Baker-Campbell-Hausdorff Theorem
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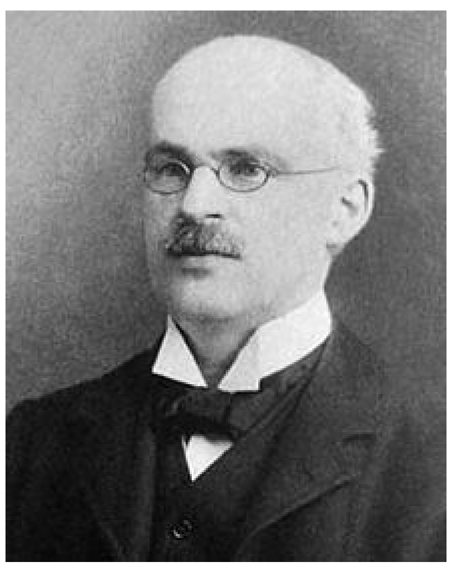
Henry Frederick Baker FRS (3 July 1866 – 17 March 1956) was a British mathematician, working mainly in algebraic geometry, but also remembered for contributions to partial differential equations (related to what would become known as solitons), and Lie groups. (http://en.wikipedia.org/wiki/H._F._Baker)

John Edward Campbell (27 May 1862, Lisburn, Ireland – 1 October 1924, Oxford, Oxfordshire, England) was a mathematician, best known for his contribution to the Baker-Campbell-Hausdorff formula.
(http://en.wikipedia.org/wiki/John_Edward_Campbell)



Felix Hausdorff (November 8, 1868 – January 26, 1942) was a German mathematician who is considered to be one of the founders of modern topology and who contributed

significantly to set theory, descriptive set theory, measure theory, function theory and functional analysis
(http://en.wikipedia.org/wiki/Felix_Hausdorff)

Here we collect the commutation relations (in particular, Baker-Campbell-Hausdorff theorem) which will be useful for the discussion on the creation and annihilation operators of simple harmonics, coherent states, squeezed state, and squeezed coherent states.

1. Creation and annihilation operators for the simple harmonics

$\hat{a}^{+}$: creation operator
$\hat{a}$: annihilation operator
$\hat{n} = \hat{a}^{+}\hat{a}$, number operator

$$\hat{a} | n >= \sqrt{n} | n-1 >,$$

$$\hat{a}^{+} | n >= \sqrt{n+1} | n+1 >,$$

$$\hat{n} | n >= n | n >,$$

$$| n >= \frac{1}{\sqrt{n!}} (\hat{a}^{+})^{n} | 0 >,$$

$$[\hat{n}, \hat{a}] = -\hat{a},$$

$$[\hat{n}, \hat{a}^{+}] = \hat{a}^{+}.$$

2. Commutation relations (I)

Without any proof we use the following relations,

$$[\hat{a}, f(\hat{a}^+)] = \frac{\partial}{\partial \hat{a}^+} f\left(\hat{a}^+\right), \qquad [\hat{a}^+, f(\hat{a})] = -\frac{\partial}{\partial \hat{a}} f\left(\hat{a}\right).$$

where *f* is an arbitrary function.

((**Example**))

$$[\hat{a}, \hat{a}^+] = \hat{1},$$

$$[\hat{a}, (\hat{a}^+)^2] = 2\hat{a}^+,$$

$$[\hat{a}, (\hat{a}^+)^3] = 3(\hat{a}^+)^2,$$

$$[\hat{a}, (\hat{a}^+)^n] = n(\hat{a}^+)^{n-1},$$

and

$$[\hat{a}^+, (\hat{a})^2] = -2\hat{a},$$

$$[\hat{a}^+, (\hat{a})^3] = -3\left(\hat{a}\right)^2,$$

$$[\hat{a}^+, (\hat{a})^n] = -n\left(\hat{a}\right)^{n-1}.$$

3. Commutation relations (II)

In general, we have the formula for the commutation relations.

$$[\hat{a}^+, f(\hat{a}, \hat{a}^+)] = -\frac{\partial}{\partial \hat{a}} f(\hat{a}, \hat{a}^+),$$

$$[\hat{a}, f(\hat{a}, \hat{a}^+)] = \frac{\partial}{\partial \hat{a}^+} f(\hat{a}, \hat{a}^+).$$

((**Proof**)) **Walls and Milburn**

We assume that

$$f(\hat{a}, \hat{a}^+) = \sum_{r,s} f_{r,s}^{(a)} \hat{a}^r (\hat{a}^+)^s$$

$$
\begin{aligned}
[\hat{a}, f(\hat{a},\hat{a}^+)] &= \sum_{r,s}[\hat{a}, f_{r,s}^{(a)}\hat{a}^r(\hat{a}^+)^s] \\
&= \sum_{r,s} f_{r,s}^{(a)}[\hat{a}, \hat{a}^r(\hat{a}^+)^s] \\
&= \sum_{r,s} f_{r,s}^{(a)}\{[\hat{a},\hat{a}^r](\hat{a}^+)^s + \hat{a}^r[\hat{a},(\hat{a}^+)^s] \\
&= \sum_{r,s} f_{r,s}^{(a)}\hat{a}^r[\hat{a},(\hat{a}^+)^s] \\
&= \sum_{r,s} f_{r,s}^{(a)}\hat{a}^r s(\hat{a}^+)^{s-1} \\
&= \frac{\partial}{\partial \hat{a}^+} f(\hat{a},\hat{a}^+)
\end{aligned}
$$

and

$$
\begin{aligned}
[\hat{a}^+, f(\hat{a},\hat{a}^+)] &= \sum_{r,s}[\hat{a}^+, f_{r,s}^{(a)}\hat{a}^r(\hat{a}^+)^s] \\
&= \sum_{r,s} f_{r,s}^{(a)}[\hat{a}^+, \hat{a}^r(\hat{a}^+)^s] \\
&= \sum_{r,s} f_{r,s}^{(a)}\{[\hat{a}^+,\hat{a}^r](\hat{a}^+)^s + \hat{a}^r[\hat{a}^+,(\hat{a}^+)^s] \\
&= \sum_{r,s} f_{r,s}^{(a)}[\hat{a}^+,\hat{a}^r](\hat{a}^+)^s \\
&= -\sum_{r,s} f_{r,s}^{(a)} r\hat{a}^{r-1}(\hat{a}^+)^s \\
&= -\frac{\partial}{\partial \hat{a}} f(\hat{a},\hat{a}^+)
\end{aligned}
$$

Here we use the following formula for the commutators,

$$
[\hat{A},\hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}],
$$

where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are non-commuting operators.

