Mechanics of Materials

Chapter 4
Shear and Moment In Beams

4.1 *Introduction*

❑ The term beam refers to a slender bar that carries **transverse loading**; that is, the applied force are perpendicular to the bar.

❑ In a beam, the internal force system consist of a shear force and a bending moment acting on the cross section of the bar. The shear force and the bending moment usually vary continuously along the length of the beam.

❑ The internal forces give rise to **two** kinds of stresses on a transverse section of a beam: (1) normal stress that is caused by bending moment and (2) shear stress due to the shear force.

❑ Knowing the distribution of the shear force and the bending moment in a beam is essential for the computation of stresses and deformations. Which will be investigated in subsequent chapters.

4.2 *Supports and Loads*

❑Beams are classified according to their supports. A *simply supported beam*, shown in Fig. 4.1 (a). The **pin support** prevents displacement of the end of the beams, but not its rotation. The term **roller support** refers to a pin connection that is free to move parallel to the axis of the beam; this type of support suppresses only the transverse displacement.
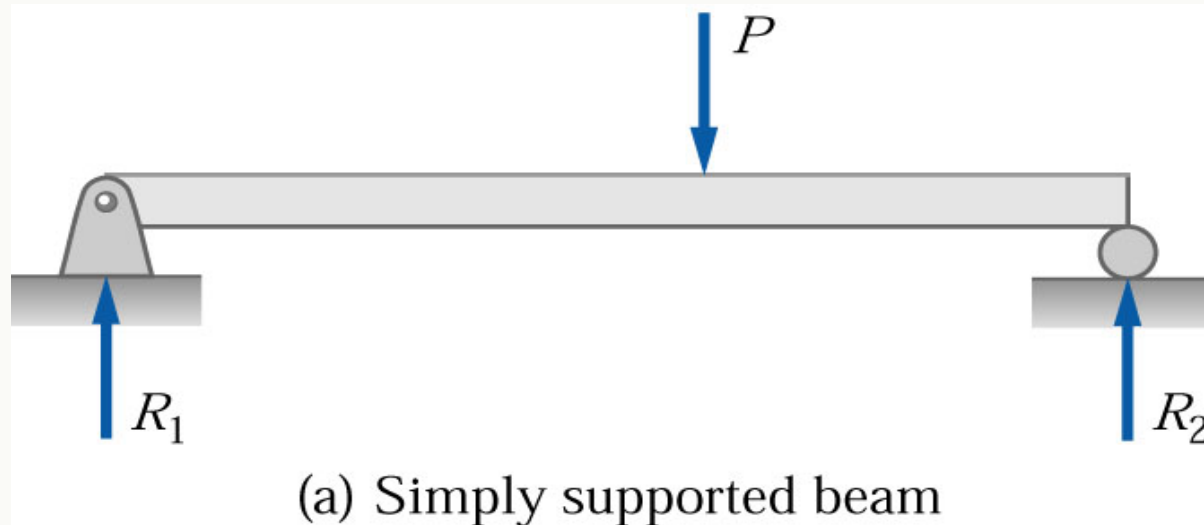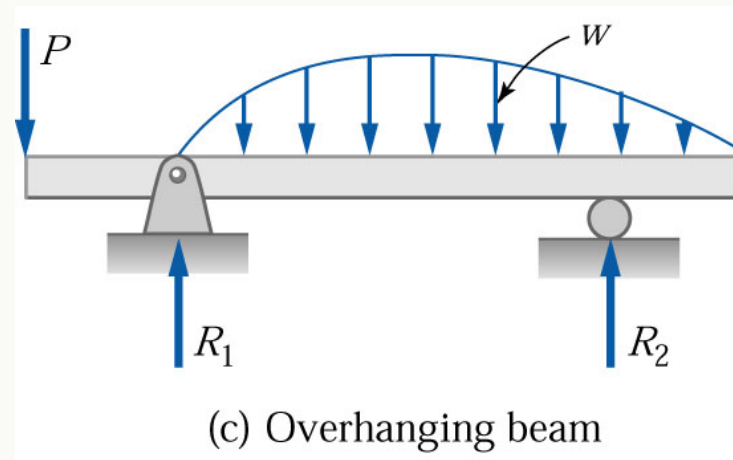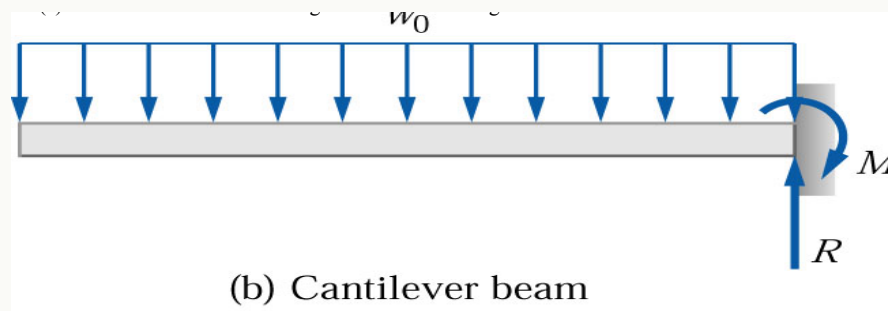
(a) Simply supported beam

Figure 4.1 (a) Statically determinate beams.

□A ***cantilever beam*** is built into a rigid support at one end, with the other end being free, as shown in Fig.4.1(b). The built-in support prevents displacements as well as rotations of the end of the beam.

□An ***overhanging beam***, illustrated in Fig.4.1(c), is supported by a pin and a roller support, with one or both ends of the beam extending beyond the supports.

□ The three types of beams are **statically determinate** because the support reactions can be found from the equilibrium equations.
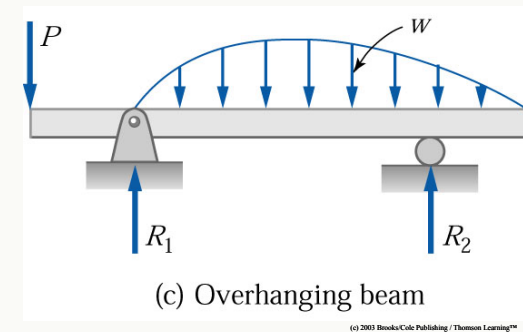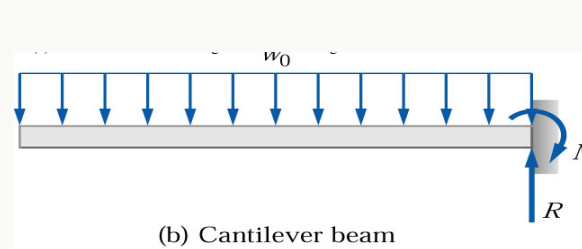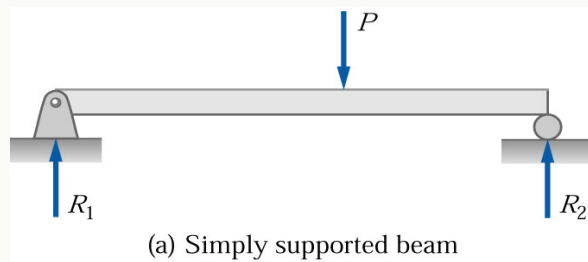

(b) Cantilever beam

(c) Overhanging beam

Figure 4.1   Statically determinate beams

❑A ***concentrated load***, such as *P* in Fig. 4.1(a). In contrast a ***distributed load*** is applied over a finite area. If the distributed load acts on a very narrow area, the load may be approximated by a *line load*.

❑The **intensity *w*** of this loading is expressed as force per unit length (lb/ft, N/m, etc.) The load distribution may be uniform, as shown in Fig.4.1(b), or it may vary with distance along the beam, as in Fig.4.1(c).

❑The weight of the beam is an example of distributed loading, but its magnitude is usually small compared to the loads applied to the beam.

(a) Simply supported beam

(b) Cantilever beam

(c) Overhanging beam

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❑Figure 4.2 shows other types of beams. These beams are *oversupported* in the sense that each beam has at least one more reaction than is necessary for support. Such beams are **statically indeterminate**; the presence of these *redundant supports* requires the use of additional equations obtained by considering the deformation of the beam. The analysis of statically indeterminate beams will be discussed in Chapter 7.
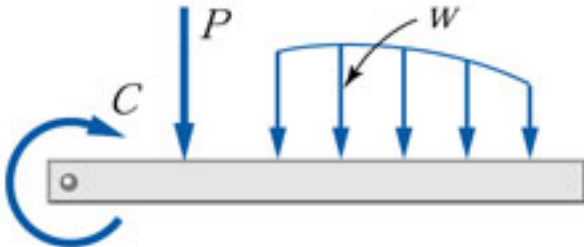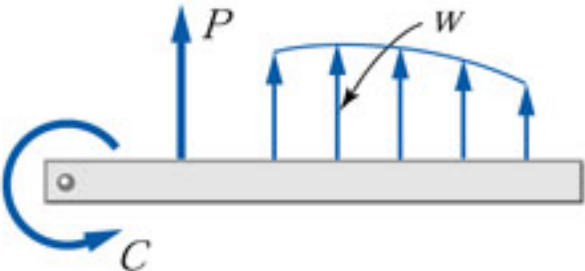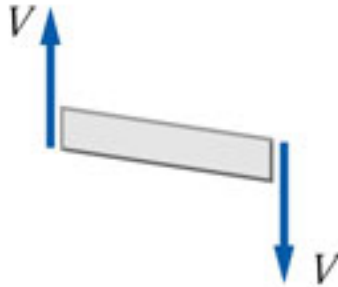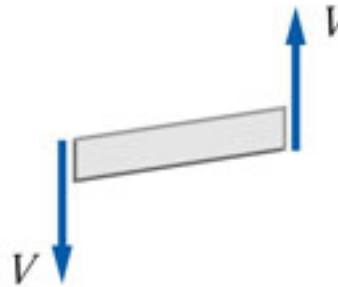

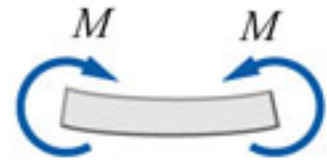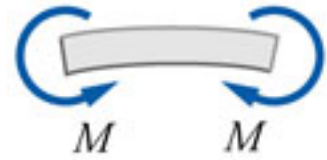(a) Propped cantilever beam

(b) Fixed or restrained beam
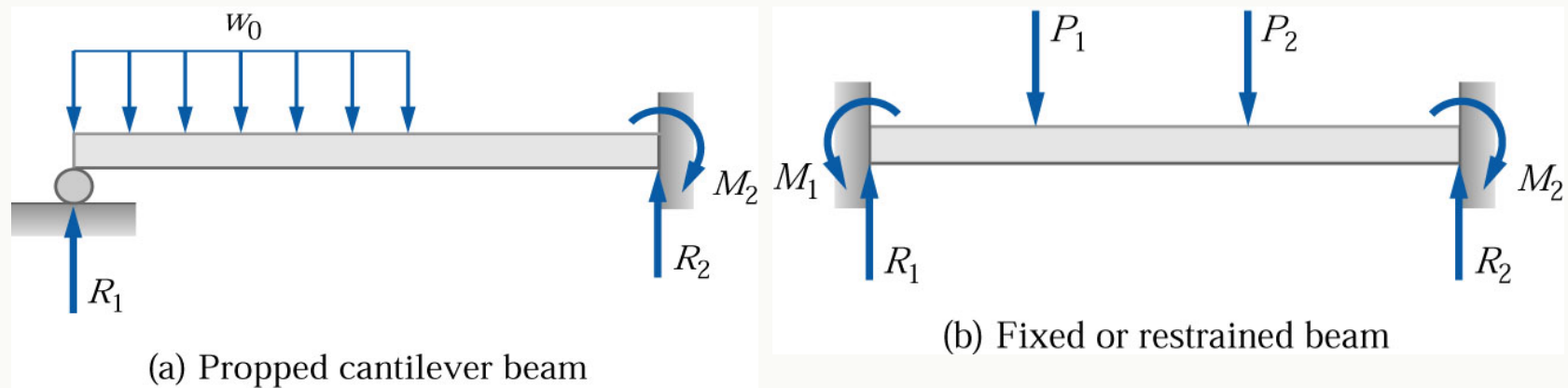
Figure 4.2   Statically indeterminate beams

4.3 Shear- Moment Equations and Shear-Moment Diagrams

❑ The determination of the internal force system acting at a *given section of a beam : draw a free-body diagram* that expose these forces and then compute the forces using equilibrium equations.

