Chapter 12. Kinetics of Particles: Newton's Second Law

Introduction

Newton's Second Law of Motion

Linear Momentum of a Particle

Systems of Units

Equations of Motion

Dynamic Equilibrium

Angular Momentum of a Particle

Equations of Motion in Radial & Transverse Components

Conservation of Angular Momentum

Newton's Law of Gravitation

Trajectory of a Particle Under a Central Force

Application to Space Mechanics

Kepler's Laws of Planetary Motion

Kinetics of Particles

©Glow Images RF

We **must analyze all of the forces acting on the racecar in order to design a good track**

©Russell Illig/Getty Images RF

As **a centrifuge reaches high velocities, the arm will experience very large forces that must be considered in design.**

Introduction

$$\Sigma \mathbf{F} = m\mathbf{a}$$

12.1 Newton's Second Law of Motion

- **If the *resultant force* acting on a particle is not zero, the particle will have an acceleration *proportional to the magnitude of resultant* and in the *direction of the resultant*.**

- **Must be expressed with respect to a *Newtonian (or inertial) frame of reference*, i.e., one that is not accelerating or rotating.**

- **This form of the equation is for a constant mass system**

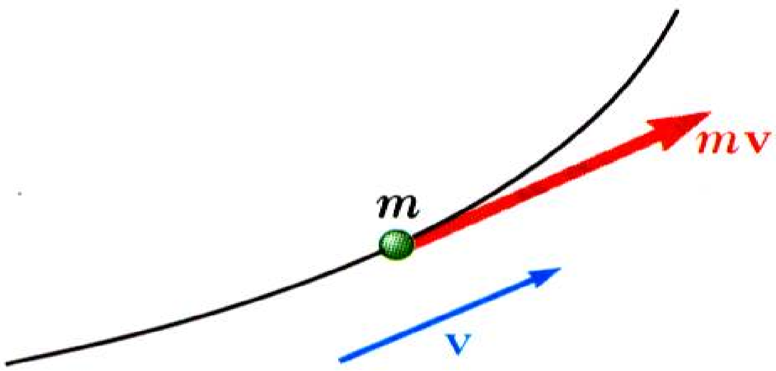
12.1 B Linear Momentum of a Particle

- *Replacing the acceleration by the derivative of the velocity yields*

$$\sum \vec{F} = m\frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}(m\vec{v}) = \frac{d\vec{L}}{dt}$$

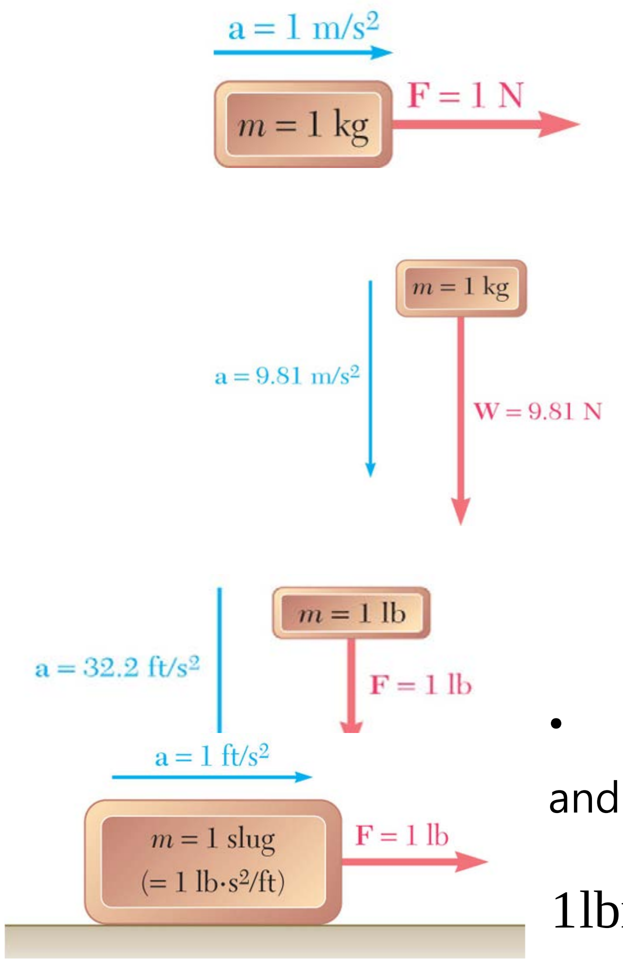
$$\vec{L} = \text{linear momentum of the particle}$$



- *Linear Momentum Conservation Principle*:

  If the resultant force on a particle is zero, the linear momentum of the particle remains constant in both magnitude and direction.

12.1C Systems of Units



- Of the units for the four primary dimensions (force, mass, length, and time), three may be chosen arbitrarily. The fourth must be compatible with Newton's 2nd Law.

International System of Units (SI Units): base units are the units of length (m), mass (kg), and time (second). The unit of force is derived,

$$1\,\mathrm{N} = (1\,\mathrm{kg})\left(1\frac{\mathrm{m}}{\mathrm{s}^2}\right) = 1\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}$$

- *U.S. Customary Units*: base units are the units of force (lb), length (m), and time (second). The unit of mass is derived,

$$1\,\mathrm{lbm} = \frac{1\,\mathrm{lb}}{32.2\,\mathrm{ft}/\mathrm{s}^2} \qquad 1\,\mathrm{slug} = \frac{1\,\mathrm{lb}}{1\,\mathrm{ft}/\mathrm{s}^2} = 1\frac{\mathrm{lb}\cdot\mathrm{s}^2}{\mathrm{ft}}$$

12.1 D Equations of Motion

- Newton's second law $$\sum \vec{F} = m\vec{a}$$

Free-body diagram ~ Kinetic diagram

- Can use scalar component equations, e.g., for rectangular components,

<u>Rectangular components</u>

$$\sum \left(F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \right) = m \left(a_x \vec{i} + a_y \vec{j} + a_z \vec{k} \right)$$

$$\sum F_x = ma_x \quad \sum F_y = ma_y \quad \sum F_z = ma_z$$

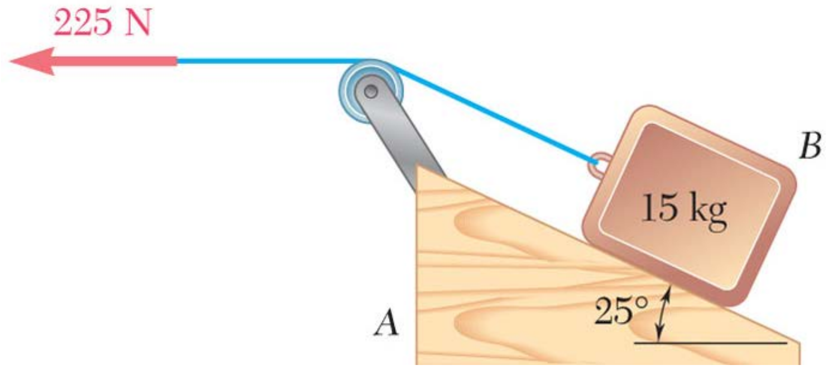
$$\sum F_x = m\ddot{x} \quad \sum F_y = m\ddot{y} \quad \sum F_z = m\ddot{z}$$

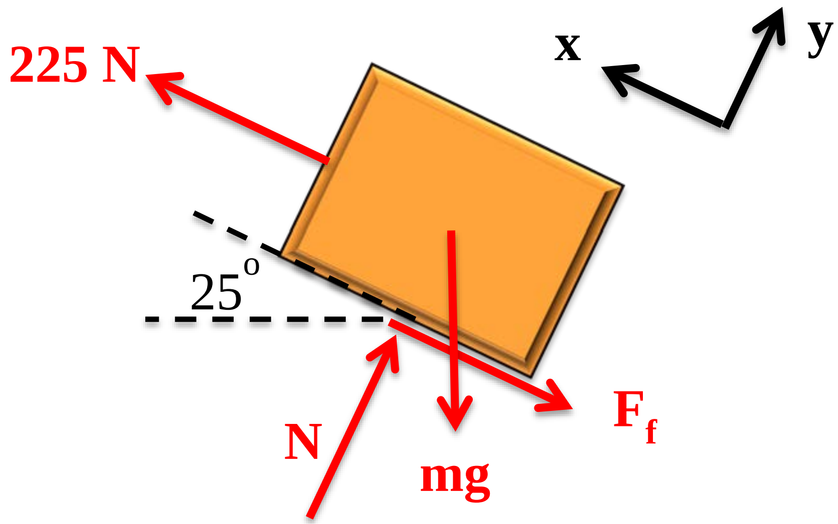
<u>Tangential and normal components, Radial and transverse components</u>

Free Body Diagrams and Kinetic Diagrams

The free body diagram is the same as you have done in statics; we will add the kinetic diagram in our dynamic analysis.

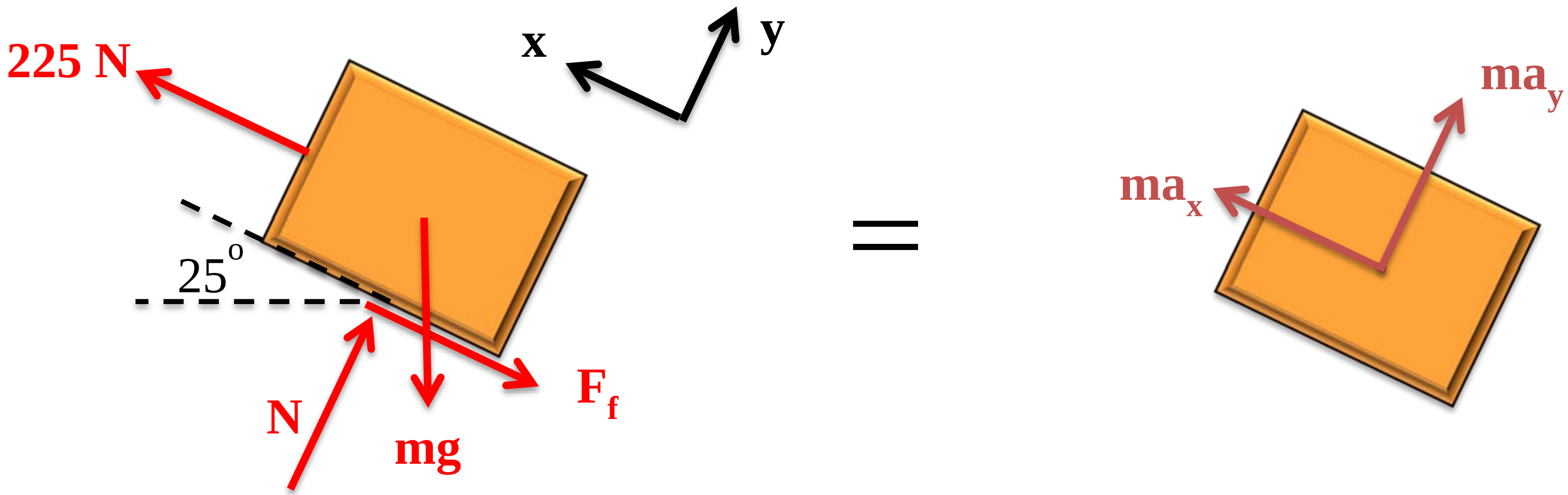
1. Isolate the body of interest (free body)
2. Draw your axis system (e.g., Cartesian, polar, path)
3. Add in applied forces (e.g., weight)
4. Replace supports with forces (e.g., reactions :normal force)
5. Draw appropriate dimensions (usually angles for particles)





Put the inertial terms for the body of interest on the kinetic diagram.

