}$$

2. Using Property 5, we can perform algebraic operations for row 2, 3, 4 and still get the same determinant as the original matrix.

$$\mathbf{2}\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 1 & 3 & 2 & 4 \\ 3 & 1 & 1 & 2 \end{vmatrix} \xrightarrow[\substack{B_2+(-5)B_1 \\ B_3+(-1)B_1 \\ B_4+(-3)B_1}]{} \mathbf{2}\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & -8 & -10 \end{vmatrix}$$

3. Now, using Property 2, we interchange the second with the third row

$$\mathbf{2}\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & -8 & -10 \end{vmatrix} = \mathbf{(2)(-1)}\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -4 & -8 & -12 \\ 0 & -5 & -8 & -10 \end{vmatrix}$$

4. Again, by using Property 5, we can perform the algebraic operations

$$\mathbf{(2)(-1)}\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -4 & -8 & -12 \\ 0 & -5 & -8 & -10 \end{vmatrix} \xrightarrow[\substack{B_3+(4)B_2 \\ B_4+(5)B_2}]{} \mathbf{(-2)}\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -12 & -12 \\ 0 & 0 & -13 & -10 \end{vmatrix}$$

5. By using Property 4, we can factorize -12 from row 3

$$(-2)\begin{vmatrix}1 & 2 & 3 & 4\\0 & 1 & -1 & 0\\0 & 0 & -12 & -12\\0 & 0 & -13 & -10\end{vmatrix} = (-2)(-12)\begin{vmatrix}1 & 2 & 3 & 4\\0 & 1 & -1 & 0\\0 & 0 & 1 & 1\\0 & 0 & -13 & -10\end{vmatrix}$$

6. Using Property 5, we can get a triangular matrix which can easily give us the determinant value.

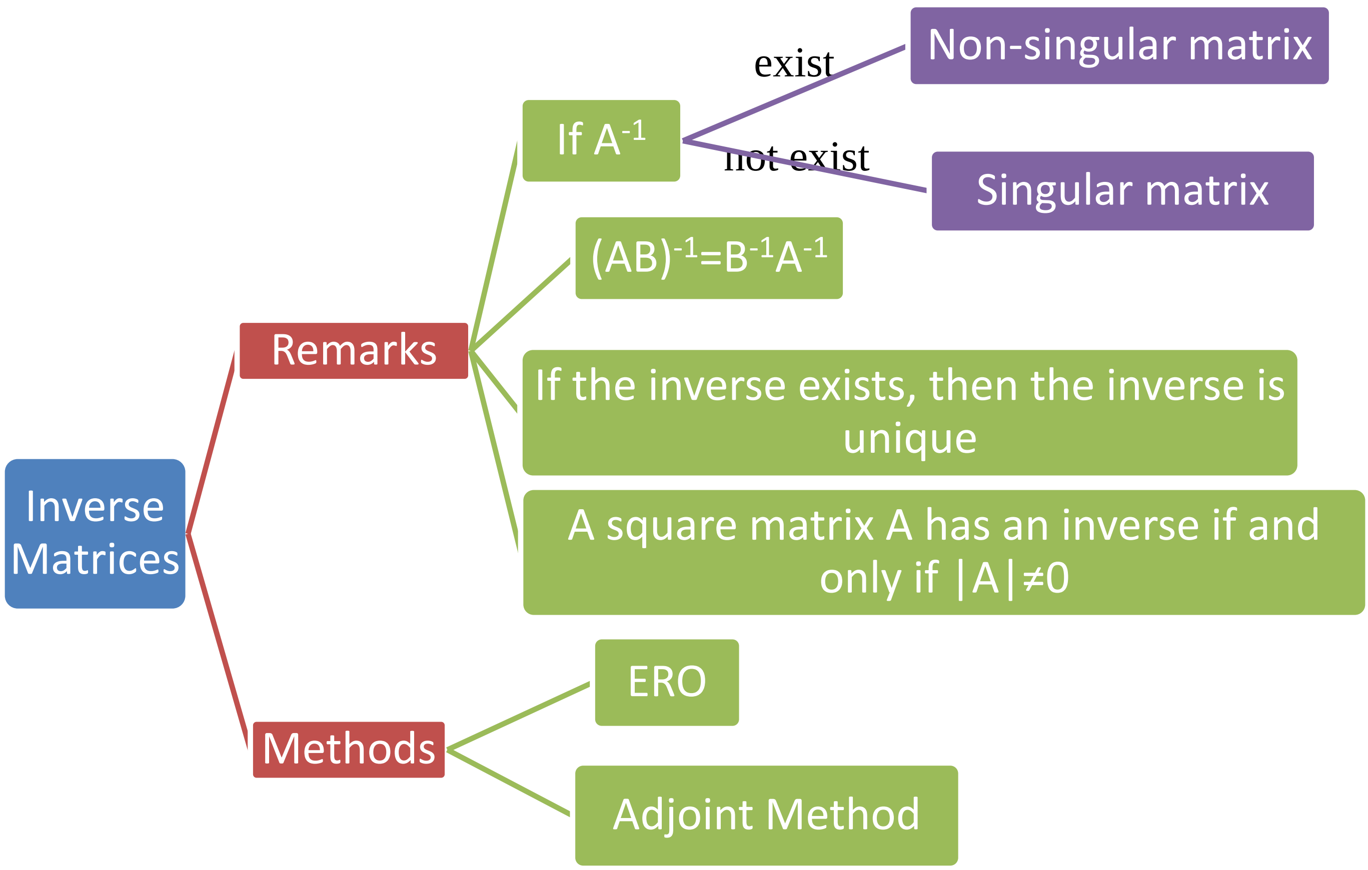
$$(24)\begin{vmatrix}1 & 2 & 3 & 4\\0 & 1 & -1 & 0\\0 & 0 & 1 & 1\\0 & 0 & -13 & -10\end{vmatrix} = (24)\begin{vmatrix}1 & 2 & 3 & 4\\0 & 1 & -1 & 0\\0 & 0 & 1 & 1\\0 & 0 & 0 & 3\end{vmatrix} = (24)3 = 72$$

