Bartle - Introduction to Real Analysis - Chapter 8 Solutions

Section 8.1

Problem 8.1-2. *Show that* $\lim(nx/(1+n^2x^2)) = 0$ *for all* $x \in \mathbb{R}$.

Solution: For $x = 0$, we have $\lim(nx/(1+n^2x^2)) = \lim(0/1) = 0$, so $f(0) = 0$. For $x \in \mathbb{R}\backslash\{0\}$, observe that $0 < nx/(1+n^2x^2) < nx/(n^2x^2) = 1/(nx)$. By the Squeeze Theorem, $\lim(nx/(1+n^2x^2)) = 0$. Therefore, $f(x) = 0$ for all $x \in \mathbb{R}$.

Problem 8.1-3. *Evaluate* $\lim(nx/(1+nx))$ *for* $x \in \mathbb{R}$*,* $x \geq 0$.

Solution: For $x = 0$, we have $\lim(nx/(1+nx)) = \lim(0/1) = 0$, so $f(0) = 0$.

For $x \in (0, \infty)$, we have:

$$\lim\left(\frac{nx}{1+nx}\right) = \lim\left(\frac{1}{1/nx+1}\right) = \frac{1}{1/x\lim(1/n)+1} = 1,$$

from which it follows that $f(x) = 1$ for $x \in (0, \infty)$. Therefore,

$$f(x) = \begin{cases} 0 & \text{for} \quad x = 0 \\ 1 & \text{for} \quad x > 0. \end{cases}$$

Problem 8.1-4. *Evaluate* $\lim(x^n/(1+x^n))$ *for* $x \in \mathbb{R}$*,* $x \geq 0$.

Solution: For $0 \leq x < 1$, we have $\lim(x^n/(1+x^n)) = 0/1 = 0$ by Example 3.1.11(b), so $f(x) = 0$. For $x = 1$, we have $\lim(x^n/(1+x^n)) = 1/2$, so $f(1) = 1/2$. For $x > 1$, we have $\lim(x^n/(1+x^n)) = \lim(1/(1+1/x^n) = 1$, so $f(x) = 1$.* Accordingly,

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x < 1 \\ \frac{1}{2} & \text{for} \quad x = 1 \\ 1 & \text{for} \quad x > 1. \end{cases}$$

* Note that for $1/x^n$ with fixed x, given $\epsilon > 0$, if $K(\epsilon) = \log_x(2/\epsilon)$, then for $n \geq K(\epsilon)$, we have $|1/x^n| = 1/x^n < 1/(2/\epsilon) = \epsilon/2 < \epsilon$. Therefore, $\lim(1/x^n) = 0$.

Problem 8.1-9. *Show that* $\lim(x^2e^{-nx}) = 0$ *and that* $\lim(n^2x^2e^{-nx}) = 0$ *for* $x \in \mathbb{R}$*,* $x \geq 0$.

Solution: Part (i): For $x = 0$, we have $\lim(x^2e^{-nx}) = \lim(0 \cdot 1^n) = 0$, so $f(0) = 0$. For $x > 0$, observe that $0 < e^{-x} < 1$. From Example 3.1.11(b), it follows that $\lim(x^2e^{-nx}) = x^2\lim(e^{-x})^n = 0$. As a result, $f(x) = 0$ for $x \geq 0$.

Part (ii): We can establish limit of $(f_n) = (n^2x^2e^{-nx})$ using L'Hopîtal's Rule and the Sequential Criterion for limits of functions. Let $g(m) = m^2x^2e^{-mx} = m^2x^2/e^{mx}$. For $x \in (0, \infty)$, the limit as $m \to \infty$ is in ∞/∞ indeterminate form, so we apply L'Hopîtal's Rule twice:

$$\lim_{n\to\infty}\frac{m^2x^2}{e^{mx}} = \lim_{m\to\infty}\frac{2mx^2}{xe^{mx}} = \lim_{m\to\infty}\frac{2m^2}{m^2e^{mx}} = \lim_{m\to\infty}\frac{2}{e^{mx}} = 0.$$

By the Sequential Criterion for limits of functions (Theorem 4.1.8), the limit of g above implies that for any sequence (y_n) on $(0, \infty)$ that converges to infinity, the sequence $(g(y_n))$ converges to 0. If $y_n = n$ for all $n \in \mathbb{N}$, then $(g(y_n)) = (n^2x^2e^{-nx})$, which is equal to (f_n). It follows that if $x > 0$, then $\lim n^2x^2e^{-nx} = 0$.

For $x = 0$, clearly $\lim n^2x^2e^{-nx} = \lim 0 = 0$. Accordingly, if $x \in [0, \infty)$, then $(n^2x^2e^{-nx})$ converges to $f(x) = 0$.

Problem 8.1-10. *Show that* $\lim(\cos(\pi x)^{2n})$ *exists for all* $x \in \mathbb{R}$*. What is its limit?*

Solution: If $x \in \mathbb{Z}$, then $\cos(\pi x)^{2n} = (\pm 1)^{2n} = 1$, so $\lim(\cos(\pi x)^{2n}) = 1$. Therefore, $f(x) = 1$.

