Chapter 6

Basics of Combinatorial Topology

6.1 Simplicial and Polyhedral Complexes

In order to study and manipulate complex shapes it is convenient to discretize these shapes and to view them as the union of simple building blocks glued together in a "clean fashion". The building blocks should be simple geometric objects, for example, points, lines segments, triangles, tehrahedra and more generally simplices, or even convex polytopes. We will begin by using simplices as building blocks. The material presented in this chapter consists of the most basic notions of combinatorial topology, going back roughly to the 1900-1930 period and it is covered in nearly every algebraic topology book (certainly the "classics"). A classic text (slightly old fashion especially for the notation and terminology) is Alexandrov [1], Volume 1 and another more "modern" source is Munkres [30]. An excellent treatment from the point of view of computational geometry can be found is Boissonnat and Yvinec [8], especially Chapters 7 and 10. Another fascinating book covering a lot of the basics but devoted mostly to three-dimensional topology and geometry is Thurston [41].

Recall that a simplex is just the convex hull of a finite number of affinely independent points. We also need to define faces, the boundary, and the interior of a simplex.

Definition 6.1 Let $\mathcal{E}$ be any normed affine space, say $\mathcal{E} = \mathbb{E}^m$ with its usual Euclidean norm. Given any $n+1$ affinely independent points $a_0, \ldots, a_n$ in $\mathcal{E}$, the *n-simplex (or simplex) σ defined by* $a_0, \ldots, a_n$ is the convex hull of the points $a_0, \ldots, a_n$, that is, the set of all convex combinations $\lambda_0 a_0 + \cdots + \lambda_n a_n$, where $\lambda_0 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$ for all i, $0 \leq i \leq n$. We call n the *dimension* of the n-simplex σ, and the points $a_0, \ldots, a_n$ are the *vertices* of σ. Given any subset $\{a_{i_0}, \ldots, a_{i_k}\}$ of $\{a_0, \ldots, a_n\}$ (where $0 \leq k \leq n$), the k-simplex generated by $a_{i_0}, \ldots, a_{i_k}$ is called a *k-face* or simply a *face* of σ. A face s of σ is a *proper face* if $s \neq \sigma$ (we agree that the empty set is a face of any simplex). For any vertex a_i, the face generated by $a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ (i.e., omitting a_i) is called the *face opposite* a_i. Every face that is an $(n-1)$-simplex is called a *boundary face* or *facet*. The union of the boundary faces is the *boundary of* σ, denoted by $\partial\sigma$, and the complement of $\partial\sigma$ in σ is the *interior* $\mathrm{Int}\,\sigma = \sigma - \partial\sigma$ of σ. The interior $\mathrm{Int}\,\sigma$ of σ is sometimes called an *open simplex*.

It should be noted that for a 0-simplex consisting of a single point $\{a_0\}$, $\partial\{a_0\} = \emptyset$, and $\mathrm{Int}\,\{a_0\} = \{a_0\}$. Of course, a 0-simplex is a single point, a 1-simplex is the line segment (a_0, a_1), a 2-simplex is a triangle (a_0, a_1, a_2) (with its interior), and a 3-simplex is a tetrahedron (a_0, a_1, a_2, a_3) (with its interior). The inclusion relation between any two faces σ and τ of some simplex, s, is written $\sigma \preceq \tau$.

We now state a number of properties of simplices, whose proofs are left as an exercise. Clearly, a point x belongs to the boundary $\partial\sigma$ of σ iff at least one of its barycentric coordinates $(\lambda_0, \ldots, \lambda_n)$ is zero, and a point x belongs to the interior $\mathrm{Int}\,\sigma$ of σ iff all of its barycentric coordinates $(\lambda_0, \ldots, \lambda_n)$ are positive, i.e., $\lambda_i > 0$ for all i, $0 \leq i \leq n$. Then, for every $x \in \sigma$, there is a unique face s such that $x \in \mathrm{Int}\,s$, the face generated by those points a_i for which $\lambda_i > 0$, where $(\lambda_0, \ldots, \lambda_n)$ are the barycentric coordinates of x.

A simplex σ is convex, arcwise connected, compact, and closed. The interior $\mathrm{Int}\,\sigma$ of a simplex is convex, arcwise connected, open, and σ is the closure of $\mathrm{Int}\,\sigma$.

We now put simplices together to form more complex shapes, following Munkres [30]. The intuition behind the next definition is that the building blocks should be "glued cleanly".

Definition 6.2 A *simplicial complex in* $\mathbb{E}^m$ (for short, a *complex* in $\mathbb{E}^m$) is a set K consisting of a (finite or infinite) set of simplices in $\mathbb{E}^m$ satisfying the following conditions:

(1) Every face of a simplex in K also belongs to K.

(2) For any two simplices σ_1 and σ_2 in K, if $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

Every k-simplex, $\sigma \in K$, is called a *k-face* (or *face*) of K. A 0-face $\{v\}$ is called a *vertex* and a 1-face is called an *edge*. The *dimension* of the simplicial complex K is the maximum of the dimensions of all simplices in K. If $\dim K = d$, then every face of dimension d is called a *cell* and every face of dimension $d - 1$ is called a *facet*.

Condition (2) guarantees that the various simplices forming a complex intersect nicely. It is easily shown that the following condition is equivalent to condition (2):

(2′) For any two distinct simplices σ_1, σ_2, $\mathrm{Int}\,\sigma_1 \cap \mathrm{Int}\,\sigma_2 = \emptyset$.

Remarks:

1. A simplicial complex, K, is a combinatorial object, namely, a *set* of simplices satisfying certain conditions but not a subset of $\mathbb{E}^m$. However, every complex, K, yields a subset of $\mathbb{E}^m$ called the geometric realization of K and denoted $|K|$. This object will be defined shortly and should not be confused with the complex. Figure 6.1 illustrates this aspect of the definition of a complex. For clarity, the two triangles (2-simplices) are drawn as disjoint objects even though they share the common edge, (v_2, v_3) (a 1-simplex) and similarly for the edges that meet at some common vertex.

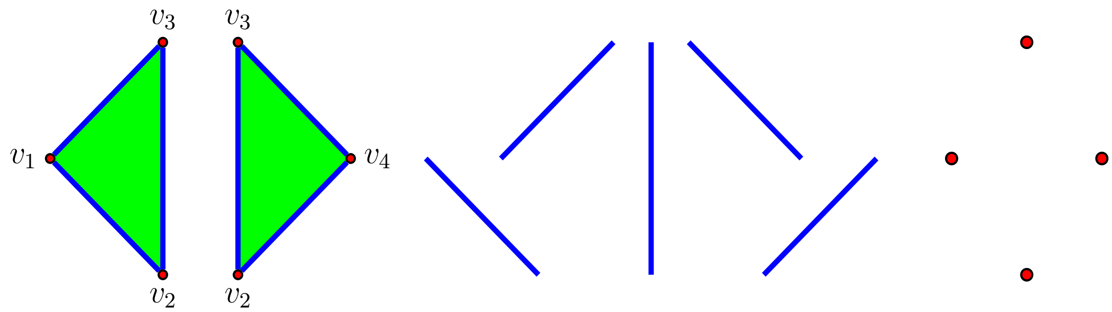

Figure 6.1: A set of simplices forming a complex

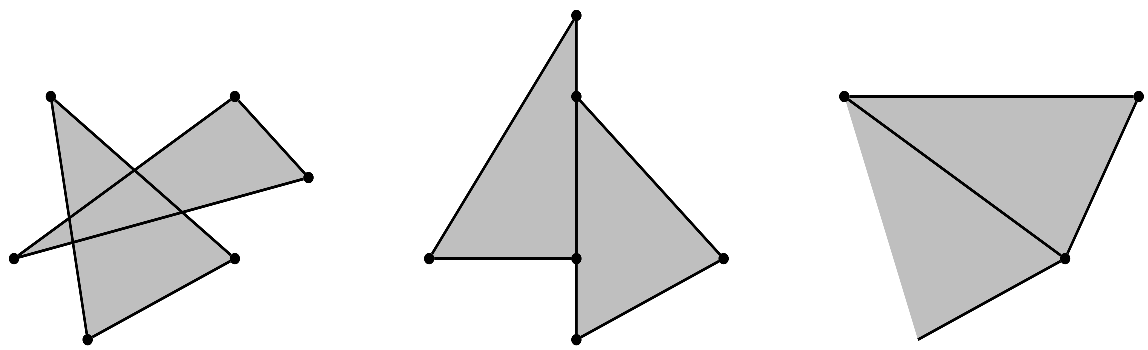

Figure 6.2: Collections of simplices not forming a complex

2. Some authors define a *facet* of a complex, K, of dimension d to be a d-simplex in K, as opposed to a $(d-1)$-simplex, as we did. This practice is not consistent with the notion of facet of a polyhedron and this is why we prefer the terminology *cell* for the d-simplices in K.

3. It is important to note that in order for a complex, K, of dimension d to be realized in $\mathbb{E}^m$, the dimension of the "ambient space", m, must be big enough. For example, there are 2-complexes that can't be realized in $\mathbb{E}^3$ or even in $\mathbb{E}^4$. There has to be enough room in order for condition (2) to be satisfied. It is not hard to prove that $m = 2d+1$ is always sufficient. Sometimes, $2d$ works, for example in the case of surfaces (where $d = 2$).

Some collections of simplices violating some of the conditions of Definition 6.2 are shown in Figure 6.2. On the left, the intersection of the two 2-simplices is neither an edge nor a vertex of either triangle. In the middle case, two simplices meet along an edge which is not an edge of either triangle. On the right, there is a missing edge and a missing vertex.

Some "legal" simplicial complexes are shown in Figure 6.4.

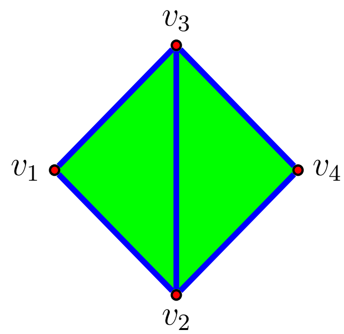

Figure 6.3: The geometric realization of the complex of Figure 6.1

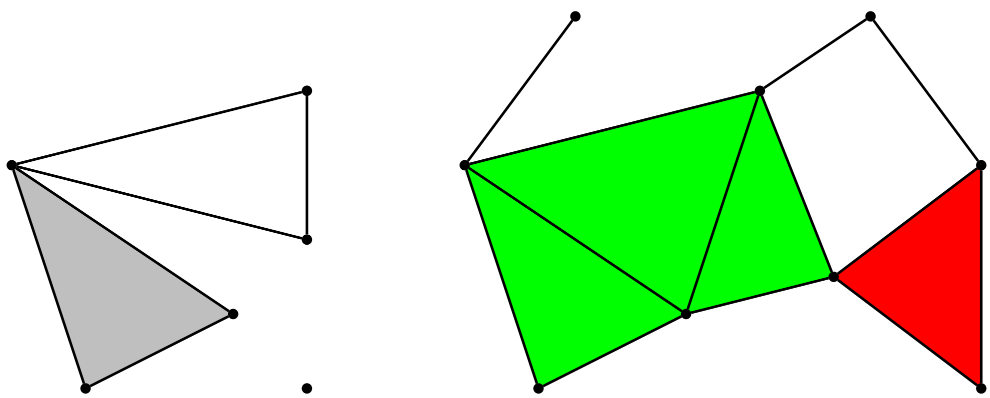

Figure 6.4: Examples of simplicial complexes

The union $|K|$ of all the simplices in K is a subset of $\mathbb{E}^m$. We can define a topology on $|K|$ by defining a subset F of $|K|$ to be closed iff $F \cap \sigma$ is closed in σ for every face $\sigma \in K$. It is immediately verified that the axioms of a topological space are indeed satisfied. The resulting topological space $|K|$ is called the *geometric realization of* K. The geometric realization of the complex from Figure 6.1 is shown in Figure 6.3.

Obviously, $|\sigma| = \sigma$ for every simplex, σ. Also, note that distinct complexes may have the same geometric realization. In fact, all the complexes obtained by subdividing the simplices of a given complex yield the same geometric realization.

A *polytope* is the geometric realization of some simplicial complex. A polytope of dimension 1 is usually called a *polygon*, and a polytope of dimension 2 is usually called a *polyhedron*. When K consists of infinitely many simplices we usually require that K be *locally finite*, which means that every vertex belongs to finitely many faces. If K is locally finite, then its geometric realization, $|K|$, is locally compact.

In the sequel, we will consider only finite simplicial complexes, that is, complexes K

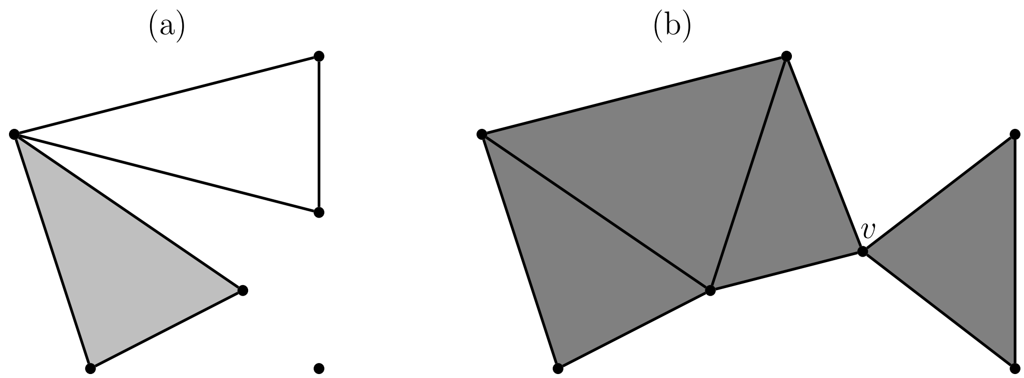

Figure 6.5: (a) A complex that is not pure. (b) A pure complex

consisting of a finite number of simplices. In this case, the topology of $|K|$ defined above is identical to the topology induced from $\mathbb{E}^m$. Also, for any simplex σ in K, $\mathrm{Int}\,\sigma$ coincides with the interior $\mathring{\sigma}$ of σ in the topological sense, and $\partial\sigma$ coincides with the boundary of σ in the topological sense.