7. Therefore,

$$\begin{vmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\\2 & 6 & 4 & 8\\3 & 1 & 1 & 2\end{vmatrix} = 72$$

7.3 INVERSE MATRICES

Definition 7.6: Inverse Matrix
If A and B are $n \times n$ matrices, then the matrix B is the inverse of matrix A (or vice versa) if and only if $AB = BA = I$.



7.3.1 Finding Inverse Matrices using ERO

STEP 1:
Write AI in the form of augmented matrix $(A|I)$.

STEP 2:
Perform ERO until we get the new augmented matrix $(I|B)$.

STEP 3:
Therefore $A^{-1} = B$.

Example 7.11:

Calculate the inverse of the following matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 3 & 5 & 1 \\ 6 & 4 & 2 \end{pmatrix}$$

Solution:

STEP 1:

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 & 1 & 0 \\ 6 & 4 & 2 & 0 & 0 & 1 \end{array}\right)$$

STEP 2:

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 & 1 & 0 \\ 6 & 4 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[B_3+(-6)B_1]{B_2+(-3)B_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 11 & -8 & -3 & 1 & 0 \\ 0 & 16 & -16 & -6 & 0 & 1 \end{array}\right) \xrightarrow{} {}^{B_2}/_{11} \quad {}^{B_3}/_{16}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -{}^{8}/_{11} & -{}^{3}/_{11} & {}^{1}/_{11} & 0 \\ 0 & 1 & -1 & -{}^{3}/_{8} & 0 & {}^{1}/_{16} \end{array}\right) \xrightarrow[B_3+(-1)B_2]{B_1+(2)B_2} \left(\begin{array}{ccc|ccc} 1 & 0 & {}^{17}/_{11} & {}^{5}/_{11} & {}^{2}/_{11} & 0 \\ 0 & 1 & -{}^{8}/_{11} & -{}^{3}/_{11} & {}^{1}/_{11} & 0 \\ 0 & 0 & -{}^{3}/_{11} & -{}^{9}/_{88} & -{}^{1}/_{11} & {}^{1}/_{16} \end{array}\right)$$

$$\xrightarrow[-{}^{11B_3}/_{3}]{} \left(\begin{array}{ccc|ccc} 1 & 0 & {}^{17}/_{11} & {}^{5}/_{11} & {}^{2}/_{11} & 0 \\ 0 & 1 & -{}^{8}/_{11} & -{}^{3}/_{11} & {}^{1}/_{11} & 0 \\ 0 & 0 & 1 & {}^{3}/_{8} & {}^{1}/_{3} & -{}^{11}/_{48} \end{array}\right) \xrightarrow[B_2+({}^{8}/_{11})B_3]{B_1+(-{}^{17}/_{11})B_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -{}^{1}/_{8} & -{}^{1}/_{3} & {}^{17}/_{48} \\ 0 & 1 & 0 & 0 & {}^{1}/_{3} & -{}^{1}/_{6} \\ 0 & 0 & 1 & {}^{3}/_{8} & {}^{1}/_{3} & -{}^{11}/_{48} \end{array}\right)$$

STEP 3:

$$A^{-1} = \begin{pmatrix} -{}^{1}/_{8} & -{}^{1}/_{3} & {}^{11}/_{48} \\ 0 & {}^{1}/_{3} & -{}^{1}/_{6} \\ {}^{3}/_{8} & {}^{1}/_{3} & -{}^{11}/_{48} \end{pmatrix} = \frac{1}{48}\begin{pmatrix} -6 & -16 & 17 \\ 0 & 16 & -8 \\ 18 & 16 & -11 \end{pmatrix}$$