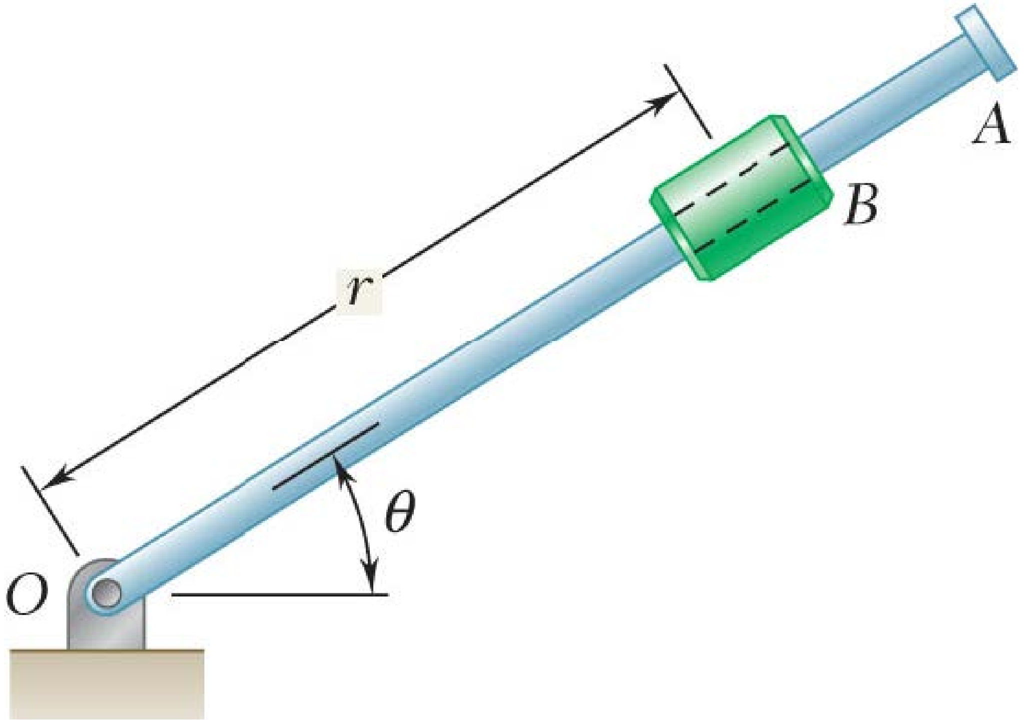
1. **Isolate the body of interest (free body)**
2. **Draw in the mass times acceleration of the particle; if unknown, do this in the positive direction according to your chosen axes**

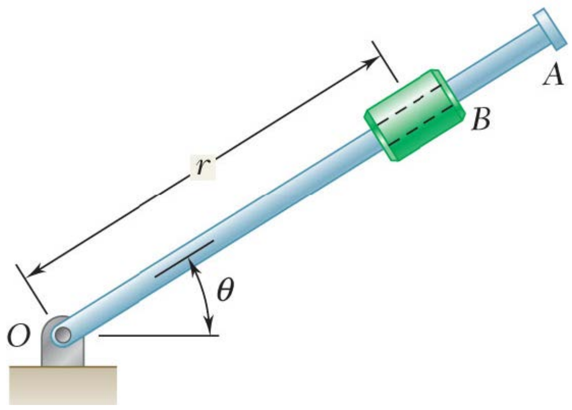


$$\Sigma \mathbf{F} = m\mathbf{a}$$

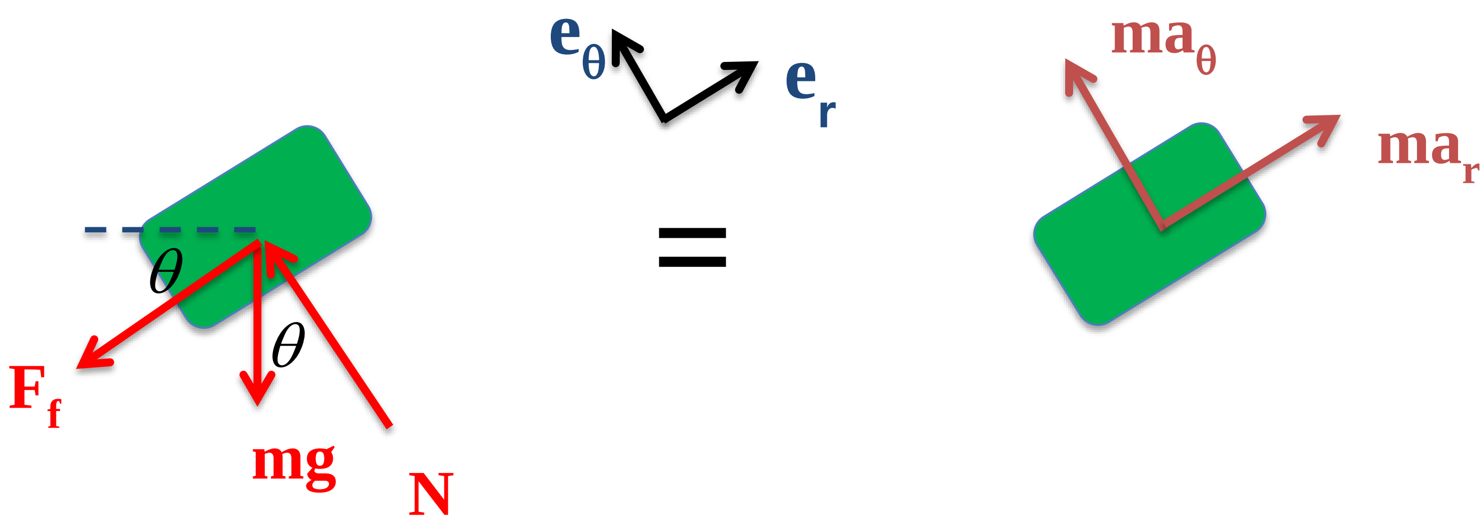
Draw the FBD and KD for block A (<u>note that the massless, frictionless pulleys</u> are attached to block A and should be included in the system).

Draw the FBD and KD for the collar B. Assume there is friction acting between the rod and collar, motion is in the vertical plane, and *q* is increasing

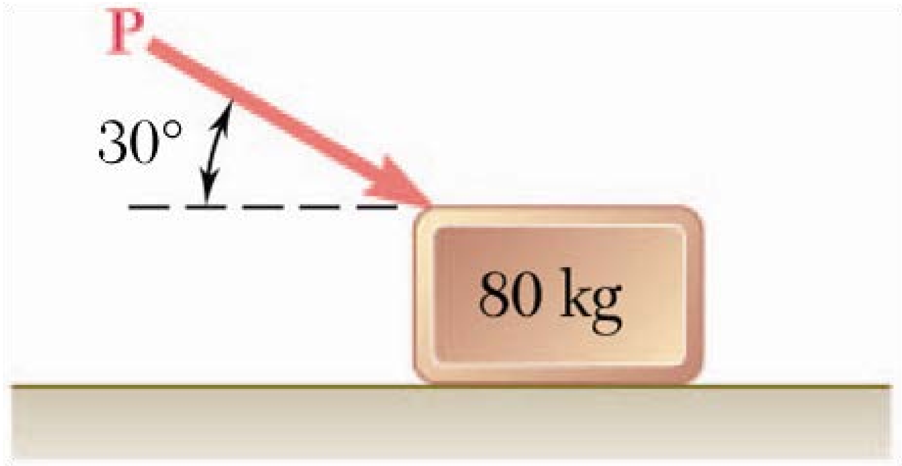




1. Isolate body
2. Axes
3. Applied forces
4. Replace supports with forces
5. Dimensions
6. Kinetic diagram



Sample Problem 12.1

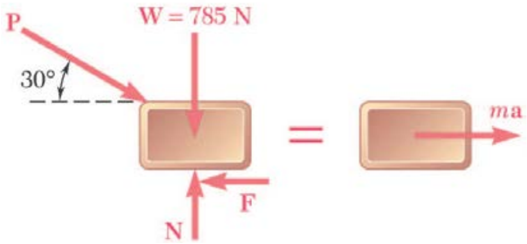


A 80-kg block rests on a horizontal plane. Find the magnitude of the force *P* required to give the block an acceleration of 2.5 m/s^2 to the right. The coefficient of kinetic friction between the block and plane is m_k = 0.25.

STRATEGY:

- Resolve the equation of motion for the block into two rectangular component equations.
- Unknowns consist of the applied force *P* and the normal reaction *N* from the plane. The two equations may be solved for these unknowns.

$$W = (80\,\text{kg})\left(9.81\frac{\text{m}}{\text{s}^2}\right) = 785\,\text{N}$$



$$m = 80\,\text{kg}$$

$$F = \mu_k N$$
$$= 0.25N$$

MODELING and ANALYSIS:

- **Resolve the equation of motion for the block into two rectangular component equations.**

$$\sum F_x = ma:$$

$$P\cos 30° - 0.25N = (80\,\text{kg})(2.5\,\text{m/s}^2)$$
$$= 200\,\text{N}$$

$$\sum F_y = 0:$$

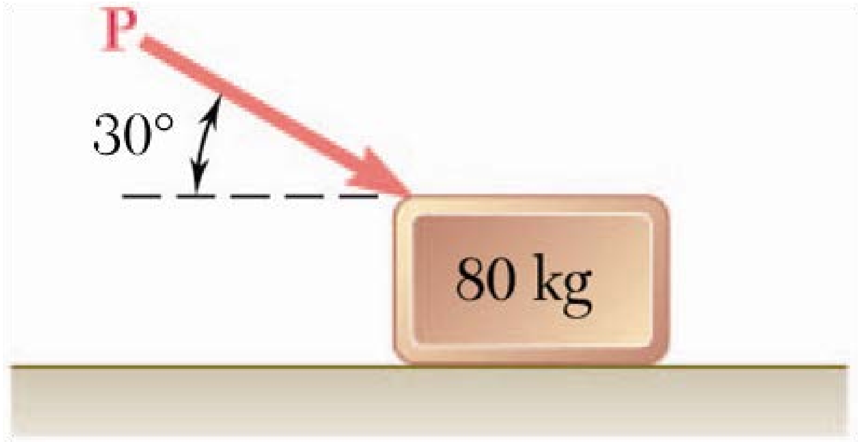
$$N - P\sin 30° - 785\,\text{N} = 0$$

Unknowns consist of the applied force *P* and the normal reaction *N* from the plane. The two equations may be solved for these unknowns.