Definition 6.3 Given any complex, K_2, a subset $K_1 \subseteq K_2$ of K_2 is a *subcomplex* of K_2 iff it is also a complex. For any complex, K, of dimension d, for any i with $0 \leq i \leq d$, the subset

$$K^{(i)} = \{\sigma \in K \mid \dim \sigma \leq i\}$$

is called the *i-skeleton* of K. Clearly, $K^{(i)}$ is a subcomplex of K. We also let

$$K^{i} = \{\sigma \in K \mid \dim \sigma = i\}.$$

Observe that K^0 is the set of vertices of K and K^i is not a complex. A simplicial complex, K_1 is a *subdivision* of a complex K_2 iff $|K_1| = |K_2|$ and if every face of K_1 is a subset of some face of K_2. A complex K of dimension d is *pure* (or *homogeneous*) iff every face of K is a face of some d-simplex of K (i.e., some cell of K). A complex is *connected* iff $|K|$ is connected.

It is easy to see that a complex is connected iff its 1-skeleton is connected. The intuition behind the notion of a pure complex, K, of dimension d is that a pure complex is the result of gluing pieces all having the same dimension, namely, d-simplices. For example, in Figure 6.5, the complex on the left is not pure but the complex on the right is pure of dimension 2.

Most of the shapes that we will be interested in are well approximated by pure complexes, in particular, surfaces or solids. However, pure complexes may still have undesirable "singularities" such as the vertex, v, in Figure 6.5(b). The notion of link of a vertex provides a technical way to deal with singularities.

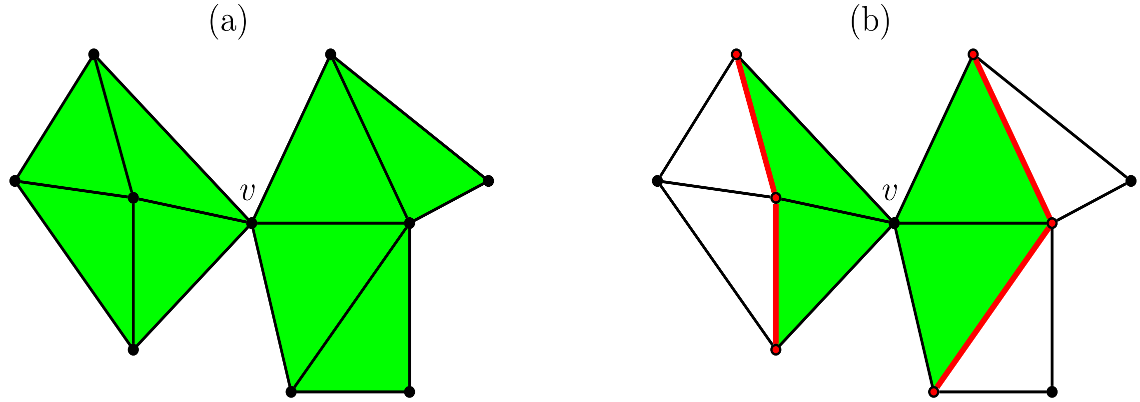

Figure 6.6: (a) A complex. (b) Star and Link of v

Definition 6.4 Let K be any complex and let σ be any face of K. The *star*, $\mathrm{St}(\sigma)$ (or if we need to be very precise, $\mathrm{St}(\sigma, K)$), of σ is the subcomplex of K consisting of all faces, τ, containing σ and of all faces of τ, *i.e.*,

$$\mathrm{St}(\sigma) = \{s \in K \mid (\exists \tau \in K)(\sigma \preceq \tau \quad \text{and} \quad s \preceq \tau)\}.$$

The *link*, $\mathrm{Lk}(\sigma)$ (or $\mathrm{Lk}(\sigma, K)$) of σ is the subcomplex of K consisting of all faces in $\mathrm{St}(\sigma)$ that do not intersect σ, i.e.,

$$\mathrm{Lk}(\sigma) = \{\tau \in K \mid \tau \in \mathrm{St}(\sigma) \quad \text{and} \quad \sigma \cap \tau = \emptyset\}.$$

To simplify notation, if $\sigma = \{v\}$ is a vertex we write $\mathrm{St}(v)$ for $\mathrm{St}(\{v\})$ and $\mathrm{Lk}(v)$ for $\mathrm{Lk}(\{v\})$. Figure 6.6 shows:

(a) A complex (on the left).

(b) The star of the vertex v, indicated in gray and the link of v, shown as thicker lines.

If K is pure and of dimension d, then $\mathrm{St}(\sigma)$ is also pure of dimension d and if $\dim \sigma = k$, then $\mathrm{Lk}(\sigma)$ is pure of dimension $d - k - 1$.

For technical reasons, following Munkres [30], besides defining the complex, $\mathrm{St}(\sigma)$, it is useful to introduce the *open star* of σ, denoted $\mathrm{st}(\sigma)$, defined as the subspace of $|K|$ consisting of the union of the interiors, $\mathrm{Int}(\tau) = \tau - \partial\tau$, of all the faces, τ, containing, σ. According to this definition, the open star of σ is not a complex but instead a subset of $|K|$.

Note that

$$\overline{\mathrm{st}(\sigma)} = |\mathrm{St}(\sigma)|,$$

that is, the closure of $\mathrm{st}(\sigma)$ is the geometric realization of the complex $\mathrm{St}(\sigma)$. Then, $\mathrm{lk}(\sigma) = |\mathrm{Lk}(\sigma)|$ is the union of the simplices in $\mathrm{St}(\sigma)$ that are disjoint from σ. If σ is a vertex, v, we have

$$\mathrm{lk}(v) = \overline{\mathrm{st}(v)} - \mathrm{st}(v).$$

However, beware that if σ is not a vertex, then $\mathrm{lk}(\sigma)$ is properly contained in $\overline{\mathrm{st}(\sigma)} - \mathrm{st}(\sigma)$!

One of the nice properties of the open star, $\mathrm{st}(\sigma)$, of σ is that it is open. To see this, observe that for any point, $a \in |K|$, there is a unique smallest simplex, $\sigma = (v_0, \ldots, v_k)$, such that $a \in \mathrm{Int}(\sigma)$, that is, such that

$$a = \lambda_0 v_0 + \cdots + \lambda_k v_k$$

with $\lambda_i > 0$ for all i, with $0 \leq i \leq k$ (and of course, $\lambda_0 + \cdots + \lambda_k = 1$). (When $k = 0$, we have $v_0 = a$ and $\lambda_0 = 1$.) For every arbitrary vertex, v, of K, we define $t_v(a)$ by

$$t_v(a) = \begin{cases} \lambda_i & \text{if } v = v_i, \text{ with } 0 \leq i \leq k, \\ 0 & \text{if } v \notin \{v_0, \ldots, v_k\}. \end{cases}$$

Using the above notation, observe that

$$\mathrm{st}(v) = \{a \in |K| \mid t_v(a) > 0\}$$

and thus, $|K| - \mathrm{st}(v)$ is the union of all the faces of K that do not contain v as a vertex, obviously a closed set. Thus, $\mathrm{st}(v)$ is open in $|K|$. It is also quite clear that $\mathrm{st}(v)$ is path connected. Moreover, for any k-face, σ, of K, if $\sigma = (v_0, \ldots, v_k)$, then

$$\mathrm{st}(\sigma) = \{a \in |K| \mid t_{v_i}(a) > 0, \quad 0 \leq i \leq k\},$$

that is,

$$\mathrm{st}(\sigma) = \mathrm{st}(v_0) \cap \cdots \cap \mathrm{st}(v_k).$$

Consequently, $\mathrm{st}(\sigma)$ is open and path connected.

Unfortunately, the "nice" equation

$$\mathrm{St}(\sigma) = \mathrm{St}(v_0) \cap \cdots \cap \mathrm{St}(v_k)$$

is false! (and anagolously for $\mathrm{Lk}(\sigma)$.) For a counter-example, consider the boundary of a tetrahedron with one face removed.

Recall that in $\mathbb{E}^d$, the *(open) unit ball, B^d*, is defined by

$$B^d = \{x \in \mathbb{E}^d \mid \|x\| < 1\},$$

the *closed unit ball, $\overline{B}^d$*, is defined by

$$\overline{B}^d = \{x \in \mathbb{E}^d \mid \|x\| \leq 1\},$$

and the $(d-1)$-*sphere*, S^{d-1}, by

$$S^{d-1} = \{x \in \mathbb{E}^d \mid \|x\| = 1\}.$$

Obviously, S^{d-1} is the boundary of $\overline{B}^d$ (and B^d).

Definition 6.5 Let K be a pure complex of dimension d and let σ be any k-face of K, with $0 \leq k \leq d-1$. We say that σ is *nonsingular* iff the geometric realization, $\mathrm{lk}(\sigma)$, of the link of σ is homeomorphic to either S^{d-k-1} or to $\overline{B}^{d-k-1}$; this is written as $\mathrm{lk}(\sigma) \approx S^{d-k-1}$ or $\mathrm{lk}(\sigma) \approx \overline{B}^{d-k-1}$, where $\approx$ means homeomorphic.

In Figure 6.6, note that the link of v is not homeomorphic to S^1 or B^1, so v is singular.

It will also be useful to express $\mathrm{St}(v)$ in terms of $\mathrm{Lk}(v)$, where v is a vertex, and for this, we define yet another notion of cone.

Definition 6.6 Given any complex, K, in $\mathbb{E}^n$, if $\dim K = d < n$, for any point, $v \in \mathbb{E}^n$, such that v does not belong to the affine hull of $|K|$, the *cone on K with vertex* v, denoted, $v * K$, is the complex consisting of all simplices of the form $(v, a_0, \ldots, a_k)$ and their faces, where $(a_0, \ldots, a_k)$ is any k-face of K. If $K = \emptyset$, we set $v * K = v$.

It is not hard to check that $v * K$ is indeed a complex of dimension $d+1$ containing K as a subcomplex.

Remark: Unfortunately, the word "cone" is overloaded. It might have been better to use the locution *pyramid* instead of cone as some authors do (for example, Ziegler). However, since we have been following Munkres [30], a standard reference in algebraic topology, we decided to stick with the terminology used in that book, namely, "cone".

The following proposition is also easy to prove:

Proposition 6.1 *For any complex, K, of dimension d and any vertex, $v \in K$, we have*

$$\mathrm{St}(v) = v * \mathrm{Lk}(v).$$

More generally, for any face, σ, of K, we have

$$\overline{\mathrm{st}(\sigma)} = |\mathrm{St}(\sigma)| \approx \sigma \times |v * \mathrm{Lk}(\sigma)|,$$

for every $v \in \sigma$ and

$$\overline{\mathrm{st}(\sigma)} - \mathrm{st}(\sigma) = \partial\, \sigma \times |v * \mathrm{Lk}(\sigma)|,$$

for every $v \in \partial\, \sigma$.

Figure 6.7 shows a 3-dimensional complex. The link of the edge (v_6, v_7) is the pentagon $P = (v_1, v_2, v_3, v_4, v_5) \approx S^1$. The link of the vertex v_7 is the cone $v_6 * P \approx B^2$. The link of (v_1, v_2) is $(v_6, v_7) \approx B^1$ and the link of v_1 is the union of the triangles (v_2, v_6, v_7) and (v_5, v_6, v_7), which is homeomorphic to B^2.

Given a pure complex, it is necessary to distinguish between two kinds of faces.

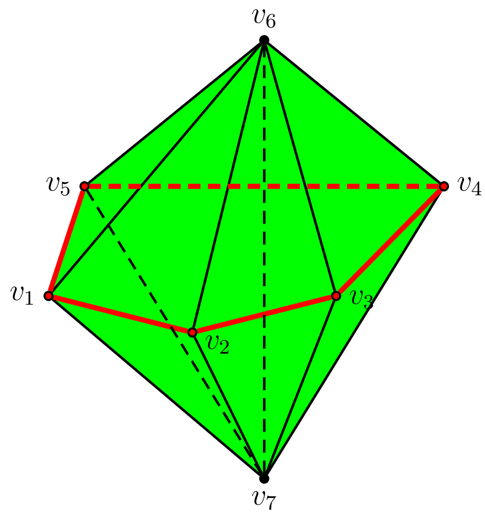

Figure 6.7: More examples of links and stars

Definition 6.7 Let K be any pure complex of dimension d. A k-face, σ, of K is a *boundary* or *external* face iff it belongs to a single cell (i.e., a d-simplex) of K and otherwise it is called an *internal* face ($0 \leq k \leq d-1$). The *boundary* of K, denoted $\mathrm{bd}(K)$, is the subcomplex of K consisting of all boundary facets of K together with their faces.

It is clear by definition that $\mathrm{bd}(K)$ is a pure complex of dimension $d-1$. Even if K is connected, $\mathrm{bd}(K)$ is not connected, in general. For example, if K is a 2-complex in the plane, the boundary of K usually consists of several simple closed polygons (i.e, 1 dimensional complexes homeomorphic to the circle, S^1).