7.3.2 Finding Inverse Matrices using Adjoint Method

Definition 7.7: Adjoint of a Matrix

The **adjoints of a square matrix *A*** is the transpose of cofactor matrix which can be obtained by interchanging every element a_{ij} with the cofactor c_{ij} and denoted as

$$adj(\boldsymbol{A}) = [c_{ij}]^T = [c_{ij}].$$

If $|A| \neq 0$, then A^{-1} exists. Therefore the inverse matrix is,

$$A^{-1} = \frac{1}{|A|}\text{adj}(A).$$

STEPS TO FIND THE INVERSE MATRIX USING ADJOINT METHOD.

STEP 1: Calculate the determinant of A.

i) If $|A| = 0$, stop the calculation because the inverse does not exist.

ii) If $|A| \neq 0$, continue to STEP 2.

STEP 2: Calculate the cofactor matrix $[c_{ij}]$.

STEP 3: Find the adjoint matrix A by finding the transpose of the cofactor matrix $[c_{ij}]$, that is

$$\text{adj}(A) = [c_{ij}]^T = [c_{ij}].$$

STEP 4: Substitute the results from STEP 1 to STEP 3 in the formula

$$A^{-1} = \frac{1}{|A|}\text{adj}(A).$$

Example 7.12:

Calculate the inverse of the following matrix

$$A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & -6 & 3 \\ -7 & 5 & 0 \end{bmatrix}$$

Solution:

Step 1: Calculate the determinant of A.

$$|A| = -154 \neq 0$$

Step 2: Find the cofactor matrix.

$$C_{11} = (-1)^2 \begin{vmatrix} -6 & 3 \\ 5 & 0 \end{vmatrix} = -15 \quad C_{12} = (-1)^3 \begin{vmatrix} -2 & 3 \\ -7 & 0 \end{vmatrix} = -21 \quad C_{13} = (-1)^4 \begin{vmatrix} -2 & -6 \\ -7 & 5 \end{vmatrix} = -52$$

$$C_{21} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 0 \end{vmatrix} = 5 \quad C_{22} = (-1)^4 \begin{vmatrix} 4 & 1 \\ -7 & 0 \end{vmatrix} = 7 \quad C_{23} = (-1)^5 \begin{vmatrix} 4 & 2 \\ -7 & 5 \end{vmatrix} = -34$$

$$C_{31} = (-1)^4 \begin{vmatrix} 2 & 1 \\ -6 & 3 \end{vmatrix} = 12 \quad C_{32} = (-1)^5 \begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} = -14 \quad C_{33} = (-1)^6 \begin{vmatrix} 4 & 2 \\ -2 & -6 \end{vmatrix} = -20$$

$\therefore$ Matrix of cofactor, $C = \begin{pmatrix} -15 & -21 & -52 \\ 5 & 7 & -34 \\ 12 & -14 & -20 \end{pmatrix}$

Step 3: Adjoint of A

$$Adj(A) = \begin{pmatrix} -15 & -21 & -52 \\ 5 & 7 & -34 \\ 12 & -14 & -20 \end{pmatrix}^T = \begin{pmatrix} -15 & 5 & 12 \\ -21 & 7 & -14 \\ -52 & -34 & -20 \end{pmatrix}$$

Step 4: Find A^{-1}

$$A^{-1} = -\frac{1}{154}\begin{pmatrix} -15 & 5 & 12 \\ -21 & 7 & -14 \\ -52 & -34 & -20 \end{pmatrix}$$

EXERCISE:

1. Calculate the inverse of the following matrices by using
   (i) Elementary Row Operations (ERO) methods
   (ii) Adjoint Method

(a) $\begin{pmatrix} -3 & -1 & 6 \\ 2 & 1 & -4 \\ -5 & -2 & 11 \end{pmatrix}$

b) $\begin{pmatrix} -3 & 1 & 2 \\ 2 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -4 \\ -5 & 2 & 1 \end{pmatrix}$