$$N = P\sin 30° + 785\ \text{N}$$

$$P\cos 30° = 0.25\,(P\sin 30° + 785\ \text{N}) + 200\ \text{N}$$

$$\boxed{P = 535\ \text{N}}$$

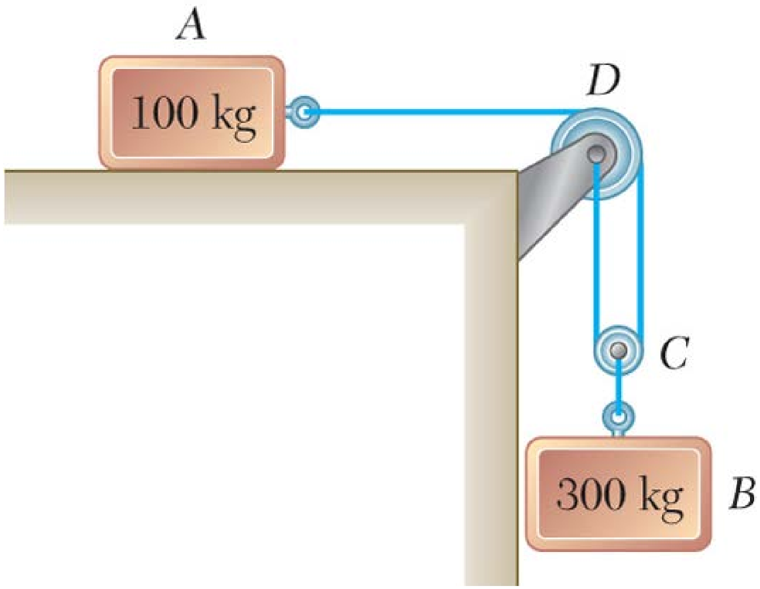


REFLECT and THINK

When you begin pushing on an object, you first have to overcome the static friction force ($F = \mu_s N$) before the object will move.

Also note that the downward component of force P increases the normal force N, which in turn increases the friction force F that you must overcome.

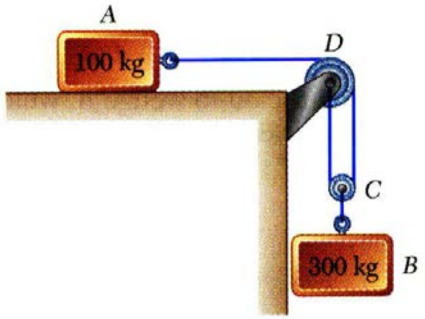
Sample Problem 12.3



The two blocks shown start from rest. The horizontal plane and the pulley are frictionless, and the pulley is assumed to be of negligible mass. Determine the acceleration of each block and the tension in the cord.

STRATEGY:

- Write the kinematic relationships for the dependent motions and accelerations of the blocks.
- Write the equations of motion for the blocks and pulley.
- Combine the kinematic relationships with the equations of motion to solve for the accelerations and cord tension.

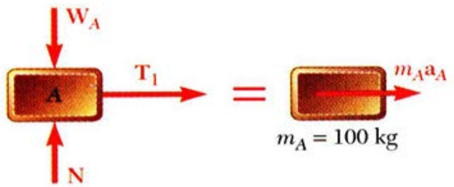


MODELING and ANALYSIS:

- Write the kinematic relationships for the dependent motions and accelerations of the blocks.

$$y_B = \frac{1}{2}x_A \qquad a_B = \frac{1}{2}a_A$$

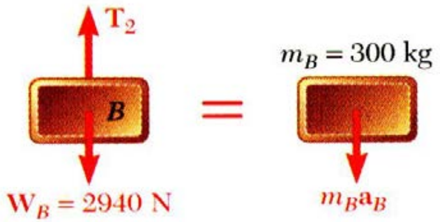
Write equations of motion for blocks and pulley.



$$\sum F_x = m_A a_A :$$

$$T_1 = (100\,\mathrm{kg})a_A$$

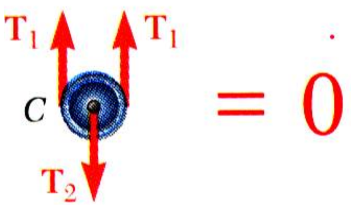
$$\sum F_y = m_B a_B :$$



$$m_B g - T_2 = m_B a_B$$

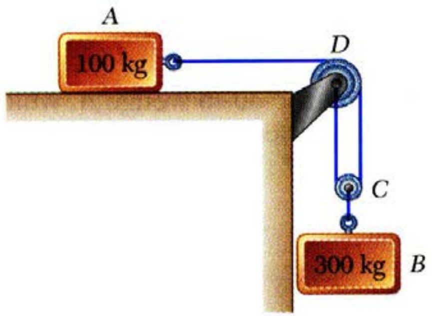
$$(300\,\mathrm{kg})\left(9.81\,\mathrm{m/s^2}\right) - T_2 = (300\,\mathrm{kg})a_B$$

$$T_2 = 2940\,\mathrm{N} - (300\,\mathrm{kg})a_B$$



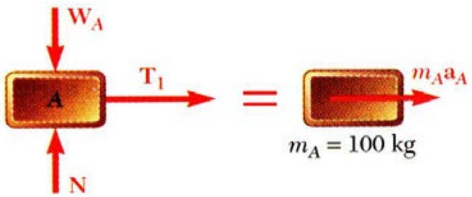
$$\sum F_y = m_C a_C = 0 :$$

$$T_2 - 2T_1 = 0$$



Combine kinematic relationships with equations of motion to solve for accelerations and cord tension.

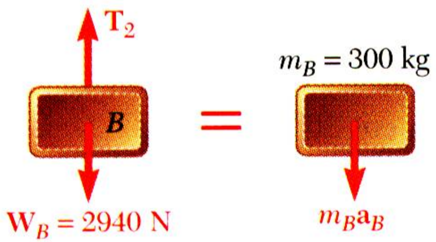
$$y_B = \tfrac{1}{2} x_A \qquad a_B = \tfrac{1}{2} a_A$$



$$T_1 = (100\,\text{kg})a_A$$

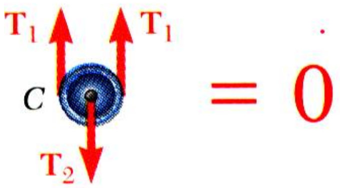
$$T_2 = 2940\,\text{N} - (300\,\text{kg})a_B$$

$$= 2940\,\text{N} - (300\,\text{kg})\left(\tfrac{1}{2} a_A\right)$$

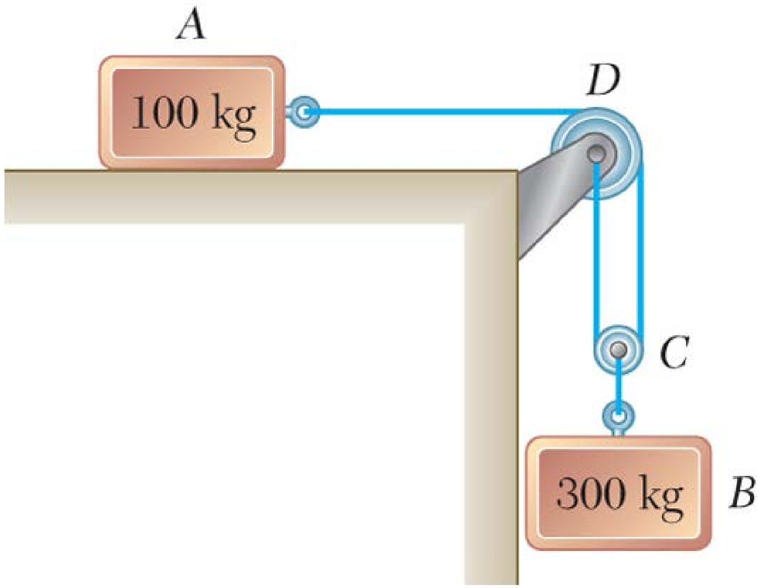


$$T_2 - 2T_1 = 0$$

$$2940\,\text{N} - (150\,\text{kg})a_A - 2(100\,\text{kg})a_A = 0$$



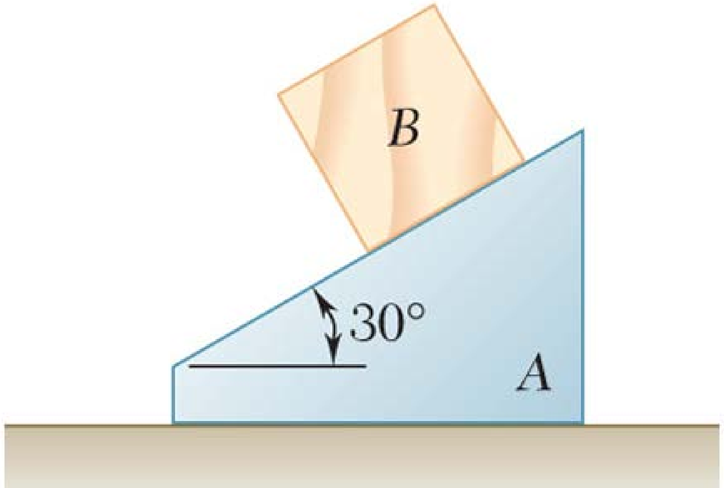
$$\boxed{\begin{aligned} a_A &= 8.40\,\text{m/s}^2 \\ a_B &= \tfrac{1}{2} a_A = 4.20\,\text{m/s}^2 \\ T_1 &= (100\,\text{kg})\, a_A = 840\,\text{N} \\ T_2 &= 2T_1 = 1680\,\text{N} \end{aligned}}$$



REFLECT and THINK

- Note that the value obtained for T_2 is *not* equal to the weight of block B. Rather than choosing B and the pulley as separate systems, you could have chosen the system to be B and the pulley. In this case, T_2 would have been an internal force.

Sample Problem 12.5

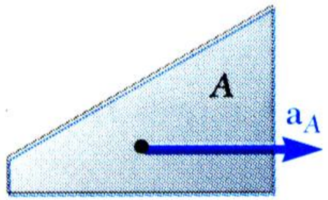


The 6-kg block *B* starts from rest and slides on the 15-kg wedge *A*, which is supported by a horizontal surface.

Neglecting friction, determine *(a)* the acceleration of the wedge, and *(b)* the acceleration of the block relative to the wedge.

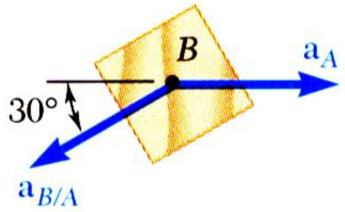
STRATEGY:

- The block is constrained to slide down the wedge. Therefore, their motions are dependent. Express the acceleration of block as the acceleration of wedge plus the acceleration of the block relative to the wedge.
- Write the equations of motion for the wedge and block.
- Solve for the accelerations.