Proposition 6.2 *Let K be any pure complex of dimension d. For any k-face, σ, of K the boundary complex,* $\mathrm{bd}(\mathrm{Lk}(\sigma))$*, is nonempty iff σ is a boundary face of K ($0 \leq k \leq d-2$). Furthermore,* $\mathrm{Lk}_{\mathrm{bd}(K)}(\sigma) = \mathrm{bd}(\mathrm{Lk}(\sigma))$ *for every face, σ, of* $\mathrm{bd}(K)$*, where* $\mathrm{Lk}_{\mathrm{bd}(K)}(\sigma)$ *denotes the link of σ in* $\mathrm{bd}(K)$.

Proof. Let F be any facet of K containing σ. We may assume that $F = (v_0, \ldots, v_{d-1})$ and $\sigma = (v_0, \ldots, v_k)$, in which case, $F' = (v_{k+1}, \ldots, v_{d-1})$ is a $(d-k-2)$-face of K and by definition of $\mathrm{Lk}(\sigma)$, we have $F' \in \mathrm{Lk}(\sigma)$. Now, every cell (i.e., d-simplex), s, containing F is of the form $s = \mathrm{conv}(F \cup \{v\})$ for some vertex, v, and $s' = \mathrm{conv}(F' \cup \{v\})$ is a $(d-k-1)$-face in $\mathrm{Lk}(\sigma)$ containing F'. Consequently, F' is an external face of $\mathrm{Lk}(\sigma)$ iff F is an external facet of K, establishing the proposition. The second statement follows immediately from the proof of the first. $\square$

Proposition 6.2 shows that if every face of K is nonsingular, then the link of every internal face is a sphere whereas the link of every external face is a ball. The following proposition shows that for any pure complex, K, nonsingularity of all the vertices is enough to imply that every open star is homeomorphic to B^d:

Proposition 6.3 *Let K be any pure complex of dimension d. If every vertex of K is nonsingular, then $\mathrm{st}(\sigma) \approx B^d$ for every k-face, σ, of K ($1 \leq k \leq d-1$).*

Proof. Let σ be any k-face of K and assume that σ is generated by the vertices $v_0, \ldots, v_k$, with $1 \leq k \leq d-1$. By hypothesis, $\mathrm{lk}(v_i)$ is homeomorphic to either S^{d-1} or $\overline{B}^{d-1}$. Then, it is easy to show that in either case, we have

$$|v_i * \mathrm{Lk}(v_i)| \approx \overline{B}^d,$$

and by Proposition 6.1, we get

$$|\mathrm{St}(v_i)| \approx \overline{B}^d.$$

Consequently, $\mathrm{st}(v_i) \approx B^d$. Furthermore,

$$\mathrm{st}(\sigma) = \mathrm{st}(v_0) \cap \cdots \cap \mathrm{st}(v_k) \approx B^d$$

and so, $\mathrm{st}(\sigma) \approx B^d$, as claimed. $\square$

Here are more useful propositions about pure complexes without singularities.

Proposition 6.4 *Let K be any pure complex of dimension d. If every vertex of K is nonsingular, then for every point, $a \in |K|$, there is an open subset, $U \subseteq |K|$, containing a such that $U \approx B^d$ or $U \approx B^d \cap \mathbb{H}^d$, where $\mathbb{H}^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_d \geq 0\}$.*

Proof. We already know from Proposition 6.3 that $\mathrm{st}(\sigma) \approx B^d$, for every $\sigma \in K$. So, if $a \in \sigma$ and σ is not a boundary face, we can take $U = \mathrm{st}(\sigma) \approx B^d$. If σ is a boundary face, then $|\sigma| \subseteq |\mathrm{bd}(\mathrm{St}(\sigma))|$ and it can be shown that we can take $U = B^d \cap \mathbb{H}^d$. $\square$

Proposition 6.5 *Let K be any pure complex of dimension d. If every facet of K is nonsingular, then every facet of K, is contained in at most two cells (d-simplices).*

Proof. If $|K| \subseteq \mathbb{E}^d$, then this is an immediate consequence of the definition of a complex. Otherwise, consider $\mathrm{lk}(\sigma)$. By hypothesis, either $\mathrm{lk}(\sigma) \approx B^0$ or $\mathrm{lk}(\sigma) \approx S^0$. As $B^0 = \{0\}$, $S^0 = \{-1, 1\}$ and $\dim \mathrm{Lk}(\sigma) = 0$, we deduce that $\mathrm{Lk}(\sigma)$ has either one or two points, which proves that σ belongs to at most two d-simplices. $\square$

Proposition 6.6 *Let K be any pure and connected complex of dimension d. If every face of K is nonsingular, then for every pair of cells (d-simplices), σ and σ', there is a sequence of cells, $\sigma_0, \ldots, \sigma_p$, with $\sigma_0 = \sigma$ and $\sigma_p = \sigma'$, and such that σ_i and σ_{i+1} have a common facet, for $i = 0, \ldots, p-1$.*

Proof. We proceed by induction on d, using the fact that the links are connected for $d \geq 2$. $\square$

Proposition 6.7 *Let K be any pure complex of dimension d. If every facet of K is nonsingular, then the boundary,* $\mathrm{bd}(K)$*, of K is a pure complex of dimension $d-1$ with an empty boundary. Furthermore, if every face of K is nonsingular, then every face of* $\mathrm{bd}(K)$ *is also nonsingular.*

Proof. Left as an exercise. $\square$

The building blocks of simplicial complexes, namely, simplicies, are in some sense mathematically ideal. However, in practice, it may be desirable to use a more flexible set of building blocks. We can indeed do this and use convex polytopes as our building blocks.

Definition 6.8 A *polyhedral complex in* $\mathbb{E}^m$ (for short, a *complex* in $\mathbb{E}^m$) is a set, K, consisting of a (finite or infinite) set of convex polytopes in $\mathbb{E}^m$ satisfying the following conditions:

(1) Every face of a polytope in K also belongs to K.

(2) For any two polytopes σ_1 and σ_2 in K, if $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

Every polytope, $\sigma \in K$, of dimension k, is called a *k-face* (or *face*) of K. A 0-face $\{v\}$ is called a *vertex* and a 1-face is called an *edge*. The *dimension* of the polyhedral complex K is the maximum of the dimensions of all polytopes in K. If $\dim K = d$, then every face of dimension d is called a *cell* and every face of dimension $d-1$ is called a *facet*.

Remark: Since the building blocks of a polyhedral complex are convex *polytopes* it might be more appropriate to use the term "polytopal complex" rather than "polyhedral complex" and some authors do that. On the other hand, most of the traditional litterature uses the terminology *polyhedral complex* so we will stick to it. There is a notion of complex where the building blocks are cones but these are called *fans*.

Every convex polytope, P, yields two natural polyhedral complexes:

(i) The polyhedral complex, $\mathcal{K}(P)$, consisting of P together with all of its faces. This complex has a single cell, namely, P itself.

(ii) The *boundary complex*, $\mathcal{K}(\partial P)$, consisting of all faces of P other than P itself. The cells of $\mathcal{K}(\partial P)$ are the facets of P.

The notions of k-skeleton and pureness are defined just as in the simplicial case. The notions of star and link are defined for polyhedral complexes just as they are defined for simplicial complexes except that the word "face" now means face of a polytope. Now, by Theorem 4.7, every polytope, σ, is the convex hull of its vertices. Let $\mathrm{vert}(\sigma)$ denote the set of vertices of σ. Then, we have the following crucial observation: Given any polyhedral complex, K, for every point, $x \in |K|$, there is a *unique* polytope, $\sigma_x \in K$, such that $x \in \mathrm{Int}(\sigma_x) = \sigma_x - \partial\,\sigma_x$. We define a function, $t\colon V \to \mathbb{R}_+$, that tests whether x belongs to the interior of any face (polytope) of K having v as a vertex as follows: For every vertex, v, of K,

$$t_v(x) = \begin{cases} 1 & \text{if } v \in \mathrm{vert}(\sigma_x) \\ 0 & \text{if } v \notin \mathrm{vert}(\sigma_x), \end{cases}$$

where σ_x is the unique face of K such that $x \in \mathrm{Int}(\sigma_x)$.

Now, just as in the simplicial case, the open star, $\mathrm{st}(v)$, of a vertex, $v \in K$, is given by

$$\mathrm{st}(v) = \{x \in |K| \mid t_v(x) = 1\}$$

and it is an open subset of $|K|$ (the set $|K| - \mathrm{st}(v)$ is the union of the polytopes of K that do not contain v as a vertex, a closed subset of $|K|$). Also, for any face, σ, of K, the open star, $\mathrm{st}(\sigma)$, of σ is given by

$$\mathrm{st}(\sigma) = \{x \in |K| \mid t_v(x) = 1, \text{ for all } v \in \mathrm{vert}(\sigma)\} = \bigcap_{v \in \mathrm{vert}(\sigma)} \mathrm{st}(v).$$

Therefore, $\mathrm{st}(\sigma)$ is also open in $|K|$.

The next proposition is another result that seems quite obvious, yet a rigorous proof is more involved that we might think. This proposition states that a convex polytope can always be cut up into simplices, that is, it can be subdivided into a simplicial complex. In other words, every convex polytope can be triangulated. This implies that simplicial complexes are as general as polyhedral complexes.

One should be warned that even though, in the plane, every bounded region (not necessarily convex) whose boundary consists of a finite number of closed polygons (polygons homeomorphic to the circle, S^1) can be triangulated, this is no longer true in three dimensions!

Proposition 6.8 *Every convex d-polytope, P, can be subdivided into a simplicial complex without adding any new vertices, i.e., every convex polytope can be triangulated.*

Proof sketch. It would be tempting to proceed by induction on the dimension, d, of P but we do not know any correct proof of this kind. Instead, we proceed by induction on the number, p, of vertices of P. Since $\dim(P) = d$, we must have $p \geq d + 1$. The case $p = d + 1$ corresponds to a simplex, so the base case holds.

For $p > d + 1$, we can pick some vertex, $v \in P$, such that the convex hull, Q, of the remaining $p - 1$ vertices still has dimension d. Then, by the induction hypothesis, Q, has a simplicial subdivision. Now, we say that a facet, F, of Q is *visible from* v iff v and the interior of Q are strictly separated by the supporting hyperplane of F. Then, we add the d-simplices, $\mathrm{conv}(F \cup \{v\}) = v * F$, for every facet, F, of Q visible from v to those in the triangulation of Q. We claim that the resulting collection of simplices (with their faces) constitutes a simplicial complex subdividing P. This is the part of the proof that requires a careful and somewhat tedious case analysis, which we omit. However, the reader should check that everything really works out! $\square$

With all this preparation, it is now quite natural to define combinatorial manifolds.

6.2 Combinatorial and Topological Manifolds

The notion of pure complex without singular faces turns out to be a very good "discrete" approximation of the notion of (topological) manifold because of its highly computational nature. This motivates the following definition:

Definition 6.9 A *combinatorial d-manifold* is any space, X, homeomorphic to the geometric realization, $|K| \subseteq \mathbb{E}^n$, of some pure (simplicial or polyhedral) complex, K, of dimension d whose faces are all nonsingular. If the link of every k-face of K is homeomorphic to the sphere S^{d-k-1}, we say that X is a combinatorial manifold *without boundary*, else it is a combinatorial manifold *with boundary*.

Other authors use the term *triangulation* for what we call a combinatorial manifold.

It is easy to see that the connected components of a combinatorial 1-manifold are either simple closed polygons or simple chains ("simple" means that the interiors of distinct edges are disjoint). A combinatorial 2-manifold which is connected is also called a *combinatorial surface* (with or without boundary). Proposition 6.7 immediately yields the following result:

Proposition 6.9 *If X is a combinatorial d-manifold with boundary, then* $\mathrm{bd}(X)$ *is a combinatorial $(d-1)$-manifold without boundary.*

Now, because we are assuming that X sits in some Euclidean space, $\mathbb{E}^n$, the space X is Hausdorff and second-countable. (Recall that a topological space is second-countable iff there is a countable family, $\{U_i\}_{i\geq 0}$, of open sets of X such that every open subset of X is the union of open sets from this family.) Since it is desirable to have a good match between manifolds and combinatorial manifolds, we are led to the definition below.

Recall that

$$\mathbb{H}^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_d \geq 0\}.$$

Definition 6.10 For any $d \geq 1$, a *(topological) d-manifold with boundary* is a second-countable, topological Hausdorff space M, together with an open cover, $(U_i)_{i\in I}$, of open sets in M and a family, $(\varphi_i)_{i\in I}$, of homeomorphisms, $\varphi_i \colon U_i \to \Omega_i$, where each Ω_i is some open subset of $\mathbb{H}^d$ in the subset topology. Each pair (U, φ) is called a *coordinate system*, or *chart*, of M, each homeomorphism $\varphi_i \colon U_i \to \Omega_i$ is called a *coordinate map*, and its inverse $\varphi_i^{-1} \colon \Omega_i \to U_i$ is called a *parameterization* of U_i. The family $(U_i, \varphi_i)_{i\in I}$ is often called an *atlas* for M. A *(topological) bordered surface* is a connected 2-manifold with boundary. If for every homeomorphism, $\varphi_i \colon U_i \to \Omega_i$, the open set $\Omega_i \subseteq \mathbb{H}^d$ is actually an open set in $\mathbb{R}^d$ (which means that $x_d > 0$ for every $(x_1, \ldots, x_d) \in \Omega_i$), then we say that M is a *d-manifold*.

Note that a d-manifold is also a d-manifold with boundary.