((**Proof**))

$$[\hat{A},\hat{B}\hat{C}]=\hat{A}\hat{B}\hat{C}-\hat{B}\hat{C}\hat{A}$$
$$=\hat{A}\hat{B}\hat{C}-\hat{B}\hat{A}\hat{C}+\hat{B}\hat{A}\hat{C}-\hat{B}\hat{C}\hat{A}$$
$$=[\hat{A},\hat{B}]\hat{C}+\hat{B}[\hat{A},\hat{C}]$$

4. Commutation relations (III)

$$[(\hat{a})^2,(\hat{a}^+)^2]=2(\hat{a}^+\hat{a}+\hat{a}\hat{a}^+)=2(2\hat{a}^+\hat{a}+\hat{1}),$$

((**Proof**)) We use the formula; $[\hat{A},\hat{B}\hat{C}]=[\hat{A},\hat{B}]\hat{C}+\hat{B}[\hat{A},\hat{C}]$

$$[(\hat{a})^2,(\hat{a}^+)^2]=[(\hat{a})^2,\hat{a}^+]\hat{a}^+ +\hat{a}^+[(\hat{a})^2,\hat{a}^+]$$
$$=-[\hat{a}^+,(\hat{a})^2]\hat{a}^+ -\hat{a}^+[\hat{a}^+,(\hat{a})^2]$$
$$=2\hat{a}\hat{a}^+ +2\hat{a}^+\hat{a}$$

$$[(\hat{a}^+)^2,(\hat{a})^2]=-2(\hat{a}^+\hat{a}+\hat{a}\hat{a}^+)=-2(2\hat{a}^+\hat{a}+\hat{1}),$$

$$[\hat{a}^2,(\hat{a}^+)^3]=2\left(\hat{a}^+\right)^2\hat{a}+2\hat{a}\left(\hat{a}^+\right)^2+2\hat{a}^+\hat{a}\hat{a}^+.$$

((**Proof**))

$$[\hat{a}^2,(\hat{a}^+)^3]=[\hat{a}^2,(\hat{a}^+)^2\hat{a}^+]$$
$$=[\hat{a}^2,(\hat{a}^+)^2]\hat{a}^+ +(\hat{a}^+)^2[\hat{a}^2,\hat{a}^+]$$
$$=2(\hat{a}\hat{a}^+ +\hat{a}^+\hat{a})\hat{a}^+ +2(\hat{a}^+)^2\hat{a}^+\hat{a}$$
$$=2(\hat{a}^+)^3\hat{a}+2\hat{a}(\hat{a}^+)^2+2\hat{a}^+\hat{a}\hat{a}^+$$

In general.

$[\hat{a}^p,\hat{n}]=p\hat{a}^p$, $\quad [\hat{a}^{+p},\hat{n}]=-p\hat{a}^{+p}$ (**Messiah, p.460**).

$[\hat{n},\hat{a}^p]=-p\hat{a}^p$, $\quad [\hat{a}^{+p},\hat{n}]=-p\hat{a}^{+p}$.

((**Proof**))

$$[\hat{a}^p, \hat{n}] = \hat{a}^p \hat{a}^+ \hat{a} - \hat{a}^+ \hat{a} \hat{a}^p$$
$$= (\hat{a}^p \hat{a}^+ - \hat{a}^+ \hat{a}^p)\hat{a}$$
$$= -[\hat{a}^+, \hat{a}^p]\hat{a}$$
$$= p\hat{a}^p$$

$$[\hat{a}^{+p}, \hat{n}] = \hat{a}^{+p} \hat{a}^+ \hat{a} - \hat{a}^+ \hat{a} \hat{a}^{+p}$$
$$= -\hat{a}^+ [\hat{a}, \hat{a}^{+p}]$$
$$= -p(\hat{a}^+)^p$$

5. Commutation relation(I): Baker-Hausdorff lemma

Here we show that the operator

$$f(x) = \exp(x\hat{A})\hat{B}\exp(-x\hat{A}),$$

can be expanded as

$$f(x) = \exp(x\hat{A})\hat{B}\exp(-x\hat{A}) = \hat{B} + \frac{x}{1!}[\hat{A}, \hat{B}] + \frac{x^2}{2!}[\hat{A},[\hat{A}, \hat{B}]] + \frac{x^3}{3!}[\hat{A},[\hat{A},[\hat{A}, \hat{B}]]] + ...$$

(1)

where x is a real variable. We can prove this by using a Taylor expansion of $f(x)$ as

$$f(x) = f(0) + \frac{x}{1!} f^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \frac{x^3}{3!} f^{(3)}(0) + ...$$

with

$$f'(x) = \hat{A}\exp(\hat{A}x)\hat{B}\exp(-\hat{A}x) - \exp(\hat{A}x)\hat{B}\exp(-\hat{A}x)\hat{A} = [\hat{A}, f(x)],$$

$$f''(x) = [\hat{A}, f'(x)],$$

$$f^{(3)}(x) = [\hat{A}, f^{(2)}(x)].$$

In general,

$$f^{(n)}(x) = [\hat{A}, f^{(n-1)}(x)].$$

From these relations we have

$$f(x = 0) = \hat{B},$$

$$f'(0) = [\hat{A}, f(0)] = [\hat{A}, \hat{B}],$$

$$f''(0) = [\hat{A}, f'(0)] = [\hat{A}, [\hat{A}, \hat{B}]],$$

$$f^{(3)}(0) = [\hat{A}, f^{(3)}(0)] = [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]],$$

$$f^{(4)}(0) = [\hat{A}, f^{(3)}(0)] = [\hat{A}, [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]]],$$

……………………………………………………………….