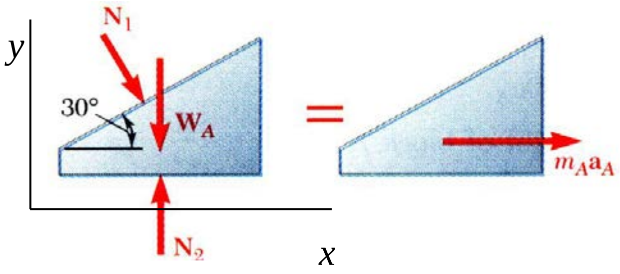
MODELING and ANALYSIS:

- **The block is constrained to slide down the wedge. Therefore, their motions are dependent.**



$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

- **Write equations of motion for wedge A and block B.**



A:

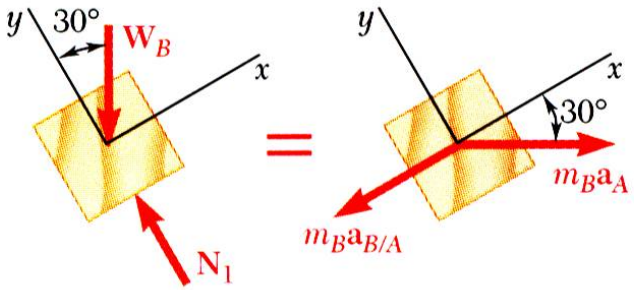
$$\sum F_x = m_A a_A :$$

$$N_1 \sin 30° = m_A a_A$$

-→ $$0.5 N_1 = (m_A) a_A \;--(1)$$

B

$$\sum F_x = m_B a_x$$

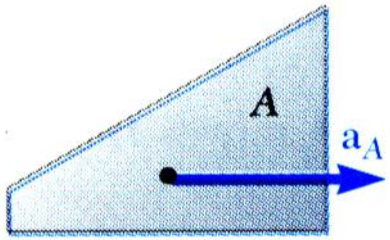


$$-m_B g \sin 30° = m_B\left(a_A \cos 30° - a_{B/A}\right)$$

$$a_{B/A} = a_A \cos 30° + g \sin 30°$$

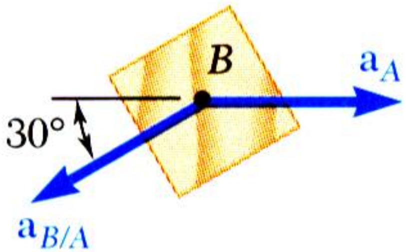
(2)

Solve for the accelerations.



$$\sum F_y = m_B a_y$$

$$N_1 - m_B g \cos 30° = -\left(m_B\right) a_A \sin 30° \quad --(3)$$



a. Acceleration of Wedge A

Substitute for N_1 from (1) into (3)

$$2m_A a_A - m_B g\cos 30° = -(m_B) a_A \sin 30°$$

Solve for $\vec{a}_A$ and substitute the numerical data

$$\vec{a}_A = 1.545 m/s^2$$

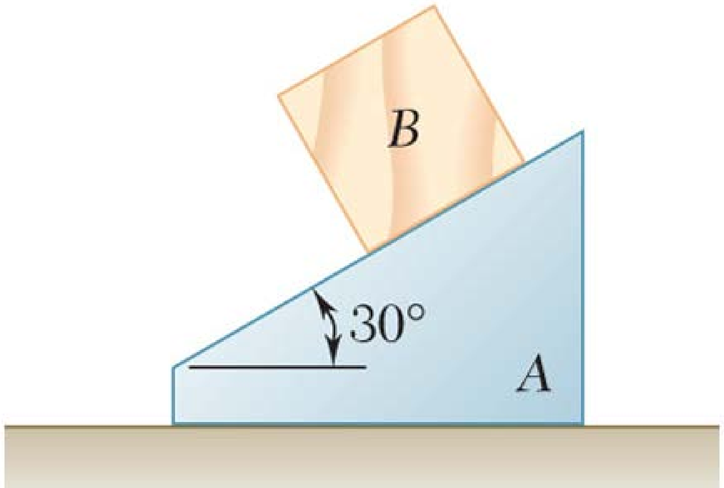
b. Acceleration of Block B Relative to A

Substitute a_A into (2)->

$$a_{B/A} = a_A \cos 30° + g\sin 30°$$

$$a_{B/A} = \left(1.545\,\mathrm{m/s^2}\right)\cos 30° + \left(9.81\,\mathrm{m/s^2}\right)\sin 30°$$

$$\boxed{a_{B/A} = 6.24\,\mathrm{m/s^2}}$$



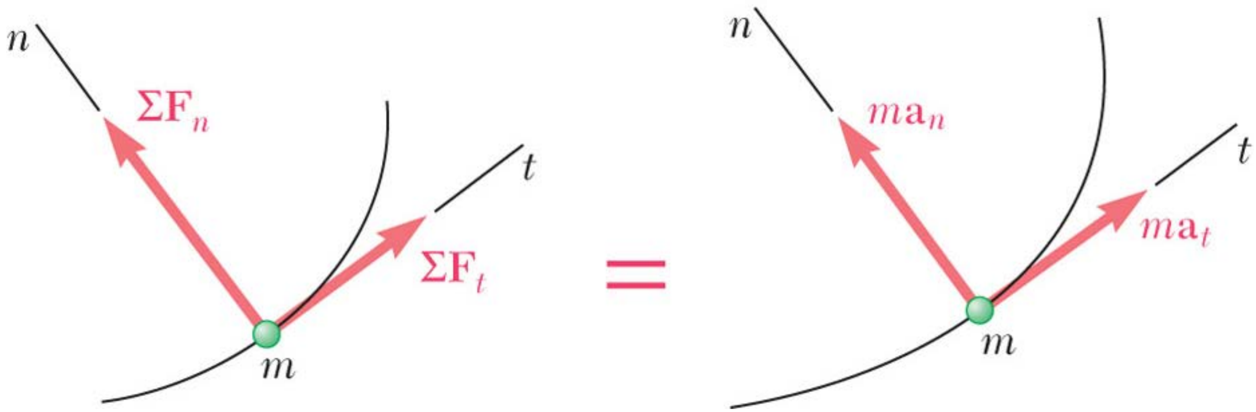
REFLECT and THINK

Many students are tempted to draw the acceleration of block *B* down the incline in the kinetic diagram. It is important to recognize that this is the direction of the *relative* acceleration. Rather than the kinetic diagram you used for block *B*, you could have simply put unknown accelerations in the *x* and *y* directions and then used your relative motion equation to obtain more scalar equations.

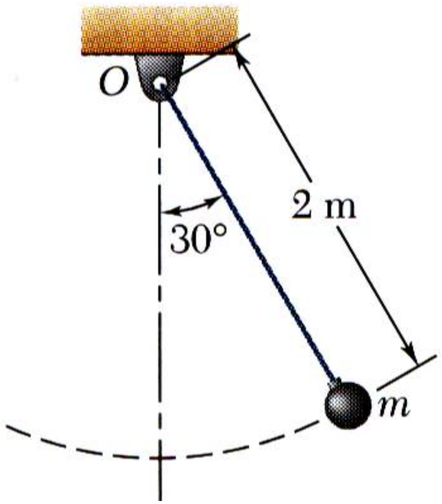
For tangential and normal components, $\sum \vec{F} = m\vec{a}$

$$\sum F_t = ma_t \qquad \sum F_n = ma_n$$

$$\sum F_t = m\frac{dv}{dt} \qquad \sum F_n = m\frac{v^2}{\rho}$$



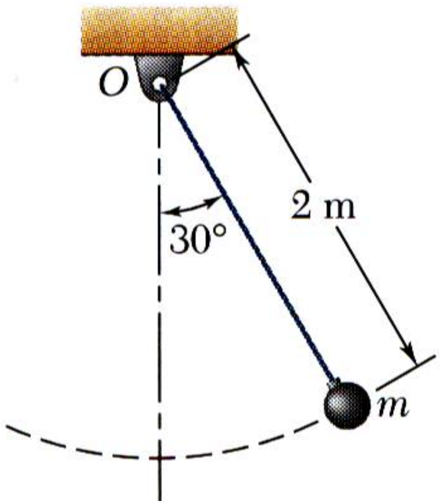
Sample Problem 12.6



The bob of a 2-m pendulum describes an arc of a circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown, find the velocity and acceleration of the bob in that position.

STRATEGY:

- **Resolve the equation of motion for the bob into tangential and normal components.**
- **Solve the component equations for the normal and tangential accelerations.**
- **Solve for the velocity in terms of the normal acceleration.**

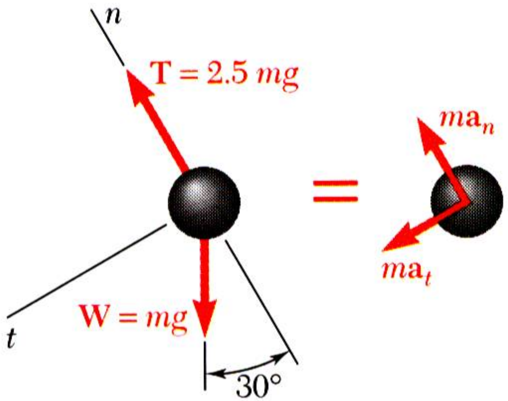


MODELING and ANALYSIS:

- Resolve the equation of motion for the bob into tangential and normal components.
- Solve the component equations for the normal and tangential accelerations.

$$\sum F_t = ma_t : \quad mg\sin 30° = ma_t$$
$$a_t = g\sin 30°$$

$$\boxed{a_t = 4.9\,\mathrm{m/s^2}}$$



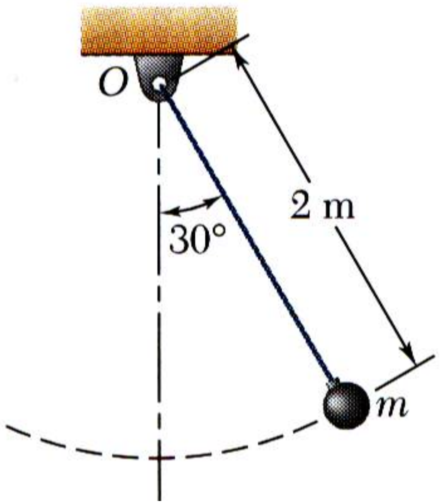
$$\sum F_n = ma_n : \quad 2.5mg - mg\cos 30° = ma_n$$
$$a_n = g\left(2.5 - \cos 30°\right)$$

$$\boxed{a_n = 16.03\,\mathrm{m/s^2}}$$

- Solve for velocity in terms of normal acceleration.

$$a_n = \frac{v^2}{\rho} \qquad v = \sqrt{\rho a_n} = \sqrt{(2\,\mathrm{m})\left(16.03\,\mathrm{m/s^2}\right)}$$

$$\boxed{v = \pm 5.66\,\mathrm{m/s}}$$

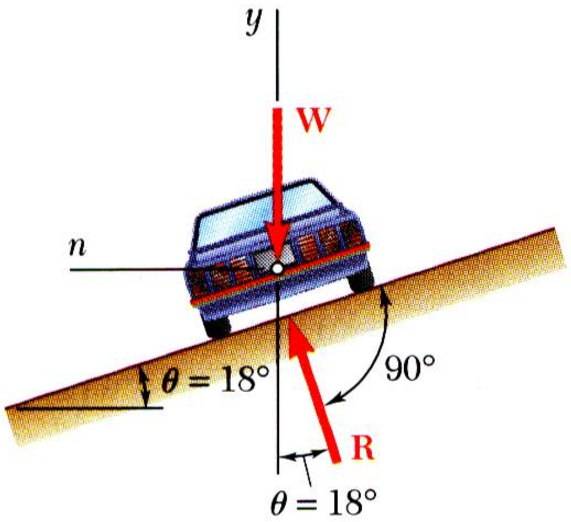


REFLECT and THINK:

- **If you look at these equations for an angle of zero instead of 30°, you will see that when the bob is straight below point O, the tangential acceleration is zero, and the velocity is a maximum.**

  The normal acceleration is not zero because the bob has a velocity at this point.