If $\varphi_i \colon U_i \to \Omega_i$ is some homeomorphism onto some open set Ω_i of $\mathbb{H}^d$ in the subset topology, some $p \in U_i$ may be mapped into $\mathbb{R}^{d-1} \times \mathbb{R}_+$, or into the "boundary" $\mathbb{R}^{d-1} \times \{0\}$

of $\mathbb{H}^d$. Letting $\partial\mathbb{H}^d = \mathbb{R}^{d-1} \times \{0\}$, it can be shown using homology that if some coordinate map, φ, defined on p maps p into $\partial\mathbb{H}^d$, then every coordinate map, ψ, defined on p maps p into $\partial\mathbb{H}^d$.

Thus, M is the disjoint union of two sets ∂M and $\mathrm{Int}\, M$, where ∂M is the subset consisting of all points $p \in M$ that are mapped by some (in fact, all) coordinate map, φ, defined on p into $\partial\mathbb{H}^d$, and where $\mathrm{Int}\, M = M - \partial M$. The set ∂M is called the *boundary* of M, and the set $\mathrm{Int}\, M$ is called the *interior* of M, even though this terminology clashes with some prior topological definitions. A good example of a bordered surface is the Möbius strip. The boundary of the Möbius strip is a circle.

The boundary ∂M of M may be empty, but $\mathrm{Int}\, M$ is nonempty. Also, it can be shown using homology that the integer d is unique. It is clear that $\mathrm{Int}\, M$ is open and a d-manifold, and that ∂M is closed. If $p \in \partial M$, and φ is some coordinate map defined on p, since $\Omega = \varphi(U)$ is an open subset of $\partial\mathbb{H}^d$, there is some open half ball B^d_{o+} centered at $\varphi(p)$ and contained in Ω which intersects $\partial\mathbb{H}^d$ along an open ball B^{d-1}_o, and if we consider $W = \varphi^{-1}(B^d_{o+})$, we have an open subset of M containing p which is mapped homeomorphically onto B^d_{o+} in such that way that every point in $W \cap \partial M$ is mapped onto the open ball B^{d-1}_o. Thus, it is easy to see that ∂M is a $(d-1)$-manifold.

Proposition 6.10 *Every combinatorial d-manifold is a d-manifold with boundary.*

Proof. This is an immediate consequence of Proposition 6.4. $\square$

Is the converse of Proposition 6.10 true?

It turns out that answer is yes for $d = 1, 2, 3$ but **no** for $d \geq 4$. This is not hard to prove for $d = 1$. For $d = 2$ and $d = 3$, this is quite hard to prove; among other things, it is necessary to prove that triangulations exist and this is very technical. For $d \geq 4$, not every manifold can be triangulated (in fact, this is undecidable!).

What if we assume that M is a triangulated manifold, which means that $M \approx |K|$, for some pure d-dimensional complex, K?

Surprisingly, for $d \geq 5$, there are triangulated manifolds whose links are not spherical (i.e., not homeomorphic to $\overline{B}^{d-k-1}$ or S^{d-k-1}), see Thurston [41].

Fortunately, we will only have to deal with $d = 2, 3$! Another issue that must be addressed is orientability.

Assume that we fix a total ordering of the vertices of a complex, K. Let $\sigma = (v_0, \ldots, v_k)$ be any simplex. Recall that every permutation (of $\{0, \ldots, k\}$) is a product of *transpositions*, where a transposition swaps two distinct elements, say i and j, and leaves every other element fixed. Furthermore, for any permutation, π, the parity of the number of transpositions needed to obtain π only depends on π and it called the *signature* of π. We say that *two permutations are equivalent* iff they have the same signature. Consequently, there are two equivalence classes of permutations: Those of even signature and those of odd signature.

Then, an *orientation* of σ is the choice of one of the two equivalence classes of permutations of its vertices. If σ has been given an orientation, then we denote by $-\sigma$ the result of assigning the other orientation to it (we call it the *opposite orientation*).

For example, $(0, 1, 2)$ has the two orientation classes:

$$\{(0, 1, 2), (1, 2, 0), (2, 0, 1)\} \quad \text{and} \quad \{(2, 1, 0), (1, 0, 2), (0, 2, 1)\}.$$

Definition 6.11 Let $X \approx |K|$ be a combinatorial d-manifold. We say that X is *orientable* if it is possible to assign an orientation to all of its cells (d-simplices) so that whenever two cells σ_1 and σ_2 have a common facet, σ, the two orientations induced by σ_1 and σ_2 on σ are opposite. A combinatorial d-manifold together with a specific orientation of its cells is called an *oriented manifold*. If X is not orientable we say that it is *non-orientable*.

Remark: It is possible to define the notion of orientation of a manifold but this is quite technical and we prefer to avoid digressing into this matter. This shows another advantage of combinatorial manifolds: The definition of orientability is simple and quite natural.

There are non-orientable (combinatorial) surfaces, for example, the Möbius strip which can be realized in $\mathbb{E}^3$. The Möbius strip is a surface with boundary, its boundary being a circle. There are also non-orientable (combinatorial) surfaces such as the Klein bottle or the projective plane but they can only be realized in $\mathbb{E}^4$ (in $\mathbb{E}^3$, they must have singularities such as self-intersection). We will only be dealing with orientable manifolds and, most of the time, surfaces.

One of the most important invariants of combinatorial (and topological) manifolds is their *Euler(-Poincaré) characteristic*. In the next chapter, we prove a famous formula due to Poincaré giving the Euler characteristic of a convex polytope. For this, we will introduce a technique of independent interest called *shelling*.

Chapter 7

Shellings, the Euler-Poincaré Formula for Polytopes, the Dehn-Sommerville Equations and the Upper Bound Theorem

7.1 Shellings

The notion of shellability is motivated by the desire to give an inductive proof of the Euler-Poincaré formula in any dimension. Historically, this formula was discovered by Euler for three dimensional polytopes in 1752 (but it was already known to Descartes around 1640). If f_0, f_1 and f_2 denote the number of vertices, edges and triangles of the three dimensional polytope, P, (i.e., the number of i-faces of P for $i = 0, 1, 2$), then the *Euler formula* states that

$$f_0 - f_1 + f_2 = 2.$$

The proof of Euler's formula is not very difficult but one still has to exercise caution. Euler's formula was generalized to arbitrary d-dimensional polytopes by Schläfli (1852) but the first correct proof was given by Poincaré. For this, Poincaré had to lay the foundations of algebraic topology and after a first "proof" given in 1893 (containing some flaws) he finally gave the first correct proof in 1899. If f_i denotes the number of i-faces of the d-dimensional polytope, P, (with $f_{-1} = 1$ and $f_d = 1$), the *Euler-Poincaré formula* states that:

$$\sum_{i=0}^{d-1} (-1)^i f_i = 1 - (-1)^d,$$

which can also be written as

$$\sum_{i=0}^{d} (-1)^i f_i = 1,$$

by incorporating $f_d = 1$ in the first formula or as

$$\sum_{i=-1}^{d} (-1)^i f_i = 0,$$

by incorporating both $f_{-1} = 1$ and $f_d = 1$ in the first formula.

Earlier inductive "proofs" of the above formula were proposed, notably a proof by Schläfli in 1852, but it was later observed that all these proofs assume that the boundary of every polytope can be built up inductively in a nice way, what is called *shellability*. Actually, counter-examples of shellability for various simplicial complexes suggested that polytopes were perhaps not shellable. However, the fact that polytopes are shellable was finally proved in 1970 by Bruggesser and Mani [12] and soon after that (also in 1970) a striking application of shellability was made by McMullen [29] who gave the first proof of the so-called "upper bound theorem".

As shellability of polytopes is an important tool and as it yields one of the cleanest inductive proof of the Euler-Poincaré formula, we will sketch its proof in some details. This Chapter is heavily inspired by Ziegler's excellent treatment [45], Chapter 8. We begin with the definition of shellability. It's a bit technical, so please be patient!

Definition 7.1 Let K be a pure polyhedral complex of dimension d. A *shelling of* K is a list, $F_1, \ldots, F_s$, of the cells (i.e., d-faces) of K such that either $d = 0$ (and thus, all F_i are points) or the following conditions hold:

(i) The boundary complex, $\mathcal{K}(\partial F_1)$, of the first cell, F_1, of K has a shelling.

(ii) For any j, $1 < j \leq s$, the intersection of the cell F_j with the previous cells is nonempty and is an initial segment of a shelling of the $(d-1)$-dimensional boundary complex of F_j, that is

$$F_j \cap \left(\bigcup_{i=1}^{j-1} F_i \right) = G_1 \cup G_2 \cup \cdots \cup G_r,$$

for some shelling $G_1, G_2, \ldots, G_r, \ldots, G_t$ of $\mathcal{K}(\partial F_j)$, with $1 \leq r \leq t$. As the intersection should be the initial segment of a shelling for the $(d-1)$-dimensional complex, ∂F_j, it has to be pure $(d-1)$-dimensional and connected for $d > 1$.

A polyhedral complex is *shellable* if it is pure and has a shelling.

Note that shellabiliy is only defined for pure complexes. Here are some examples of shellable complexes:

(1) Every 0-dimensional complex, that is, evey set of points, is shellable, by definition.

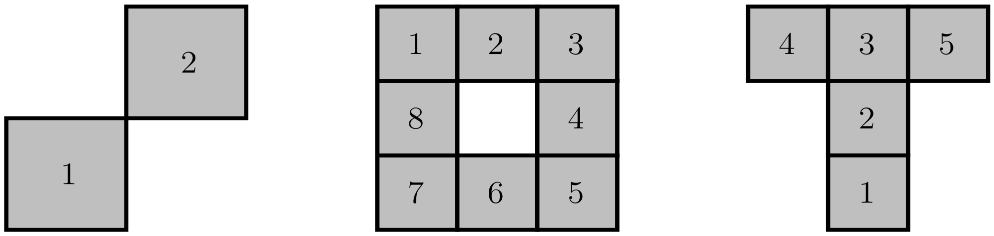

Figure 7.1: Non shellable and Shellable 2-complexes

(2) A 1-dimensional complex is a graph without loops and parallel edges. A 1-dimensional complex is shellable iff it is connected, which implies that it has no isolated vertices. Any ordering of the edges, $e_1, \ldots, e_s$, such that $\{e_1, \ldots, e_i\}$ induces a connected subgraph for every i will do. Such an ordering can be defined inductively, due to the connectivity of the graph.

(3) Every simplex is shellable. In fact, any ordering of its facets yields a shelling. This is easily shown by induction on the dimension, since the intersection of any two facets F_i and F_j is a facet of both F_i and F_j.

(4) The d-cubes are shellable. By induction on the dimension, it can be shown that every ordering of the $2d$ facets $F_1, \ldots, F_{2d}$ such that F_1 and F_{2d} are opposite (that is, $F_{2d} = -F_1$) yields a shelling.

However, already for 2-complexes, problems arise. For example, in Figure 7.1, the left and the middle 2-complexes are not shellable but the right complex is shellable.

The problem with the left complex is that cells 1 and 2 intersect at a vertex, which is not 1-dimensional, and in the middle complex, the intersection of cell 8 with its predecessors is not connected. In contrast, the ordering of the right complex is a shelling. However, observe that the reverse ordering is not a shelling because cell 4 has an empty intersection with cell 5!

Remarks:

1. Condition (i) in Definition 7.1 is redundant because, as we shall prove shortly, every polytope is shellable. However, if we want to use this definition for more general complexes, then condition (i) is necessary.

2. When K is a simplicial complex, condition (i) is of course redundant, as every simplex is shellable but condition (ii) can also be simplified to:

   (ii') For any j, with $1 < j \leq s$, the intersection of F_j with the previous cells is nonempty and pure $(d-1)$-dimensional. This means that for every $i < j$ there is some $l < j$ such that $F_i \cap F_j \subseteq F_l \cap F_j$ and $F_l \cap F_j$ is a facet of F_j.

The following proposition yields an important piece of information about the local structure of shellable simplicial complexes:

Proposition 7.1 *Let K be a shellable simplicial complex and say $F_1, \ldots, F_s$ is a shelling for K. Then, for every vertex, v, the restriction of the above sequence to the link,* $\mathrm{Lk}(v)$*, and to the star,* $\mathrm{St}(v)$*, are shellings.*

Since the complex, $\mathcal{K}(P)$, associated with a polytope, P, has a single cell, namely P itself, note that by condition (i) in the definition of a shelling, $\mathcal{K}(P)$ is shellable iff the complex, $\mathcal{K}(\partial P)$, is shellable. We will say simply say that "P is shellable" instead of "$\mathcal{K}(\partial P)$ is shellable".

We have the following useful property of shellings of polytopes whose proof is left as an exercise (use induction on the dimension):

Proposition 7.2 *Given any polytope, P, if $F_1, \ldots, F_s$ is a shelling of P, then the reverse sequence $F_s, \ldots, F_1$ is also a shelling of P.*

Proposition 7.2 generally fails for complexes that are not polytopes, see the right 2-complex in Figure 7.1.

We will now present the proof that every polytope is shellable, using a technique invented by Bruggesser and Mani (1970) known as *line shelling* [12]. This is quite a simple and natural idea if one is willing to ignore the technical details involved in actually checking that it works. We begin by explaining this idea in the 2-dimensional case, a convex polygon, since it is particularly simple.