Therefore, we get

$$f(x) = \hat{B} + \frac{x}{1!}[\hat{A}, \hat{B}] + \frac{x^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{x^3}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \frac{x^4}{4!}[\hat{A}, [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]]] + ...$$

This is known as the Baker- Hausdorff lemma. When

$$[\hat{A}, [\hat{A}, \hat{B}]] = 0,$$

we get the simple formula

$$f(x) = \hat{B} + x[\hat{A}, \hat{B}].$$

6. Commutation relation (II)

If the commutator of two operators $\hat{A}$ and $\hat{B}$ commutes with each of them ($\hat{A}$ and $\hat{B}$)

$$[\hat{A}, [\hat{A}, \hat{B}]] = \hat{0},$$

$$[\hat{B}, [\hat{A}, \hat{B}]] = \hat{0}.$$

One has an identity

$$\exp(\hat{A} + \hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp(-\frac{1}{2}[\hat{A}, \hat{B}]). \qquad (2)$$

This is known as the Baker-Campbell-Hausdorff theorem.

((**Proof by Glauber**))

Roy J. Glauber (**A. Messiah, Quantum Mechanics p.422**)

We start with

$$f(x) = \exp(x\hat{A})\exp(x\hat{B}).$$

Taking a derivative of $f(x)$ with respect to x, we get

$$\begin{aligned}\frac{df(x)}{dx} &= \hat{A}\exp(x\hat{A})\exp(x\hat{B}) + \exp(x\hat{A})\hat{B}\exp(x\hat{B}) \\ &= (\hat{A} + \exp(x\hat{A})\hat{B}\exp(-x\hat{A}))\exp(x\hat{A})\exp(x\hat{B}) \\ &= [\hat{A} + \exp(x\hat{A})\hat{B}\exp(-x\hat{A})]f(x)\end{aligned}$$

or

$$\begin{aligned}\frac{df(x)}{dx} &= \exp(x\hat{A})\exp(x\hat{B})\hat{B} + \hat{A}\exp(x\hat{A})\exp(x\hat{B}) \\ &= \exp(x\hat{A})\exp(x\hat{B})[\hat{B} + \exp(-x\hat{B})\hat{A}\exp(x\hat{B})] \\ &= f(x)[\hat{B} + \exp(-x\hat{B})\hat{A}\exp(x\hat{B})]\end{aligned}$$

We note the commutation relations which is derived above. If

$$[\hat{A},[\hat{A},\hat{B}]] = \hat{0}, \qquad \text{and} \qquad [\hat{B},[\hat{A},\hat{B}]] = \hat{0},$$

then we have

$$\hat{A} + \exp(x\hat{A})\hat{B}\exp(-x\hat{A}) = \hat{A} + \hat{B} + [\hat{A},\hat{B}]x$$

.

$$\hat{B} + \exp(-x\hat{B})\hat{A}\exp(x\hat{B}) = \hat{A} + \hat{B} + [\hat{A},\hat{B}]x$$

Using this relation, we get

$$\frac{df(x)}{dx} = (\hat{A} + \hat{B} + [\hat{A},\hat{B}]x)f(x) = f(x)(\hat{A} + \hat{B} + [\hat{A},\hat{B}]x),$$

with $f(x=0) = \hat{1}$. The operators $\hat{A} + \hat{B}$ and $[\hat{A},\hat{B}]$ commute,

$$[\hat{A} + \hat{B},[\hat{A},\hat{B}]] = [\hat{A},[\hat{A},\hat{B}]] + [\hat{B},[\hat{A},\hat{B}]] = 0.$$

The function $f(x)$ and $\hat{A} + \hat{B} + [\hat{A},\hat{B}]x$ commutes. Then they can be considered as quantities of ordinary algebra,

$$\int \frac{1}{f} df = \int (\hat{A} + \hat{B} + [\hat{A}, \hat{B}] x) dx ,$$

or

$$\ln(f) = (\hat{A} + \hat{B}) x + \frac{x^2}{2} [\hat{A}, \hat{B}] ,$$

or

$$f(x) = \exp(\hat{A}x) \exp(\hat{B}x) = \exp[(\hat{A} + \hat{B}) x + \frac{x^2}{2} [\hat{A}, \hat{B}]) .$$

When $x = 1$,

$$\exp(\hat{A}) \exp(\hat{B}) = \exp(\hat{A} + \hat{B} + \frac{1}{2} [\hat{A}, \hat{B}]) ,$$

or

$$\exp[(\hat{A} + \hat{B}) = \exp(-\frac{1}{2} [\hat{A}, \hat{B}]) \exp(\hat{A}) \exp(\hat{B})$$

where

$$[\hat{A}, [\hat{A}, \hat{B}]] = 0 , \qquad [\hat{B}, [\hat{A}, \hat{B}]] = 0$$

In general case, we have

$$\exp(\hat{A}) \exp(\hat{B}) = \exp[(\hat{A} + \hat{B}) + \frac{1}{2} [\hat{A}, \hat{B}] + \frac{1}{12} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{12} [\hat{B}, [\hat{B}, \hat{A}]] + ...]$$

7. Commutation relation (III):

In general, we have

$$\exp(x\hat{A}) \hat{B} \exp(-x\hat{A}) = \hat{B} + \frac{x}{1!} [\hat{A}, \hat{B}] + \frac{x^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{x^3}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + ...$$