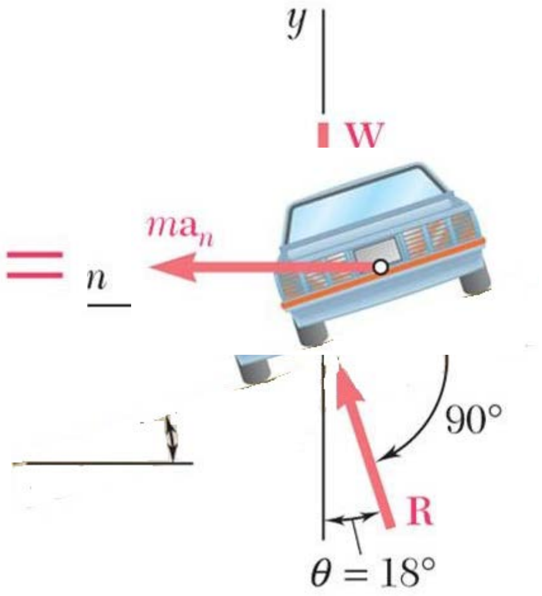
Sample Problem 12.7



Determine the rated speed of a highway curve of radius r = 120 m banked through an angle q = 18^{o}. The rated speed of a banked highway curve is the speed at which a car should travel if no lateral friction force is to be exerted at its wheels.

STRATEGY:

- The car travels in a horizontal circular path with a normal component of acceleration directed toward the center of the path. The forces acting on the car are its weight and a normal reaction from the road surface.
- Resolve the equation of motion for the car into vertical and normal components.
- Solve for the vehicle speed.



SOLUTION:

MODELING and ANALYSIS:

- **The car travels in a horizontal circular path with a normal component of acceleration directed toward the center of the path. The forces acting on the car are its weight and a normal reaction from the road surface.**

- **Resolve the equation of motion for the car into vertical and normal components.**

$$\sum F_y = 0: \qquad R\cos\theta - W = 0$$

$$R = \frac{W}{\cos\theta}$$

$$\sum F_n = ma_n: \qquad R\sin\theta = \frac{W}{g}a_n$$

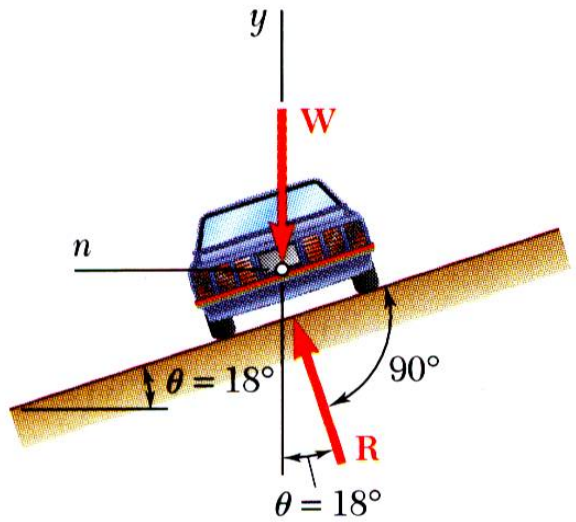
$$\frac{W}{\cos\theta}\sin\theta = \frac{W}{g}\frac{v^2}{\rho}$$

- **Solve for the vehicle speed.**

$$v^2 = g\rho\tan\theta$$

$$= \left(9.81\,\mathrm{m/s^2}\right)\left(120\,\mathrm{m}\right)\tan 18^\circ$$

$$\boxed{v = 19.56\,\mathrm{m/s} = 70.4\,\mathrm{km/h}}$$



REFLECT and THINK:

- **For a highway curve, this seems like a reasonable speed for avoiding a spin-out. If the roadway were banked at a larger angle, would the rated speed be larger or smaller than this calculated value?**
- **For this problem, the tangential direction is into the page; since you were not asked about forces or accelerations in this direction, you did not need to analyze motion in the tangential direction.**

Kinetics: Radial and Transverse Coordinates

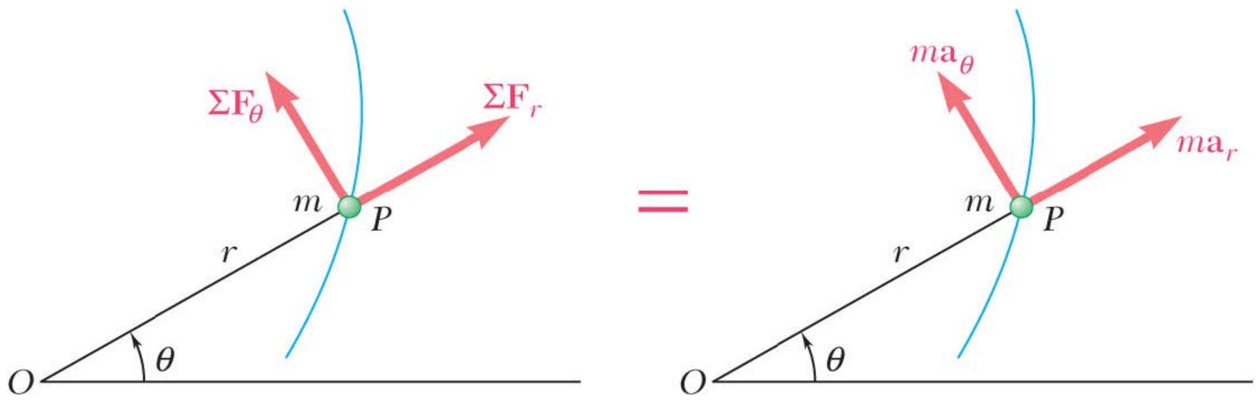
Hydraulic actuators, extending robotic arms, and centrifuges as shown below are often analyzed using radial and transverse coordinates.

©Russell Illig/Getty Images RF

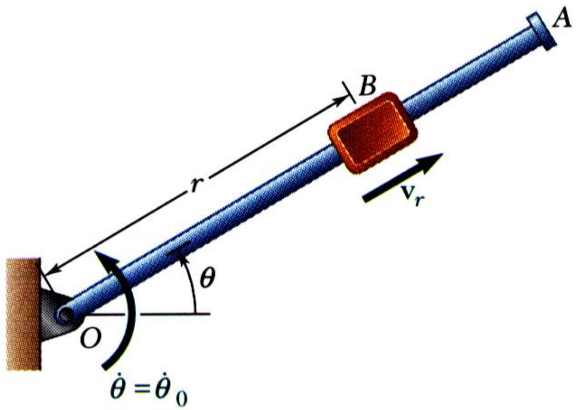
Eqs of Motion in Radial & Transverse Components

- **Consider particle in polar coordinates,**

$$\sum F_r = ma_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$$
$$\sum F_\theta = ma_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$



Sample Problem 12.10



A **block** B **of mass** m **can slide freely on a frictionless arm** OA **which rotates in a horizontal plane at a constant rate**

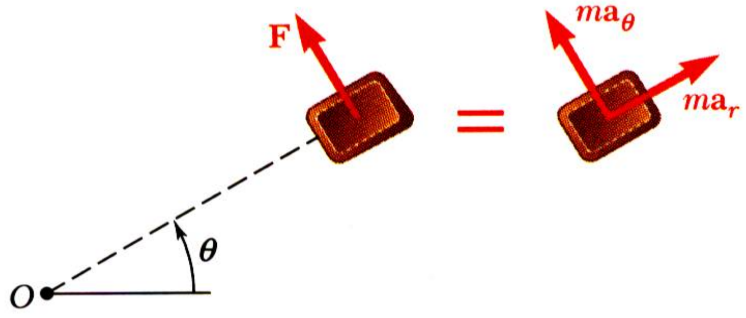
Knowing that B **is released at a distance** r_0 **from** O, **express as a function of** r

a) **the component** v_r **of the velocity of** B **along** OA**, and**

b) **the magnitude of the horizontal force exerted on** B **by the arm** OA**.**

STRATEGY:

- **Write the radial and transverse equations of motion for the block.**
- **Integrate the radial equation to find an expression for the radial velocity.**
- **Substitute known information into the transverse equation to find an expression for the force on the block.**



MODELING and ANALYSIS:

- Write the radial and transverse equations of motion for the block.

$$\sum F_r = m a_r : \; 0 = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

$$\sum F_\theta = m a_\theta : F = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

- Integrate the radial equation to find an expression for the radial velocity.

$$\ddot{r} = \dot{v}_r = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r \frac{dv_r}{dr}$$

$$\ddot{r} = \dot{v}_r = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r \frac{dv_r}{dr}$$

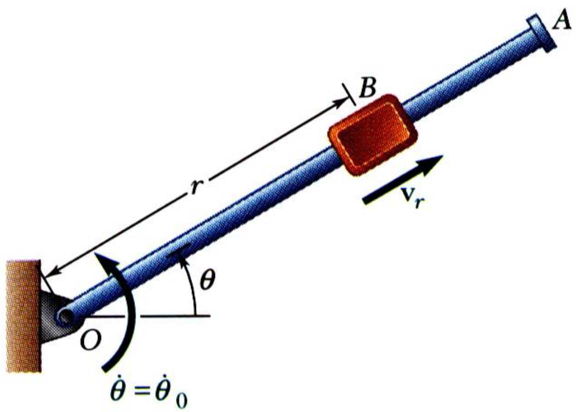
$$v_r dv_r = r\dot{\theta}^2 dr = r\dot{\theta}_0^2 dr$$

$$\int_0^{v_r} v_r dv_r = \dot{\theta}_0^2 \int_{r_0}^{r} r\, dr$$

$$\boxed{v_r^2 = \theta_0^2\left(r^2 - r_0^2\right)}$$

- Substitute known information into the transverse equation to find an expression for the force on the block.