❑ The goal of the beam analysis —determine the shear force V and the bending moment M at *every* cross section of the beam.

❑ To derive the expressions for V and M in terms of the distance x measured along the beam. By plotting these expressions to scale, obtain the ***shear force and bending moment diagrams*** for the beam.

❑ The shear force and bending moment diagrams are convenient visual references to the internal forces in a beam; in particular, they identify the maximum values of V and M.

a. Sign conventions

	Positive	Negative
External loads		
Shear force		
Bending moment		

Figure 4.3 Sign conventions for external loads; shear force, and bending moment.

b. *Procedure for determining shear force and bending moment diagrams*

・ Compute the support reactions from the free-body diagram (FBD) of the entire beam.

・ Divide the beam into segment so that the loading within each segment is continuous. Thus, the end-points of the segments are discontinuities of loading, including concentrated loads and couples.

■ Perform the following steps for each segment of the beam:

▪ Introduce an imaginary cutting plane within the segment, located at a distance x from the left end of the beam, that cuts the beam into two parts.

▪ Draw a FBD for the part of the beam lying either to the left or to the right of the cutting plane, whichever is more convenient. At the cut section, show V and M acting in their positive directions.
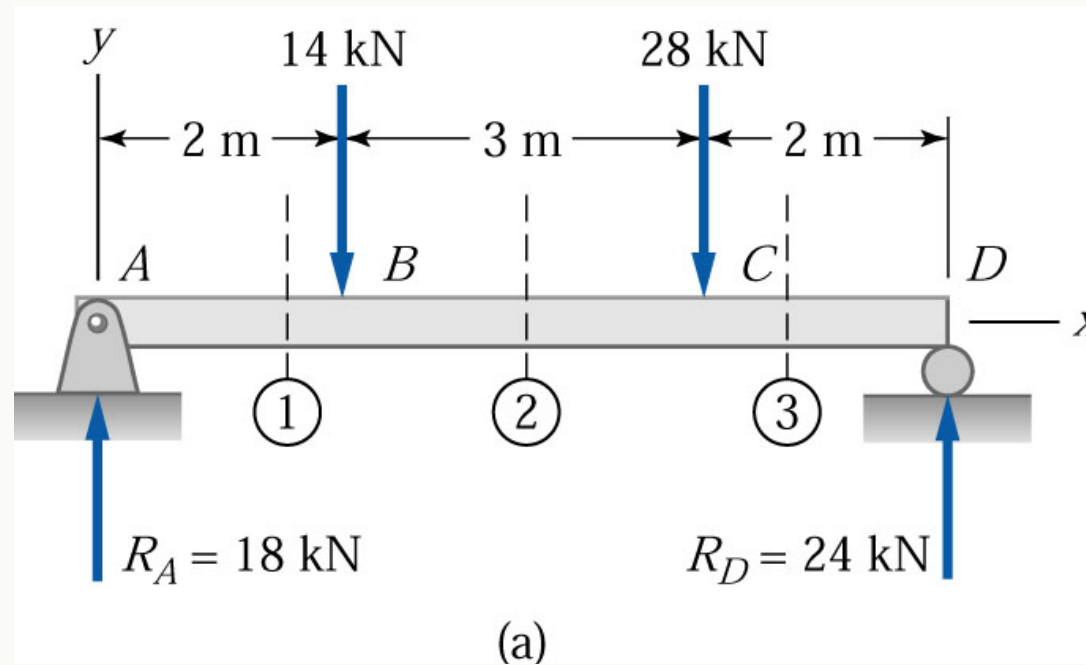
- Determine the expressions for *V* and *M* from the equilibrium equations obtainable from the FBD. These expressions, which are usually functions of x, are the shear force and bending moment equations for the segment.

- Plot the expressions for V and M for the segment. It is visually desirable to draw the *V*-diagram below the FBD of the entire beam, and then draw the *M*- diagram below the *V*-diagram.

- The bending moment and shear force diagrams of the beam are composites of the *V* and *M* diagrams of the segments. These diagrams are usually discontinuous, or have discontinuous slopes. At the end-points of the segments due to discontinuities in loading.

Sample Problem 4.1

The simply supported beam in Fig. (a) carries two concentrated loads. (1) Derive the expressions for the shear force and the bending moment for each segment of the beam. (2) Sketch the shear force and bending moment diagrams. Neglect the weight of the beam. Note that the support reactions at *A* and *D* have been computed and are shown in Fig. (a).

(a)

Solution

Part 1

The determination of the expressions for *V* and *M* for each of the three beam segments (*AB*,*BC*, and *CD*) is explained below.

Segment AB $(0 < x < 2$ m$)$
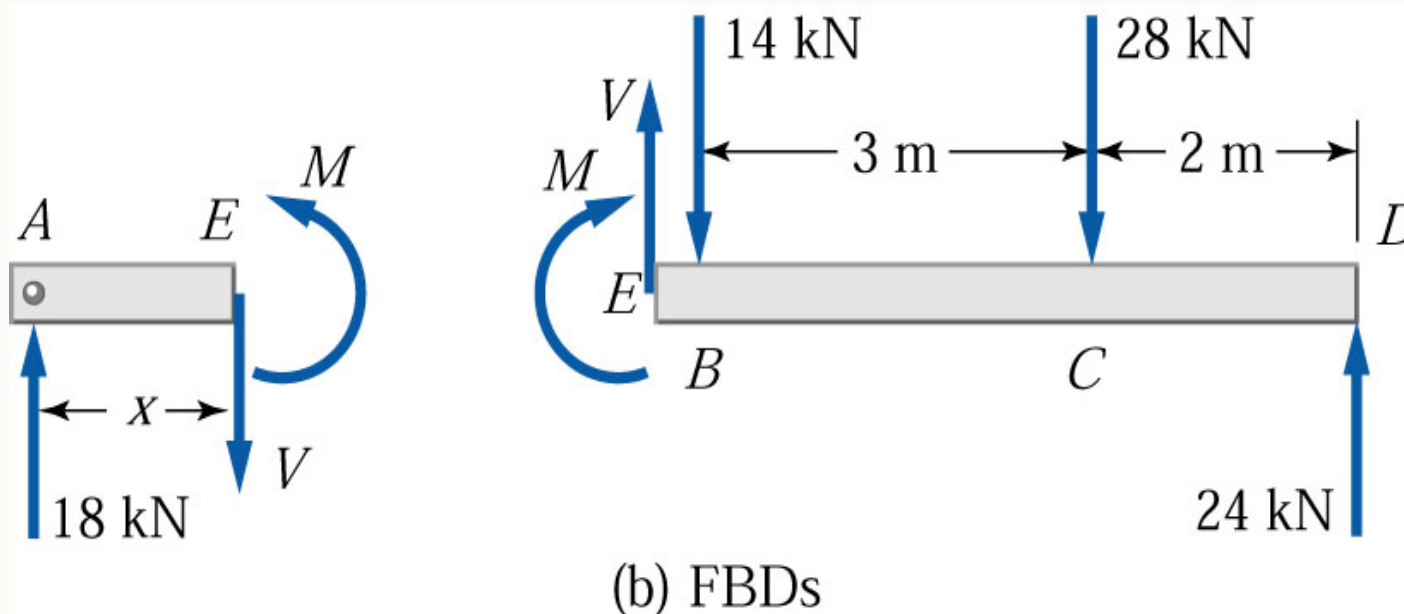
$$\Sigma F_y = 0 + \uparrow \qquad 18 - V = 0$$

$$V = +18 \text{ kN} \qquad\qquad Answer$$

$$\Sigma M_E = 0 + \circlearrowleft \qquad -18x + M = 0$$

$$M = +18x \text{ kN} \cdot \text{m} \qquad\qquad Answer$$

(b) FBDs

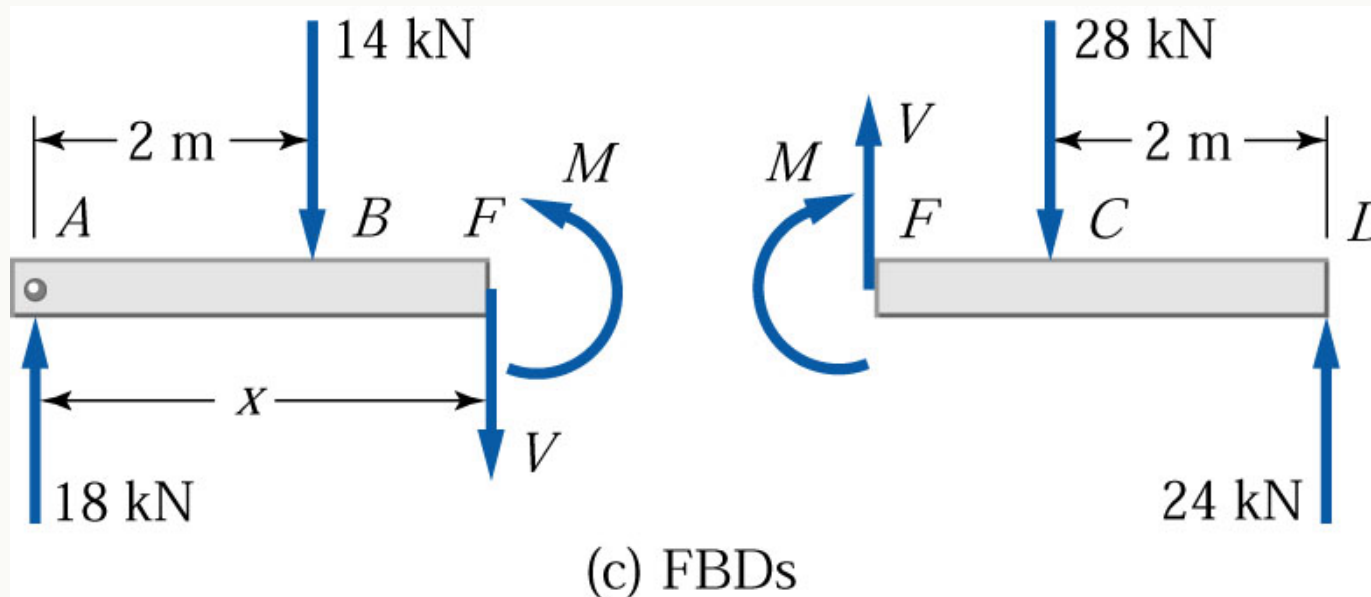## Segment AB ($2 < x < 5$ m)

$\Sigma F_y = 0 + \uparrow$ $18 - 14 - V = 0$

$$V = +18 - 14 = +4 \text{ kN} \qquad \textit{Answer}$$

$\Sigma M_E = 0 + \circlearrowleft$ $-18x + 14(x-2) + M = 0$

$$M = +18x - 14(x-2) = 4x + 28 \text{ kN} \cdot \text{m} \qquad \textit{Answer}$$

(c) FBDs

Segment CD (5 m$<$x$<$7 m)

$\Sigma F_y = 0 + \uparrow$ $18\text{-}14\text{---}28\text{-}V = 0$

   $V = +18\text{-}14\text{-}28 = \text{-}24$ kN   *Answer*

$\Sigma M_G = 0 + \circlearrowleft$ $-18x + 14(x\text{-}2) + 28(x\text{-}5) + M = 0$

   $M = +18x\text{-}14(x\text{-}2) - (x\text{-}5) = \text{-}24x + 168$ kN·m   *Answer*