If $x \in \mathbb{R}\backslash\mathbb{Z}$, then $0 \leq \cos^2(\pi x) < 1$, so by Example 3.1.11(b), $\lim[\cos^2(\pi x)]^n = 0$. Therefore:

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in \mathbb{Z} \\ 1 & \text{for} \quad x \in \mathbb{R}\backslash\mathbb{Z}. \end{cases}$$

Problem 8.1-10. *Show that if $a > 0$, then the convergence of the sequence in Exercise 1 is uniform on the interval $[0, a]$, but is not uniform on the interval $[0, \infty)$.*

Solution: Let $a > 0$ and $A = [0, a]$. Because f_n is continuous, it is bounded on A by Theorem 5.3.2. Suppose $f(x) = 0$ for $x \in A$. Then $\|f_n - 0\|_A = \sup\{x/(x+n) : x \in A\} = a/(a+n)$ because f_n is increasing on A. Therefore, $\lim \|f_n - 0\|_A = \lim a/(a+n) = 0$. By Lemma 8.1.8, (f_n) converges uniformly to $f(x) = 0$ on A.

Now let $A = [0, \infty]$. As shown in Theorem 1 below, if (f_n) is uniformly convergent on A, then it must converge uniformly to $f(x) = 0$ because this sequence is pointwise convergent to that function on A. We see that $\|f_n - 0\|_A = \sup\{|x/(x+n)| : x \geq 0\} = 1$. This is because $0 < x/(x+n) < 1$, and for $0 < \delta < 2$, if $x = n(2/\delta - 1)$, then $1 - \delta < x/(x+n) = 1 - \delta/2 < 1$. Therefore, 1 is the supremum of $\{|x/(x+n)| : x \geq 0\}$. Consequently, $\lim \|f_n - 0\|_A = 1$. By Lemma 8.1.8, (f_n) does not uniformly converge to any f on $[0, \infty)$.

Theorem 1. *Suppose (f_n) converges pointwise to f on $A \subseteq \mathbb{R}$. If (f_n) does not uniformly converge to f on A, then (f_n) does not uniformly converge to any function on A.*

Proof. Suppose there is a function $f' : A \to \mathbb{R}$ to which (f_n) converges uniformly on A. Now assume that $f' \neq f$. It follows that (f_n) must converges pointwise to f' on A. However, by Theorem 3.1.4, the limit function f is uniquely determined, so we have a contradiction if $f' \neq f$. Therefore, it must be that $f = f'$. Accordingly, if (f_n) does not converge uniformly to f on A, it does not converge uniformly to any function on A. □

Problem 8.1-12. *Show that if $a > 0$, then the convergence of the sequence in Exercise 2 is uniform on the interval $[a, \infty]$, but is not uniform on the interval $[0, \infty)$.*

Solution: Let $a > 0$ and $A = [a, \infty)$. From Exercise 8.1.2, (f_n) converges pointwise to $f(x) = 0$ on A. Observe that $f_n'(x) = n(1 - n^2x^2)/(1 + n^2x^2)^2$. We see that if $x_0 = 1/\sqrt{n}$, then $f_n'(x_0) = 0$. On either side of this point, $f_n'(x) > 0$ for $x < x_0$ and $f_n'(x) < 0$ for $x > x_0$. By Theorem 6.2.8, the maximum of f_n on $[0, \infty)$ is at x_0. Moreover, it is clear that f_n' decreases on (x_0, ∞). It follows that the maximum of f_n on A is $b = \sup\{1/\sqrt{n}, a\}$. We then have:

$$\left\| \frac{nx}{1 + n^2x^2} - 0 \right\|_A = \sup\left\{ \left| \frac{nx}{1 + n^2x^2} \right| : x \in A \right\} = \frac{bn}{1 + n^2b^2},$$

Since $0 < bn/(1 + n^2b^2) < bn/(n^2b^2) = 1/(nb)$ and $\lim 1/(nb) = 0$, it follows from the Squeeze Theorem that $\lim \|f_n - f\|_A = 0$. By Lemma 8.1.8, (f_n) uniformly converges to $f(x) = 0$ on $[a, \infty)$. (Note that the limit above holds true even though b may be a function of n. For the sake of brevity, I cavalierly omitted this point in applying the Squeeze Theorem.)

Now suppose $A = [0, \infty)$. Let (x_k) be a sequence on A where $x_k = 1/k$ and $n_k = k$. For $\epsilon = 1/4$:

$$|f_{n_k}(x_k) - f(x_k)| = \frac{k(1/k)}{1 + k^2(1/k^2)} = \frac{1}{2} > \epsilon.$$

From Lemma 8.1.5, it follows that (f_n) does not converge uniformly on $[0, \infty)$.

Problem 8.1-13. *Show that if $a > 0$, then the convergence of the sequence in Exercise 3 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.*

Solution: Let $a > 0$ and $A = [a, \infty)$. We know that (f_n) converges pointwise to $f(x) = 1$ on A. Observe that:

$$\|f_n(x) - 1)\|_A = \sup\left\{ \left| \frac{nx}{1 + nx} - 1 \right| = \left| \frac{-1}{1 + nx} \right| = \frac{1}{1 + nx} : x \in A \right\} = \frac{1}{1 + an}. \tag{1}$$

It follows that $\lim \|f_n - f\|_A = 0$. Therefore, (f_n) uniformly converges to $f(x) = 1$ on $[a, \infty)$.

Now let $A = [0, \infty)$. We then have:

$$\|f_n(x) - f(x)\|_A = \sup\left(\left\{\left|\frac{nx}{1+nx} - 1\right| : x \in (0,\infty)\right\} \cup \left\{\left|\frac{n \cdot 0}{1 + n \cdot 0} - 0\right|\right\}\right) = 1. \tag{2}$$

Consequently, $\lim \|f_n - f\|_A = 1$, so by Lemma 8.1.7 (f_n) does not converge uniformly on A.