$$\boxed{F = 2m\theta_0^2\left(r^2 - r_0^2\right)^{1/2}}$$

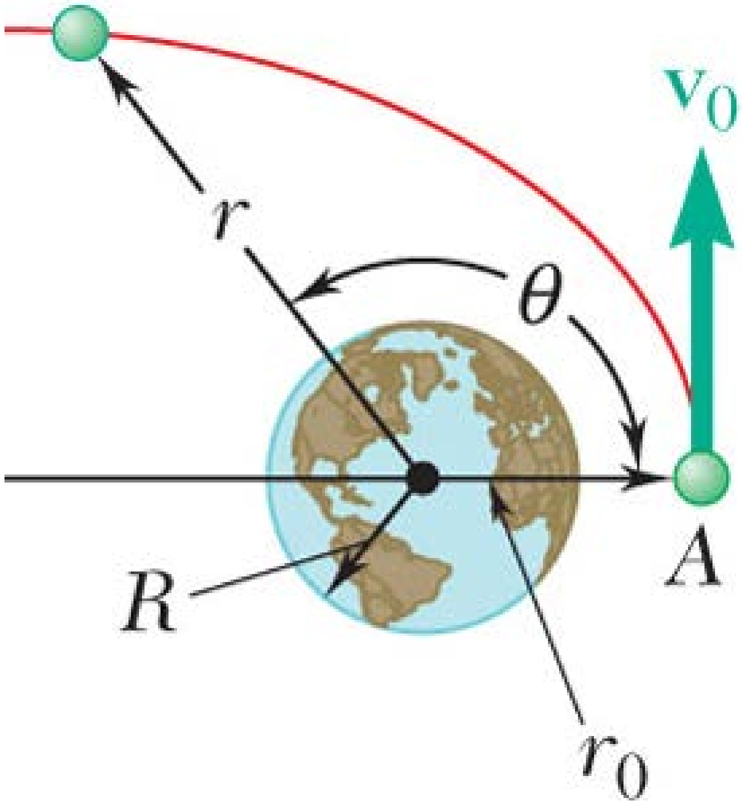


REFLECT and THINK:

- Introducing radial and transverse components of force and acceleration involves using components of velocity as well in the computations. But this is still much simpler and more direct than trying to use other coordinate systems.
- Even though the radial acceleration is zero, the rod accelerates relative to the rod with acceleration $\ddot{r}$.

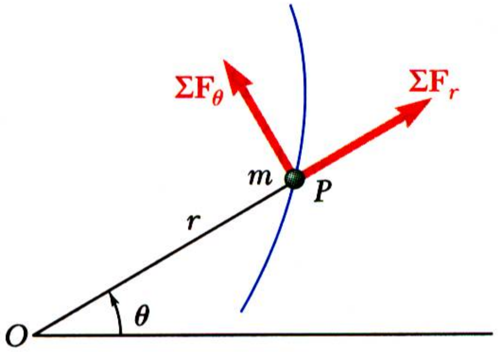
12.2 Angular Momentum and Orbital Motion

NASA/JSC



Satellite orbits are analyzed using conservation of angular momentum.

Eqs of Motion in Radial & Transverse Components

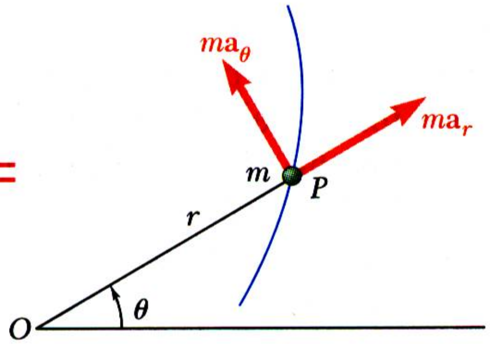


Consider particle in polar coordinates,

$$\sum F_r = ma_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

$$\sum F_\theta = ma_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

=



- This result may also be derived from conservation of angular momentum,

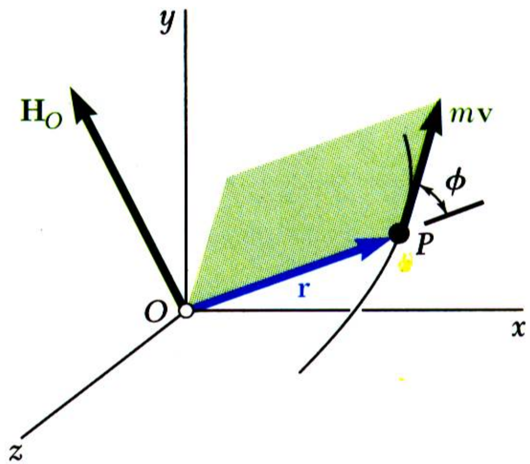
$$H_O = mr^2\dot{\theta}$$

$$r\sum F_\theta = \frac{d}{dt}\left(mr^2\dot{\theta}\right)$$

$$= m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right)$$

$$\sum F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

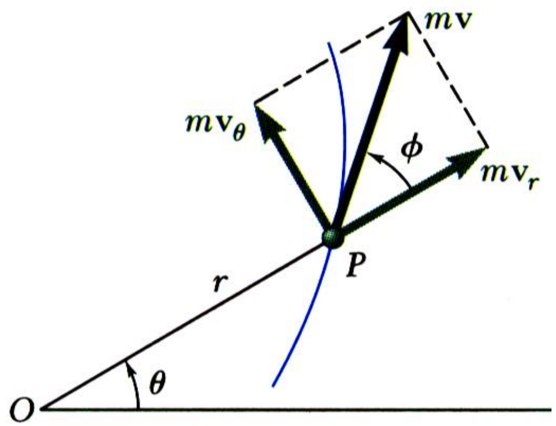
A. Angular Momentum of a Particle



- $\vec{H}_O = \vec{r} \times m\vec{V} =$ *moment of momentum* or the *angular momentum* of the particle about *O*.
- $\vec{H}_O$ is perpendicular to plane containing $\vec{r}$ and $m\vec{V}$

$$H_O = rmV\sin\phi$$
$$= rmv_\theta$$
$$= mr^2\dot{\theta}$$

$$\vec{H}_O = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

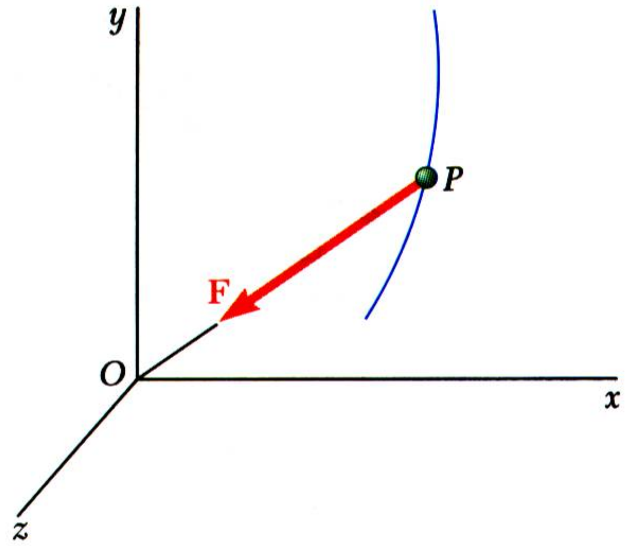


- Derivative of angular momentum with respect to time,

$$\dot{\vec{H}}_O = \dot{\vec{r}} \times m\vec{V} + \vec{r} \times m\dot{\vec{V}} = \vec{V} \times m\vec{V} + \vec{r} \times m\vec{a}$$
$$= r \times \sum \vec{F}$$
$$= \sum \vec{M}_O$$

- It follows from Newton's second law that the sum of the moments about *O* of the forces acting on the particle is equal to the rate of change of the angular momentum of the particle about *O*.

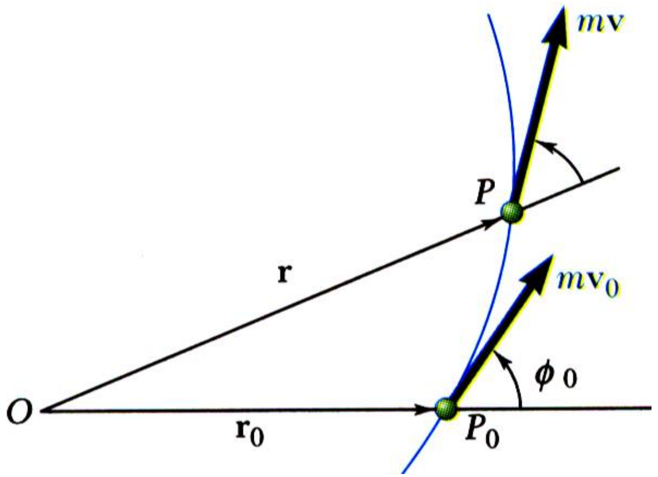
B. Conservation of Angular Momentum



- When only force acting on particle is directed toward or away from a fixed point *O*, the particle is said to be *moving under a central force*.
- Since the line of action of the central force passes through *O*, $\sum \vec{M}_O = \dot{\vec{H}}_O = 0$ and

$$\vec{r} \times m\vec{V} = \vec{H}_O = \text{constant} \quad \textbf{(12.22)}$$

- Position vector and motion of particle are in a plane perpendicular to $\vec{H}_O$.



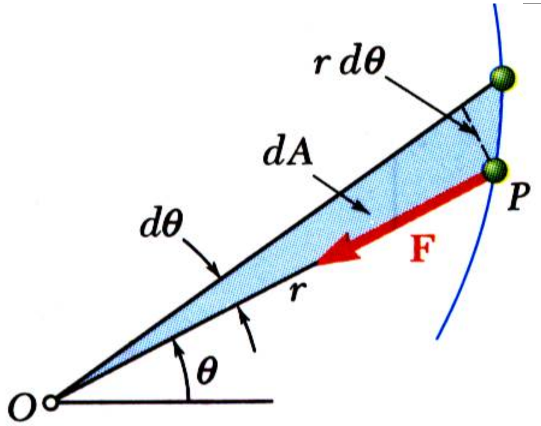
* Magnitude of angular momentum,

$$H_O = rmV\sin\phi = \text{constant}$$
$$= r_0 m V_0 \sin\phi_0 \quad \textbf{(12.23)}$$

or

$$H_O = mr^2\dot{\theta} = \text{constant}$$

$$\frac{H_O}{m} = r^2\dot{\theta} = h = \frac{\text{angular momentum}}{\text{unit mass}} \quad \textbf{(12.24)}$$



- **Radius vector *OP* sweeps infinitesimal area**

$$dA = \tfrac{1}{2} r^2 d\theta$$

- **Define *areal velocity***

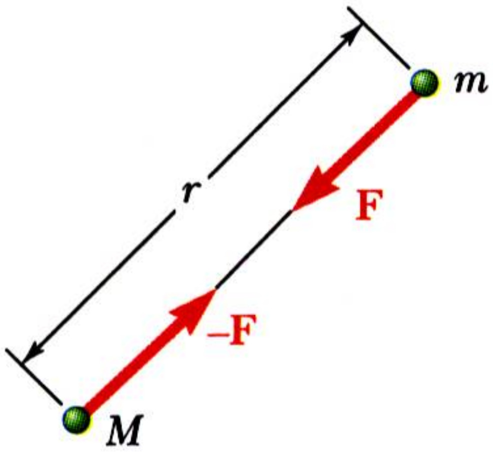
$$\frac{dA}{dt} = \tfrac{1}{2} r^2 \frac{d\theta}{dt} = \tfrac{1}{2} r^2 \dot{\theta} =$$

● **Recall, for a body moving under a central force,**

$$h = r^2\dot{\theta} = \text{constant}$$

- **When a particle moves under a central force, its areal velocity is constant.**

C Newton's Law of Gravitation



*Gravitational force exerted by the sun on a planet or by the earth on a satellite is an important example of gravitational force.