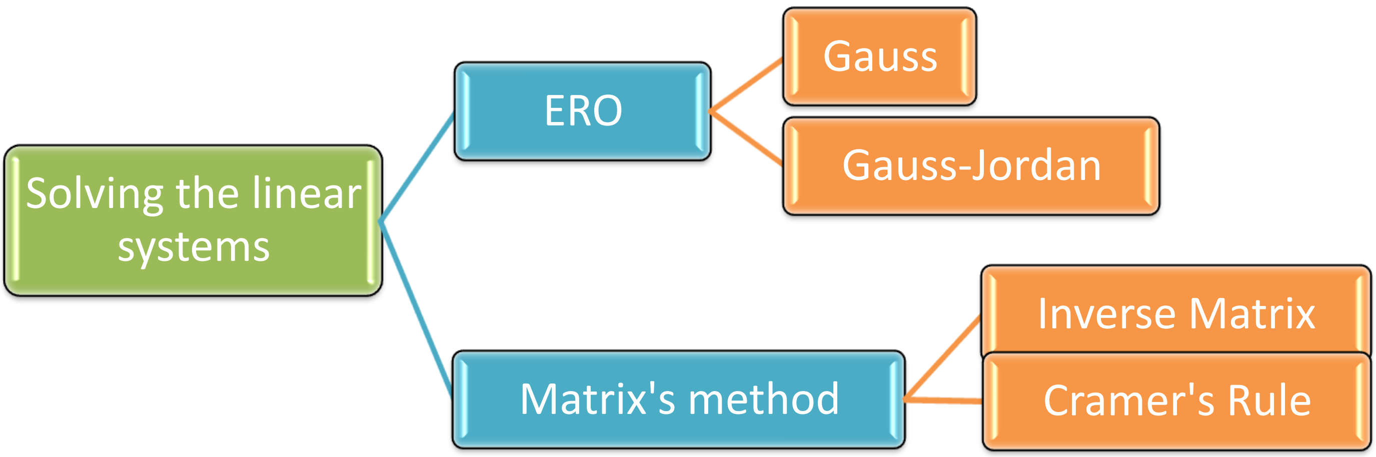
7.4 SYSTEMS OF LINEAR EQUATIONS

❖ A system of linear equations with m linear equations and n number of variables can be written as,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

❖ A solution to a linear system are real values of $x_1, x_2, x_3, \ldots, x_n$ which satisfy every equations in the linear systems.

❖ If the solution does not exist, then the system is inconsistent.



Non-homogeneous system

- *p(A)=p(A|b)* = number of variables ↦ the system has a unique solution.
- *p(A)* = *p(A|b)* < number of variables ↦ the system has many solutions.
- *p(A)* < *p(A|b)* = the number of variables ↦ the system has no solution.

Homogeneous system

- *p(A)* = number of variables ↦ the system has trivial solution.
- *p(A)* < the number of variables ↦ the system has many solutions.

7.4.1 Gauss Elimination Method

Gauss Elimination is a method of solving a linear system $A\mathbf{x}=\mathbf{b}$ by bringing the augmented matrix

$$[A : b] = \left(\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & b_3 \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_4 \end{array}\right)$$

to an ***echelon matrix***

$$\left(\begin{array}{cccc|c} 1 & c_{21} & \cdots & c_{1n} & d_1 \\ 0 & 1 & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & d_3 \\ 0 & 0 & \cdots & 1 & d_4 \end{array}\right).$$

Then the solution is found by using back substitution.

Example 7.13:

Solve the following system by using Gauss Elimination method.

$$2x_1 - 3x_2 - x_3 + 2x_4 + 3x_5 = 4$$
$$4x_1 - 4x_2 - x_3 + 4x_4 + 11x_5 = 4$$
$$2x_1 - 5x_2 - 2x_3 + 2x_4 - x_5 = 9$$
$$2x_2 + x_3 + 4x_5 = -5$$

Solution:

STEP 1: Construct the augmented matrix

$$\left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{array}\right)$$

STEP 2: Use ERO to transform this matrix into the following echelon matrix

$$\left(\begin{array}{ccccc|c} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

STEP 3: Solve using back substitution

$$x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 + x_4 + \frac{3}{2}x_5 = 2$$
$$x_2 + \frac{1}{2}x_3 + \frac{5}{2}x_5 = -2$$
$$x_5 = 1$$

Set $x_3 = s$ and $x_4 = t$,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -(25/4) - (1/4)s - t \\ -(9/2) - (1/2)s \\ s \\ t \\ 1 \end{pmatrix}$$

7.4.2 Gauss-Jordan Elimination Method

Gauss Elimination is a method of solving a linear system $A\mathbf{x} = \mathbf{b}$ by bringing the augmented matrix

$$[A : b] = \left(\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & b_3 \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_4 \end{array}\right)$$

to a ***reduced echelon form.*** Then the solution is found by using back substitution.

<u>Example 7.14:</u>

By using the same matrix in Example 7.13, find the solution for the linear system by using Gauss-Jordan Elimination method.