Problem 8.1-14. *Show that if $0 < b < 1$, then the convergence of the sequence in Exercise 4 is uniform on the interval $[0, b]$, but is not uniform on the interval $[0, 1]$.*

Solution: Let $b \in (0, 1)$ be given and $A = [0, b]$. We then have:

$$\left\|\frac{x_n}{1+x^n} - 0\right\|_A = \sup\left\{\left|\frac{x^n}{1+x^n}\right| : x\ inA\right\} = \frac{b^n}{1+b^n},$$

because f_n is increasing on A (since $f_n' > 0$ on that interval). Clearly $\lim \|x_n/(1+x^n) - 0\|_A = (\lim b^n)/(1+\lim b^n) = 0$ since $n \in (0, 1)$. Therefore, (f_n) converges uniformly to $f(x) = 0$ on $x \in [0, b]$.

Now let $A = [0, 1]$. Let (x_k) be a sequence in A where $x_k = 2^{-1/k}$ and $n_k = k$. It follows that for any $\epsilon > 0$ where $\epsilon < 1/3$:

$$|f_{n_k}(x_k) - f(x_k)| = \frac{(2^{-1/k})^k}{1 + (2^{-1/k})^k} = \frac{1/2}{1 + 1/2} = 1/3 > \epsilon.$$

By Lemma 8.1.5, (f_n) does not uniformly converge on $[0, 1]$.

Problem 8.1-15. *Show that if $a > 0$, then the convergence of the sequence in Exercise 5 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.*

Solution: Suppose $a > 0$ and $A = [a, \infty)$. From Exercise 8, we know that (f_n) converges pointwise to $f(x) = 0$ on A. It is clear that $|\sin(nx)| \leq 1$. Therefore, $0 \leq \sup\{|\sin(nx)/(1 + nx) - 0| : x \in A\} \leq 1/(1 + an) < 1/an$. Since $\lim 1/an = 0$, it follows from the Squeeze Theorem that $\lim \|f_n - f\|_A = 0$. By Lemma 8.1.8, (f_n) converges uniformly to $f(x) = 0$.

Now let $A = [0, \infty)$. Let (x_k) be a sequence on A where $x_k = \pi/(2k)$ and $n_k = k$. For any positive ϵ where $\epsilon < 1/(1 + \pi/2)$:

$$\left|\frac{\sin(k(\pi/(2k)))}{1 + k(\pi/(2k))} - 0\right| = \left|\frac{\sin \pi/2}{1 + \pi/2}\right| = \frac{1}{1 + \pi/2} > \epsilon.$$

It follows that (f_n) does not uniformly converge on $[0, \infty)$.

Problem 8.1-19. *Show that the sequence $(x^2 e^{-nx})$ converges uniformly on $[0, \infty)$.*

Solution: Let $A = [0, \infty)$. From Exercise 8.1.9, $(f_n) = (x^2 e^{-nx})$ converges pointwise to $f(x) = 0$. Note that $f_n'(x) = e^{-nx}(2x - nx^2)$. The roots of f_n' are at 0 and $2/n$. A simple calculation shows that $f_n(2/n) = 4/(en)^2 > f_n(0) = 0$. In addition, $f_n'(x) > 0$ for $0 < x < 4/n$ and $f_n'(x) < 0$ for $x > 4/n$. By Theorem 6.2.8, f_n is at an absolute maximum at $x = 4/n$.

As a result, $0 < \|f_n - f\|_A = f_n(2/n) = 4/(en)^2 < 4/n$. Because $\lim(4/n) = 0$, it follows from the Squeeze Theorem that $\lim \|f_n - f\|_A = 0$. Therefore, (f_n) converges uniformly to $f(x) = 0$ on A.

Problem 8.1-21. *Show that if (f_n), (g_n) converge uniformly on the set A to f, g, respectively, then $(f_n + g_n)$ converges uniformly on A to $f + g$.*

Solution: For $\epsilon > 0$, there are $K'(\epsilon/2)$, $K''(\epsilon/2) > 0$ such that if $n \geq K'(\epsilon/2)$, then $\|f_n - f\|_A < \epsilon/2$, and if $n \geq K''(\epsilon/2)$, then $\|g_n - g\|_A < \epsilon/2$. Let $K(\epsilon) = \sup\{K'(\epsilon/2), K''(\epsilon/2)\}$. By the Triangle Inequality and Theorem 2 below, for $n \geq K(\epsilon)$:

$$|\|(f_n+g_n)-(f+g)\|_A - 0| \le \|f_n - f\|_A + \|g_n - g\|_A < \epsilon/2 + \epsilon/2 = \epsilon.$$

It follows that $\lim \|(f_n+g_n)-(f+g)\|_A = 0$, so (f_n+g_n) converges uniformly on A to $f+g$.