If the operators satisfy the relation (β: constant),

$$[\hat{A}, [\hat{A}, \hat{B}]] = \beta \hat{B} ,$$

then we get

$$\exp(x\hat{A})\hat{B}\exp(-x\hat{A}) = \hat{B} + \frac{x}{1!}[\hat{A},\hat{B}] + \frac{x^2}{2!}\beta\hat{B} + \frac{x^3}{3!}\beta[\hat{A},\hat{B}] + \frac{x^4}{4!}\beta^2\hat{B} + ...$$

$$= \hat{B}(1 + \frac{(x\sqrt{\beta})^2}{2!} + \frac{(x\sqrt{\beta})^4}{4!} + ...)$$

$$+ \frac{1}{\sqrt{\beta}}[\hat{A},\hat{B}]\{\frac{x\sqrt{\beta}}{1!} + \frac{\left(x\sqrt{\beta}\right)^3}{3!} + ...\}$$

$$= \hat{B}\cosh(x\sqrt{\beta}) + \frac{1}{\sqrt{\beta}}[\hat{A},\hat{B}]\sinh(x\sqrt{\beta})$$

or

$$\exp(x\hat{A})\hat{B}\exp(-x\hat{A}) = \hat{B}\cosh(x\sqrt{\beta}) + \frac{1}{\sqrt{\beta}}[\hat{A},\hat{B}]\sinh(x\sqrt{\beta}) . \qquad (3)$$

where

$$[\hat{A},[\hat{A},\hat{B}]] = \beta\hat{B} .$$

Suppose that $\beta = 0$, $[\hat{A},[\hat{A},\hat{B}]] = 0$. Then we get

$$\exp(x\hat{A})\hat{B}\exp(-x\hat{A}) = \hat{B} + x[\hat{A},\hat{B}] ,$$

since

$$\lim_{\beta \to 0} \frac{\sinh(x\sqrt{\beta})}{\sqrt{\beta}} = x .$$

8. Example-1: Displacement operator $\hat{D}_\alpha$

Suppose that

$$\hat{A} = -\alpha\hat{a}^+ + \alpha^*\hat{a} , \qquad \hat{B} = \hat{a} ,$$

where α is a complex number.

$$[\hat{A},\hat{B}] = [-\alpha\hat{a}^+ + \alpha^*\hat{a},\hat{a}] = -\alpha[\hat{a}^+,\hat{a}] = \alpha\hat{1} .$$

In other words,

$$[\hat{A},[\hat{A},\hat{B}]] = \hat{0}, \qquad [\hat{B},[\hat{A},\hat{B}]] = \hat{0}.$$

Then we use the above formula,

$$\exp(\hat{A})\hat{B}\exp(-\hat{A}) = \hat{B} + [\hat{A},\hat{B}],$$

or

$$\exp(-\alpha\hat{a}^{+} + \alpha^{*}\hat{a})\hat{a}\exp(\alpha\hat{a}^{+} - \alpha^{*}\hat{a}) = \hat{a} + \alpha\hat{1},$$

or

$$\hat{D}_{\alpha}{}^{+}\hat{a}\hat{D}_{\alpha} = \alpha\hat{1} + \hat{a},$$

where $\hat{D}_{\alpha}$ is called the displacement operator,

$$\hat{D}_{\alpha} = \exp(\alpha\hat{a}^{+} - \alpha^{*}\hat{a}).$$

Note that

$$\hat{D}_{\alpha}{}^{+} = \hat{D}_{-\alpha} = \exp(-\alpha\hat{a}^{+} + \alpha^{*}\hat{a}),$$

$$\hat{D}_{\alpha}{}^{+}\hat{a}^{+}\hat{D}_{\alpha} = \alpha^{*}\hat{1} + \hat{a}^{+},$$

$$\hat{D}_{\alpha}{}^{+}\hat{D}_{\alpha} = \hat{1}. \quad \text{(Unitary operator).}$$

9. Properties of $\hat{D}_{\alpha}$

Using the above theorem, we can derive the following formula.

$$\exp(\alpha\hat{a}^{+})\exp(-\alpha^{*}\hat{a}) = \exp(\alpha\hat{a}^{+} - \alpha^{*}\hat{a} + \frac{1}{2}|\alpha|^{2})$$
$$= \exp(\frac{1}{2}|\alpha|^{2})\exp(\alpha\hat{a}^{+} - \alpha^{*}\hat{a})$$

We use the above theorem,

$$\hat{A} = \alpha\hat{a}^{+}, \qquad \hat{B} = -\alpha^{*}\hat{a},$$

$$[\hat{A},\hat{B}] = [\alpha\hat{a}^{+}, -\alpha^{*}\hat{a}] = |\alpha|^{2}[\hat{a},\hat{a}^{+}] = |\alpha|^{2}\hat{1},$$

with

$$[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0 .$$

Then we have

$$\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp(-\frac{1}{2}[\hat{A},\hat{B}]) .$$

Thus we get

$$\hat{D}_\alpha = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a}) = \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a}),$$

where

$$[\hat{a},\exp(\alpha\hat{a}^+)] = \alpha\exp(\alpha\hat{a}^+) ,$$

$$[\hat{a}^+,\exp(\alpha\hat{a}),] = -\alpha\exp(\alpha\hat{a}) .$$

10. Examnple-2: Squeezed state

We define the operator

$$\hat{S}_\varsigma = \exp(\frac{1}{2}\varsigma^*\hat{a}^2 - \frac{1}{2}\varsigma\hat{a}^{+2}),$$

where ζ is a complex number,

$$\varsigma = se^{i\vartheta} .$$

For any operator $\hat{A}$ and $\hat{B}$, it is known that

$$\exp(x\hat{A})\hat{B}\exp(-x\hat{A}) = \hat{B} + \frac{x}{1!}[\hat{A},\hat{B}] + \frac{x^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{x^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + ...$$

Suppose that

$$\hat{A} = \frac{1}{2}\varsigma^*\hat{a}^2 - \frac{1}{2}\varsigma\hat{a}^{+2}, \qquad \hat{B} = \hat{a} .$$