Solution:

From STEP 2 in Example 7.13, we can use ERO to find the reduced echelon matrix for the augmented matrix.

$$\left(\begin{array}{ccccc|c} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[B_2+\left(-\frac{5}{2}\right)B_3]{B_1+\left(\frac{3}{2}\right)B_2} \left(\begin{array}{ccccc|c} 1 & 0 & 1/4 & 1 & 21/4 & -1 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[B_1+\left(-\frac{21}{4}\right)B_3]{} \left(\begin{array}{ccccc|c} 1 & 0 & 1/4 & 1 & 0 & -25/4 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

From the reduced echelon matrix, we will get the following equations

$x_5 = 1$

$x_2 = -(9/2) - (1/2)x_3$

$x_1 = -(25/4) - (1/4)x_3 - x_4$

By setting $x_3 = s$ and $x_4 = t$,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -(25/4) - (1/4)s - t \\ -(9/2) - (1/2)s \\ s \\ t \\ 1 \end{pmatrix}$$

EXERCISE:

1. Solve the linear system by using
   (i) Gauss elimination method
   (ii) Gauss-Jordan elimination method

   a) $y + z = 2,$
   $2x + 3z = 5,$
   $x + y + z = 3$

   b) $x - 2y + 3z = -2,$
   $-x + y - 2z = 3,$
   $2x - y + 3z = 1$

7.4.3 Inverse Matrix Method

If $|A| \neq 0$ and $A\mathbf{x} = \mathbf{b}$ represents the linear equations where A is an $n \times n$ matrix and B is an $n \times 1$ matrix, then the solution for the system is given as

$$\mathbf{x} = A^{-1}\mathbf{b}$$

<u>Example 7.15:</u>

Use the method of inverse matrix to determine the solution to the following system of linear equations.

$$3x_1 - x_2 + 5x_3 = -2$$
$$-4x_1 + x_2 + 7x_3 = 10$$
$$2x_1 + 4x_2 - x_3 = 3$$

<u>Solution:</u>

STEP 1: Check whether $|A| \neq 0$.

$$\underbrace{\begin{bmatrix} 3 & -1 & 5 \\ -4 & 1 & 7 \\ 2 & 4 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -2 \\ 10 \\ 3 \end{bmatrix}}_{\mathbf{b}}$$

$$\begin{aligned} |A| &= (3)(1)(-1) + (-1)(7)(2) + (5)(-4)(4) \\ &\quad -(-1)(-4)(-1) - (2)(1)(5) - (4)(7)(3) \\ &= -187 \neq 0 \end{aligned}$$

STEP 2: Find A^{-1}.by using Adjoint Method or ERO.

i) Matrix of cofactor and adj(A),

$$C = \begin{pmatrix} \begin{vmatrix} 1 & 7 \\ 4 & -1 \end{vmatrix} & -\begin{vmatrix} -4 & 7 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} -4 & 1 \\ 2 & 4 \end{vmatrix} \\ -\begin{vmatrix} -1 & 5 \\ 4 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix} \\ \begin{vmatrix} -1 & 5 \\ 1 & 7 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ -4 & 7 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ -4 & 1 \end{vmatrix} \end{pmatrix}$$

$$C = \begin{pmatrix} -29 & 10 & -18 \\ 19 & -13 & -14 \\ -12 & -41 & -1 \end{pmatrix}, \text{adj}(A) = C^T = \begin{pmatrix} -29 & 19 & -12 \\ 10 & -13 & -41 \\ -18 & -14 & -1 \end{pmatrix}$$

ii) $A^{-1} = \frac{1}{-187}\begin{pmatrix} -29 & 19 & -12 \\ 10 & -13 & -41 \\ -18 & -14 & -1 \end{pmatrix}$

$$= \begin{pmatrix} \frac{29}{187} & -\frac{19}{187} & \frac{12}{187} \\ -\frac{10}{187} & \frac{13}{187} & \frac{41}{187} \\ \frac{18}{187} & \frac{14}{187} & \frac{1}{187} \end{pmatrix}$$

STEP 3: Solution for **x** is given by

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{28}{187} & -\frac{19}{187} & \frac{12}{187} \\ -\frac{10}{187} & \frac{13}{187} & \frac{41}{187} \\ \frac{18}{187} & \frac{14}{187} & \frac{1}{187} \end{bmatrix} \begin{bmatrix} -2 \\ 10 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{212}{187} \\ \frac{273}{187} \\ \frac{107}{187} \end{bmatrix}$$

EXERCISE

1) Solve the following system linear equations by using Inverse Matrix Method

(a) $x_1 + x_2 + 2x_3 = 7$

$x_1 - x_2 - 3x_3 = -6$

$2x_1 + 3x_2 + x_3 = 4$

(b) $2x_1 + 3x_2 + x_3 = 11$

$2x_1 - 2x_2 - 3x_3 = 5$

$3x_1 - 5x_2 + 2x_3 = -3$

7.4.4 Cramer's Rule

Given the system of linear equations $A\mathbf{x} = \mathbf{b}$, where A is an n x n matrix, $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ matrices. If $|A| \neq 0$, then the solution to the system is given by,

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \dots, x_n = \frac{|A_n|}{|A|}$$

for $i = 1,2,\dots,n$ where A_i is the matrix found by replacing the i^{th} column of A with $\mathbf{b}$.