Theorem 2. *Given any ϕ, $\psi : A \to \mathbb{R}$ on $A \subseteq \mathbb{R}$, $\|\phi+\psi\|_A \le \|\phi\|_A + \|\psi\|_A$.*

Proof. Let $s = \|\phi+\psi\|_A$, in which case $s = \sup\{|\phi(x)+\psi(x)| : x \in A\}$. Now let $t_1 = \sup\{|\psi(x)| : x \in A\}$ and $t_2 = \sup\{|\psi(x)| : x \in A\}$. If $x \in A$, then $t_1 + t_2 \ge |\phi(x)| + |\psi(x)| \ge |\phi(x)+\psi(x)|$. It follows that t_1+t_2 is an upper bound of $\{|\phi(x)+\psi(x)| : x \in A\}$ and therefore is greater than or equal to s. Since $t_1 + t_2 = \|\phi\|_A + \|\psi\|_A$, we have shown the desired inequality. □

Problem 8.1-22. *Show that if $f_n(x) := x + 1/n$ and $f(x) := x$, then (f_n) converges uniformly on $\mathbb{R}$ to f, but the sequence (f_n^2) does not converge uniformly on $\mathbb{R}$. (Thus the product of uniformly convergent sequences of functions may not coverge uniformly.)*

Solution: Note that in contrast to the functions in Exercise 8.1-23, f_n is not bounded on $\mathbb{R}$.

For (f_n), we have $\lim \|x + 1/n - x\|_A = \lim \|1/n\|_A = \lim 1/n = 0$. Therefore, (f_n) converges uniformly to f on $\mathbb{R}$.

Note that $\lim f_n^2 = \lim(x^2 + 2x/n + 1/n^2) = x^2 + \lim(2x/n + 1/n^2) = x^2$. Accordingly, (f_n^2) converges pointwise to f^2 on $\mathbb{R}$.

We will now show that (f_n^2) does not uniformly converge on $\mathbb{R}$. Let (x_k) be a sequence on $\mathbb{R}$ such that $x_k = k$ and $n_k = k$. We then have for all $k \in \mathbb{N}$:

$$\left|f_n^2 - f^2\right| = \left|\frac{2x_k}{k} + \frac{1}{k^2}\right| = \left|2 - \frac{1}{k^2}\right| \ge 1.$$

For positive ϵ where $\epsilon \le 1$, we have $\left|f_n^2 - f^2\right| \ge \epsilon$. By Lemma 8.1.5, (f_n^2) does not uniformly converge on $\mathbb{R}$.

Problem 8.1-23. *Let (f_n), (g_n) be sequences of bounded functions on A that converge uniformly on A to f, g, respectively. Show that $(f_n g_n)$ converges uniformly on A to fg.*

Solution: In order to show the uniform convergence of $(f_n g_n)$ on A, we can rewrite the limit we seek under Lemma 8.1.8 as:

$$\lim \|f_n g_n - fg\|_A = \lim \|(f_n g_n - f g_n) - (fg - f g_n)\|_A = \lim \|g_n(f_n - f) + f(g_n - g)\|_A.$$

Since f_n is bounded on A, it must be that f is bounded on A by some $M_1 \ge 0$. We can prove this by contradiction. If we assume this were not true, then for some $x \in A$, it would follow that $|f_n(x) - f(x)|$ is unbounded, resulting in the contradiction that the supremum of $\{|f_n(x) - f(x)| : x \in A\}$ does not exist.

It must also be that (g_n) is bounded by some $M_2 \ge 0$ for all $x \in A$. This follows from the fact that $\lim g_n$ must exist because (g_n) must pointwise converge to g on A. Theorem 3.2.2 then requires that, (g_n) be bounded.

We may now establish boundaries on $\|f_n g_n - fg\|_A = \sup\{|f_n g_n - fg| : x \in A\}$. We have from the Triangle Inequality:

$$|f_n g_n - fg| = |g_n(f_n - f) + f(g_n - g)| \le M_2 |f_n - f| + M_1 |g_n - g|.$$

It follows that:

$$0 \le \sup\{|f_n g_n - fg| : x \in A\} < M_2 \|f_n - f\|_A + M_1 \|g_n - g\|_A.$$

By hypothesis:

$$\lim (M_2 \|f_n - f\|_A + M_1 \|g_n - g\|_A) = M_2 \lim \|f_n - f\|_A + M_1 \lim \|g_n - g\|_A = M_2 \cdot 0 + M_1 \cdot 0 = 0.$$

By the Squeeze Theorem, $\lim \|f_n g_n - fg\|_A = 0$. Therefore, $(f_n g_n)$ converges uniformly to fg on A.

Problem 8.1-24. *Let (f_n) be a sequence of functions that converges uniformly to f on A and that satisfies $|f_n(x)| \le M$ for all $n \in \mathbb{N}$ and for all $x \in A$. If g is continuous on the interval $[-M, M]$, show that the sequence $(g \circ f_n)$ converges uniformly to $(g \circ f)$ on A.*

Solution: Let $\epsilon > 0$ be given. Since g is continuous on $[-M, M]$, there is a $\delta > 0$ such that if $y \in [-M, M]$ and $|y - c| < \delta$, then $|g(y) - g(c)| < \epsilon/2$. We will let f_n and f take the place of y and c to establish our result. Because (f_n) uniformly converges to f on A, there is a $K(\delta) > 0$ such that if $n \geq K(\delta)$, then:

$$0 \leq \|f_n - f\|_A = \sup\{|f_n(x) - f(x)| : x \in A\} < \delta.$$

Accordingly, if $n \geq K(\delta)$ and $x \in A$, then $|f_n(x) - f(x)| < \delta$. It follows from the continuity of g that:

$$|(g \circ f_n)(x) - (g \circ f)(x)| = |g(f_n(x)) - g(f(x)| < \epsilon/2.$$

Because this is true for all $x \in A$, we have established $\epsilon/2$ as an upper bound of $\|g \circ f_n - g \circ f\|_A$ for $n \geq K(\delta)$. Consequently, $|\|g \circ f_n - g \circ f\|_A - 0| \leq \epsilon/2 < \epsilon$ for $n \geq K(\delta)$, from which it follows that $\lim \|g \circ f_n - g \circ f\|_A = 0$. We conclude that $(g \circ f_n)$ uniformly converges to $g \circ f$ on A.