Then we get

$$[A,\hat{B}]=[\frac{1}{2}\varsigma^*\hat{a}^2-\frac{1}{2}\varsigma\hat{a}^{+2},\hat{a}]=\frac{1}{2}\varsigma[\hat{a},\hat{a}^{+2}]=\varsigma\hat{a}^+ ,$$

with

$$[\hat{a},\hat{a}^{+2}]=2\hat{a}^+ , \qquad [\hat{a}^+,\hat{a}^2]=-2\hat{a} .$$

We also get

$$\begin{aligned}[\hat{A},[\hat{A},\hat{B}]] &= [\hat{A},\varsigma\hat{a}^+] \\ &= \varsigma[\frac{1}{2}\varsigma^*\hat{a}^2-\frac{1}{2}\varsigma\hat{a}^{+2},\hat{a}^+] \\ &= -\varsigma\varsigma^*\frac{1}{2}[\hat{a}^+,\hat{a}^2] \\ &= |\varsigma|^2\hat{a}\end{aligned}$$

$$[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]=[\hat{A},|\varsigma|^2\hat{a}]=|\varsigma|^2[\hat{A},\hat{a}]=\varsigma|\varsigma|^2\hat{a}^+ .$$

Then

$$\begin{aligned}\exp(x\hat{A})\hat{B}\exp(-x\hat{A}) &= \hat{a}+\frac{x}{1!}\varsigma\hat{a}^+ +\frac{x^2}{2!}|\varsigma|^2\hat{a}+\frac{x^3}{3!}\varsigma|\varsigma|^2\hat{a}^+ +\frac{x^4}{4!}|\varsigma|^4\hat{a}... \\ &= \hat{a}(1+\frac{x^2}{2!}|\varsigma|^2+\frac{x^4}{4!}|\varsigma|^4+...)\hat{a} \\ &\quad +\frac{\varsigma}{|\varsigma|}(\frac{x|\varsigma|}{1!}\hat{a}^+ +\frac{x^3}{3!}|\varsigma|^3\hat{a}^+ +\frac{x^5}{4!}|\varsigma|^5+...)\hat{a}^+ \\ &= \hat{a}\cosh(|\varsigma|x)+\frac{\varsigma}{|\varsigma|}\hat{a}^+\sinh(|\varsigma|x)\end{aligned}$$

When x = -1, we have

$$\begin{aligned}\exp(-\hat{A})\hat{B}\exp(\hat{A}) &= \hat{a}\cosh(-|\varsigma|)+\frac{\varsigma}{|\varsigma|}\hat{a}^+\sinh(-|\varsigma|) \\ &= \hat{a}\cosh(|\varsigma|)-\frac{\varsigma}{|\varsigma|}\hat{a}^+\sinh(|\varsigma|)\end{aligned}$$

We note that

$$\exp(\hat{A})=\exp[\frac{1}{2}\varsigma^*\hat{a}^2-\frac{1}{2}\varsigma\hat{a}^{+2}]=\hat{S}(\varsigma) ,$$

$$\exp(-\hat{A}) = \exp[-(\frac{1}{2}\varsigma^* \hat{a}^2 - \frac{1}{2}\varsigma \hat{a}^{+2})] = \hat{S}^+(\varsigma),$$

which leads to the relation

$$\hat{S}_\varsigma{}^+ \hat{a} \hat{S}_\varsigma = \hat{a}\cosh(|\varsigma|) - \frac{\varsigma}{|\varsigma|}\hat{a}^+ \sinh(|\varsigma|)$$
$$= \hat{a}\cosh(s) - e^{i\vartheta}\hat{a}^+ \sinh(s)$$

The Hermitian conjugate of this expression is

$$\hat{S}_\varsigma{}^+ \hat{a}^+ \hat{S}_\varsigma = \hat{a}^+ \cosh(s) - e^{-i\vartheta}\hat{a}\sinh(s).$$

Note that

$$\cosh(s) = \frac{e^s + e^{-s}}{2} = 1 + \frac{1}{2!}s^2 + \frac{1}{4!}s^4 + ...$$

$$\sinh(s) = \frac{e^s - e^{-s}}{2} = s + \frac{1}{3!}s^3 + \frac{1}{5!}s^5 + ...$$

((Proof))

We can give a proof for the above equation when ζ *is a real number*.

$$\hat{G}(\varsigma) = \hat{S}_\varsigma{}^+ \hat{a} \hat{S}_\varsigma .$$

We take a derivative of $\hat{G}(\varsigma)$ with respect to ζ,

$$\frac{d\hat{G}(\varsigma)}{d\varsigma} = \frac{d\hat{S}_\varsigma{}^+}{d\varsigma}\hat{a}\hat{S}_\varsigma + \hat{S}_\varsigma{}^+ \hat{a}\frac{d\hat{S}_\varsigma}{d\varsigma}$$
$$= -\hat{S}_\varsigma{}^+ \frac{1}{2}(\hat{a}^2 - \hat{a}^{+2})\hat{a}\hat{S}_\varsigma + \hat{S}\varsigma^+ \hat{a}\frac{1}{2}(\hat{a}^2 - \hat{a}^{+2})\hat{S}_\varsigma$$
$$= \hat{S}_\varsigma{}^+[\hat{a}, \frac{1}{2}(\hat{a}^2 - \hat{a}^{+2})]\hat{S}_\varsigma$$
$$= -\frac{1}{2}\hat{S}_\varsigma{}^+[\hat{a}, \hat{a}^{+2}]\hat{S}_\varsigma$$
$$= -\hat{S}_\varsigma{}^+ \hat{a}^+ \hat{S}_\varsigma = -\hat{G}^+(\varsigma)$$