(d) FBDs

Shear force and bending moment diagrams
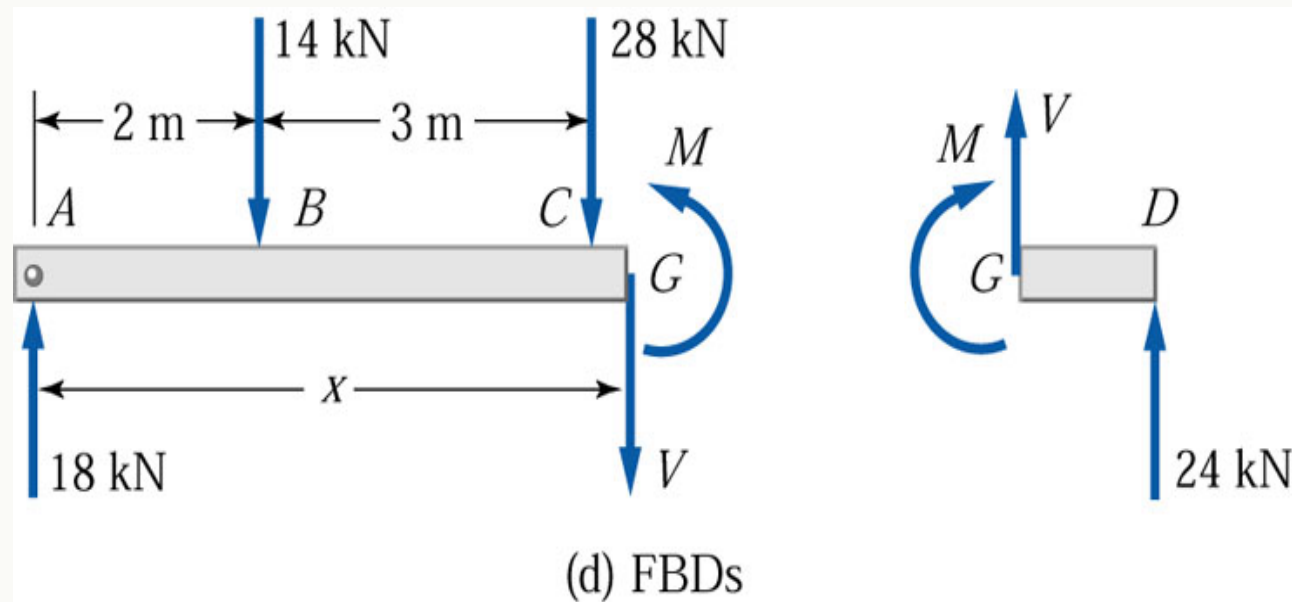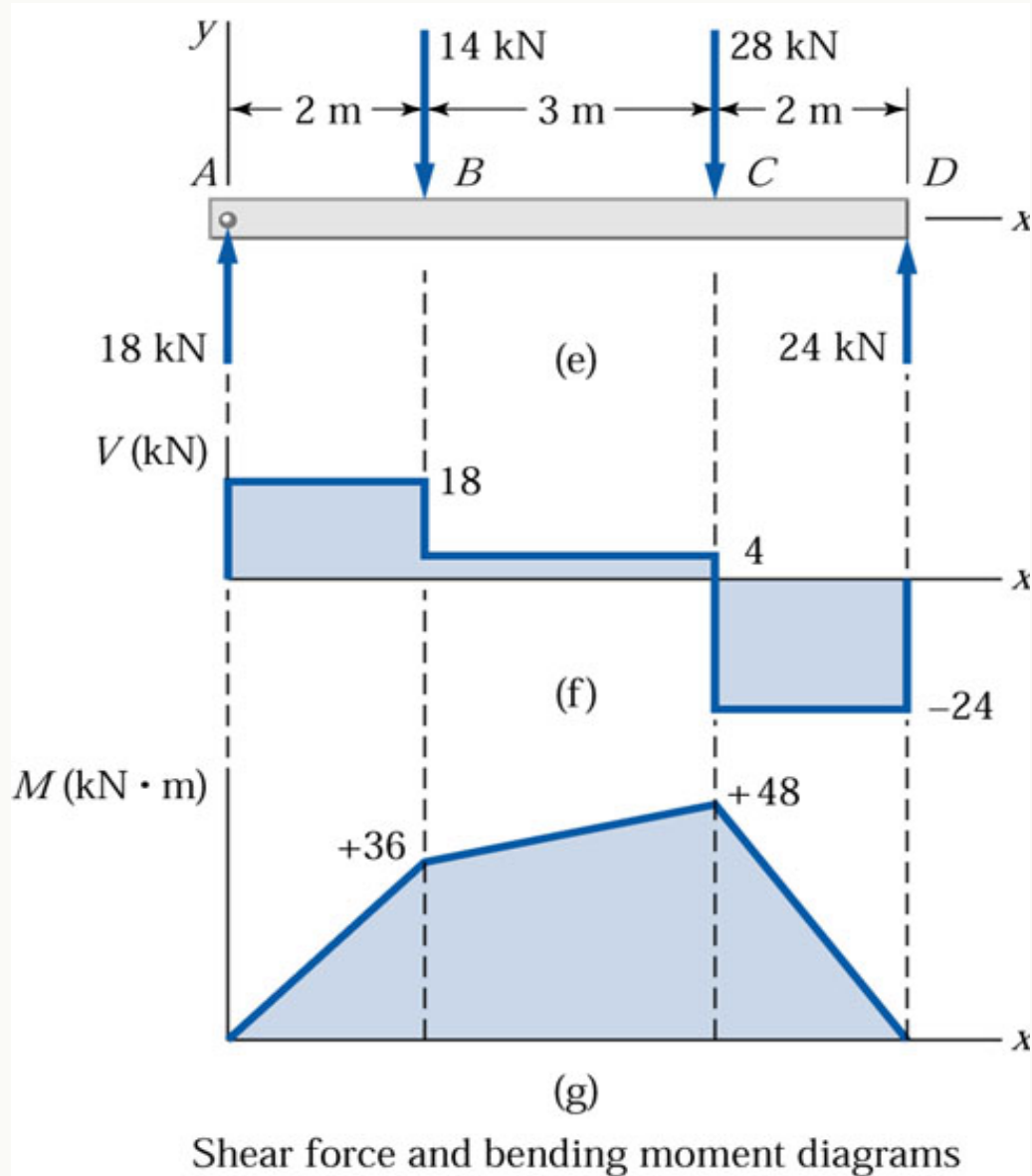
Part 2

❑ The *V*-diagram reveals that the largest shear force in the beam is -24 kN : segment *CD*

❑ The *M*-diagram reveals that the maximum bending moment is +48 kN·m : the 28-kN load at *C*.

❑ Note that at each concentrated force the *V*-diagram **"jumps"** by an amount equal to the force.

❑ There is a discontinuity in the slope of the *M*-diagram at each concentrated force.

Sample problem 4.2

The simply supported beam in Fig. (a) is loaded by the clockwise couple C_0 at B. (1) Derive the shear and bending moment equations. And (2) draw the shear force and bending moment diagrams. Neglect the weight of the beam. The support reactions A and C have been computed, and their values are shown in Fig. (a).

Solution

Part 1

Due to the presence of the couple C_0, We must analyze segments AB and BC separately.

(a)

$R_A = \dfrac{C_0}{L}$

$R_C = \dfrac{C_0}{L}$
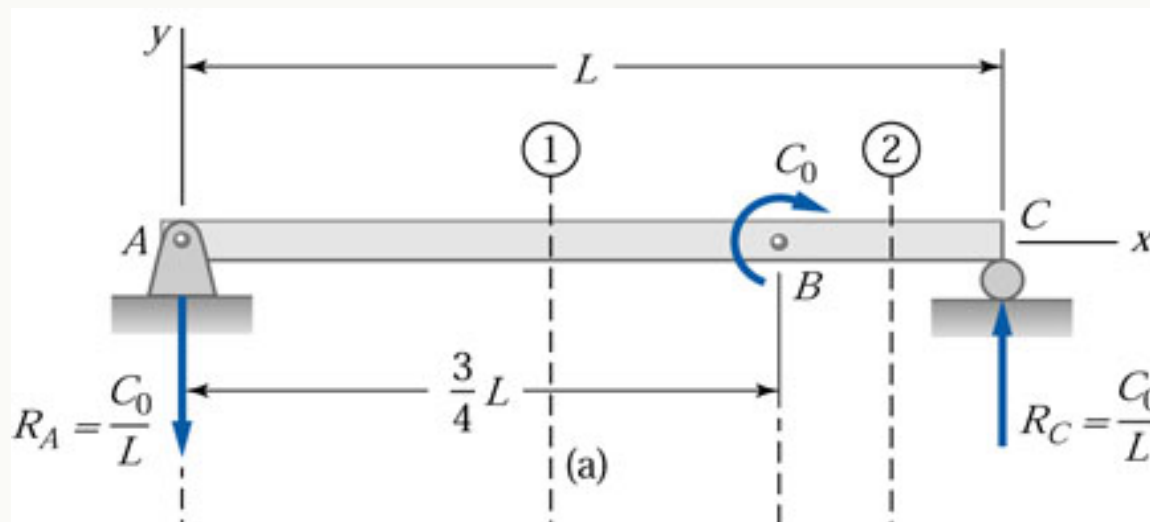
Segment AB $(0<x<3L/4)$

$$\sum F_y = 0+\uparrow \quad -\frac{C_0}{L} - V = 0$$

$$V = -\frac{C_0}{L} \qquad Answer$$

$$\sum M_D = 0+ \quad \circlearrowleft \quad \frac{C_0}{L}x + M = 0$$
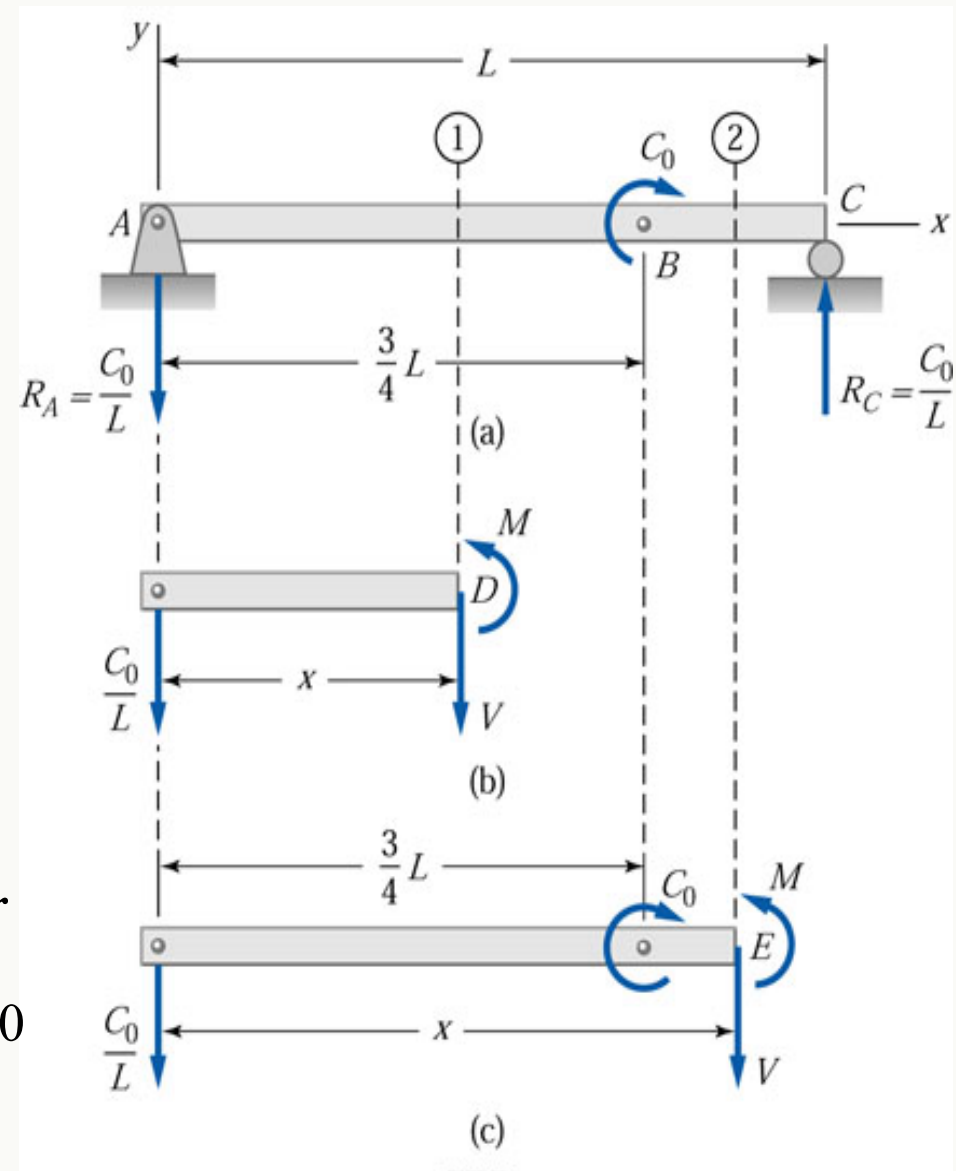
$$M = -\frac{C_0}{L}x \qquad Answer$$

Segment BC $(3L/4<x<L)$

$$\sum F_y = 0+\uparrow \quad -\frac{C_0}{L} - V = 0$$

$$V = -\frac{C_0}{L} \qquad Answer$$

$$\sum M_E = 0+ \circlearrowleft \quad \frac{C_0}{L}x - C_0 + M = 0$$

$$M = -\frac{C_0}{L}x + C_0 \qquad Answer$$

(a)

(b)

(c)

