Chapter 6

Applications of Definite Integrals

Chapter 6

In this chapter we will see some of the many additional applications of definite integrals.

We will use the definite integral to define and find volumes, lengths of plane curves, and areas of surfaces of revolution.

We will see how integrals are used to solve physical problems involving the work done by a force, and how they give the location of an object's center of mass.

The integral arises in these and other applications in which we can approximate a desired quantity by Riemann sums. The limit of those Riemann sums, which is the quantity we seek, is given by a definite integral.

Section 6.1

Volumes Using
Cross-Sections

A cross-section of a solid S is the planar region formed by intersecting S with a plane。

We present three different methods for obtaining the cross-sections appropriate to finding the volume of a particular solid:
1.the method of slicing,
2.the disk method,
3.the washer method.

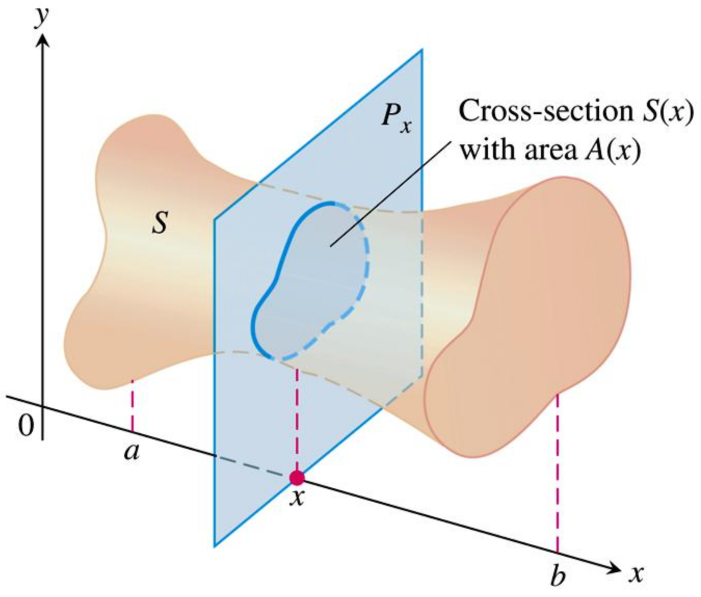


FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

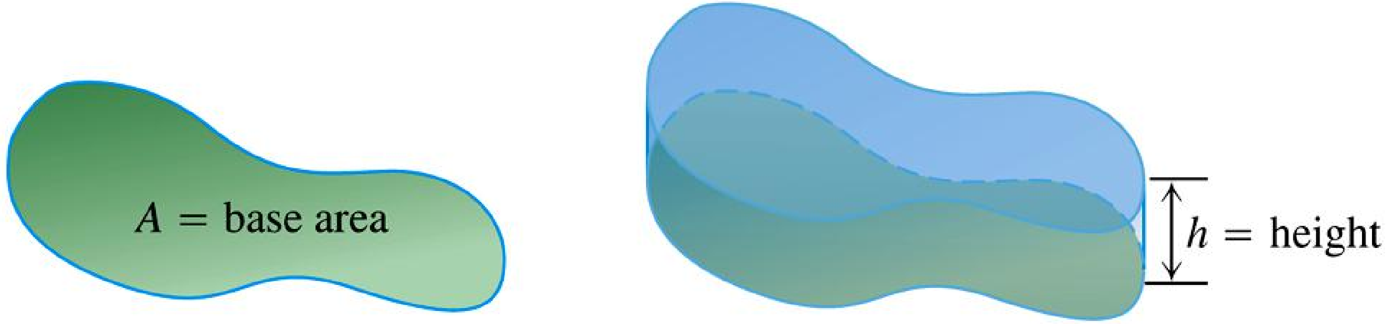
Suppose that we want to find the volume of a solid S like the one pictured in Figure 6.1.

At each point x in the interval [a, b] we form a cross-section S(x) by intersecting S with a plane perpendicular to the x-axis through the point x, which gives a planar region whose area is A(x).

We will show that if A is a continuous function of x, then the volume of the solid S is the definite integral of A(x). This method of computing volumes is known as the method of slicing.

Three different methods:

1. the method of slicing,
2. the disk method,
3. the washer method.



Plane region whose area we know

Cylindrical solid based on region
Volume = base area × height = Ah

FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.

We need to extend the definition of a cylinder from the usual cylinders of classical geometry (which have circular, square, or other regular bases) to cylindrical solids that have more general bases.