Similarly,

$$\frac{d\hat{G}^{+}(\varsigma)}{d\varsigma} = -\hat{G}(\varsigma) .$$

From these two equations, we get the second order differential equation

$$\frac{d^2\hat{G}(\varsigma)}{d\varsigma^2} = \hat{G}(\varsigma) ,$$

with the boundary condition

$$\hat{G}(\varsigma = 0) = \hat{a} , \qquad \frac{d\hat{G}(\varsigma)}{d\varsigma}\Big|_{\varsigma=0} = -\hat{a}^{+} .$$

The solution of this differential equation is

$$\hat{G}(\varsigma) = \frac{1}{2}(\hat{a} - \hat{a}^{+})e^{\varsigma} + \frac{1}{2}(\hat{a} + \hat{a}^{+})e^{-\varsigma} = \hat{a}\cosh\varsigma - \hat{a}^{+}\sinh\varsigma .$$

Similarly

$$\hat{G}^{+}(\varsigma) = \hat{a}^{+}\cosh\varsigma - \hat{a}\sinh\varsigma .$$

11. Example-3: squeezed coherent state

The squeezed state is obtained by first squeezing the vacuum state and then displacing it.

$$\left|\alpha,\varsigma\right\rangle = \hat{D}_{\alpha}\hat{S}_{\varsigma}\left|0\right\rangle ,$$

where

$$\alpha = |\alpha|e^{i\theta} , \qquad \varsigma = se^{i\vartheta} .$$

$$\begin{aligned}
\hat{S}_{\varsigma}{}^{+}\hat{D}_{\alpha}{}^{+}\hat{a}\hat{D}_{\alpha}\hat{S}_{\varsigma} &= \hat{S}_{\varsigma}{}^{+}(\hat{a} + \alpha\hat{1})\hat{S}_{\varsigma} \\
&= \hat{S}_{\varsigma}{}^{+}\hat{a}\hat{S}_{\varsigma} + \alpha\hat{1} \\
&= \hat{a}\cosh(s) - e^{i\vartheta}\hat{a}^{+}\sinh(s) + \alpha\hat{1}
\end{aligned}$$

where

$$\hat{S}_{\varsigma}{}^{+}\hat{a}\hat{S}_{\varsigma} = \hat{a}\cosh(s) - e^{i\vartheta}\hat{a}^{+}\sinh(s) ,$$

$$\hat{D}_{\alpha}{}^{+}\hat{a}\hat{D}_{\alpha} = \hat{a} + \alpha\hat{1} .$$

Similarly we get

$$\hat{S}_\varsigma^{\ +}\hat{D}_\alpha^{\ +}\hat{a}^+\hat{D}_\alpha\hat{S}_\varsigma = \hat{S}_\varsigma^{\ +}(\alpha^*\hat{1}+\hat{a}^+)\hat{S}_\varsigma$$
$$= \hat{a}^+\cosh(s) - e^{-i\vartheta}\hat{a}\sinh(s) + \alpha^*\hat{1}$$

where

$$\hat{D}_\alpha^{\ +}\hat{a}^+\hat{D}_\alpha = \alpha^*\hat{1}+\hat{a}^+.$$

12. Weyl commutator (from Englert)

We found interesting topics related to the Baker-Hausdorff theorem in a book written by Englert;
B-G. Englert, Lectures on Quantum Mechanics Vol.2 Simple Systems (World Scientific, 2006)

(a) Weyl commutator

We start with the operator

$$\hat{A} = \exp(\frac{ix'\hat{p}}{\hbar})\exp(\frac{ip'\hat{x}}{\hbar})$$

The matrix element:

$$\langle x|\hat{A}|p\rangle = \langle x|\exp(\frac{ix'\hat{p}}{\hbar})\exp(\frac{ip'\hat{x}}{\hbar})|p\rangle$$
$$= \int dp''\int dx''\langle x|\exp(\frac{ix'\hat{p}}{\hbar})|p''\rangle\langle p''|\exp(\frac{ip'\hat{x}}{\hbar})|x''\rangle\langle x''|p\rangle$$
$$= \int dp''\int dx''\exp(\frac{ix'p''}{\hbar})\exp(\frac{ip'x''}{\hbar})\langle x|p''\rangle\langle p''|x''\rangle\langle x''|p\rangle$$
$$= \frac{1}{(2\pi\hbar)^{3/2}}\int dp''\int dx''\exp(\frac{ip''x'}{\hbar}+\frac{ip'x''}{\hbar}+\frac{ip''x}{\hbar}-\frac{ip''x''}{\hbar}+\frac{ipx''}{\hbar})$$

or

$$\langle x|\hat{A}|p\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\int \exp[\frac{ip''(x+x')}{\hbar}]dp''\int dx''\exp[\frac{i(p'-p''+p)x''}{\hbar}]$$
$$= \frac{1}{(2\pi\hbar)^{1/2}}\int \exp[\frac{ip''(x+x')}{\hbar}]dp''\delta(p'-p''+p)$$
$$= \frac{1}{(2\pi\hbar)^{1/2}}\exp[\frac{i(p+p')(x+x')}{\hbar}]$$

where

$$\int dx''\exp[\frac{i(p'-p''+p)x''}{\hbar}] = 2\pi\delta[\frac{p'-p''+p}{\hbar}]$$
$$= 2\pi\hbar\delta[p'-p''+p]$$

Using the transformation function

$$\langle x|p\rangle = \frac{1}{(2\pi\hbar)^{1/2}}\exp(\frac{ipx}{\hbar})$$

we get

$$\frac{\langle x|\hat{A}|p\rangle}{\langle x|p\rangle} = \exp[\frac{i(xp'+x'p+x'p')}{\hbar}]$$

Next, we introduce a new operator,

$$\hat{B} = \exp(\frac{ip'\hat{x}}{\hbar})\exp(\frac{ix'\hat{p}}{\hbar})$$

The matrix element:

$$\frac{\langle x|\hat{B}|p\rangle}{\langle x|p\rangle} = \frac{\langle x|\exp(\frac{ip'\hat{x}}{\hbar})\exp(\frac{i\hat{p}x'}{\hbar})|p\rangle}{\langle x|p\rangle}$$
$$= \exp[\frac{i(p'x+px')}{\hbar}]$$