Section 8.2

Problem 8.2-2. *Prove that the sequence in Example 8.2.1(c) is an example of a sequence of continuous functions that converges nonuniformly to a continuous limit.*

Solution: If $x = 0$, then $\lim f_n(x) = \lim 0 = 0$. If $x \in [0, 2]$, then let $\epsilon > 0$ and $K(\epsilon, x) = 2/x$. For $n = K(\epsilon, x)$, we have $|f_n(x) - 0| = \left|n^2(x - 2/n)\right| = 0 < \epsilon$. If $n > K(\epsilon, x)$, then $f_n(x) = 0$, from which follows that $|f_n(x) - 0| < \epsilon$. We then infer that $\lim f_n(x) = 0$.

We will now prove that (f_n) is does not converge uniformly on $[0, 2]$. Let (x_k) be a sequence on $[0, 2]$ where $x_k = 1/k$ and let $n_k = k$, in each case for all $k \in \mathbb{N}$. For a given $0 < \epsilon < 1$, we have $|f_{n_k}(x_k) - 0| = k^2(1/k) = k \geq 1 > \epsilon$. By Lemma 8.1.5, (f_n) does not converge uniformly on $[0, 2]$.

Problem 8.2-4. *Suppose (f_n) is a sequence of continuous functions on an interval I that converges uniformly on I to a function f. If $(x_n) \subseteq I$ converges to $x_o \in I$, show that $\lim(f_n(x_n)) = \lim f(x_0)$.*

Solution: By Theorem 8.2.2, f is continuous on I, so $\lim_{x \to x_0} f(x) = f(x_0)$. Applying the Sequential Criterion (Theorem 5.1.3), since (x_n) converges to x_0, it follows that $\lim f(x_n) = f(x_0)$. For a given $\epsilon > 0$, there is a $K'(\epsilon/2) \in \mathbb{N}$ such that if $n \geq K'(\epsilon/2)$, then:

$$|f(x_n) - f(x_0)| < \frac{\epsilon}{2}.$$

Because (f_n) converges uniformly on I, there is also a $K''(\epsilon/2) \in \mathbb{N}$ such that if $n \geq K''(\epsilon/2)$, then for $x = x_n$ for any $n \in \mathbb{N}$:

$$|f_n(x_n) - f(x_n)| < \frac{\epsilon}{2}.$$

Let $K(\epsilon) = \sup\{K'(\epsilon/2), K''(\epsilon/2)\}$, We then have for $n \geq K(\epsilon)$:

$$|f_n(x_n) - f(x_0)| = |(f_n(x_n) - f(x_n)) + (f(x_n) - f(x_0)|$$

$$\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, $\lim f_n(x_n) = f(x_0)$.

Problem 8.2-5. *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$ and let $f_n(x) := f(x + 1/n)$ for $x \in \mathbb{R}$. Show that (f_n) converges uniformly on $\mathbb{R}$ to f.*

Solution: Because f is uniformly continuous on $\mathbb{R}$, for any given $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that for any $x, y \in \mathbb{R}$, if $|x - u| < \delta(\epsilon)$, then $|f(x) - f(u)| < \epsilon$. Let $K(\epsilon) = 2/\delta(\epsilon)$. If $n \geq K(\epsilon)$, then $|(x + 1/n) - x| \leq \delta(\epsilon)/2 < \delta(\epsilon)$, in which case $|f_n(x) - f(x)| = |f(x + 1/n) - f(x)| < \epsilon$. Therefore:

$$\|f_n - f\|_{\mathbb{R}} = \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} \le \epsilon.$$

Since ϵ is arbitrary, $\lim \|f_n - f\|_{\mathbb{R}} = 0$, and (f_n) uniformly converges to f on $\mathbb{R}$.

Problem 8.2-7. *Suppose the sequence (f_n) converges uniformly to f on the set A, and suppose that each f_n is bounded on A. (That is, for each n there is a constant M_n such that $|f_n(x)| \le M_n$ for all $x \in A$.) Show that the function f is bounded on A.*

Solution: For any $\epsilon > 0$, there is a $K \in \mathbb{N}$ such that if $n \ge K$, then $\sup\{|f_n(x) - f(x)| : x \in A\} < \epsilon$. Consequently, ϵ is an upper bound on this set, so $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Since ϵ is arbitrary, $f_n(x) = f(x)$ for $n \ge K$. Because f_K is bounded on A, it follows that $|f(x)| \le M_K$ for all $x\ inA$. The function f is therefore bounded on A.

Problem 8.2-10. *Let $g_n(x) := e^{-nx}/n$ for $x \ge 0$, $n \in \mathbb{N}$. Examine the relationship between $\lim(g_n)$ and $\lim(g_n')$.*

Solution: Observe that $0 < e^{-nx} \le 1$ for all $x \in [0, \infty)$, so $0 < e^{-nx}/n \le 1/n$. Therefore, $\lim e^{-nx}/n = 0$ for $x \in [0, \infty)$. Because g_n is bounded above by $1/n$, it follows from the Squeeze Theorem that $\lim \|e^{-nx}/n - 0\|_{\mathbb{R}_{\ge 0}} = 0$. Therefore, (g_n) converges uniformly to $g(x) = 0$ on $[0, \infty)$.