Shear force and bending moment diagrams
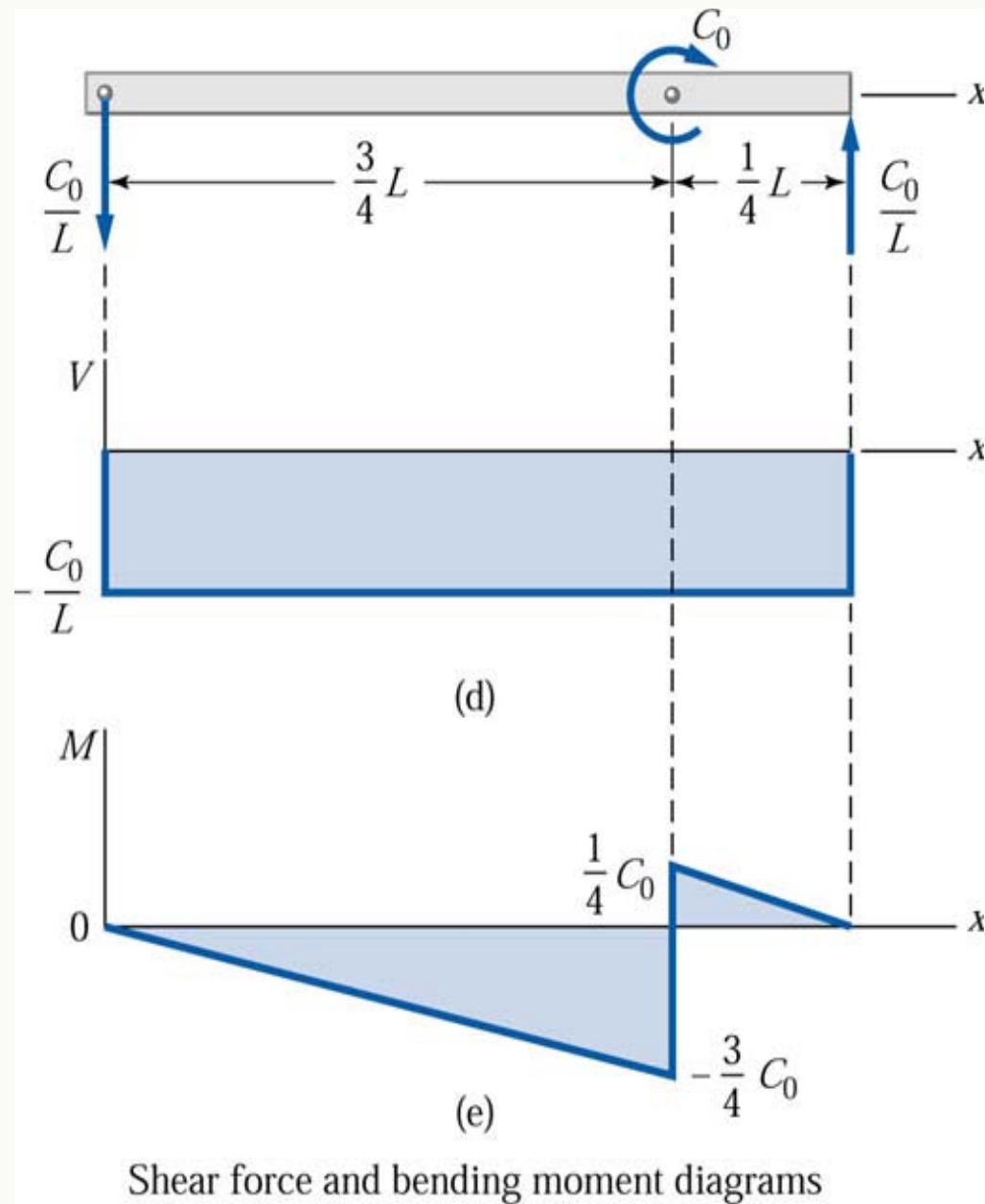
Part 2

From the *V*-diagram, the shear force is the same for all cross sections of the beam. The M-diagram shows jump of magnitude C_0 at the point of application of the couple.

Sample Problem 4.3

The cantilever beam in Fig.(a) carries a triangular load. The intensity of which varies from zero at the left end to 360 lb/ft at the right end. In addition, a 1000-lb upward vertical load acts at the free end of the beam. (1) Derive the shear force and bending moment equations. And (2) draw the shear force and bending moment diagrams. Neglect the weight of the beam.

(a)
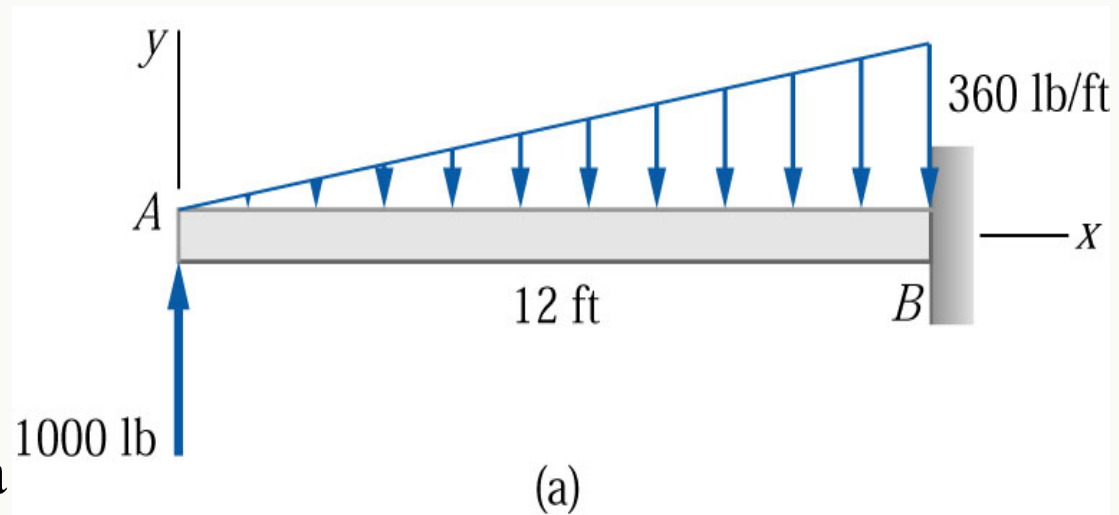
Solution

Note that the triangular load has been replaced by is resultant, which is the force 0.5 (12) (360) = 2160 lb (area under the loading diagram) acting at the centroid of the loading diagram.

❑ Because the loading is continuous, the beam does not have to be divided into segment.

❑ $w/x = 360/12$, or
  $w = 30x$ lb/ft.

8 ft | **2160 lb** | **360 lb/ft**

$M_B = 3360$ lb · ft

A ... B

1000 lb | 12 ft | $R_B = 1160$ lb

(b)

$\frac{x}{3}$

$15x^2$ lb | $w = 30x$ lb/ft

M

A ... C

1000 lb | x | V

(c)

Part 1

$\sum F_y = 0 + \uparrow$ $1000 - 15x^2 - V = 0$

$V = 1000 - 15x^2$ lb   *Answer*

$\sum M_C = 0 + \circlearrowleft$ $-1000x + 15x^2 (x/3) + M = 0$

$M = 1000x - 5x^3$ lb· ft   *Answer*
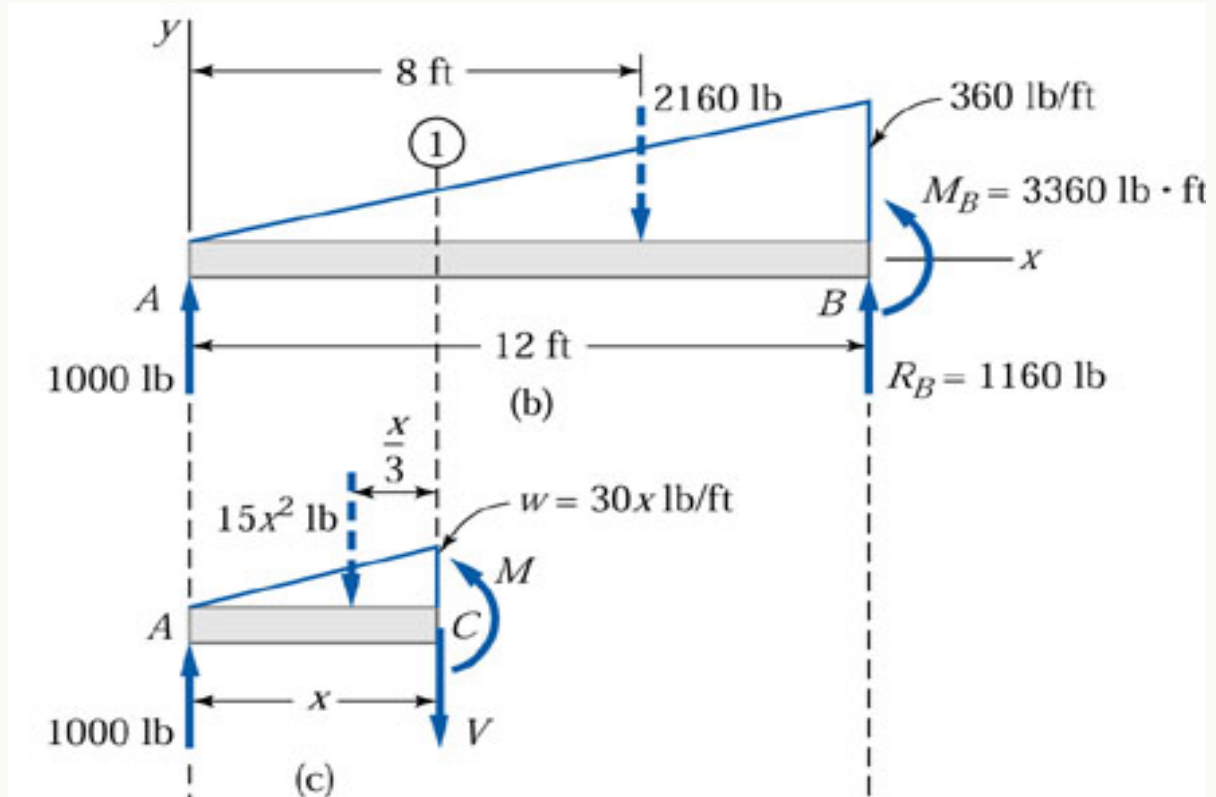
Part 2

The location of the section where the shear force is **zero** is found from
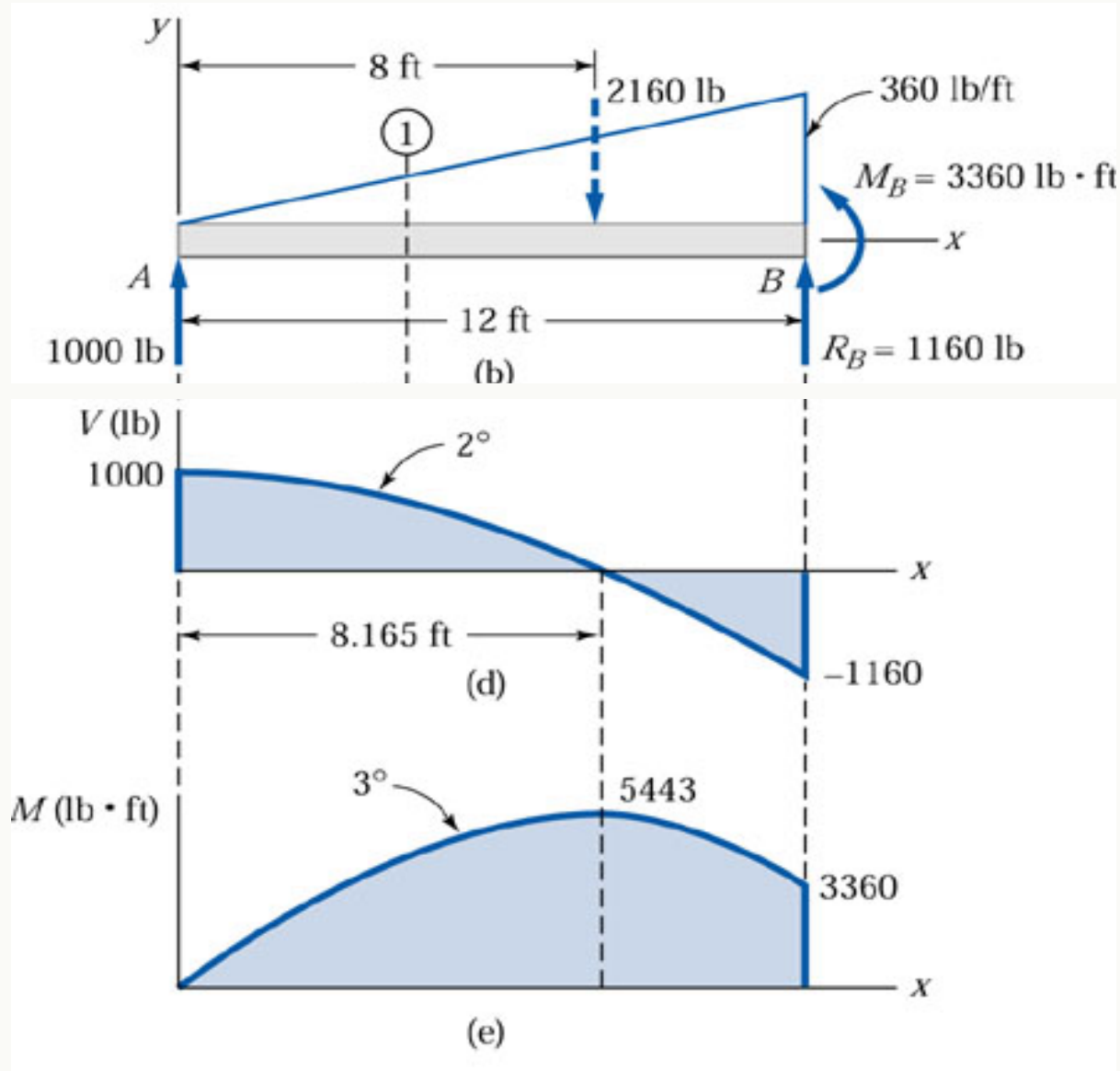
$$V = 1000 - 15x^2 = 0$$