Thus, we have the relation

$$\frac{\langle x|\hat{A}|p\rangle}{\langle x|p\rangle} = \exp(\frac{ip'x'}{\hbar})\frac{\langle x|\hat{B}|p\rangle}{\langle x|p\rangle}$$

leading to

$$\hat{B} = \exp(-\frac{ip'x'}{\hbar})\hat{A}$$

or

$$\exp(\frac{ip'\hat{x}}{\hbar})\exp(\frac{ix'\hat{p}}{\hbar}) = \exp(-\frac{ip'x'}{\hbar})\exp(\frac{ix'\hat{p}}{\hbar})\exp(\frac{ip'\hat{x}}{\hbar})$$

This is Hermann K.H. Weyl's commutation relation for the basic unitary operators associated with $\hat{x}$ and $\hat{p}$.

Here we use the formula

$$[\hat{x}, \exp[ig(\hat{p})] = -\hbar g'(\hat{p})]\exp[ig(\hat{p})],$$

$$[\hat{p}, \exp[if(\hat{x})] = \hbar f'(\hat{x})]\exp[if(\hat{x})],$$

which are equivalent to

$$\hat{x} - \hbar g'(\hat{p}) = \exp[-ig(\hat{p})]\hat{x}\exp[ig(\hat{p})],$$

$$\hat{p} - \hbar f'(\hat{x}) = \exp[if(\hat{x})]\hat{p}\exp[-if(\hat{x})].$$

We consider the relation

$$\exp[i\frac{(p'\hat{x} + x'\hat{p}}{\hbar})] = \exp[i\frac{p'}{\hbar}(\hat{x} + \frac{x'}{p'}\hat{p})] \qquad (p' \neq 0)$$

Using the commutation relation, we get

$$\hat{x} + \frac{x'}{p'}\hat{p} = \hat{x} - \hbar g'(\hat{p}) = \exp[-ig(\hat{p})]\hat{x}\exp[ig(\hat{p})],$$

where

$$g'(\hat{p}) = -\frac{x'\hat{p}}{\hbar p'}, \qquad g(\hat{p}) = -\frac{x'}{2\hbar p'}\hat{p}^2,$$

Thus, we obtain

$$\hat{x} + \frac{x'}{p'}\hat{p} = \exp[i\frac{x'}{2\hbar p'}\hat{p}^2]\hat{x}\exp[-i\frac{x'}{2\hbar p'}\hat{p}^2].$$

Using this, we have

$$\exp[i\frac{(p'\hat{x}+x'\hat{p})}{\hbar})] = \exp[i\frac{p'}{\hbar}(\hat{x}+\frac{x'}{p'}\hat{p})]$$

$$= \exp[i\frac{p'}{\hbar}\exp(i\frac{x'}{2\hbar p'}\hat{p}^2)\hat{x}\exp(-i\frac{x'}{2\hbar p'}\hat{p}^2)]$$

$$= \exp[\exp(i\frac{x'}{2\hbar p'}\hat{p}^2)(i\frac{p'}{\hbar}\hat{x})\exp(-i\frac{x'}{2\hbar p'}\hat{p}^2)]$$

This relation can be derived as follows. We put, for convenience

$$\hat{A}_1 = \exp(i\frac{x'\hat{p}^2}{2\hbar p'}), \qquad \hat{A}_1^{-1} = \exp(-i\frac{x'\hat{p}^2}{2\hbar p'}),$$

with

$$\hat{A}_1\hat{A}_1^{-1} = \hat{A}_1^{-1}\hat{A}_1 = \hat{1}$$

$$\exp[i\frac{(p'\hat{x}+x'\hat{p})}{\hbar}] = \exp[\hat{A}_1(i\frac{p'\hat{x}}{\hbar})\hat{A}_1^{-1}]$$

$$= \sum_n \frac{1}{n!}[\hat{A}_1(i\frac{p'\hat{x}}{\hbar})\hat{A}_1^{-1}]^n$$

$$= \hat{A}_1\sum_n \frac{1}{n!}(i\frac{p'\hat{x}}{\hbar})^n\hat{A}_1^{-1}$$

$$= \hat{A}_1\exp(i\frac{p'\hat{x}}{\hbar})\hat{A}_1^{-1}$$

$$= \exp(i\frac{x'\hat{p}^2}{2\hbar p'})\exp(i\frac{p'\hat{x}}{\hbar})\exp(-i\frac{x'\hat{p}^2}{2\hbar p'})$$

Thus, we get

$$\exp[i\frac{(p'\hat{x}+x'\hat{p})}{\hbar}] = \exp(i\frac{x'\hat{p}^2}{2\hbar p'})[\exp(i\frac{p'\hat{x}}{\hbar})\exp(-i\frac{x'\hat{p}^2}{2\hbar p'})$$

$$\exp(-i\frac{p'\hat{x}}{\hbar})]\exp(i\frac{p'\hat{x}}{\hbar})$$

Here we note that

$$\exp(i\frac{p'}{\hbar}\hat{x})\exp(-i\frac{x'\hat{p}^2}{2\hbar p'})\exp(-i\frac{p'\hat{x}}{\hbar}) = \exp\{-\frac{ix'}{2\hbar p'}[\exp(i\frac{p'\hat{x}}{\hbar})\hat{p}\exp(-i\frac{p'\hat{x}}{\hbar})]^2\}$$

where

$$\exp(i\frac{p'\hat{x}}{\hbar})\exp(-i\frac{x'\hat{p}^2}{2\hbar p'})\exp(-i\frac{p'\hat{x}}{\hbar}) = \exp(i\frac{p'\hat{x}}{\hbar})\sum_n \frac{1}{n!}(-i\frac{x'}{2\hbar p'})^n \hat{p}^{2n}\exp(-i\frac{p'\hat{x}}{\hbar})$$