We then have $g_n'(x) = -e^{-nx}$. If $x = 0$, then $\lim[-e^{-nx}] = -1$. If $x \in (0, \infty)$, then because $0 < e^{-x} < 1$, it follows from the Squeeze Theorem that $\lim[-e^{-nx}] = -\lim(e^{-x})^n) = 0$. Therefore, (g_n') converges to $g'(0) = -1$ and $g'(x) = 0$ for $x \in (0, \infty)$. Note that g' is discontinuous at $x = 0$.

Now let ϵ be given where $0 < \epsilon < 1/2$. Suppose (x_k) is a sequence on $[0, \infty)$ where $x_k = -\ln(2\epsilon)/k$ (note that the allowed range of ϵ ensures $x_k > 0$ for all $k \in \mathbb{N}$) and $n_k = k$. Then $\left|-e^{-kx_k}\right| = e^{\ln 2\epsilon} = 2\epsilon > \epsilon$ for all $k \in \mathbb{N}$. By Lemma 8.1.5, (g_n') does not uniformly converge on $[0, \infty)$.

Problem 8.2-11. *Let $I := [a, b]$ and let (f_n) be a sequence of functions on $I \to \mathbb{R}$ that converges on I to f. Suppose that each derivative f_n' is continuous on I and that the sequence (f_n') is uniformly convergent to g on I. Prove that $f(x) - f(a) = \int_a^b g(t)dt$ and that $f'(x) = g(x)$ for all $x \in I$.*

Solution: Because (f_n) converges to f on the bounded interval I and (f_n') exists for $n \in \mathbb{N}$ and converges uniformly to g, it follows from Theorem 8.2.3 that (f_n) converges uniformly to some function. This function must be f because the limit of (f_n) is unique. It further follows from Theorem 8.2.3 that $f'(x) = g(x)$ for all $x \in I$.

Now let $x \in [a, b]$. Given that (f_n') converges uniformly on I, it must converge uniformly on $[a, x]$ (since this result has not yet been proven, see Theorem 3 below). Because each f_n' is continuous on I, by the Lebesque Cirterion $f_n \in \mathcal{R}[a, x]$. Applying Theorem 8.2.4 and the fact that (f_n') converges to g by hypothesis, we have:

$$\int_a^x g = \lim \int_a^x f_n' = \int_a^x f',$$

and $g \in \mathcal{R}[a, x]$.

Since f' exists on all of I, f' is continuous on I. Applying the Fundamental Theorem of Calculus, we get:

$$\int_a^x g = \int_a^x f' = f(x) - f(a).$$

Theorem 3. *Suppose (f_n) converges uniformly to f on $[a, b]$. If $\gamma \in [a, b]$, then (f_n) also converges uniformly to f on $[a, \gamma]$.*

Proof. By hypothesis, $\lim \|f_n - f\|_{[a,b]} = 0$. Observe that $0 \le \|f_n - f\|_{[a,\gamma]} \le \|f_n - f\|_{[a,b]}$ for all $n \in \mathbb{N}$. By the Squeeze Theorem, $\lim \|f_n - f\|_{[a,\gamma]} = 0$. The sequence (f_n) therefore uniformly converges to f on $[a, \gamma]$. □

Problem 8.2-15. *Let $g_n(x) := nx(1 - x)^n$ for $x \in [0, 1]$, $n \in \mathbb{N}$. Discuss the convergence of (g_n) and $(_i nt_0^1 g_n dx$.*

Solution: Observe that $g_n(0) = g_n(1) = 0$ for all $n \in \mathbb{N}$. Now let $x \in (0, 1)$. There is a $y > 0$ such that $1 - x = 1/(1 + y)$. By the Binomial Theorem:

$$(1+y)^n = \binom{n}{0}1 + \binom{n}{1}y + \binom{n}{2}y^2 + \cdots + \binom{n}{n}y^n > \frac{1}{2}n(n-1)y^2.$$

It follows that for $n \geq 2$:

$$0 < nx(1-x)^n = \frac{nx}{(1+y)^n} < \frac{nx}{(1/2)n(n-1)y^2} = \frac{2x}{y^2(n-1)}.$$

By the Squeeze Theorem, the 2-tail of (g_n) converges to zero. By Theorem 3.1.9, $\lim(g_n - 0) = 0$, so (g_n) converges to $g(x) = 0$ on $x \in [0,1]$.

We see that $g_n'(x) = n(1-x)^{n-1}[1-(n+1)x]$. Setting $g_n(x_0) = 0$, we see that g_n is at an absolute maximum at $x_0 = 1/(n+1)$. We then have $g_n(x_0) = (n/(n+1))^{n+1} < 1$. Therefore, $0 \leq g_n(x) < 1$ for all $x \in [0,1]$, from which it follows that $\|g_n\|_{[0,1]} < 1$ for all $n \in \mathbb{N}$. Since g_n is continuous on $[0,1]$, each $g_n \in \mathcal{R}[0,1]$; further, $g \in \mathcal{R}[0,1]$. Applying Theorem 8.2.5, we infer that $\int_0^1 g = 0 = \lim \int_0^1 g_n$.