$$x = 8.165 \text{ ft}$$

$$\frac{dM}{dx} = 1000 - 15x^2 = 0$$

$$x = 8.165 \text{ ft }.$$

the maximum bending moment is

$$M_{max} = 1000(8.165) - 5(8.165)^3 = 5443 \text{ lb} \cdot \text{ft}$$

4.4 *Area Method* for Drawing Shear- Moment Diagrams

❑ Useful relationships between the loading, shear force, and bending moment can be derived from the equilibrium equations.

❑ These relationships enable us to plot the shear force diagram directly from the load diagram, and then construct the bending moment diagram from the shear force diagram. This technique, called the *area method*, allows us to draw the shear force and bending moment diagrams without having to derive the equations for V and M.

❑ First consider beam subjected to distributed loading and then discuss concentrated forces and couples.

(a)

(b)
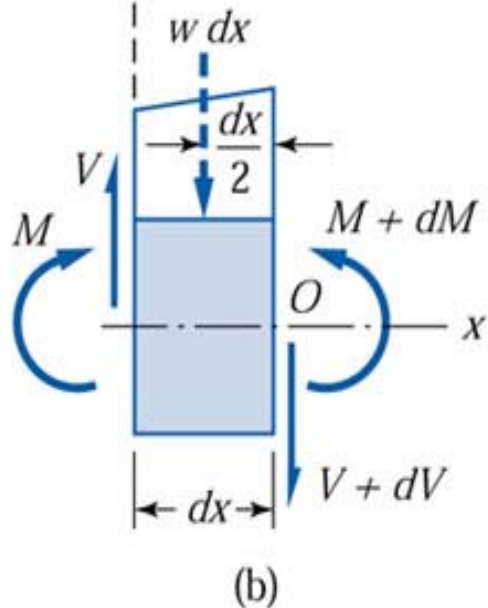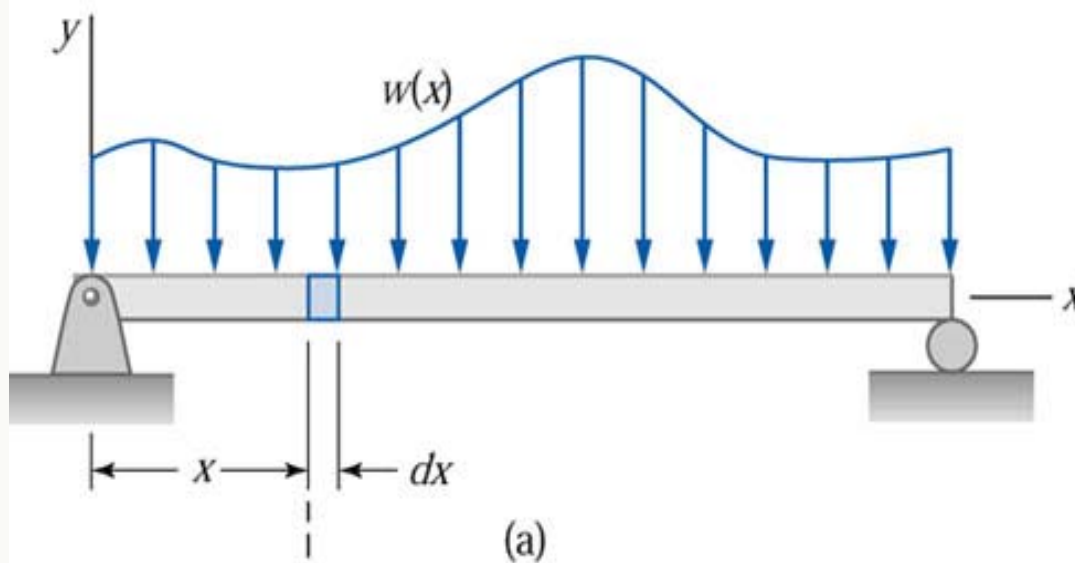
4.4 Figures (a) Simply supported beam carrying distributed loading; (b) free-body diagram of an infinitesimal beam segment.

a.   *Distributed loading*

Consider the beam in Fig. 4.4 (a) that is subjected to a the distributed load $w(x)$ is assumed to be a continuous function. The free-body diagram of an infinitesimal element of the beam, loaded at the distance x from the left end, is shown in Fig. 4.4 (b)

The force equation of equilibrium is

$$\Sigma F_y = 0 + \uparrow \quad V - w\,dx - (V + dV) = 0$$

From which we get

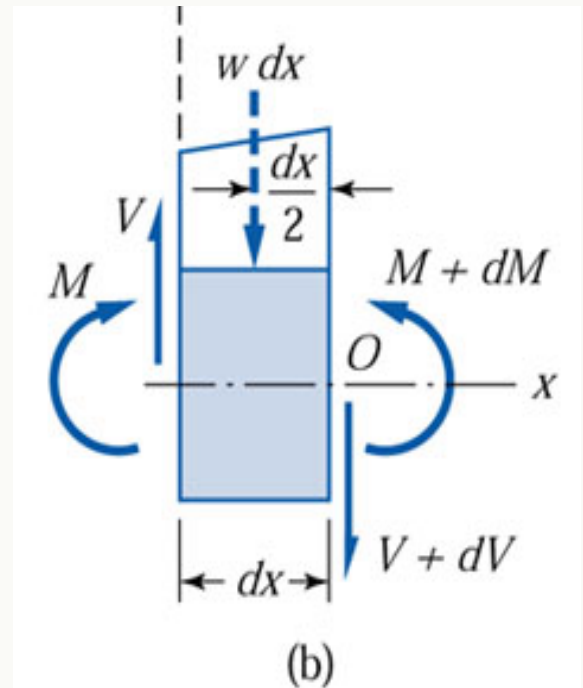$$\boxed{w = -\frac{dV}{dx}} \qquad (4.1)$$

(b)

The moment equation of equilibrium yields

$$\Sigma M_0 = 0 + \circlearrowleft \quad -M - V\,dx + (M + dM) + w\,dx(dx/2) = 0$$

After canceling M and dividing by dx, we get

$$-V + \frac{dM}{dx} + \frac{w\,dx}{2} = 0 \qquad \boxed{V = \frac{dM}{dx}} \qquad (4.2)$$

Equations (4.1) and (4.2) are called the ***differential equations of equilibrium*** for beams.

The following five theorems relating the load, the shear force, and the bending moment diagrams follow from these equations.

1. The load intensity w at any section of a beam is equal to the negative of the slope of the shear force diagram at the section.

   Proof — follows directly from Eq. (4.1).

   $$w = -\frac{dV}{dx}$$

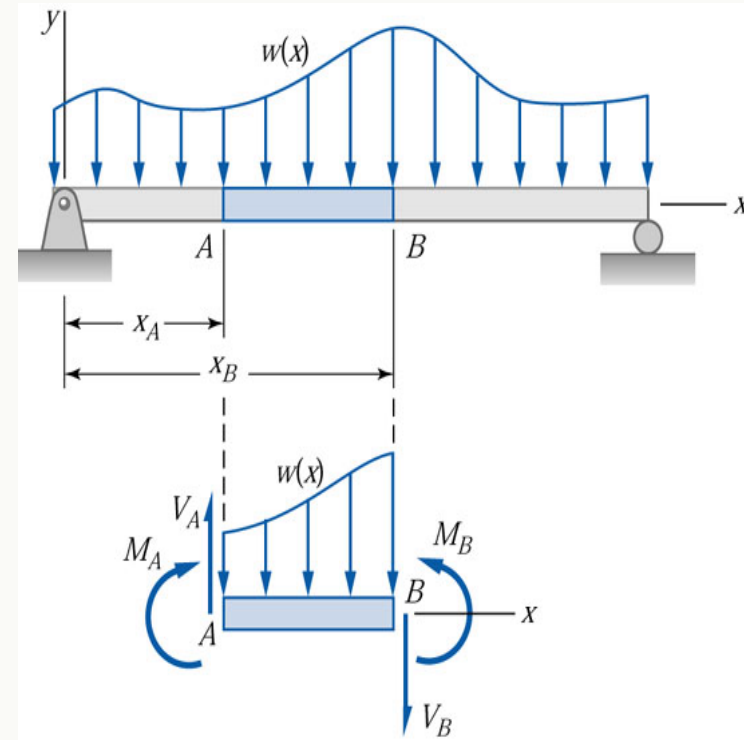2. The shear force V at any section is equal to the slope of the bending moment diagram at that section.

   Proof — follows directly from Eq. (4.2).

   $$V = \frac{dM}{dx}$$

3. The difference between the shear forces at two sections of a beam is equal to the negative of the area under the load diagram between those two sections.

Proof — integrating Eq. (4.1) between sections A and B in Fig. 4.5, we obtain

$$w = -\frac{dV}{dx} \qquad \int_{xA}^{xB} \frac{dV}{dx}\,dx = V_B - V_A = -\int_{xA}^{xB} w\,dx$$
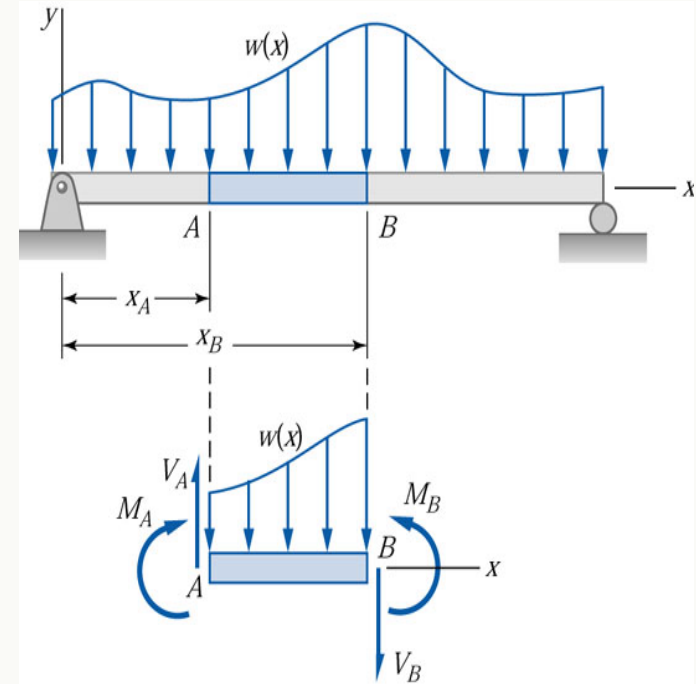
$$V_B - V_A = -\text{area of } w\text{-diagram}]_A^B$$

$$V_B = V_A - \text{area of } w\text{-diagram}]_A^B \qquad\qquad (4.3)$$

Note that the signs in Eq. (4.3) are correct only if $x_B > x_A$.

4. The difference between the bending moments at two sections of a beam is equal to the area of the shear force diagram between these two sections.