Consider the 2-polytope, P, shown in Figure 7.2 (a polygon) whose faces are labeled F_1, F_2, F_3, F_4, F_5. Pick any line, ℓ, intersecting the interior of P and intersecting the supporting lines of the facets of P (*i.e.*, the edges of P) in distinct points labeled z_1, z_2, z_3, z_4, z_5 (such a line can always be found, as will be shown shortly). Orient the line, ℓ, (say, upward) and travel on ℓ starting from the point of P where ℓ leaves P, namely, z_1. For a while, only face F_1 is visible but when we reach the intersection, z_2, of ℓ with the supporting line of F_2, the face F_2 becomes visible and F_1 becomes invisible as it is now hidden by the supporting line of F_2. So far, we have seen the faces, F_1 and F_2, *in that order*. As we continue traveling along ℓ, no new face becomes visible but for a more complicated polygon, other faces, F_i, would become visible one at a time as we reach the intersection, z_i, of ℓ with the supporting line of F_i and the order in which these faces become visible corresponds to the ordering of the z_i's along the line ℓ. Then, we imagine that we travel very fast and when we reach "$+\infty$" in the upward direction on ℓ, we instantly come back on ℓ from below at "$-\infty$". At this point, we only see the face of P corresponding to the lowest supporting line of faces of P, i.e., the line corresponding to the smallest z_i, in our case, z_3. At this stage, the only visible face is F_3. We continue traveling upward on ℓ and we reach z_3, the intersection of the supporting line of F_3 with ℓ. At this moment, F_4 becomes visible and F_3 disappears as it is now hidden by the supporting line of F_4. Note that F_5 is not visible at this stage. Finally, we reach z_4, the intersection of the supporting line of F_4 with ℓ and at this moment, the last facet, F_5, becomes visible (and F_4 becomes invisible, F_3 being also invisible). Our trip stops when we reach z_5, the intersection of F_5 and ℓ. During the second phase of our trip, we saw F_3, F_4 and F_5 and the entire trip yields the sequence F_1, F_2, F_3, F_4, F_5, which is easily seen to be a shelling of P.

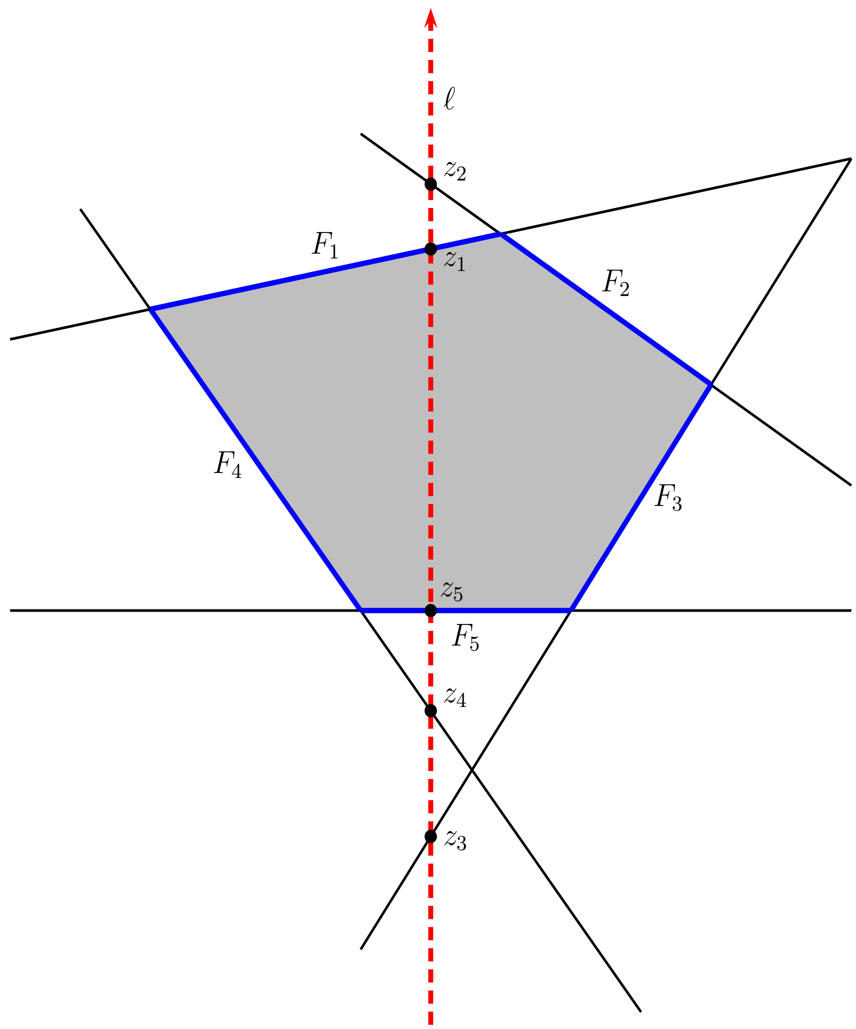

Figure 7.2: Shelling a polygon by travelling along a line

This is the crux of the Bruggesser-Mani method for shelling a polytope: We travel along a suitably chosen line and record the order in which the faces become visible during this trip. This is why such shellings are called *line shellings*.

In order to prove that polytopes are shellable we need the notion of points and lines in "general position". Recall from the equivalence of $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes that a polytope, P, in $\mathbb{E}^d$ with nonempty interior is cut out by t irredundant hyperplanes, H_i, and by picking the origin in the interior of P the equations of the H_i may be assumed to be of the form

$$a_i \cdot z = 1$$

where a_i and a_j are not proportional for all $i \neq j$, so that

$$P = \{z \in \mathbb{E}^d \mid a_i \cdot z \leq 1,\ 1 \leq i \leq t\}.$$

Definition 7.2 Let P be any polytope in $\mathbb{E}^d$ with nonempty interior and assume that P is cut out by the irredudant hyperplanes, H_i, of equations $a_i \cdot z = 1$, for $i = 1, \ldots, t$. A point, $c \in \mathbb{E}^d$, is said to be in *general position* w.r.t. P is c does not belong to any of the H_i, that is, if $a_i \cdot c \neq 1$ for $i = 1, \ldots, t$. A line, ℓ, is said to be in *general position* w.r.t. P if ℓ is not parallel to any of the H_i and if ℓ intersects the H_i in distinct points.

The following proposition showing the existence of lines in general position w.r.t. a polytope illustrates a very useful technique, the "perturbation method". The "trick" behind this particular perturbation method is that polynomials (in one variable) have a finite number of zeros.

Proposition 7.3 *Let P be any polytope in $\mathbb{E}^d$ with nonempty interior. For any two points, x and y in $\mathbb{E}^d$, with x outside of P; y in the interior of P; and x in general position w.r.t. P, for $\lambda \in \mathbb{R}$ small enough, the line, ℓ_λ, through x and y_λ with*

$$y_\lambda = y + (\lambda, \lambda^2, \ldots, \lambda^d),$$

intersects P in its interior and is in general position w.r.t. P.

Proof. Assume that P is defined by t irredundant hyperplanes, H_i, where H_i is given by the equation $a_i \cdot z = 1$ and write $\Lambda = (\lambda, \lambda^2, \ldots, \lambda^d)$ and $u = y - x$. Then the line ℓ_λ is given by

$$\ell_\lambda = \{x + s(y_\lambda - x) \mid s \in \mathbb{R}\} = \{x + s(u + \Lambda) \mid s \in \mathbb{R}\}.$$

The line, ℓ_λ, is not parallel to the hyperplane H_i iff

$$a_i \cdot (u + \Lambda) \neq 0, \quad i = 1, \ldots, t$$

and it intersects the H_i in distinct points iff there is no $s \in \mathbb{R}$ such that

$$a_i \cdot (x + s(u + \Lambda)) = 1 \quad \text{and} \quad a_j \cdot (x + s(u + \Lambda)) = 1 \quad \text{for some } i \neq j.$$

Observe that $a_i \cdot (u + \Lambda) = p_i(\lambda)$ is a nonzero polynomial in λ of degree at most d. Since a polynomial of degree d has at most d zeros, if we let $Z(p_i)$ be the (finite) set of zeros of p_i we can ensure that ℓ_λ is not parallel to any of the H_i by picking $\lambda \notin \bigcup_{i=1}^{t} Z(p_i)$ (where

$\bigcup_{i=1}^{t} Z(p_i)$ is a finite set). Now, as x is in general position w.r.t. P, we have $a_i \cdot x \neq 1$, for $i = 1 \ldots, t$. The condition stating that ℓ_λ intersects the H_i in distinct points can be written

$$a_i \cdot x + s a_i \cdot (u + \Lambda) = 1 \quad \text{and} \quad a_j \cdot x + s a_j \cdot (u + \Lambda) = 1 \quad \text{for some } i \neq j,$$

or

$$s p_i(\lambda) = \alpha_i \quad \text{and} \quad s p_j(\lambda) = \alpha_j \quad \text{for some } i \neq j,$$

where $\alpha_i = 1 - a_i \cdot x$ and $\alpha_j = 1 - a_j \cdot x$. As x is in general position w.r.t. P, we have $\alpha_i, \alpha_j \neq 0$ and as the H_i are irredundant, the polynomials $p_i(\lambda) = a_i \cdot (u + \Lambda)$ and $p_j(\lambda) = a_j \cdot (u + \Lambda)$ are not proportional. Now, if $\lambda \notin Z(p_i) \cup Z(p_j)$, in order for the system

$$\begin{aligned} s p_i(\lambda) &= \alpha_i \\ s p_j(\lambda) &= \alpha_j \end{aligned}$$

to have a solution in s we must have

$$q_{ij}(\lambda) = \alpha_i p_j(\lambda) - \alpha_j p_i(\lambda) = 0,$$

where $q_{ij}(\lambda)$ is not the zero polynomial since $p_i(\lambda)$ and $p_j(\lambda)$ are not proportional and $\alpha_i, \alpha_j \neq 0$. If we pick $\lambda \notin Z(q_{ij})$, then $q_{ij}(\lambda) \neq 0$. Therefore, if we pick

$$\lambda \notin \bigcup_{i=1}^{t} Z(p_i) \cup \bigcup_{i \neq j}^{t} Z(q_{ij}),$$

the line ℓ_λ is in general position w.r.t. P. Finally, we can pick λ small enough so that $y_\lambda = y + \Lambda$ is close enough to y so that it is in the interior of P. $\square$

It should be noted that the perturbation method involving $\Lambda = (\lambda, \lambda^2, \ldots, \lambda^d)$ is quite flexible. For example, by adapting the proof of Proposition 7.3 we can prove that for any two distinct facets, F_i and F_j of P, there is a line in general position w.r.t. P intersecting F_i and F_j. Start with x outside P and very close to F_i and y in the interior of P and very close to F_j.

Finally, before proving the existence of line shellings for polytopes, we need more terminology. Given any point, x, strictly outside a polytope, P, we say that a facet, F, of P is *visible from* x iff for every $y \in F$ the line through x and y intersects F only in y (equivalently, x and the interior of P are strictly separared by the supporting hyperplane of F). We now prove the following fundamental theorem due to Bruggesser and Mani [12] (1970):

Theorem 7.4 *(Existence of Line Shellings for Polytopes) Let P be any polytope in $\mathbb{E}^d$ of dimension d. For every point, x, outside P and in general position w.r.t. P, there is a shelling of P in which the facets of P that are visible from x come first.*

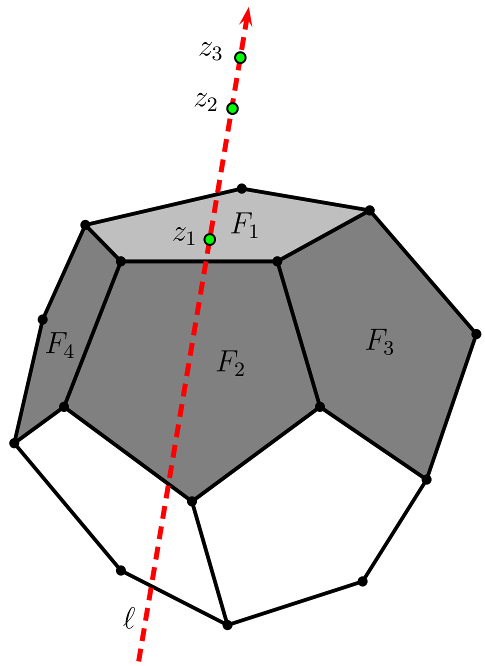

Figure 7.3: Shelling a polytope by travelling along a line, ℓ

Proof. By Proposition 7.3, we can find a line, ℓ, through x such that ℓ is in general position w.r.t. P and ℓ intersects the interior of P. Pick one of the two faces in which ℓ intersects P, say F_1, let $z_1 = \ell \cap F_1$, and orient ℓ from the inside of P to z_1. As ℓ intersects the supporting hyperplanes of the facets of P in distinct points, we get a linearly ordered list of these intersection points along ℓ,

$$z_1,\ z_2,\ \cdots,\ z_m,\ z_{m+1}, \cdots,\ z_s,$$

where z_{m+1} is the smallest element, z_m is the largest element and where z_1 and z_s belong to the faces of P where ℓ intersects P. Then, as in the example illustrated by Figure 7.2, by travelling "upward" along the line ℓ starting from z_1 we get a total ordering of the facets of P,

$$F_1, F_2, \ldots, F_m, F_{m+1}, \ldots, F_s$$

where F_i is the facet whose supporting hyperplane cuts ℓ in z_i.

We claim that the above sequence is a shelling of P. This is proved by induction on d. For $d = 1$, P consists a line segment and the theorem clearly holds.

Consider the intersection $\partial F_j \cap (F_1 \cup \cdots \cup F_{j-1})$. We need to show that this is an initial segment of a shelling of ∂F_j. If $j \leq m$, i.e., if F_j become visible before we reach ∞, then the above intersection is exactly the set of facets of F_j that are visible from $z_j = \ell \cap \mathrm{aff}(F_j)$.

Therefore, by induction on the dimension, these facets are shellable and they form an initial segment of a shelling of the whole boundary ∂F_j.