<u>Example 7.16:</u>

Use Cramer's rule to determine the solution to the following system of linear equations.

$$3x_1 - x_2 + 5x_3 = -2$$
$$-4x_1 + x_2 + 7x_3 = 10$$
$$2x_1 + 4x_2 - x_3 = 3$$

<u>Solution:</u>

1. Test whether $|A| \neq 0$, or not.

$$\underbrace{\begin{bmatrix} 3 & -1 & 5 \\ -4 & 1 & 7 \\ 2 & 4 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -2 \\ 10 \\ 3 \end{bmatrix}}_{\mathbf{b}}$$

$$\begin{aligned} |A| &= (3)(1)(-1) + (-1)(7)(2) + (5)(-4)(4) \\ &\quad -(-1)(-4)(-1) - (2)(1)(5) - (4)(7)(3) \\ &= -187 \neq 0 \end{aligned}$$

By using the Cramer's rule,

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} \boxed{-2} & -1 & 5 \\ \boxed{10} & 1 & 7 \\ \boxed{3} & 4 & -1 \end{vmatrix}}{-187} = -\frac{212}{187}$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 3 & \boxed{-2} & 5 \\ -4 & \boxed{10} & 7 \\ 2 & \boxed{3} & -1 \end{vmatrix}}{-187} = \frac{273}{187}$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 3 & -1 & \boxed{-2} \\ -4 & 1 & \boxed{10} \\ 2 & 4 & \boxed{3} \end{vmatrix}}{-187} = \frac{107}{187}$$

EXERCISE:

Solve the following system linear equations by using Cramer's Rule Method.

(a)

$$x_1 + x_2 + 2x_3 = 7$$
$$x_1 - x_2 - 3x_3 = -6$$
$$2x_1 + 3x_2 + x_3 = 4$$

(b)

$$2x_1 + 3x_2 + x_3 = 11$$
$$2x_1 - 2x_2 - 3x_3 = 5$$
$$3x_1 - 5x_2 + 2x_3 = -3$$

7.5 EIGENVALUES & EIGENVECTORS

7.5.1 Eigenvalues & Eigenvectors

Definition 7.8: Eigenvalues & Eigenvectors

Let A be an n x n matrix and the scalar λ is called an eigenvalue of A if there is a non zero vector $\boldsymbol{x}$ such that

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

The scalar λ is called an **eigenvalue** of A corresponding to the eigenvector $\boldsymbol{x}$**.**

Example 7.17:

Show that $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$. Hence, find the corresponding eigenvalue.

Solution:

$$A\boldsymbol{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\boldsymbol{x}.$$

Therefore, the corresponding eigenvalue is 3.

Definition 7.9: Eigenvalues

The eigenvalues of an n x n matrix A are the n zeroes of the polynomial $P(\lambda) = |A - \lambda I|$ or equivalently the n roots of the n^{th} degree polynomial equation $|A - \lambda I| = 0$.

Example 7.18:

Determine the eigenvalues and eigenvector for the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$

Solution:

Step 1: Write down the characteristic equation.

$$\left| \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix} \right| = 0$$

$$P(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Step 2: Find the roots/eigenvalues

By using trial and error, we can take $\lambda = 1$ and it will give

$$P(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 0$$

Thus $(\lambda - 1)$ is a factor for $P(\lambda)$.
By using long division, the other two factors are $(\lambda - 2)$ and $(\lambda - 3)$.
Therefore,

$$P(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

Hence, the eigenvalues of matrix A are $\lambda = 1, 2, 3$.

Step 3: Use the eigenvalues to find the eigenvectors using formula $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

When $\lambda = 1$:

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (1)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Using ERO

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow[B_1+(-1)B_3]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow[B_3+(1)B_1]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[B_3+(-1)B_2]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} x_1 &= 0 \\ x_2 &= -2x_3 = -2k \\ x_3 &= k \end{aligned}$$

Therefore,

$\boldsymbol{x} = \begin{pmatrix} 0 \\ -2k \\ k \end{pmatrix} = k\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ and the corresponding eigenvector is $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

<u>When $\lambda = 2$:</u>

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (2)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Using ERO

$$\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix} \xrightarrow[B_3+(-1)B_1]{} \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{gathered} 2x_3 = -x_3 = 0 \ \Rightarrow \ x_3 = 0 \\ x_2 = k \\ -x_1 + x_2 + 2x_3 = 0 \Rightarrow x_1 = x_2 = k \end{gathered}$$

Therefore

$\boldsymbol{x} = \begin{pmatrix} k \\ k \\ 0 \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and the corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

When $\lambda = 3$:

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Using ERO

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \xrightarrow[B_3+(-\frac{1}{2})B_1]{} \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & \frac{1}{2} & -1 \end{pmatrix} \xrightarrow[B_3+(\frac{1}{2})B_2]{} \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$x_3 = \text{k}$$

$$-x_2 + 2x_3 = 0 \Rightarrow x_2 = 2k$$

$$-2x_1 + x_2 + 2x_3 = 0 \Rightarrow x_1 = 2k$$

Therefore

$\boldsymbol{x} = \begin{pmatrix} 2k \\ 2k \\ k \end{pmatrix} = k\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and the corresponding eigenvector is $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

7.5.2 Vector Space

Definition 7.10: Vector Space

A vector space is a set V on which two operations called vector addition and scalar multiplication are defined so that for any elements $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in V and any scalar α and β, the sum $\mathbf{u}+\mathbf{v}$ and the scalar multiple $\alpha\mathbf{u}$ are unique elements of V, and satisfy the following properties.

Properties of Vector Space

(1) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

(2) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{v}+(\mathbf{u}+\mathbf{w})$.

(3) There is an element $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.

(4) There is an element $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

(5) $(1)\mathbf{u}=\mathbf{u}$.

(6) $(\alpha\beta)\mathbf{u}=\alpha(\beta\mathbf{u})$.

(7) $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$.

(8) $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}$

7.5.3 Linear Combinations and Span

Definition 7.11: Linear Combinations

A vector $\mathbf{v}$ is a linear combination of a vector in a subset S of a vector space V if there exist $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \dots, \mathbf{v_n}$ in S and scalars $\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3, \dots, \mathrm{c}_\mathrm{n}$ such that

$$\mathbf{v}=\mathrm{c}_1\mathbf{v_1}+\mathrm{c}_2\mathbf{v_2}+\mathrm{c}_3\mathbf{v_3}+\cdots+\mathrm{c}_\mathrm{n}\mathbf{v_n}.$$

The scalars are called the coefficients of the linear combination.

Definition 7.12: Span

The span of a non-empty subset of S of a vector space V is the set of all linear combinations of vectors in S. This set is denoted by $\mathbf{Span}(S)$.
If $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \dots, \mathbf{v_n}\} \in V$, then

$$\mathbf{Span}(S) = \mathbf{Span}(\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \dots, \mathbf{v_n}\}).$$

Example 7.19:

Let $V = \mathbb{R}^2$, for the following question, find if $\mathbf{y}$ is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$. If yes, write out the linear combination and determine whether $\mathbf{y} \in \text{span}(\mathbf{v_1}, \mathbf{v_2})$.

a) $\mathbf{y} = (2,1), \mathbf{v_1} = (1,1),\ \mathbf{v_2} = (2,2)$
b) $\mathbf{y} = (2,1), \mathbf{v_1} = (1,1),\ \mathbf{v_2} = (1,3)$

Solution:

a) Since $\mathbf{y}$ is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ if $\mathbf{y} = c_1\mathbf{v_1} + c_2\mathbf{v_2}$,

$$\begin{aligned}(2,1) &= c_1(1,1) + c_2(2,2)\\ &= (c_1 + 2c_2, c_1 + 2c_2)\end{aligned}$$

This gives
$c_1 + 2c_2 = 2$
$c_1 + 2c_2 = 1$

By solving the system of linear equation

$$\left(\begin{array}{cc|c}1 & 2 & 2\\ 1 & 2 & 1\end{array}\right) \xrightarrow[B_2+(-1)B_1]{} \left(\begin{array}{cc|c}1 & 2 & 2\\ 0 & 0 & -1\end{array}\right)$$

The second row implies that the system of linear equation is inconsistent. Therefore c_1 and c_2 do not exist and $\mathbf{y}$ is not a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ and $\mathbf{y}$ is not $\text{span}(\mathbf{v_1}, \mathbf{v_2})$.

b) Since $\mathbf{y}$ is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ if $\mathbf{y} = c_1\mathbf{v_1} + c_2\mathbf{v_2}$,

$$\begin{aligned}(2,1) &= c_1(1,1) + c_2(1,3)\\ &= (c_1 + c_2, c_1 + 3c_2)\end{aligned}$$

This gives

$c_1 + c_2 = 2$

$c_1 + 3c_2 = 1$

By solving the system of linear equation

$$\left(\begin{array}{cc|c}1 & 1 & 2\\ 1 & 3 & 1\end{array}\right) \xrightarrow[B_2+(-1)B_1]{} \left(\begin{array}{cc|c}1 & 1 & 2\\ 0 & 2 & -1\end{array}\right) \xrightarrow[\left(\frac{1}{2}\right)B_2]{} \left(\begin{array}{cc|c}1 & 1 & 2\\ 0 & 1 & -1/2\end{array}\right) \xrightarrow[B_1+(-1)B_2]{}$$

$$\left(\begin{array}{cc|c}1 & 0 & 5/2\\ 0 & 1 & -1/2\end{array}\right).$$

Therefore $\mathbf{y}$ is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ where $c_1 = \frac{5}{2}$ and $c_2 = -\frac{1}{2}$. Therefore $\mathbf{y} \in \text{span}(\mathbf{v_1}, \mathbf{v_2})$.