Proof — integrating Eq. (4.2) between sections A and B in (see Fig. 4.5),

$$V = \frac{dM}{dx}$$

$$\int_{XA}^{XB} \frac{dM}{dx} dx = M_B - M_A = \int_{XA}^{XB} V dx$$

$$M_B - M_A = \text{area of } V\text{-diagram}]_A^B \quad \text{Q.E.D}$$

$$M_B = M_A + \text{area of } V\text{-diagram}]_A^B \qquad (4.4)$$

The signs in Eq. (4.4) are correct only if $x_B > x_A$.

5. If the load W diagram is a polynomial of degree n, then shear force V diagram is polynomial of degree ($n+1$), and the bending moment M diagram is polynomial of degree ($n+2$).

Proof — followings directly from the integration of Eqs. (4.1) and (4.2).

$$w = -\frac{dV}{dx}$$

$$V = \frac{dM}{dx}$$

❑ Consider the beam segment shown in Fig. 4.6 (a), which is 2 m long and is subjected to a uniformly distributed load $w = 300$ N /m. Figure 4.6 (a) shows the shear force and the bending moment at the left end are $V_A = +1000$ N and $M_A = +3000$ N· m.
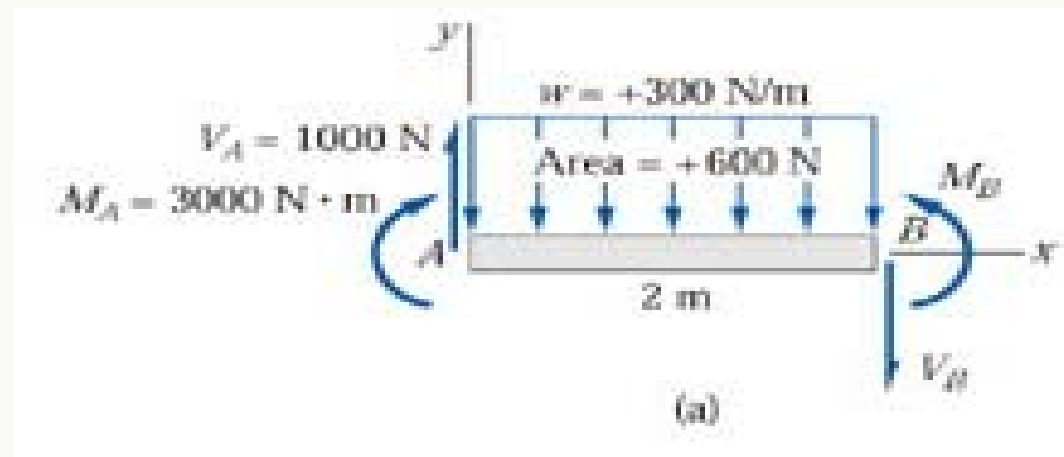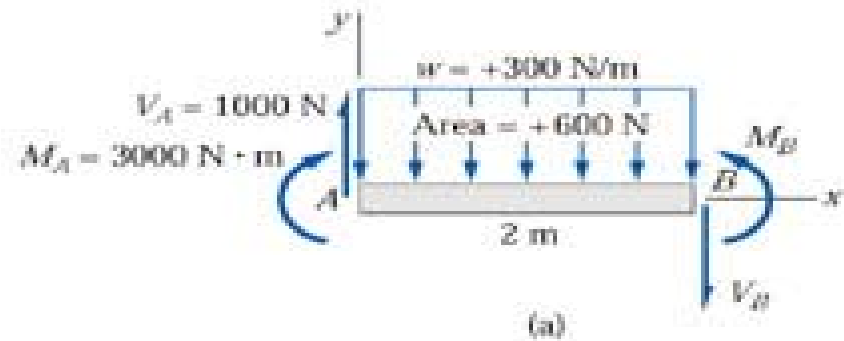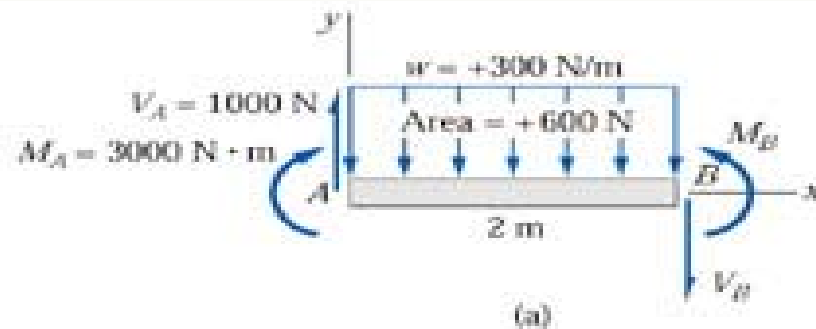

(a)

Figure 4.6 (a) Free-body diagram of a beam segment carrying uniform loading;

(a)

$M_A = +3000 \text{ N} \cdot \text{m}$ (given)	3000 • $M(\text{N} \cdot \text{m})$ A B
$M_B = M_A + \text{area of } V\text{-diagram}]_A^B$ $= 3000 + 1400 = +4400 \text{ N} \cdot \text{m}$	• 4400 3000 • $M(\text{N} \cdot \text{m})$ A B
$(dM/dx)_A = V_A = +1000 \text{ N}$ $(dM/dx)_B = V_B = +400 \text{ N}$	$M(\text{N} \cdot \text{m})$ A B
M-diagram is a parabola	4400 3000 $M(\text{N} \cdot \text{m})$ A B

Figure 4.6(b) constructing shear force and bending moment diagrams for the beam segment.
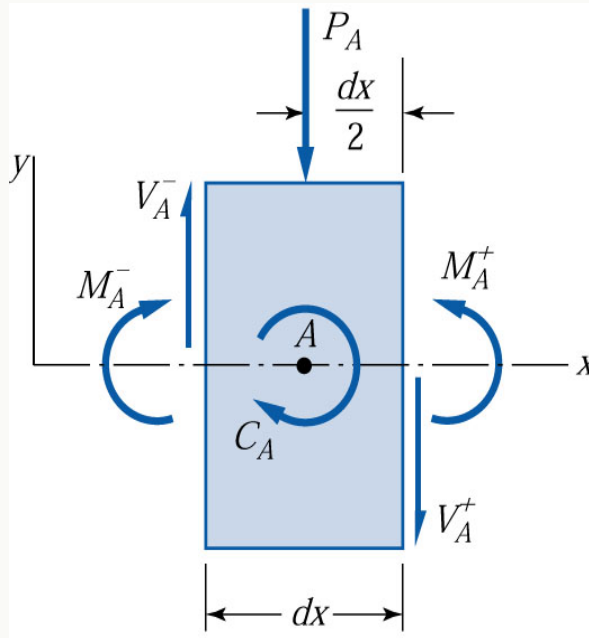
Figure 4.7 Free-body diagram of an infinitesimal beam element carrying a concentrated force P_A and a concentrated couple C_A.

b. *Concentrated forces and couples*.

❑ The force equilibrium equation

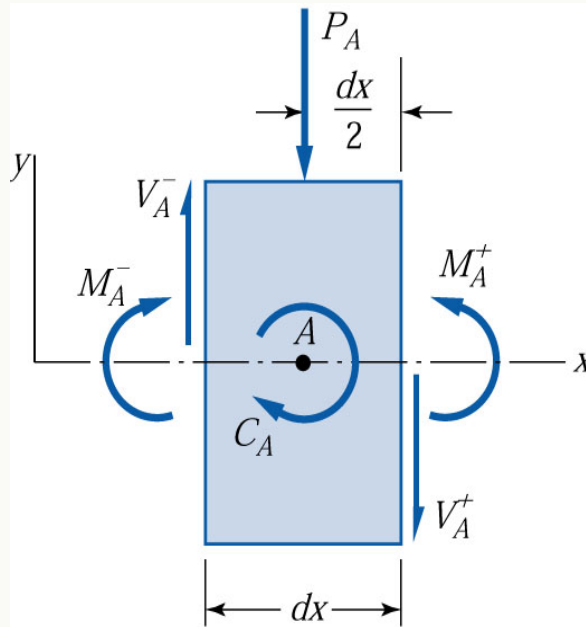$$\Sigma F_y = 0 + \uparrow \qquad V_A^- - P_A - V_A^+ = 0 \qquad \boxed{V_A^+ = V_A^- - P_A \qquad (4.5)}$$

Equation (4.5) indicates that a positive concentrated force causes a negative jump discontinuity in the shear force diagram at A (a concentrated couple does not affect the shear force diagram).

The moment equilibrium equation yields

$$\Sigma M_A = 0 + \circlearrowleft \qquad M_A^+ - M_A^- - C_A - V_A^+ \frac{dx}{2} - V_A^- \frac{dx}{2} = 0$$

$$\boxed{M_A^+ = M_A^- + C_A}$$

Thus, a positive concentrated couple causes a positive jump in the bending moment diagram.

Figure 4.7 Free-body diagram of an infinitesimal beam element carrying a concentrated force P_A and a concentrated couple C_A.

c. Summary

The area method is useful only if the area under the load and shear force diagrams can be easily computed.

$$w = -\frac{dV}{dx} \qquad\qquad (4.1)$$

$$V = \frac{dM}{dx} \qquad\qquad (4.2)$$

$$V_B = V_A - \text{area of w-diagram}]_A^{\ B} \qquad\qquad (4.3)$$

$$M_B = M_A + \text{area of w-diagram}]_A^{\ B} \qquad\qquad (4.4)$$

$$V_A^+ = V_A^- - P_A \qquad\qquad (4.5)$$

$$M_A^+ = M_A^- + C_A \qquad\qquad (4.6)$$

Procedure for the Area Method

❑ Compute the support reactions force the free-body diagrams (FBD) of the entire beam.

❑ Draw the load diagram of the beam (which is essentially a FBD) showing the values of the loads, including the support reactions. Use the sign conventions in Fig. 4.3 to determine the correct sign of each load.

❑ Working from left to right, construct the V-and M-diagram for each segment of the beam using Eqs. (4.1)-(4.6).

❑ When reach the right end of the beam, check to see whether the computed values of V-and M are consistent with the end conditions. If they are not, you made an error in the computations.

Sample Problem 4.4

The simply supported beam in Fig. (a) supports 30-kN concentrated force at B and a 40-kN·m couple at D. Sketch the shear force and beading moment diagrams by the area method. Neglect the weight of the beam.

(a)
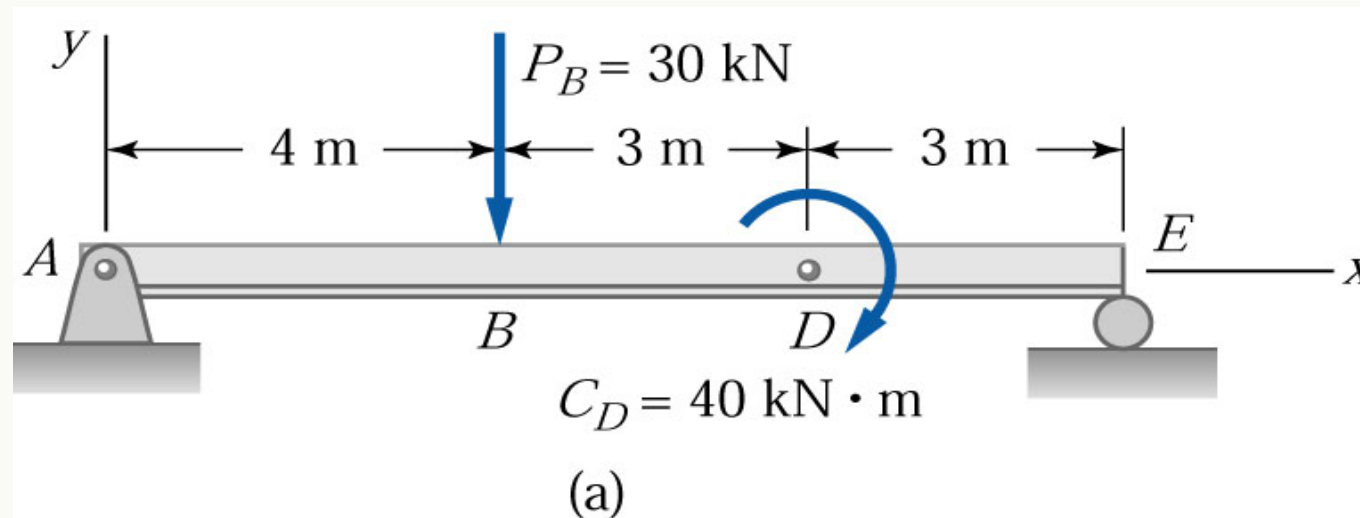
Solution

Load Diagram

The load diagram for the beam is shown in Fig. (b). The reactions at *A* and *E* are found from equilibrium analysis. Indicating its sign as established by the sign conventions in Fig. 4.3.