Note, however, that (g_n) does not uniformly converge on $[0,1]$ (hence the power of Theorem 8.2.5). Suppose $0 < \epsilon < 1/e$. Let (x_k) be a sequence on $[0,1]$ where $x_k = 1/k$ and $n_k = k$ for all $k \in \mathbb{N}$. We then have:

$$|g_{n_k}(x_k)| = n_k x_k (1-x_k)^n_k = \left(1 - \frac{1}{k}\right)^k = \frac{1}{e} > \epsilon,$$

where we have used the result from Exercise 3.3.12(d). By Lemma 8.1.5, (g_n) does not uniformly converge on $[0,1]$.

Problem 8.2-17. *Let $f_n(x) := 1$ for $x \in (0, 1/n)$ and $f_n(x) := 0$ elsewhere on $[0,1]$. Show that (f_n) is a decreasing sequence of discontinuous functions that converge to a continuous limit function, but the convergence is not uniform on $[0,1]$*

Solution: Clearly f_n is discontinuous at $x = 0$ and $x = 1/n$ for all $n \in \mathbb{N}$.

Let $x \in [0,1]$ and $n \in \mathbb{N}$. Note that $1/(n+1) < 1/n$. If $x \in (0, 1/(n+1))$, then $f_n(x) = f_{n+1}(x) = 1$. If $x \in [0,1] \backslash (0, 1/n)$, then $f_n(x) = f_{n+1}(x) = 0$. If $x \in [1/(n+1), 1/n)$, then $f_n(x) = 1 > f_{n+1}(x) = 0$. Therefore, (f_n) is a decreasing sequence of discontinuous functions.

Let $x \in [0,1]$ and $\epsilon > 0$ be given. Suppose $K(\epsilon) = 2/x$. If $n \geq K(\epsilon)$, then $1/n = x/2 < x$, so $f_n(x) = 0$. Therefore, $|f_n(x) - 0| = 0 < \epsilon$ for all $x \in [0,1]$. It follows that (f_n) converges pointwise to $f(x) = 0$ on $[0,1]$.

The sequence does not, however, converge uniformly on this interval. Suppose (x_k) is a sequence on $[0,1]$ where $x_k = 1/(2k)$ and $n_k = k$ for $k \in \mathbb{N}$. Let $\epsilon = 1/2$. Then $|f_{n_k}(x_k) - 0| = f_k(1/(2k) = 1 > \epsilon$ because $0 < 1/(2k) < 1/k$ for all $k \in \mathbb{N}$. By Lemma 8.1.5, (f_n) does not converge uniformly on $[0,1]$.

Problem 8.2-18. *Let $f_n(x) := x^n$ for $x \in [0,1]$, $n \in \mathbb{N}$. Show that (f_n) is a decreasing sequence of continuous functions that converges to a function that is not continuous, but the convergence is not uniform on $[0,1]$.*

Solution: Clearly all f_n are continuous on $[0,1]$. If $x \in [0,1]$, then $f_{n+1}(x) = x^{n+1} = x \cdot x^n = x f_n(x) \leq f_n(x)$. Accordingly, (f_n) is a decreasing sequence of continuous functions.

If $x \in [0,1)$, then $\lim f_n(x) = \lim x^n = 0$ by Example 3.1.11(b). If $x = 1$, then $\lim f_n(x) = \lim 1^n = 1$. Therefore, (f_n) converges on $[0,1]$ to:

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in [0,1) \\ 1 & \text{for} \quad x = 1. \end{cases}$$

Obviously f is discontinuous at $x = 1$.

We will show that $\|f_n - f\|_{[0,1]} = 1$ in all cases. Let $n \in \mathbb{N}$ be arbitrary. Now observe that f_n is bounded above by 1 on $[0,1]$. Clearly $|f_n(1) - f(1)| = 0$, so for the uniform norm to be greater than zero, we must look on $[0,1)$. Let $\epsilon \in (0,1)$. If we solve $1 - \epsilon < x^n$, then $x > (1-\epsilon)^{1/n}$. Since $(1-\epsilon)^{1/n} < 1$, we can choose $x_0 \in ((1-\epsilon)^{1/n}, 1)$. We then have $1 - \epsilon < f_n(x_0)$. It follows that $1 = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$. Therefore, $\lim \|f_n - f\|_{[0,1]} = 1$. We conclude that (f_n) does not uniformly converge on $[0,1]$.

Problem 8.2-19. *Let $f_n(x) := x/n$ for $x \in [0, \infty)$, $n \in \mathbb{N}$. Show that (f_n) is a decreasing sequence of continuous functions that converges to a continuous limit function, but the convergence is not uniform on $[0, \infty)$.*

Solution: Each f_n is obviously continuous on the interval of interest. Let $n \in \mathbb{N}$ and $x \in [0, \infty)$. Because $f_n(x) = x/n \geq x/(n+1) = f_{n+1}(x)$, the sequence (f_n) is a decreasing sequence of continuous functions.

Now let $\epsilon > 0$ be given. Let $K(\epsilon) = 2x/\epsilon$. If $n \geq K(\epsilon)$, then $|f_n(x) - 0| = x/n \leq \epsilon/2 < \epsilon$. Therefore, $\lim f_n(x) = 0$ for all $x \in [0, \infty)$. The sequence (f_n) thus converges pointwise to $f(x) = 0$ on $[0, \infty)$.

Each f_n is obviously unbounded on $[0, \infty)$; hence, for all $n \in \mathbb{N}$, the uniform norm $\|f_n - f\|_{[0,\infty)} = \infty$ and is therefore divergent. Consequently, (f_n) does not uniformly converge on $[0, \infty)$.