Newton's law of universal gravitation - two particles of mass M and m attract each other with equal and opposite force directed along the line connecting the particles,

$$F = G\frac{Mm}{r^2}$$

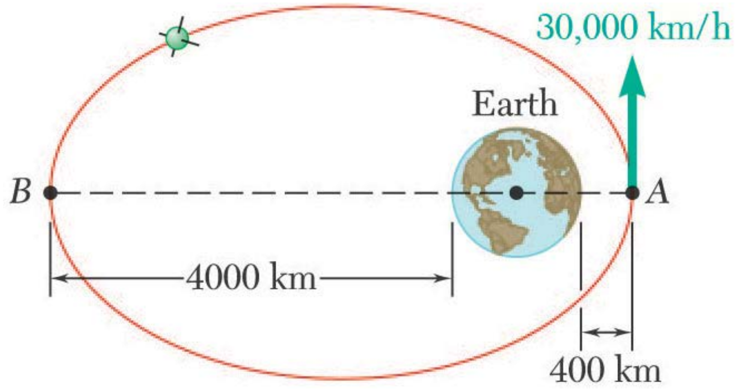
$$G = \text{constant of gravitation}$$

$$= 66.73\times10^{-12}\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2} = 34.4\times10^{-9}\frac{\text{ft}^4}{\text{lb}\cdot\text{s}^4}$$

- For particle of mass m on the earth's surface,

$$W = m\frac{MG}{R^2} = mg \qquad g = 9.81\frac{\text{m}}{\text{s}^2} = 32.2\frac{\text{ft}}{\text{s}^2}$$

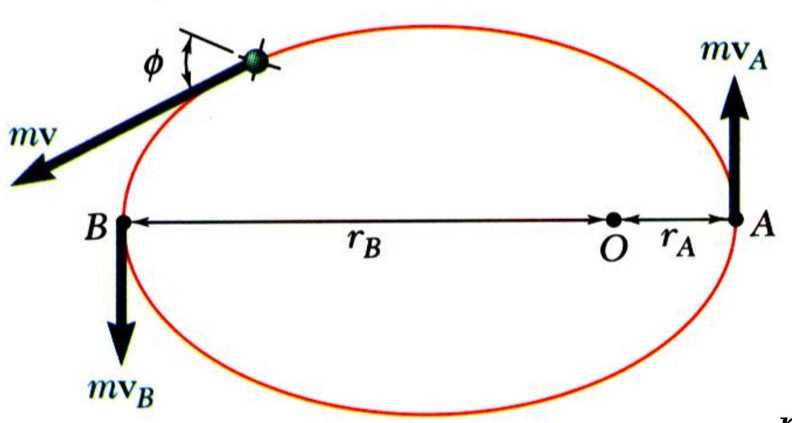
Sample Problem 12.12



A satellite is launched in a direction parallel to the surface of the earth with a velocity of 30,000 km/h from an altitude of 400 km. Determine the velocity of the satellite as it reaches it maximum altitude of 4000 km. The radius of the earth is 6370 km.

STRATEGY:

- Since the satellite is moving under a central force, its angular momentum is constant. Equate the angular momentum at *A* and *B* and solve for the velocity at *B*.



MODELING and ANALYSIS:

- Since the satellite is moving under a central force, its angular momentum is constant. Equate the angular momentum at *A* and *B* and solve for the velocity at *B*.

$$r m v \sin\phi = H_O = \text{constant}$$

$$r_A m v_A = r_B m v_B$$

$$v_B = v_A \frac{r_A}{r_B}$$

$$= (30{,}000\,\text{km/h}) \frac{(6370 + 400)\,\text{km}}{(6370 + 4000)\,\text{km}}$$

$$\boxed{v_B = 19{,}590\,\text{km/h}}$$

REFLECT and THINK:

- **Note that in order to increase velocity, a spacecraft often applies thrusters to push it closer to the earth. This central force means the spacecraft's angular momentum remains constant, its radial distance r decreases, and its velocity v increases.**

*12.3 APPLICATIONS OF CENTRAL FORCE MOTION

Trajectory of a Particle Under a Central Force

- **For particle moving under central force directed towards force center,**

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = \sum F_r = -F \qquad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = \sum F_\theta = 0$$

- **Second expression is equivalent to from which,**

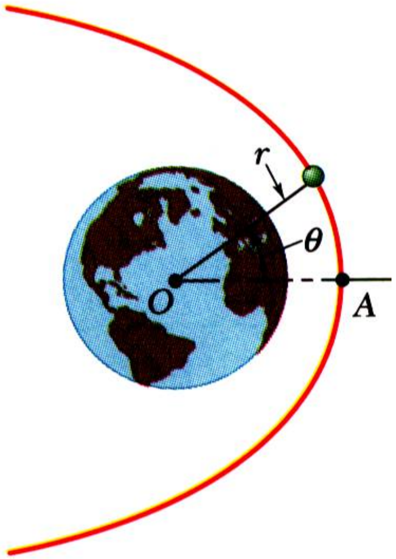
$$\dot{\theta} = \frac{h}{r^2} \quad \text{and} \quad \ddot{r} = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$$

- **After substituting into the radial equation of motion and simplifying,**

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{mh^2u^2} \quad \text{where} \quad u = \frac{1}{r}$$

- **If *F* is a known function of *r* or *u*, then particle trajectory may be found by integrating for $u = f(\theta)$, with constants of integration determined from initial conditions.**

Application to Space Mechanics



*Consider earth satellites subjected to only gravitational pull of the earth,

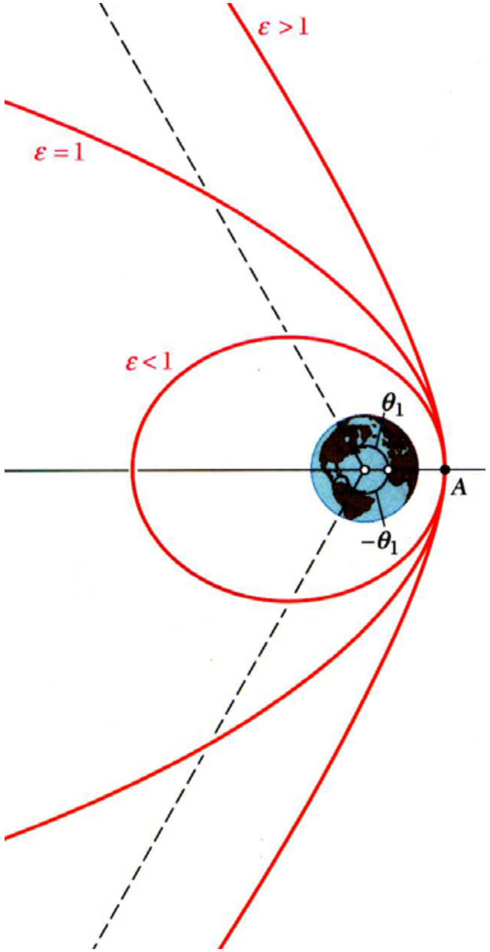
$$\frac{d^2u}{d\theta^2}+u=\frac{F}{mh^2u^2} \quad \text{where} \quad u=\frac{1}{r} \qquad F=\frac{GMm}{r^2}=GMmu^2$$

$$\frac{d^2u}{d\theta^2}+u=\frac{GM}{h^2}=\text{constant}$$

- **Solution is equation of conic section,(12.37)**

$$u=\frac{1}{r}=\frac{GM}{h^2}(1+\varepsilon\cos\theta) \qquad \varepsilon=\frac{Ch^2}{GM}=\text{eccentricity}$$

- **Origin, located at earth's center, is a focus of the conic section.**
- **Trajectory may be ellipse, parabola, or hyperbola depending on value of eccentricity.**



• **Trajectory of earth satellite is defined by**

$$\frac{1}{r} = \frac{GM}{h^2}(1+\varepsilon\cos\theta) \qquad \varepsilon = \frac{Ch^2}{GM} = \text{eccentricity}$$

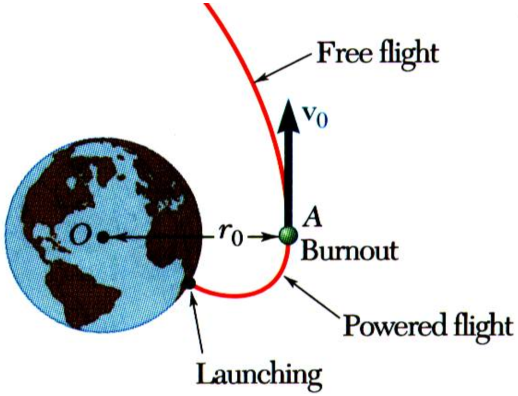
(12.37)

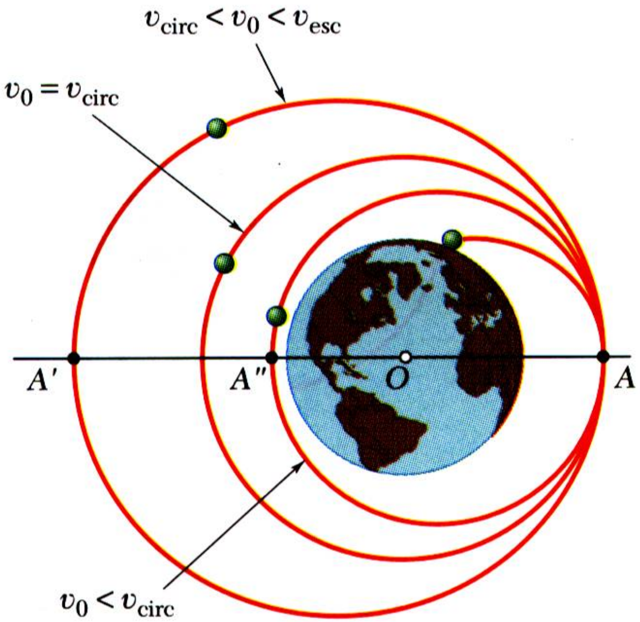
***Hyperbola*, *e* > 1 or *C* > *GM/h*²**. **The radius vector becomes infinite for**

$$1+\varepsilon\cos\theta_1 = 0 \quad \theta_1 = \pm\cos^{-1}\left(-\frac{1}{\varepsilon}\right) = \pm\cos^{-1}\left(-\frac{GM}{Ch^2}\right)$$