$P_B = 30$ kN (+)

$C_D = 40$ kN · m (+)

4 m — 3 m — 3 m

$R_A = 14$ kN (−)

$R_E = 16$ kN (−)

(b)
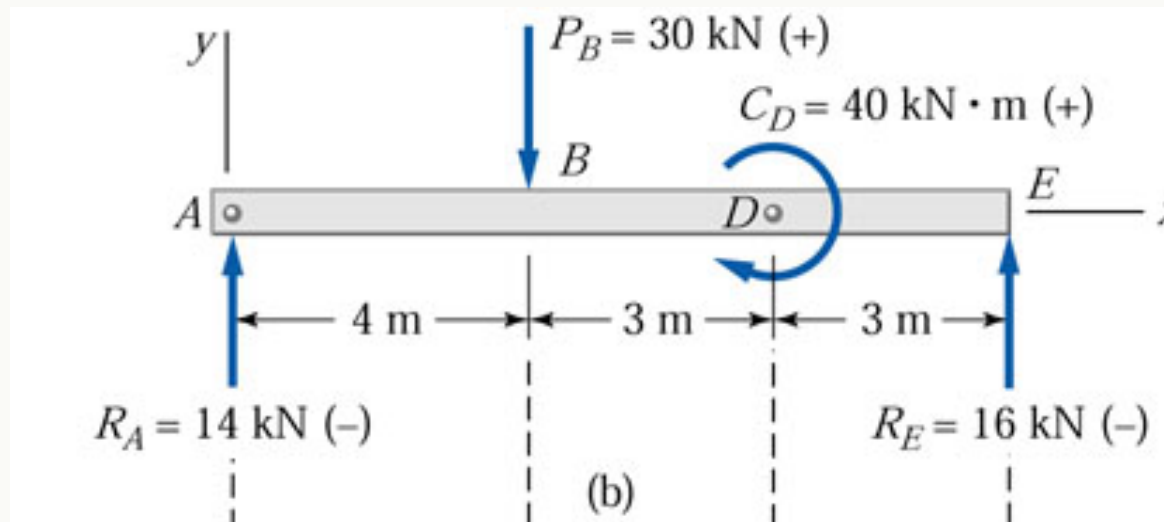
Shear Force Diagram

There are concentrated forces at A,B, noting that $V_A^- = 0$ because no load is applied to the left of A

$$V_A^+ = V_A^- - R_A = 0 - (-14) = (+14kN)$$

Plot point. ⓐ

$$V_B^- = V_A^- - \text{area of } w\text{-diagram}]_A^B$$

$$= 14-0 = 14 \text{ kN} \qquad \textit{Plot point.} \ ⓑ$$

Because $w = -dV/dx = 0$ between A and B,

Connect ⓐ *and* ⓑ *with*

a horizontal straight line

(b)

$R_A = 14 \text{ kN } (-)$

$R_E = 16 \text{ kN } (-)$

$P_B = 30 \text{ kN } (+)$

$C_D = 40 \text{ kN} \cdot \text{m } (+)$

V (kN)

$+56 \text{ kN} \cdot \text{m}$

$-48 \text{ kN} \cdot \text{m}$ $-48 \text{ kN} \cdot \text{m}$

$$V_B^+ = V_B^- - P_B = 14 - (+30) = -16kN \qquad \textit{Plot point.} \ ⓒ$$
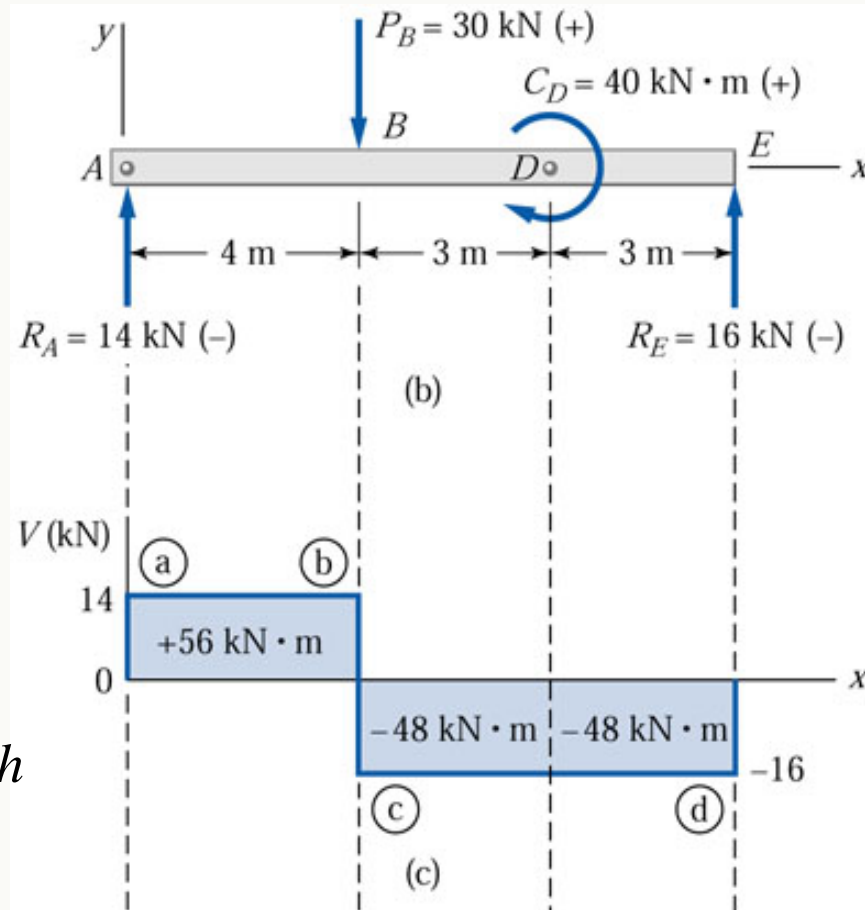
$$V_E^- = V_B^+ - \text{area of } w\text{-diagram}]_B^E = -16-0 = -16 \text{ kN} \quad \textit{Plot point.} \ ⓓ$$

Because $w = -dV/dx = 0$ between B and E
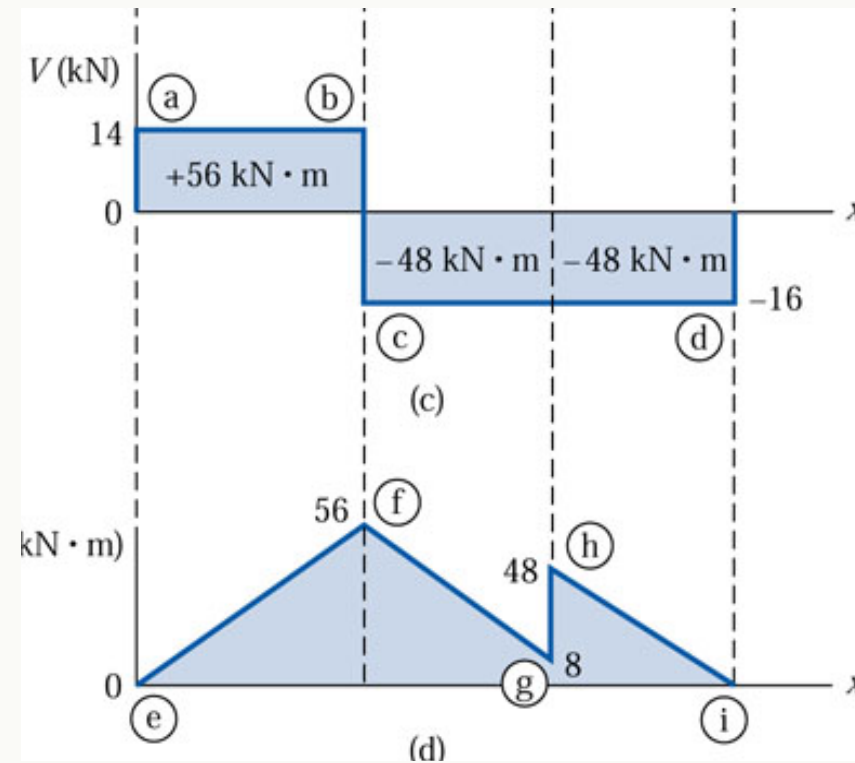
Connect ⓒ *and* ⓓ *with a horizontal line*

$$V_E^+ = V_E^- - R_E = -16 - (-16) = 0 \qquad \textit{Check !}$$

Bending Moment Diagram

● The applied couple is cause a jump in the bending moment diagram at D.

● The areas are either positive or negative, depending on the sign of the shear force in Fig. (c). $M_A = 0$ (there is no couple applied at A). point (e)

$M_B = M_A + \text{area of } V\text{-diagram}]_A^B = 0 + (+56) = 56 \text{ kN} \cdot \text{m}$, point (f)

The shear force bewteen A and B is constant and positive. The slope of the M-diagram between these two sections is also constant and positive. (recall that $V = dM/dx = 0$), connect (e) and (f) with straight line.

$M_D^- = M_B + \text{area of } V\text{-diagram}]_B^D = 56 + (-48) = 8 \text{ kN} \cdot \text{m}$, point (g)

The slope of the V diagram between B and D is negative and constant, the M-diagram has a constant, negative slope in this segment,), connect (f) and (g) with straight line.
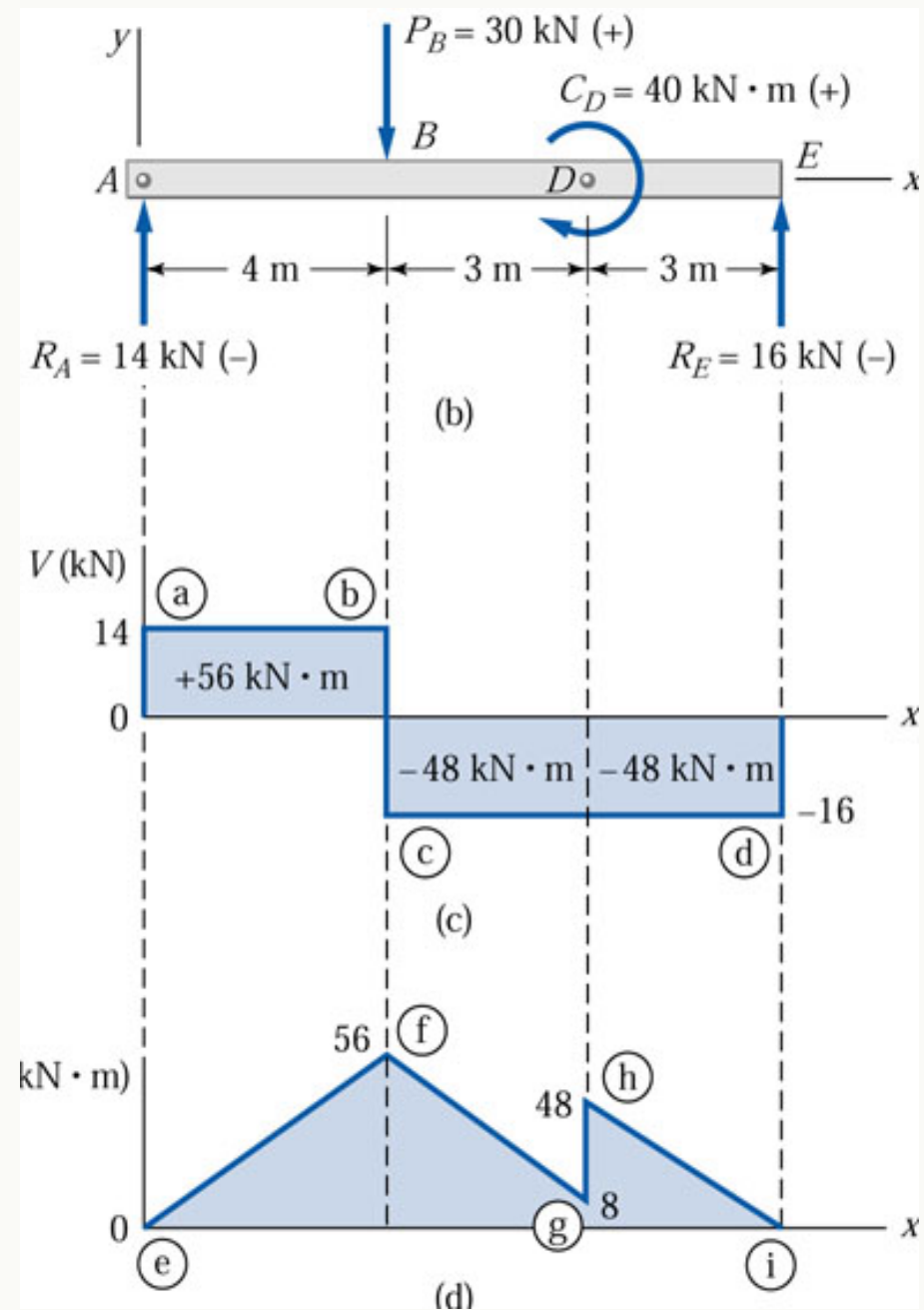
$$M_D^+ = M_D^- + C_D$$
$$= 8 + (+40) = 48 \text{ kN} \cdot \text{m}$$

Point (h), note that $M_E = 0$ (these is no couple applied at E).

$$M_E = M_D^+ + \text{area of V-diagram}]_D^E$$
$$= 48 + (-48) = 0 \quad Check \ !$$

The shear force between D and E is negative and constant, which means that the slope of the M - diagram for this segment is also constant and negative, connect (h) and (i) with straight line.