$$= \sum_n \frac{1}{n!}[\exp(i\frac{p'\hat{x}}{\hbar})(-i\frac{x'\hat{p}^2}{2\hbar p'})\exp(-i\frac{p'\hat{x}}{\hbar})]^n$$

$$= \exp[\exp(i\frac{p'\hat{x}}{\hbar})(-i\frac{x'\hat{p}^2}{2\hbar p'})\exp(-i\frac{p'\hat{x}}{\hbar})]$$

$$= \exp\{-\frac{ix'}{2\hbar p'}[\exp(i\frac{p'\hat{x}}{\hbar})\hat{p}\exp(-i\frac{p'\hat{x}}{\hbar})]^2\}$$

and

$$[\hat{p}, \exp(-i\frac{p'\hat{x}}{\hbar})] = \frac{\hbar}{i}(-\frac{ip'}{\hbar})\exp(-i\frac{p'\hat{x}}{\hbar})$$

$$= -p'\exp(-i\frac{p'\hat{x}}{\hbar})$$

or

$$\exp(i\frac{p'\hat{x}}{\hbar})\hat{p}\exp(-i\frac{p'\hat{x}}{\hbar}) = \hat{p} - p'\hat{1}$$

Thus we have

$$\exp[i\frac{(p'\hat{x} + x'\hat{p})}{\hbar}] = \exp(i\frac{x'}{2\hbar p'}\hat{p}^2)\exp\{-\frac{ix'}{2\hbar p'}[\exp(i\frac{p'}{\hbar}\hat{x})\hat{p}\exp(-i\frac{p'}{\hbar}\hat{x})]^2\}\exp(i\frac{p'}{\hbar}\hat{x})$$

$$= \exp(i\frac{x'}{2\hbar p'}\hat{p}^2)\exp\{-\frac{ix'}{2\hbar p'}(\hat{p} - p'\hat{1})^2\}\exp(i\frac{p'}{\hbar}\hat{x})$$

$$= \exp\{i\frac{x'}{2\hbar p'}[\hat{p}^2 - (\hat{p} - p'\hat{1})^2]\}\exp(i\frac{p'\hat{x}}{\hbar})$$

$$= \exp(-i\frac{p'x'}{2\hbar})\exp(i\frac{\hat{p}x'}{\hbar})\exp(i\frac{p'\hat{x}}{\hbar})$$

or

$$\exp[i\frac{(p'\hat{x} + x'\hat{p})}{\hbar}] = \exp(-i\frac{p'x'}{2\hbar})\exp(i\frac{\hat{p}x'}{\hbar})\exp(i\frac{p'\hat{x}}{\hbar})$$

13. The use of Baker-Housdorff lemma

In order to prove the previous result, we use the Baker-Hausdorff lemma.

If the commutator of two operators $\hat{A}$ and $\hat{B}$ commutes with each of them ($\hat{A}$ and $\hat{B}$)

$$[\hat{A},[\hat{A},\hat{B}]]=\hat{0},$$

$$[\hat{B},[\hat{A},\hat{B}]]=\hat{0}.$$

One has an identity

$$\exp(\hat{A}+\hat{B})=\exp(\hat{A})\exp(\hat{B})\exp(-\frac{1}{2}[\hat{A},\hat{B}]). \qquad (2)$$

This is known as the Baker-Campbell-Hausdorff theorem.

Using this lemma, we get

$$\exp(\alpha\hat{p}+\beta\hat{x})=\exp(\alpha\hat{p})\exp(\beta\hat{x})\exp(\frac{1}{2}i\hbar\alpha\beta),$$

$$\exp(\beta\hat{x}+\alpha\hat{p})=\exp(\beta\hat{x})\exp(\alpha\hat{p})\exp(-\frac{1}{2}i\hbar\alpha\beta),$$

leading to

$$\exp(\alpha\hat{p})\exp(\beta\hat{x})\exp(\frac{1}{2}i\hbar\alpha\beta)=\exp(\beta\hat{x})\exp(\alpha\hat{p})\exp(-\frac{1}{2}i\hbar\alpha\beta),$$

or

$$\exp(\alpha\hat{p})\exp(\beta\hat{x})\exp(-\alpha\hat{p})=\exp(-i\hbar\alpha\beta)\exp(\beta\hat{x}),$$

or

$$\exp(\beta\hat{x})\exp(\alpha\hat{p})\exp(-\beta\hat{x})=\exp(i\hbar\alpha\beta)\exp(\alpha\hat{p}).$$

When $\alpha=\frac{ix'}{\hbar}$, and $\beta=\frac{ip'}{\hbar}$,

(i)

$$\exp(\frac{ix'\hat{p}}{\hbar})\exp(\frac{ip'\hat{x}}{\hbar})\exp(-\frac{ix'\hat{p}}{\hbar})=\exp(\frac{ix'p'}{\hbar})\exp(\frac{ip'\hat{x}}{\hbar}).$$

(ii)

$$\exp(\frac{ip'\hat{x}}{\hbar})\exp(\frac{ix'\hat{p}}{\hbar})\exp(-\frac{ip'\hat{x}}{\hbar}) = \exp(-\frac{ix'p'}{\hbar})\exp(\frac{ix'\hat{p}}{\hbar}).$$

(iii)

$$\begin{aligned}\exp[\frac{i(x'\hat{p}+p'\hat{x})}{\hbar}] &= \exp(-\frac{ix'p'}{2\hbar})\exp(\frac{ix'\hat{p}}{\hbar})\exp(\frac{ip'\hat{x}}{\hbar}) \\ &= \exp(\frac{ix'p'}{2\hbar})\exp(\frac{ip'\hat{x}}{\hbar})\exp(\frac{ix'\hat{p}}{\hbar})\end{aligned}$$

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