If $j \geq m+1$, that is, after "passing through ∞" and reentering from $-\infty$, the intersection $\partial F_j \cap (F_1 \cup \cdots \cup F_{j-1})$ is the set of non-visible facets. By reversing the orientation of the line, ℓ, we see that the facets of this intersection are shellable and we get the reversed ordering of the facets.

Finally, when we reach the point x starting from z_1, the facets visible from x form an initial segment of the shelling, as claimed. $\square$

Remark: The trip along the line ℓ is often described as a *rocket flight* starting from the surface of P viewed as a little planet (for instance, this is the description given by Ziegler [45] (Chapter 8)). Observe that if we reverse the direction of ℓ, we obtain the reversal of the original line shelling. Thus, the reversal of a line shelling is not only a shelling but a line shelling as well.

We can easily prove the following corollary:

Corollary 7.5 *Given any polytope, P, the following facts hold:*

(1) For any two facets F and F', there is a shelling of P in which F comes first and F' comes last.

(2) For any vertex, v, of P, there is a shelling of P in which the facets containing v form an initial segment of the shelling.

Proof. For (1), we use a line in general position and intersecting F and F' in their interior. For (2), we pick a point, x, beyond v and pick a line in general position through x intersecting the interior of P. Pick the origin, O, in the interior of P. A point, x, is *beyond* v iff x and O lies on different sides of every hyperplane, H_i, supporting a facet of P containing x but on the same side of H_i for every hyperplane, H_i, supporting a facet of P **not** containing x. Such a point can be found on a line through O and v, as the reader should check. $\square$

Remark: A *plane triangulation*, K, is a pure two-dimensional complex in the plane such that $|K|$ is homeomorphic to a closed disk. Edelsbrunner proves that every plane triangulation has a shelling and from this, that $\chi(K) = 1$, where $\chi(K) = f_0 - f_1 + f_2$ is the Euler-Poincaré characteristic of K, where f_0 is the number of vertices, f_1 is the number of edges and f_2 is the number of triangles in K (see Edelsbrunner [17], Chapter 3). This result is an immediate consequence of Corollary 7.5 if one knows about the stereographic projection map, which will be discussed in the next Chapter.

We now have all the tools needed to prove the famous Euler-Poincaré Formula for Polytopes.

7.2 The Euler-Poincaré Formula for Polytopes

We begin by defining a very important topological concept, the Euler-Poincaré characteristic of a complex.

Definition 7.3 Let K be a d-dimensional complex. For every i, with $0 \leq i \leq d$, we let f_i denote the number of i-faces of K and we let

$$\mathbf{f}(K) = (f_0, \cdots, f_d) \in \mathbb{N}^{d+1}$$

be the *f-vector* associated with K (if necessary we write $f_i(K)$ instead of f_i). The *Euler-Poincaré characteristic*, $\chi(K)$, of K is defined by

$$\chi(K) = f_0 - f_1 + f_2 + \cdots + (-1)^d f_d = \sum_{i=0}^{d} (-1)^i f_i.$$

Given any d-dimensional polytope, P, the *f-vector* associated with P is the f-vector associated with $\mathcal{K}(P)$, that is,

$$\mathbf{f}(P) = (f_0, \cdots, f_d) \in \mathbb{N}^{d+1},$$

where f_i, is the number of i-faces of P (= the number of i-faces of $\mathcal{K}(P)$ and thus, $f_d = 1$), and the *Euler-Poincaré characteristic*, $\chi(P)$, of P is defined by

$$\chi(P) = f_0 - f_1 + f_2 + \cdots + (-1)^d f_d = \sum_{i=0}^{d} (-1)^i f_i.$$

Moreover, the *f-vector* associated with the boundary, ∂P, of P is the f-vector associated with $\mathcal{K}(\partial P)$, that is,

$$\mathbf{f}(\partial P) = (f_0, \cdots, f_{d-1}) \in \mathbb{N}^{d}$$

where f_i, is the number of i-faces of ∂P (with $0 \leq i \leq d-1$), and the *Euler-Poincaré characteristic*, $\chi(\partial P)$, of ∂P is defined by

$$\chi(\partial P) = f_0 - f_1 + f_2 + \cdots + (-1)^{d-1} f_{d-1} = \sum_{i=0}^{d-1} (-1)^i f_i.$$

Observe that $\chi(P) = \chi(\partial P) + (-1)^d$, since $f_d = 1$.

Remark: It is convenient to set $f_{-1} = 1$. Then, some authors, including Ziegler [45] (Chapter 8), define the *reduced Euler-Poincaré characteristic*, $\chi'(K)$, of a complex (or a polytope), K, as

$$\chi'(K) = -f_{-1} + f_0 - f_1 + f_2 + \cdots + (-1)^d f_d = \sum_{i=-1}^{d} (-1)^i f_i = -1 + \chi(K),$$

i.e., they incorporate $f_{-1} = 1$ into the formula.

A crucial observation for proving the Euler-Poincaré formula is that the Euler-Poincaré characteristic is additive, which means that if K_1 and K_2 are any two complexes such that $K_1 \cup K_2$ is also a complex, which implies that $K_1 \cap K_2$ is also a complex (because we must have $F_1 \cap F_2 \in K_1 \cap K_2$ for every face F_1 of K_1 and every face F_2 of K_2), then

$$\chi(K_1 \cup K_2) = \chi(K_1) + \chi(K_2) - \chi(K_1 \cap K_2).$$

This follows immediately because for any two sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

To prove our next theorem we will use complete induction on $\mathbb{N} \times \mathbb{N}$ ordered by the lexicographic ordering. Recall that the lexicographic ordering on $\mathbb{N} \times \mathbb{N}$ is defined as follows:

$$(m, n) < (m', n') \quad \text{iff} \quad \begin{cases} m = m' \quad \textit{and} \quad n < n' \\ \text{or} \\ m < m'. \end{cases}$$

Theorem 7.6 *(Euler-Poincaré Formula) For every polytope, P, we have*

$$\chi(P) = \sum_{i=0}^{d} (-1)^i f_i = 1 \qquad (d \geq 0),$$

and so,

$$\chi(\partial P) = \sum_{i=0}^{d-1} (-1)^i f_i = 1 - (-1)^d \qquad (d \geq 1).$$

Proof. We prove the following statement: For every d-dimensional polytope, P, if $d = 0$ then

$$\chi(P) = 1,$$

else if $d \geq 1$ then for every shelling $F_1, \ldots, F_{f_{d-1}}$, of P, for every j, with $1 \leq j \leq f_{d-1}$, we have

$$\chi(F_1 \cup \cdots \cup F_j) = \begin{cases} 1 & \text{if } 1 \leq j < f_{d-1} \\ 1 - (-1)^d & \text{if } j = f_{d-1}. \end{cases}$$

We proceed by complete induction on $(d, j) \geq (0, 1)$. For $d = 0$ and $j = 1$, the polytope P consists of a single point and so, $\chi(P) = f_0 = 1$, as claimed.

For the induction step, assume that $d \geq 1$. For $1 = j < f_{d-1}$, since F_1 is a polytope of dimension $d - 1$, by the induction hypothesis, $\chi(F_1) = 1$, as desired.

For $1 < j < f_{d-1}$, we have

$$\chi(F_1 \cup \cdots F_{j-1} \cup F_j) = \chi\left(\bigcup_{i=1}^{j-1} F_i\right) + \chi(F_j) - \chi\left(\left(\bigcup_{i=1}^{j-1} F_i\right) \cap F_j\right).$$

Since $(d, j-1) < (d, j)$, by the induction hypothesis,

$$\chi\left(\bigcup_{i=1}^{j-1} F_i\right) = 1$$

and since $\dim(F_j) = d-1$, again by the induction hypothesis,

$$\chi(F_j) = 0.$$

Now, as $F_1, \ldots, F_{f_{d-1}}$ is a shelling and $j < f_{d-1}$, we have

$$\left(\bigcup_{i=1}^{j-1} F_i\right) \cap F_j = G_1 \cup \cdots \cup G_r,$$

for some shelling $G_1, \ldots, G_r, \ldots, G_t$ of $\mathcal{K}(\partial F_j)$, with $r < t = f_{d-2}(\partial F_j)$. The fact that $r < f_{d-2}(\partial F_j)$, *i.e.*, that $G_1 \cup \cdots \cup G_r$ is not the whole boundary of F_j is a property of line shellings and also follows from Proposition 7.2. As $\dim(\partial F_j) = d-2$, and $r < f_{d-2}(\partial F_j)$, by the induction hypothesis, we have

$$\chi\left(\left(\bigcup_{i=1}^{j-1} F_i\right) \cap F_j\right) = \chi(G_1 \cup \cdots \cup G_r) = 1.$$

Consequently,

$$\chi(F_1 \cup \cdots F_{j-1} \cup F_j) = 1 + 1 - 1 = 1,$$

as claimed (when $j < f_{d-1}$).

If $j = f_{d-1}$, then we have a complete shelling of $\partial F_{f_{d-1}}$, that is,

$$\left(\bigcup_{i=1}^{f_{d-1}-1} F_i\right) \cap F_{f_{d-1}} = G_1 \cup \cdots \cup G_{f_{d-2}(F_{f_{d-1}})} = \partial F_{f_{d-1}}.$$

As $\dim(\partial F_j) = d-2$, by the induction hypothesis,

$$\chi(\partial F_{f_{d-1}}) = \chi(G_1 \cup \cdots \cup G_{f_{d-2}(F_{f_{d-1}})}) = 1 - (-1)^{d-1}$$

and it follows that

$$\chi(F_1 \cup \cdots \cup F_{f_{d-1}}) = 1 + 1 - (1 - (-1)^{d-1}) = 1 + (-1)^{d-1} = 1 - (-1)^d,$$

establishing the induction hypothesis in this last case. But then,

$$\chi(\partial P) = \chi(F_1 \cup \cdots \cup F_{f_{d-1}}) = 1 - (-1)^d$$

and

$$\chi(P) = \chi(\partial P) + (-1)^d = 1,$$

proving our theorem. $\square$

Remark: Other combinatorial proofs of the Euler-Poincaré formula are given in Grünbaum [24] (Chapter 8), Boissonnat and Yvinec [8] (Chapter 7) and Ewald [18] (Chapter 3). Coxeter gives a proof very close to Poincaré's own proof using notions of homology theory [13] (Chapter IX). We feel that the proof based on shellings is the most direct and one of the most elegant. Incidently, the above proof of the Euler-Poincaré formula is very close to Schläfli proof from 1852 but Schläfli did not have shellings at his disposal so his "proof" had a gap. The Bruggesser-Mani proof that polytopes are shellable fills this gap!

7.3 Dehn-Sommerville Equations for Simplicial Polytopes and h-Vectors

If a d-polytope, P, has the property that its faces are all simplices, then it is called a *simplicial polytope*. It is easily shown that a polytope is simplicial iff its facets are simplices, in which case, every facet has d vertices. The polar dual of a simplicial polytope is called a *simple polytope*. We see immediately that every vertex of a simple polytope belongs to d facets.

For simplicial (and simple) polytopes it turns out that other remarkable equations besides the Euler-Poincaré formula hold among the number of i-faces. These equations were discovered by Dehn for $d = 4, 5$ (1905) and by Sommerville in the general case (1927). Although it is possible (and not difficult) to prove the Dehn-Sommerville equations by "double counting", as in Grünbaum [24] (Chapter 9) or Boissonnat and Yvinec (Chapter 7, but beware, these are the dual formulae for simple polytopes), it turns out that instead of using the f-vector associated with a polytope it is preferable to use what's known as the h-vector because for simplicial polytopes the h-numbers have a natural interpretation in terms of shellings. Furthermore, the statement of the Dehn-Sommerville equations in terms of h-vectors is transparent:

$$h_i = h_{d-i},$$

and the proof is very simple in terms of shellings.

In the rest of this section, we restrict our attention to simplicial complexes. In order to motivate h-vectors, we begin by examining more closely the structure of the new faces that are created during a shelling when the cell F_j is added to the partial shelling $F_1, \ldots, F_{j-1}$.

If K is a simplicial polytope and V is the set of vertices of K, then every i-face of K can be identified with an $(i+1)$-subset of V (that is, a subset of V of cardinality $i+1$).

Definition 7.4 For any shelling, $F_1, \ldots, F_s$, of a simplicial complex, K, of dimension d, for every j, with $1 \leq j \leq s$, the *restriction*, R_j, of the facet, F_j, is the set of "obligatory" vertices

$$R_j = \{v \in F_j \mid F_j - \{v\} \subseteq F_i, \text{ for some } i \text{ with } 1 \leq i < j\}.$$

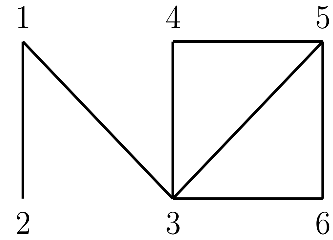

Figure 7.4: A connected 1-dimensional complex, G

The crucial property of the R_j is that the new faces, G, added at step j (when F_j is added to the shelling) are precisely the faces in the set

$$I_j = \{G \subseteq V \mid R_j \subseteq G \subseteq F_j\}.$$

The proof of the above fact is left as an exercise to the reader.

But then, we obtain a partition, $\{I_1, \ldots, I_s\}$, of the set of faces of the simplicial complex (other that K itself). Note that the empty face is allowed. Now, if we define

$$h_i = |\{j \mid |R_j| = i,\ 1 \leq j \leq s\}|,$$

for $i = 0, \ldots, d$, then it turns out that we can recover the f_k in terms of the h_i as follows:

$$f_{k-1} = \sum_{j=1}^{s} \binom{d - |R_j|}{k - |R_j|} = \sum_{i=0}^{k} h_i \binom{d-i}{k-i},$$

with $1 \leq k \leq d$.