***Parabola*, *e* = 1 or *C* = *GM/h*²**. **The radius vector becomes infinite for** $1+\cos\theta_2 = 0 \quad \theta_2 = 180°$

***Ellipse*, *e* < 1 or *C* < *GM/h*²**. **The radius vector is finite for θ and is constant, i.e., a circle, for *e* = 0.**





Integration constant C is determined by conditions at beginning of free flight, θ =0, r = r_0,

$$\frac{1}{r_0} = \frac{GM}{h^2}\left(1 + \frac{Ch^2}{GM}\cos 0°\right)$$

$$C = \frac{1}{r_0} - \frac{GM}{h^2} = \frac{1}{r_0} - \frac{GM}{(r_0 v_0)^2}$$

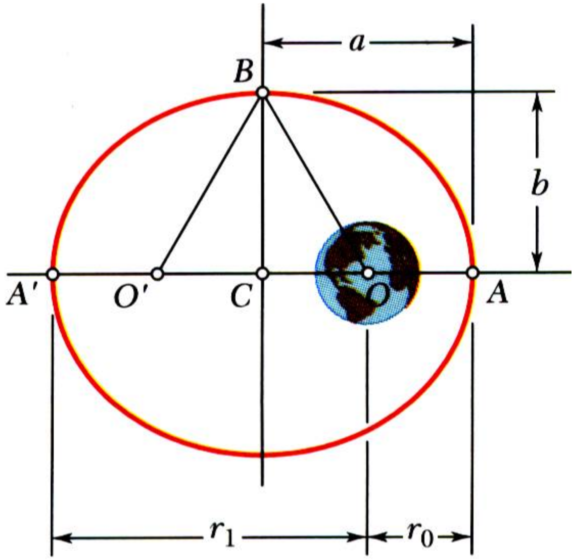
- **Satellite escapes earth orbit for**

$$\varepsilon \geq 1 \text{ or } C \geq GM/h^2 = GM/(r_0 v_0)^2$$

$$v_{esc} = v_0 = \sqrt{\frac{2GM}{r_0}}$$

- **Trajectory is elliptic for v_0 < v_{esc} and becomes circular for e = 0 or C = 0,**

$$v_{circ} = \sqrt{\frac{GM}{r_0}}$$



- **Recall that for a particle moving under a central force, the *areal velocity* is constant, i.e.,**

- ***Periodic time* or time required for a satellite to complete an orbit is equal to area within the orbit divided by *areal velocity*,**

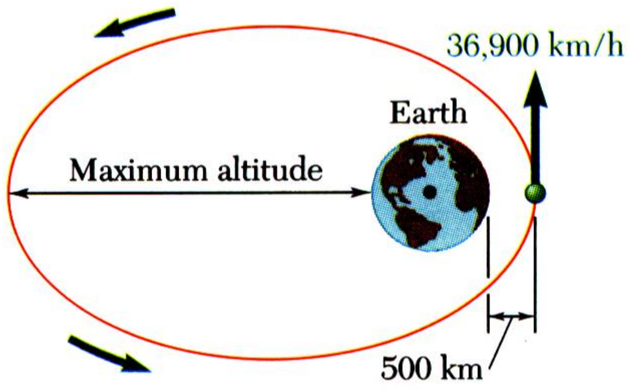
$$\frac{dA}{dt} = \tfrac{1}{2} r^2 \dot{\theta} = \tfrac{1}{2} h = \text{constant} \qquad \tau = \frac{\pi ab}{h/2} = \frac{2\pi ab}{h}$$

where

$$a = \tfrac{1}{2}\left(r_0 + r_1\right)$$

$$b = \sqrt{r_0 r_1}$$

Sample Problem 12.14



A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36,900 km/h at an altitude of 500 km.

Determine: a) the maximum altitude reached by the satellite, and b) the periodic time of the satellite.

STRATEGY:

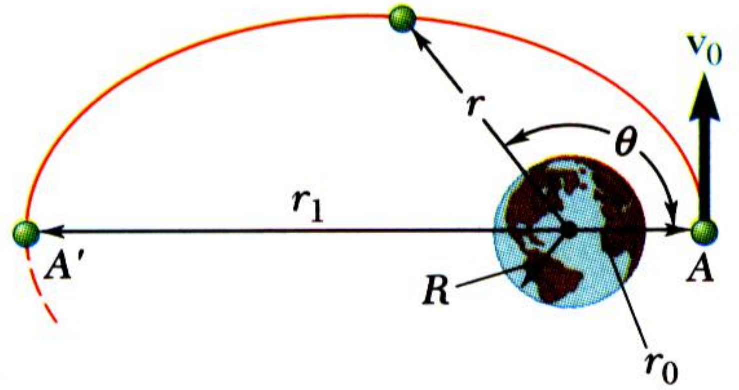
- **Trajectory of the satellite is described by**

$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta$$

Evaluate C using the initial conditions at θ = 0.

a)Determine the maximum altitude by finding r at θ = 180°.

- With the altitudes at the perigee and apogee known, the periodic time can be evaluated.



MODELING and ANALYSIS:

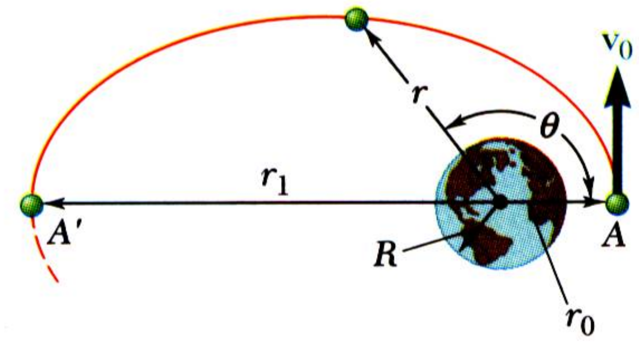
- Trajectory of the satellite is described by

$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta$$

Evaluate C using the initial conditions at $\theta = 0$.

$$r_0 = (6370 + 500)\,\text{km}$$
$$= 6.87\times10^6\,\text{m}$$
$$v_0 = 36{,}900\,\frac{\text{km}}{\text{h}} \times \frac{1000\,\text{m/km}}{3600\,\text{s/h}}$$
$$= 10.25\times10^3\,\text{m/s}$$
$$h = r_0 v_0 = \left(6.87\times10^6\text{m}\right)\left(10.25\times10^3\,\text{m/s}\right)$$
$$= 70.4\times10^9\,\text{m}^2/\text{s}$$
$$GM = gR^2 = \left(9.81\text{m/s}^2\right)\left(6.37\times10^6\text{m}\right)^2$$
$$= 398\times10^{12}\,\text{m}^3/\text{s}^2$$

$$C = \frac{1}{r_0} - \frac{GM}{h^2}$$
$$= \frac{1}{6.87\times10^6\,\text{m}} - \frac{398\times10^{12}\,\text{m}^3/\text{s}^2}{\left(70.4\,\text{m}^2/\text{s}\right)^2}$$
$$= 65.3\times10^{-9}\,\text{m}^{-1}$$

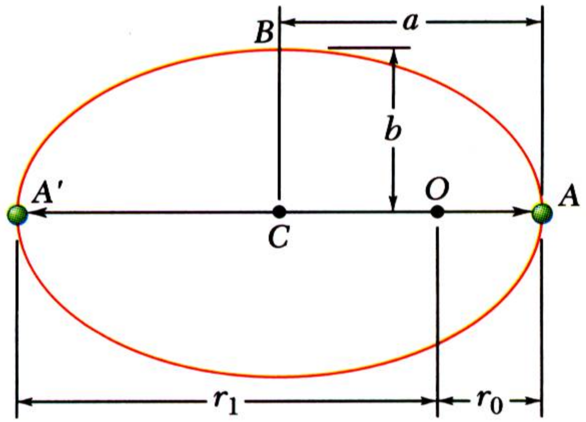


Determine the maximum altitude by finding r_1 at θ = 180°.

$$\frac{1}{r_1} = \frac{GM}{h^2} - C = \frac{398 \times 10^{12}\, \mathrm{m^3/s^2}}{\left(70.4\, \mathrm{m^2/s}\right)^2} - 65.3 \times 10^{-9}\, \frac{1}{\mathrm{m}}$$

$$r_1 = 66.7 \times 10^6\, \mathrm{m} = 66{,}700\ \mathrm{km}$$

$$\boxed{\text{max altitude} = (66{,}700\text{-}6370)\,\mathrm{km} = 60{,}300\ \mathrm{km}}$$



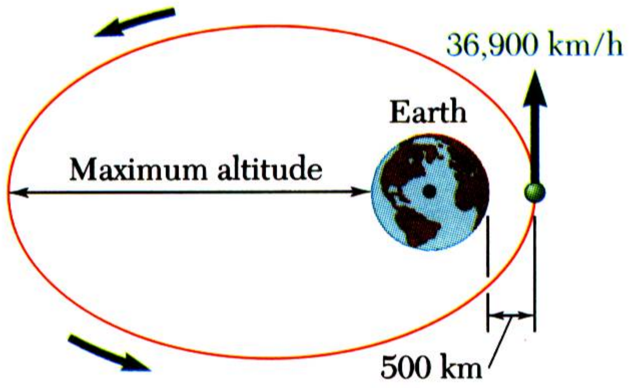
b) With the altitudes at the perigee and apogee known, the periodic time can be evaluated.

$$a = \tfrac{1}{2}(r_0 + r_1) = \tfrac{1}{2}(6.87 + 66.7) \times 10^6\, \mathrm{m} = 36.8 \times 10^6\, \mathrm{m}$$

$$b = \sqrt{r_0 r_1} = \sqrt{6.87 \times 66.7} \times 10^6\, \mathrm{m} = 21.4 \times 10^6\, \mathrm{m}$$

$$\tau = \frac{2\pi ab}{\mathrm{h}} = \frac{2\pi\left(36.8 \times 10^6\, \mathrm{m}\right)\left(21.4 \times 10^6\, \mathrm{m}\right)}{70.4 \times 10^9\, \mathrm{m^2/s}}$$

$$\boxed{\tau = 70.3 \times 10^3\, \mathrm{s} = 19\,\mathrm{h}\ 31\,\mathrm{min}}$$



REFLECT and THINK:

- The satellite takes less than one day to travel over 60,000 km from the earth and back. In this problem, you started with Eq. 12.37, but it is important to remember that this formula was the solution to a differential equation that was derived using Newton's second law.

Kepler's Laws of Planetary Motion

- Results obtained for trajectories of satellites around earth may also be applied to trajectories of planets around the sun.
- Properties of planetary orbits around the sun were determined astronomical observations by Johann Kepler (1571-1630) before Newton had developed his fundamental theory.
  - Each planet describes an ellipse, with the sun located at one of its foci.
  - The radius vector drawn from the sun to a planet sweeps equal areas in equal times.
  - The squares of the periodic times of the planets are proportional to the cubes of the semimajor axes of their orbits.