Problem 8.2-20. *Give an example of a decreasing sequence* (f_n) *of continuous functions on* $[0, 1)$ *that converges to a continuous limit function, but the convergence is not uniform on* $[0, 1)$

Solution: The sequence where:

$$f_n(x) = \frac{1}{n(1-x)},$$

for all $n \in \mathbb{N}$ fits the bill. Notice that f_n extends to infinity as $x \to 1$.

Clearly, each f_n is continuous on $[0, 1)$. In addition, for any $n \in \mathbb{N}$ and $x \in [0, 1)$, we have $f_n(x) = 1/[n(1-x)] > 1/[(n+1)(1-x)] = f_{n+1}(x)$. Therefore, (f_n) is a decreasing sequence of continuous functions. Moreover, $\lim 1/[n(1-x)] = 1/(1-x) \lim 1/n = 0$. Accordingly, (f_n) converges pointwise to $f(x) = 0$ on $[0, 1)$, which is continuous on that interval.

But the sequence does not uniformly converge on $[0, 1)$. As with the sequence in exercise 8.2-19, each f_n is unbounded. Assume for any $n \in \mathbb{N}$ there is an $M_n > 0$ such that $f_n(x) \leq M_n$. Let $x_0 = 1 - 1/(2M_n n) \in [0, 1)$. It follows that $f_n(x_0) = 2M_n > M_n$. Therefore, f_n is unbounded on $[0, 1)$. As a result, $\|f_n - f\|_{[0,1)} = \infty$, from which it follows that the limit of the uniform norm is divergent. The sequence therefore does not uniformly converge on $[0, 1)$. This result does not conflict with Dini's Theorem, however, because the interval of interest here is half-open, whereas Dini's Theorem requires a closed, bounded interval.

Section 8.3

Problem 8.3-5. *If* $x \geq 0$ *and* $n \in \mathbb{N}$, *show that:*

$$\frac{1}{x+1} = 1 - x + x^2 - x^3 + \cdots + (-x)^{n-1} + \frac{(-x)^n}{1+x}.$$

Use this to show that:

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^n}{n} + \int_0^x \frac{(-t)^n}{1+t} dt$$

and that:

$$\left| \ln(x+1) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^n}{n} \right) \right| \leq \frac{x^{n+1}}{n+1}.$$

Solution: The first part can be easily shown by multiplying the right-hand side of the expression by $(1+x)/(1+x)$.

Now let $f(x) = \ln(1+x)$, so $f'(x) = 1/(1+x)$ on $(-1, \infty)$. Clearly, f is continuous on $[0, \infty)$. If $x \geq 0$, then f' is continuous on $[0, x]$, from which it follows that $f' \in \mathcal{R}[0, x]$. By the Fundamental Theorem of Calculus:

$$f(x) - f(0) = \ln(1+x) = \int_0^x f'(x) dx$$

$$= \int_0^x \left[1 - x + x^2 - x^3 + \cdots + (-x)^{n-1}\right] dx + \int_0^x \frac{(-t)^n}{1+t} dt.$$

$$= \sum_{k=0}^{n-1} (-1)^k \frac{x^{k+1}}{k+1} + \int_0^x \frac{(-t)^n}{1+t} dt,$$

which is what we set out to prove for the second part.

For the third part, rearrange the previous expression and take the absolute value to get:

$$\left|\ln(1+x) - \sum_{k=0}^{n-1}(-1)^k \frac{x^{k+1}}{k+1}\right| = \left|\int_0^x \frac{(-t)^n}{1+t}dt\right| = \int_0^x \frac{|t|^n}{1+t}dt.$$

Observe that $|t|^n/(1+t) \leq |t|^n$ for $t \geq 0$. Applying Theorem 7.1.5(c), we have:

$$\left|\ln(1+x) - \sum_{k=0}^{n-1}(-1)^k \frac{x^{k+1}}{k+1}\right| \leq \int_0^x |t|^n \, dt = \frac{x^{n+1}}{n+1}.$$

Problem 8.3-8. *Let $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Show that there exists $K \in \mathbb{R}$ such that $f(x) = Ke^x$ for all $x \in \mathbb{R}$.*

Solution: If $f(0) \neq 0$, let $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x)/f(0)$. It follows that $g'(x) = f'(x)/f(0) = g(x)$ and $g(0) = 1$. By Theorem 8.3.4, $g(x) = e^x$ because the function with both these properties is unique. Therefore $f(x) = f(0)g(x) = Ke^x$ where $K = f(0)$.

Now suppose $f(0) = 0$. It can be easily shown by induction that $f^{(n)} = f$ for all $n \in \mathbb{N}$. Therefore, $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Let $x \in \mathbb{R}$ be given. Applying Taylor's Theorem at $x_0 = 0$, we have for any n:

$$f(x) = f(0) + \sum_{j=1}^{n} \frac{f^{(j)}(0)}{j!}x^j + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1},$$

for some c between 0 and x_0 (non-inclusive).

Now let $L > 0$ be such that $|f(x)| \leq L$ for all $x \in [0, x]$. It follows that $|f(x)| \leq K|x|^{n+1}/(n+1)!$. Since $\lim |x|^{n+1}/(n+1)! = 0$ by Exercise 3.2-19(c), we infer that $\lim |f(x)| = 0$. Since x is arbitrary, we conclude that $f(x) = 0$ for all $x \in \mathbb{R}$. Therefore, $f(x) = 0 \cdot e^x = 0$.