$P_B = 30$ kN (+)

$C_D = 40$ kN · m (+)

B

A o   D o   E ___ x

— 4 m —|— 3 m —|— 3 m —

$R_A = 14$ kN (–) $R_E = 16$ kN (–)

(b)

V (kN)
(a)   (b)
14
+56 kN · m
0 _____ x
       –48 kN · m | –48 kN · m
       (c)   (d)
                         –16
(c)

kN · m)
56 (f)
              48 (h)
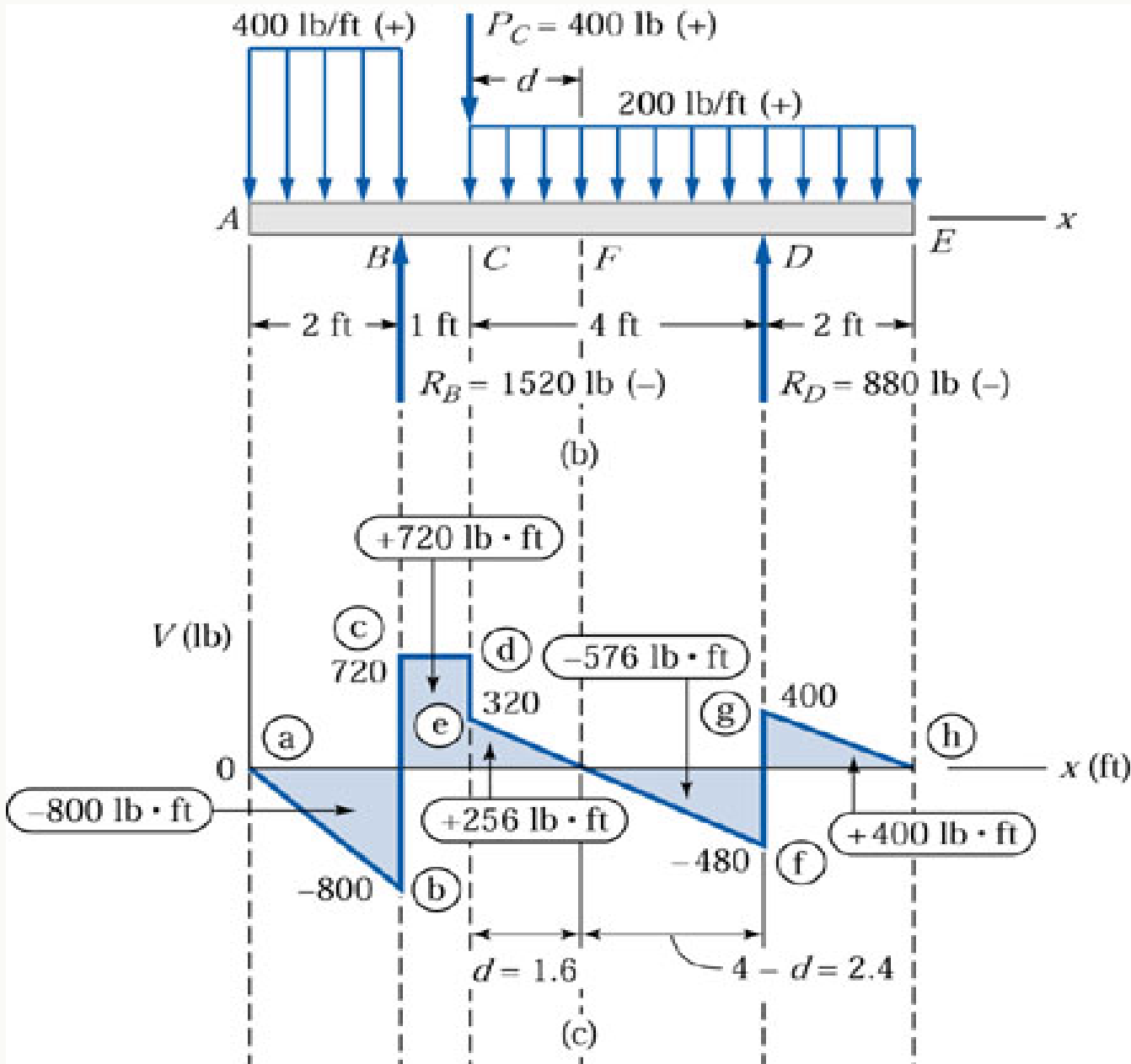   8
0 (e)   (g)   (i) ___ x
(d)

Sample Problem 4.5

The overhanging beam in Fig. (a) carries two uniformly distributed loads and a concentrated load Using the area method. Draw the shear force and bending moment diagrams for the beam.
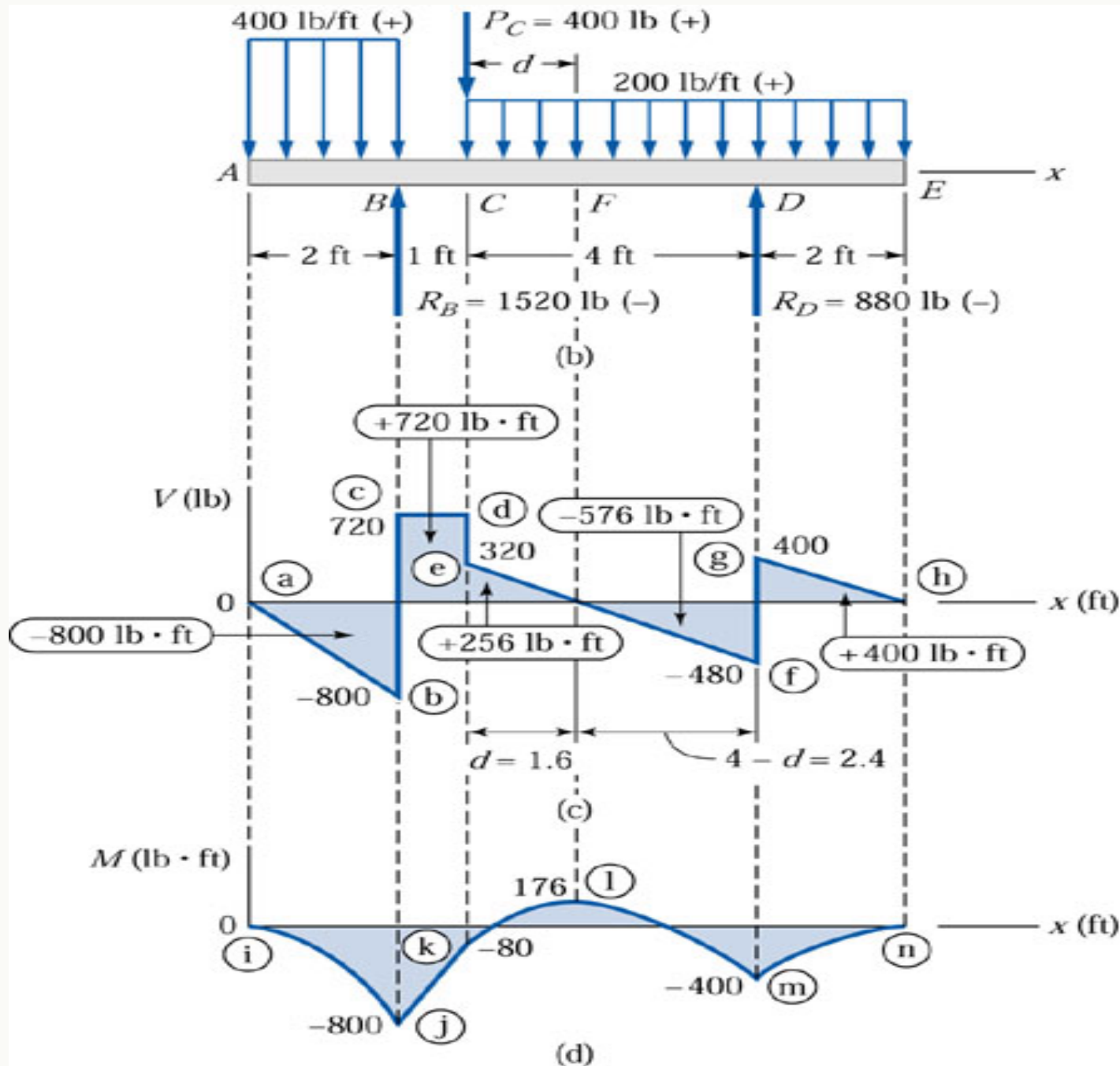

(a)
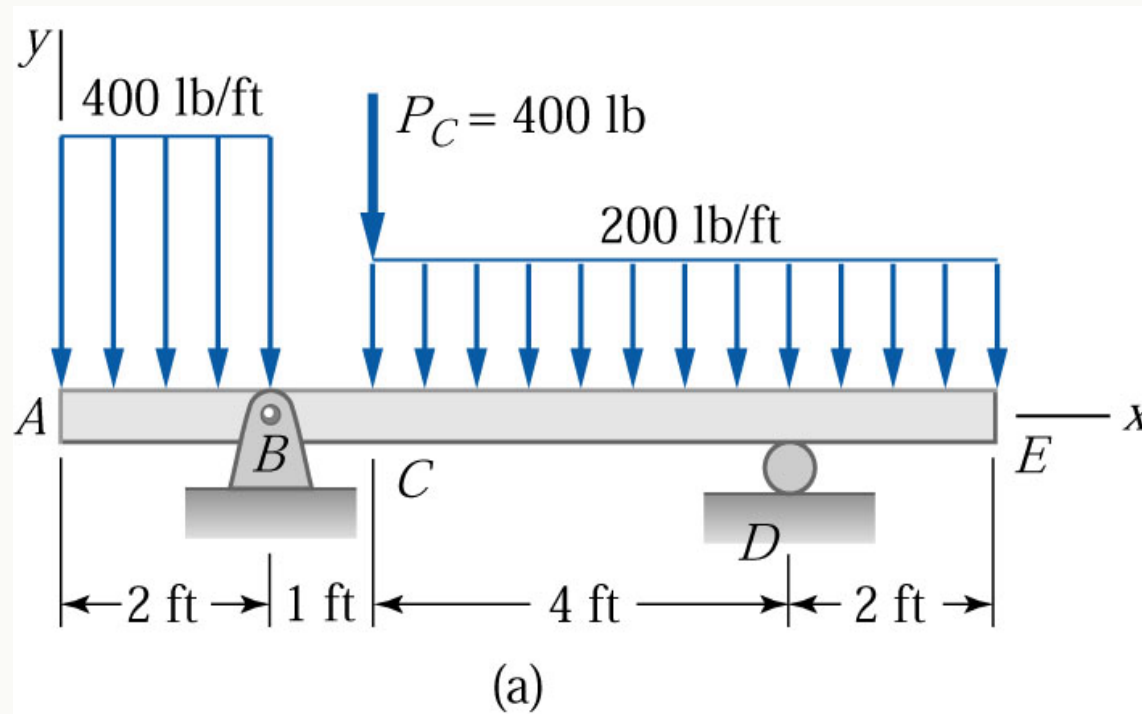
Load Diagram

The load diagram for the beam is given in Fig. (b)

Shear Force Diagram

The steps required to construct the shear force diagram in Fig. (c) are now detailed.

Bending Moment Diagram

The slope of the *M*-diagram is discontinuous at j and m.