But more is true: The above equations are invertible and the h_k can be expressed in terms of the f_i as follows:

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1},$$

with $0 \leq k \leq d$ (remember, $f_{-1} = 1$).

Let us explain all this in more detail. Consider the example of a connected graph (a simplicial 1-dimensional complex) from Ziegler [45] (Section 8.3) shown in Figure 7.4:

A shelling order of its 7 edges is given by the sequence

$$12,\ 13,\ 34,\ 35,\ 45,\ 36,\ 56.$$

The partial order of the faces of G together with the blocks of the partition $\{I_1, \ldots, I_7\}$ associated with the seven edges of G are shown in Figure 7.5, with the blocks I_j shown in boldface:

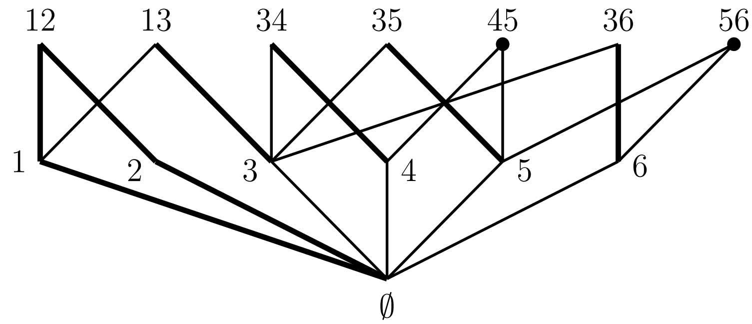

Figure 7.5: the partition associated with a shelling of G

The "minimal" new faces (corresponding to the R_j's) added at every stage of the shelling are

$$\emptyset,\ 3,\ 4,\ 5,\ 45,\ 6,\ 56.$$

Again, if h_i is the number of blocks, I_j, such that the corresponding restriction set, R_j, has size i, that is,

$$h_i = |\{j \mid |R_j| = i,\ 1 \leq j \leq s\}|,$$

for $i = 0, \ldots, d$, where the simplicial polytope, K, has dimension $d-1$, we define the *h-vector* associated with K as

$$\mathbf{h}(K) = (h_0, \ldots, h_d).$$

Then, in the above example, as $R_1 = \{\emptyset\}$, $R_2 = \{3\}$, $R_3 = \{4\}$, $R_4 = \{5\}$, $R_5 = \{4, 5\}$, $R_6 = \{6\}$ and $R_7 = \{5, 6\}$, we get $h_0 = 1$, $h_1 = 4$ and $h_2 = 2$, that is,

$$\mathbf{h}(G) = (1, 4, 2).$$

Now, let us show that if K is a shellable simplicial complex, then the f-vector can be recovered from the h-vector. Indeed, if $|R_j| = i$, then each $(k-1)$-face in the block I_j must use all i nodes in R_j, so that there are only $d-i$ nodes available and, among those, $k-i$ must be chosen. Therefore,

$$f_{k-1} = \sum_{j=1}^{s} \binom{d - |R_j|}{k - |R_j|}$$

and, by definition of h_i, we get

$$f_{k-1} = \sum_{i=0}^{k} h_i \binom{d-i}{k-i} = h_k + \binom{d-k+1}{1} h_{k-1} + \cdots + \binom{d-1}{k-1} h_1 + \binom{d}{k} h_0, \qquad (*)$$

where $1 \leq k \leq d$. Moreover, the formulae are invertible, that is, the h_i can be expressed in terms of the f_k. For this, form the two polynomials

$$f(x) = \sum_{i=0}^{d} f_{i-1} x^{d-i} = f_{d-1} + f_{d-2} x + \cdots + f_0 x^{d-1} + f_{-1} x^d$$

with $f_{-1} = 1$ and

$$h(x) = \sum_{i=0}^{d} h_i x^{d-i} = h_d + h_{d-1}x + \cdots + h_1 x^{d-1} + h_0 x^d.$$

Then, it is easy to see that

$$f(x) = \sum_{i=0}^{d} h_i (x+1)^{d-i} = h(x+1).$$

Consequently, $h(x) = f(x-1)$ and by comparing the coefficients of x^{d-k} on both sides of the above equation, we get

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}.$$

In particular, $h_0 = 1$, $h_1 = f_0 - d$, and

$$h_d = f_{d-1} - f_{d-2} + f_{d-3} + \cdots + (-1)^{d-1} f_0 + (-1)^d.$$

It is also easy to check that

$$h_0 + h_1 + \cdots + h_d = f_{d-1}.$$

Now, we just showed that if K is shellable, then its f-vector and its h-vector are related as above. But even if K is not shellable, the above suggests defining the h-vector from the f-vector as above. Thus, we make the definition:

Definition 7.5 For any $(d-1)$-dimensional simplicial complex, K, the *h-vector* associated with K is the vector

$$\mathbf{h}(K) = (h_0, \ldots, h_d) \in \mathbb{Z}^{d+1},$$

given by

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}.$$

Note that if K is shellable, then the interpretation of h_i as the number of cells, F_j, such that the corresponding restriction set, R_j, has size i shows that $h_i \geq 0$. However, for an arbitrary simplicial complex, some of the h_i can be strictly negative. Such an example is given in Ziegler [45] (Section 8.3).

We summarize below most of what we just showed:

Proposition 7.7 *Let K be a $(d-1)$-dimensional pure simplicial complex. If K is shellable, then its h-vector is nonnegative and h_i counts the number of cells in a shelling whose restriction set has size i. Moreover, the h_i do not depend on the particular shelling of K.*

There is a way of computing the h-vector of a pure simplicial complex from its f-vector reminiscent of the Pascal triangle (except that negative entries can turn up). Again, the reader is referred to Ziegler [45] (Section 8.3).

We are now ready to prove the Dehn-Sommerville equations. For $d = 3$, these are easily obtained by double counting. Indeed, for a simplicial polytope, every edge belongs to two facets and every facet has three edges. It follows that

$$2f_1 = 3f_2.$$

Together with Euler's formula

$$f_0 - f_1 + f_2 = 2,$$

we see that

$$f_1 = 3f_0 - 6 \quad \text{and} \quad f_2 = 2f_0 - 4,$$

namely, that the number of vertices of a simplicial 3-polytope determines its number of edges and faces, these being linear functions of the number of vertices. For arbitrary dimension d, we have

Theorem 7.8 *(Dehn-Sommerville Equations) If K is any simplicial d-polytope, then the components of the h-vector satisfy*

$$h_k = h_{d-k} \qquad k = 0, 1, \ldots, d.$$

Equivalently

$$f_{k-1} = \sum_{i=k}^{d} (-1)^{d-i} \binom{i}{k} f_{i-1} \qquad k = 0, \ldots, d.$$

Furthermore, the equation $h_0 = h_d$ is equivalent to the Euler-Poincaré formula.

Proof. We present a short and elegant proof due to McMullen. Recall from Proposition 7.2 that the reversal, $F_s, \ldots, F_1$, of a shelling, $F_1, \ldots, F_s$, of a polytope is also a shelling. From this, we see that for every F_j, the restriction set of F_j in the reversed shelling is equal to $R_j - F_j$, the complement of the restriction set of F_j in the original shelling. Therefore, if $|R_j| = k$, then F_j contributes "1" to h_k in the original shelling iff it contributes "1" to h_{d-k} in the reversed shelling (where $|R_j - F_j| = d - k$). It follows that the value of h_k computed in the original shelling is the same as the value of h_{d-k} computed in the reversed shelling. However, by Proposition 7.7, the h-vector is independent of the shelling and hence, $h_k = h_{d-k}$.

Define the polynomials $F(x)$ and $H(x)$ by

$$F(x) = \sum_{i=0}^{d} f_{i-1} x^i; \quad H(x) = (1-x)^d F\left(\frac{x}{1-x}\right).$$

Note that $H(x) = \sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i}$ and an easy computation shows that the coefficient of x^k is equal to

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1} = h_k.$$

Now, the equations $h_k = h_{d-k}$ are equivalent to

$$H(x) = x^d H(x^{-1}),$$

that is,

$$F(x-1) = (-1)^d F(-x).$$

As

$$F(x-1) = \sum_{i=0}^{d} f_{i-1}(x-1)^i = \sum_{i=0}^{d} f_{i-1} \sum_{j=0}^{i} \binom{i}{i-j} x^{i-j} (-1)^j,$$

we see that the coefficient of x^k in $F(x-1)$ (obtained when $i-j=k$, that is, $j = i-k$) is

$$\sum_{i=0}^{d} (-1)^{i-k} \binom{i}{k} f_{i-1} = \sum_{i=k}^{d} (-1)^{i-k} \binom{i}{k} f_{i-1}.$$

On the other hand, the coefficient of x^k in $(-1)^d F(-x)$ is $(-1)^{d+k} f_{k-1}$. By equating the coefficients of x^k, we get

$$(-1)^{d+k} f_{k-1} = \sum_{i=k}^{d} (-1)^{i-k} \binom{i}{k} f_{i-1},$$

which, by multiplying both sides by $(-1)^{d+k}$, is equivalent to

$$f_{k-1} = \sum_{i=k}^{d} (-1)^{d+i} \binom{i}{k} f_{i-1} = \sum_{i=k}^{d} (-1)^{d-i} \binom{i}{k} f_{i-1},$$

as claimed. Finally, as we already know that

$$h_d = f_{d-1} - f_{d-2} + f_{d-3} + \cdots + (-1)^{d-1} f_0 + (-1)^d$$

and $h_0 = 1$, by multiplying both sides of the equation $h_d = h_0 = 1$ by $(-1)^{d-1}$ and moving $(-1)^d(-1)^{d-1} = -1$ to the right hand side, we get the Euler-Poincaré formula. $\square$

Clearly, the Dehn-Sommerville equations, $h_k = h_{d-k}$, are linearly independent for $0 \leq k < \lfloor \frac{d+1}{2} \rfloor$. For example, for $d = 3$, we have the two independent equations

$$h_0 = h_3,\ h_1 = h_2,$$

and for $d = 4$, we also have two independent equations

$$h_0 = h_4,\ h_1 = h_3,$$

since $h_2 = h_2$ is trivial. When $d = 3$, we know that $h_1 = h_2$ is equivalent to $2f_1 = 3f_2$ and when $d = 4$, if one unravels $h_1 = h_3$ in terms of the f_i' one finds

$$2f_2 = 4f_3,$$

that is $f_2 = 2f_3$. More generally, it is easy to check that

$$2f_{d-2} = df_{d-1}$$

for all d. For $d = 5$, we find three independent equations

$$h_0 = h_5,\ h_1 = h_4,\ h_2 = h_3,$$

and so on.

It can be shown that for general d-polytopes, the Euler-Poincaré formula is the only equation satisfied by all h-vectors and for simplicial d-polytopes, the $\lfloor \frac{d+1}{2} \rfloor$ Dehn-Sommerville equations, $h_k = h_{d-k}$, are the only equations satisfied by all h-vectors (see Grünbaum [24], Chapter 9).

Remark: Readers familiar with homology and cohomology may suspect that the Dehn-Sommerville equations are a consequence of a type of Poincaré duality. Stanley proved that this is indeed the case. It turns out that the h_i are the dimensions of cohomology groups of a certain *toric variety* associated with the polytope. For more on this topic, see Stanley [37] (Chapters II and III) and Fulton [19] (Section 5.6).

As we saw for 3-dimensional simplicial polytopes, the number of vertices, $n = f_0$, determines the number of edges and the number of faces, and these are linear in f_0. For $d \geq 4$, this is no longer true and the number of facets is no longer linear in n but in fact quadratic. It is then natural to ask which d-polytopes with a prescribed number of vertices have the maximum number of k-faces. This question which remained an open problem for some twenty years was eventually settled by McMullen in 1970 [29]. We will present this result (without proof) in the next section.

7.4 The Upper Bound Theorem and Cyclic Polytopes

Given a d-polytope with n vertices, what is an upper bound on the number of its i-faces? This question is not only important from a theoretical point of view but also from a computational point of view because of its implications for algorithms in combinatorial optimization and in computational geometry.

The answer to the above problem is that there is a class of polytopes called *cyclic polytopes* such that the cyclic d-polytope, $C_d(n)$, has the maximum number of i-faces among all d-polytopes with n vertices. This result stated by Motzkin in 1957 became known as the *upper bound conjecture* until it was proved by McMullen in 1970, using shellings [29] (just after Bruggesser and Mani's proof that polytopes are shellable). It is now known as the *upper bound theorem*. Another proof of the upper bound theorem was given later by Alon and Kalai [2] (1985). A version of this proof can also be found in Ewald [18] (Chapter 3).

McMullen's proof is not really very difficult but it is still quite involved so we will only state some propositions needed for its proof. We urge the reader to read Ziegler's account of this beautiful proof [45] (Chapter 8). We begin with cyclic polytopes.

First, consider the cases $d = 2$ and $d = 3$. When $d = 2$, our polytope is a polygon in which case $n = f_0 = f_1$. Thus, this case is trivial.

For $d = 3$, we claim that $2f_1 \geq 3f_2$. Indeed, every edge belongs to exactly two faces so if we add up the number of sides for all faces, we get $2f_1$. Since every face has at least three sides, we get $2f_1 \geq 3f_2$. Then, using Euler's relation, it is easy to show that

$$f_1 \leq 6n - 3 \quad f_2 \leq 2n - 4$$

and we know that equality is achieved for simplicial polytopes.