Example 7.20:

Write the linear combination of matrix $B = \begin{pmatrix} -1 & 0 \\ -1 & 7 \end{pmatrix}$ in terms of matrices $\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ -3 & 1 \end{pmatrix}$. Determine whether B is the span(S), where $S = \left\{ \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -3 & 1 \end{pmatrix} \right\}$.

Solution:

$$\begin{pmatrix} -1 & 0 \\ -1 & 7 \end{pmatrix} = \alpha \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} + \beta \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 2 \\ -3 & 1 \end{pmatrix}$$

From above, we obtain the following system of linear equation.

$$-\alpha + 3\beta = -1$$
$$\alpha + 2\gamma = 0$$
$$-2\alpha + \beta - 3\gamma = -1$$
$$2\alpha + \beta + \gamma = 7$$

To find the coefficients, we can solve the system using simultaneous equations method or by using ERO as in previous example.
By using simultaneous equations, we will get

$$\alpha = 4, \qquad \beta = 1, \quad \gamma = -2$$

Hence,

$$\begin{pmatrix} -1 & 0 \\ -1 & 7 \end{pmatrix} = 4 \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} + 1 \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} - 2 \begin{pmatrix} 0 & 2 \\ -3 & 1 \end{pmatrix}$$

and the above expression shows that $B \in \text{span}(S)$.

Example 7.21:

Let $p(x) = 1 - 2x, q(x) = x - x^2,$ and $r(x) = -2 + 3x + x^2$.
Determine whether $s(x) = 3 - 5x - x^2$ is in span$\big(p(x), q(x), r(x)\big)$.

Solution:

Write out $s(x)$ as a linear combination of $p(x), q(x),$ and $r(x)$.

$$3 - 5x - x^2 = \alpha(1 - 2x) + \beta(x - x^2) + \gamma(-2 + 3x + x^2)$$

By comparing the coefficients of x^2, x and the constant, we obtain

$$-\beta + \gamma = -1$$
$$-2\alpha + \beta + 3\gamma = -5$$
$$\alpha - 2\gamma = 3$$

The solution of the simultaneous equations will give us non-unique solutions where

If $\gamma = t, \beta = 1 + t,$ and $\alpha = 3 + 2t$. In the linear combination form,

$$3 - 5x - x^2 = (3 + 2t)(1 - 2x) + (1 + t)(x - x^2)$$
$$+(t)(-2 + 3x + x^2).$$

Or if $\beta = t, \gamma = t - 1,$ and $\alpha = 2t + 1$. In the linear combination form,

$$3 - 5x - x^2 = (2t + 1)(1 - 2x) + (t)(x - x^2)$$
$$+(t - 1)(-2 + 3x + x^2).$$

Therefore $s(x) \in$ span$\big(p(x), q(x), r(x)\big)$

7.5.4 Linearly Independence

Definition 7.13: Linearly Independent

A set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \ldots, \mathbf{v_n}\}$ is **linearly independent** if

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} + \cdots + c_n\mathbf{v_n} = \mathbf{0}$$

for all $c_1 = c_2 = c_3 = \cdots = c_n = 0$.

If not all $c_1, c_2, c_3, \ldots, c_n$ are zero such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} + \cdots + c_n\mathbf{v_n} = \mathbf{0},$$

we say that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \ldots, \mathbf{v_n}\}$ is **linearly dependent**.

<u>Example 7.22:</u>

Determine if the following sets of vectors are linearly dependent or linearly dependent.

a) $v_1 = (3, -1)$ and $v_2 = (-2,2)$.

b) $v_1 = (2, -2,4)$, $v_2 = (3, -5,4)$ and $v_3 = (0,1,1)$

<u>Solution:</u>

a) Let c_1 and c_2 are constants such that

$$c_1(3, -1) + c_2(-2,2) = 0$$

From above, we can get the following system of linear equations

$$3c_1 - 2c_2 = 0$$
$$-c_1 + 2c_2 = 0$$

The solution of the above system is

$$c_1 = c_2 = 0.$$

Since this is the only solution so these two vectors are linearly independent.

b) Let c_1, c_2 and c_3 are constants such that

$$c_1(2,-2,4) + c_2(3,-5,4) + c_3(0,1,1) = 0$$

Therefore,

$$2c_1 + 3c_2 = 0$$
$$-2c_1 - 5c_2 + c_3 = 0$$
$$4c_1 + 4c_2 + c_3 = 0$$

The solution for this system is

$$c_1 = -\frac{3}{4}t, \quad c_2 = \frac{1}{2}t, \quad c_3 = t$$

where t is any real number.

Hence, these vectors are linearly dependent.