Let us now consider the general case. The rational curve, $c\colon \mathbb{R} \to \mathbb{R}^d$, given parametrically by

$$c(t) = (t, t^2, \ldots, t^d)$$

is at the heart of the story. This curve if often called the *moment curve* or *rational normal curve* of degree d. For $d = 3$, it is known as the *twisted cubic*. Here is the definition of the cyclic polytope, $C_d(n)$.

Definition 7.6 For any sequence, $t_1 < \ldots < t_n$, of distinct real number, $t_i \in \mathbb{R}$, with $n > d$, the convex hull,

$$C_d(n) = \mathrm{conv}(c(t_1), \ldots, c(t_n))$$

of the n points, $c(t_1), \ldots, c(t_n)$, on the moment curve of degree d is called a *cyclic polytope*.

The first interesting fact about the cyclic polytope is that it is simplicial.

Proposition 7.9 *Every $d+1$ of the points $c(t_1), \ldots, c(t_n)$ are affinely independent. Consequently, $C_d(n)$ is a simplicial polytope and the $c(t_i)$ are vertices.*

Proof. We may assume that $n = d+1$. Say $c(t_1), \ldots, c(t_n)$ belong to a hyperplane, H, given by

$$\alpha_1 x_1 + \cdots + \alpha_d x_d = \beta.$$

(Of course, not all the α_i are zero.) Then, we have the polynomial, $H(t)$, given by

$$H(t) = -\beta + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_d t^d,$$

of degree at most d and as each $c(t_i)$ belong to H, we see that each $c(t_i)$ is a zero of $H(t)$. However, there are $d+1$ distinct $c(t_i)$, so $H(t)$ would have $d+1$ distinct roots. As $H(t)$ has degree at most d, it must be the zero polynomial, a contradiction. Returing to the original $n > d+1$, we just proved every $d+1$ of the points $c(t_1), \ldots, c(t_n)$ are affinely independent. Then, every proper face of $C_d(n)$ has at most d independent vertices, which means that it is a simplex. $\square$

The following proposition already shows that the cyclic polytope, $C_d(n)$, has $\binom{n}{k}$ $(k-1)$-faces if $1 \leq k \leq \lfloor \frac{d}{2} \rfloor$.

Proposition 7.10 *For any k with $2 \leq 2k \leq d$, every subset of k vertices of $C_d(n)$ is a $(k-1)$-face of $C_d(n)$. Hence*

$$f_k(C_d(n)) = \binom{n}{k+1} \qquad \text{if} \quad 0 \leq k < \left\lfloor \frac{d}{2} \right\rfloor.$$

Proof. Consider any sequence $t_{i_1} < t_{i_2} < \cdots < t_{i_k}$. We will prove that there is a hyperplane separating $F = \mathrm{conv}(\{c(t_{i_1}), \ldots, c(t_{i_k})\})$ and $C_d(n)$. Consider the polynomial

$$p(t) = \prod_{j=1}^{k} (t - t_{i_j})^2$$

and write

$$p(t) = a_0 + a_1 t + \cdots + a_{2k} t^{2k}.$$

Consider the vector

$$a = (a_1, a_2, \ldots, a_{2k}, 0, \ldots, 0) \in \mathbb{R}^d$$

and the hyperplane, H, given by

$$H = \{x \in \mathbb{R}^d \mid x \cdot a = -a_0\}.$$

Then, for each j with $1 \leq j \leq k$, we have

$$c(t_{i_j}) \cdot a = a_1 t_{i_j} + \cdots + a_{2k} t_{i_j}^{2k} = p(t_{i_j}) - a_0 = -a_0,$$

and so, $c(t_{i_j}) \in H$. On the other hand, for any other point, $c(t_i)$, distinct from any of the $c(t_{i_j})$, we have

$$c(t_i) \cdot a = -a_0 + p(t_i) = -a_0 + \prod_{j=1}^{k} (t_i - t_{i_j})^2 > -a_0,$$

proving that $c(t_i) \in H_+$. But then, H is a supporting hyperplane of F for $C_d(n)$ and F is a $(k-1)$-face. $\square$

Observe that Proposition 7.10 shows that any subset of $\lfloor \frac{d}{2} \rfloor$ vertices of $C_d(n)$ forms a face of $C_d(n)$. When a d-polytope has this property it is called a *neighborly polytope*. Therefore, cyclic polytopes are neighborly. Proposition 7.10 also shows a phenomenon that only manifests itself in dimension at least 4: For $d \geq 4$, the polytope $C_d(n)$ has n pairwise adjacent vertices. For $n >> d$, this is counter-intuitive.

Finally, the combinatorial structure of cyclic polytopes is completely determined as follows:

Proposition 7.11 *(Gale evenness condition, Gale (1963)). Let n and d be integers with $2 \leq d < n$. For any sequence $t_1 < t_2 < \cdots < t_n$, consider the cyclic polytope*

$$C_d(n) = \operatorname{conv}(c(t_1), \ldots, c(t_n)).$$

A subset, $S \subseteq \{t_1, \ldots, t_n\}$ with $|S| = d$ determines a facet of $C_d(n)$ iff for all $i < j$ not in S, then the number of $k \in S$ between i and j is even:

$$|\{k \in S \mid i < k < j\}| \equiv 0 \pmod 2 \quad \textit{for} \quad i, j \notin S$$

Proof. Write $S = \{s_1, \ldots, s_d\} \subseteq \{t_1, \ldots, t_n\}$. Consider the polyomial

$$q(t) = \prod_{i=1}^{d} (t - s_i) = \sum_{j=0}^{d} b_j t^j,$$

let $b = (b_1, \ldots, b_d)$, and let H be the hyperplane given by

$$H = \{x \in \mathbb{R}^d \mid x \cdot b = -b_0\}.$$

Then, for each i, with $1 \leq i \leq d$, we have

$$c(s_i) \cdot b = \sum_{j=1}^{d} b_j s_i^j = q(s_i) - b_0 = -b_0,$$

so that $c(s_i) \in H$. For all other $t \neq s_i$,

$$q(t) = c(t) \cdot b + b_0 \neq 0,$$

that is, $c(t) \notin H$. Therefore, $F = \{c(s_1), \ldots, c(s_d)\}$ is a facet of $C_d(n)$ iff $\{c(t_1), \ldots, c(t_n)\} - F$ lies in one of the two open half-spaces determined by H. This is equivalent to $q(t)$ changing its sign an even number of times while, increasing t, we pass through the vertices in F. Therefore, the proposition is proved. $\square$

In particular, Proposition 7.11 shows that the combinatorial structure of $C_d(n)$ does not depend on the specific choice of the sequence $t_1 < \cdots < t_n$. This justifies our notation $C_d(n)$.

Here is the celebrated upper bound theorem first proved by McMullen [29].

Theorem 7.12 *(Upper Bound Theorem, McMullen (1970)) Let P be any d-polytope with n vertices. Then, for every k, with $1 \leq k \leq d$, the polytope P has at most as many $(k-1)$-faces as the cyclic polytope, $C_d(n)$, that is*

$$f_{k-1}(P) \leq f_{k-1}(C_d(n)).$$

Moreover, equality for some k with $\lfloor \frac{d}{2} \rfloor \leq k \leq d$ implies that P is neighborly.

The first step in the proof of Theorem 7.12 is to prove that among all d-polytopes with a given number, n, of vertices, the maximum number of i-faces is achieved by simplicial d-polytopes.

Proposition 7.13 *Given any d-polytope, P, with n-vertices, it is possible to form a simplicial polytope, P', by perturbing the vertices of P such that P' also has n vertices and*

$$f_{k-1}(P) \leq f_{k-1}(P') \qquad \textit{for} \quad 1 \leq k \leq d.$$

Furthermore, equality for $k > \lfloor \frac{d}{2} \rfloor$ can occur only if P is simplicial.

Sketch of proof. First, we apply Proposition 6.8 to triangulate the facets of P without adding any vertices. Then, we can perturb the vertices to obtain a simplicial polytope, P', with at least as many facets (and thus, faces) as P. $\square$

Proposition 7.13 allows us to restict our attention to simplicial polytopes. Now, it is obvious that

$$f_{k-1} \leq \binom{n}{k}$$

for any polytope P (simplicial or not) and we also know that equality holds if $k \leq \lfloor \frac{d}{2} \rfloor$ for neighborly polytopes such as the cyclic polytopes. For $k > \lfloor \frac{d}{2} \rfloor$, it turns out that equality can only be achieved for simplices.

However, for a *simplicial* polytope, the Dehn-Sommerville equations $h_k = h_{d-k}$ together with the equations $(*)$ giving f_k in terms of the h_i's show that $f_0, f_1, \ldots, f_{\lfloor \frac{d}{2} \rfloor}$ already determine the whole f-vector. Thus, it is possible to express the f_{k-1} in terms of $h_0, h_1, \ldots, h_{\lfloor \frac{d}{2} \rfloor}$ for $k \geq \lfloor \frac{d}{2} \rfloor$. It turns out that we get

$$f_{k-1} = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor}{}^{*} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) h_i,$$

where the meaning of the superscript $*$ is that when d is even we only take half of the last term for $i = \frac{d}{2}$ and when d is odd we take the whole last term for $i = \frac{d-1}{2}$ (for details, see Ziegler [45], Chapter 8). As a consequence if we can show that the neighborly polytopes maximize not only f_{k-1} but also h_{k-1} when $k \leq \lfloor \frac{d}{2} \rfloor$, the upper bound theorem will be proved. Indeed, McMullen proved the following theorem which is "more than enough" to yield the desired result ([29]):

Theorem 7.14 *(McMullen (1970)) For every simplicial d-polytope with $f_0 = n$ vertices, we have*

$$h_k(P) \leq \binom{n-d-1+k}{k} \qquad \text{for} \quad 0 \leq k \leq d.$$

Furthermore, equality holds for all l and all k with $0 \leq k \leq l$ iff $l \leq \lfloor \frac{d}{2} \rfloor$ and P is l-neighborly. (a polytope is l-neighborly iff any subset of l or less vertices determine a face of P.)

The proof of Theorem 7.14 is too involved to be given here, which is unfortunate, since it is really beautiful. It makes a clever use of shellings and a careful analysis of the h-numbers of links of vertices. Again, the reader is referred to Ziegler [45], Chapter 8.

Since cyclic d-polytopes are neighborly (which means that they are $\lfloor \frac{d}{2} \rfloor$-neighborly), Theorem 7.12 follows from Proposition 7.13, and Theorem 7.14.

Corollary 7.15 *For every simplicial neighborly d-polytope with n vertices, we have*

$$f_{k-1} = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor}{}^{*} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) \binom{n-d-1+i}{i} \qquad \text{for} \quad 1 \leq k \leq d.$$

This gives the maximum number of $(k-1)$-faces for any d-polytope with n-vertices, for all k with $1 \leq k \leq d$. In particular, the number of facets of the cyclic polytope, $C_d(n)$, is

$$f_{d-1} = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor}{}^{*} 2\binom{n-d-1+i}{i}$$

and, more explicitly,

$$f_{d-1} = \binom{n - \lfloor \frac{d+1}{2} \rfloor}{n-d} + \binom{n - \lfloor \frac{d+2}{2} \rfloor}{n-d}.$$

Corollary 7.15 implies that the number of facets of any d-polytope is $O(n^{\lfloor \frac{d}{2} \rfloor})$. An unfortunate consequence of this upper bound is that the complexity of any convex hull algorithms for n points in $\mathbb{E}^d$ is $O(n^{\lfloor \frac{d}{2} \rfloor})$.

The $O(n^{\lfloor \frac{d}{2} \rfloor})$ upper bound can be obtained more directly using a pretty argument using shellings due to R. Seidel [36]. Consider any shelling of any simplicial d-polytope, P. For every facet, F_j, of a shelling either the restriction set R_j or its complement $F_j - R_j$ has at most $\lfloor \frac{d}{2} \rfloor$ elements. So, either in the shelling or in the reversed shelling, the restriction set of F_j has at most $\lfloor \frac{d}{2} \rfloor$ elements. Moreover, the restriction sets are all distinct, by construction. Thus, the number of facets is at most twice the number of k-faces of P with $k \leq \lfloor \frac{d}{2} \rfloor$. It follows that

$$f_{d-1} \leq 2 \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{n}{i}$$

and this rough estimate yields a $O(n^{\lfloor \frac{d}{2} \rfloor})$ bound.

Remark: There is also a *lower bound theorem* due to Barnette (1971, 1973) which gives a lower bound on the f-vectors all d-polytopes with n vertices. In this case, there is an analog of the cyclic polytopes called *stacked polytopes*. These polytopes, $P_d(n)$, are simplicial polytopes obtained from a simplex by building shallow pyramids over the facets of the simplex. Then, it turns out that if $d \geq 2$, then

$$f_k \geq \begin{cases} \binom{d}{k} n - \binom{d+1}{k+1} k & \text{if } 0 \leq k \leq d-2 \\ (d-1)n - (d+1)(d-2) & \text{if } k = d-1. \end{cases}$$

There has been a lot of progress on the combinatorics of f-vectors and h-vectors since 1971, especially by R. Stanley, G. Kalai and L. Billera and K. Lee, among others. We recommend two excellent surveys:

1. Bayer and Lee [4] summarizes progress in this area up to 1993.

2. Billera and Björner [7] is a more advanced survey which reports on results up to 1997.

In fact, many of the chapters in Goodman and O'Rourke [22] should be of interest to the reader.

Generalizations of the Upper Bound Theorem using sophisticated techniques (face rings) due to Stanley can be found in Stanley [37] (Chapters II) and connections with toric varieties can be found in Stanley [37] (Chapters III) and Fulton [19].