If the cylindrical solid has a base whose area is A and its height is h, then the volume of the cylindrical solid is

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

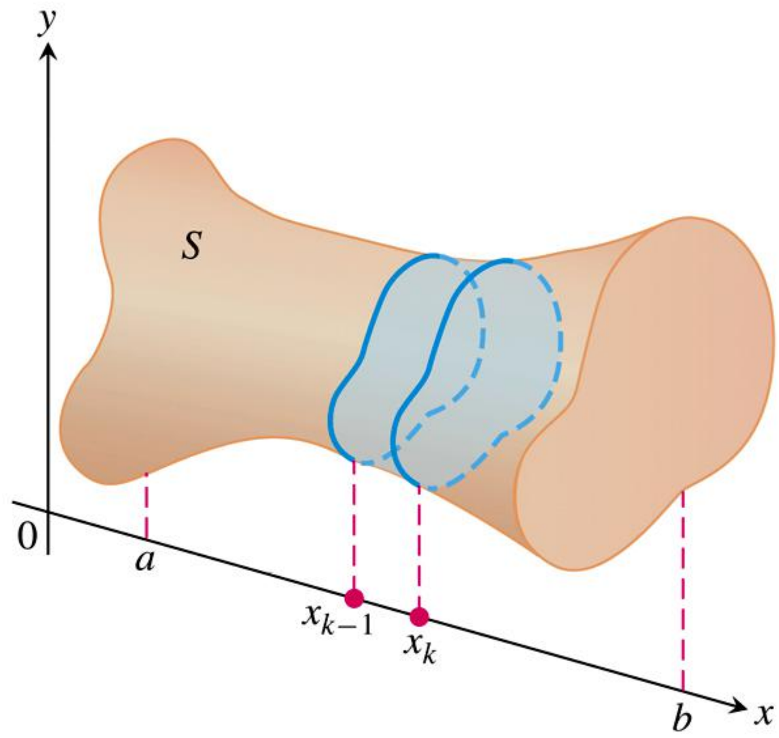


FIGURE 6.3 A typical thin slab in the solid S.

In the method of slicing, the base will be the cross-section of S that has area $A(x)$, and the height will correspond to the width Δx_k of subintervals formed by partitioning the interval $[a, b]$ into finitely many subintervals $[x_{k-1}, x_k]$.

Slicing by Parallel Planes

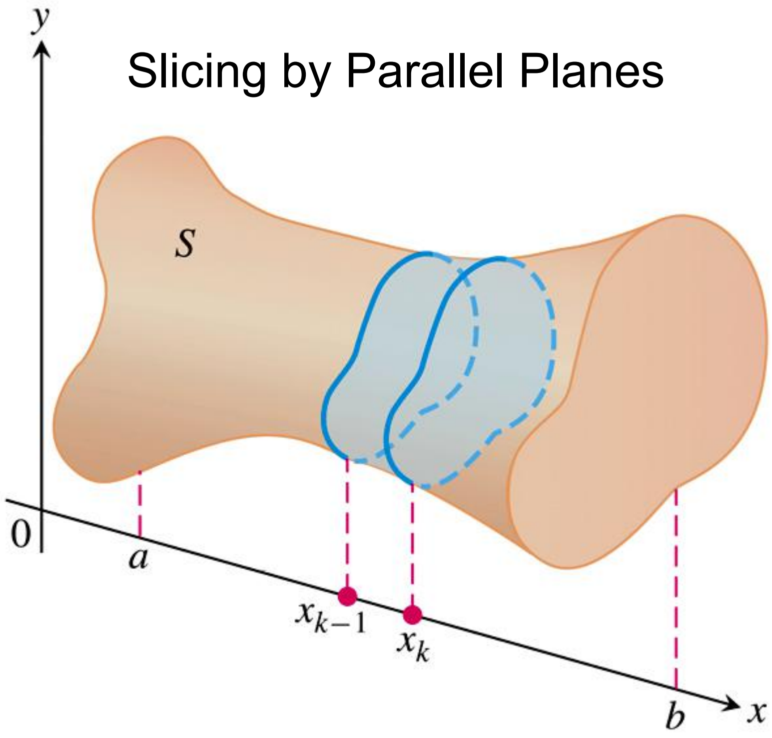


FIGURE 6.3 A typical thin slab in the solid S.

$$a = x_0 < x_1 < \cdots < x_n = b.$$

$$\text{Volume of the } k\text{th slab} \approx V_k = A(x_k)\,\Delta x_k.$$

slab 英[slæb]美[slæb] 厚板，平板

$$V \approx \sum^{n} V_k = \sum^{n} A(x_k)\,\Delta x_k.$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k = \int_a^b A(x)\,dx.$$

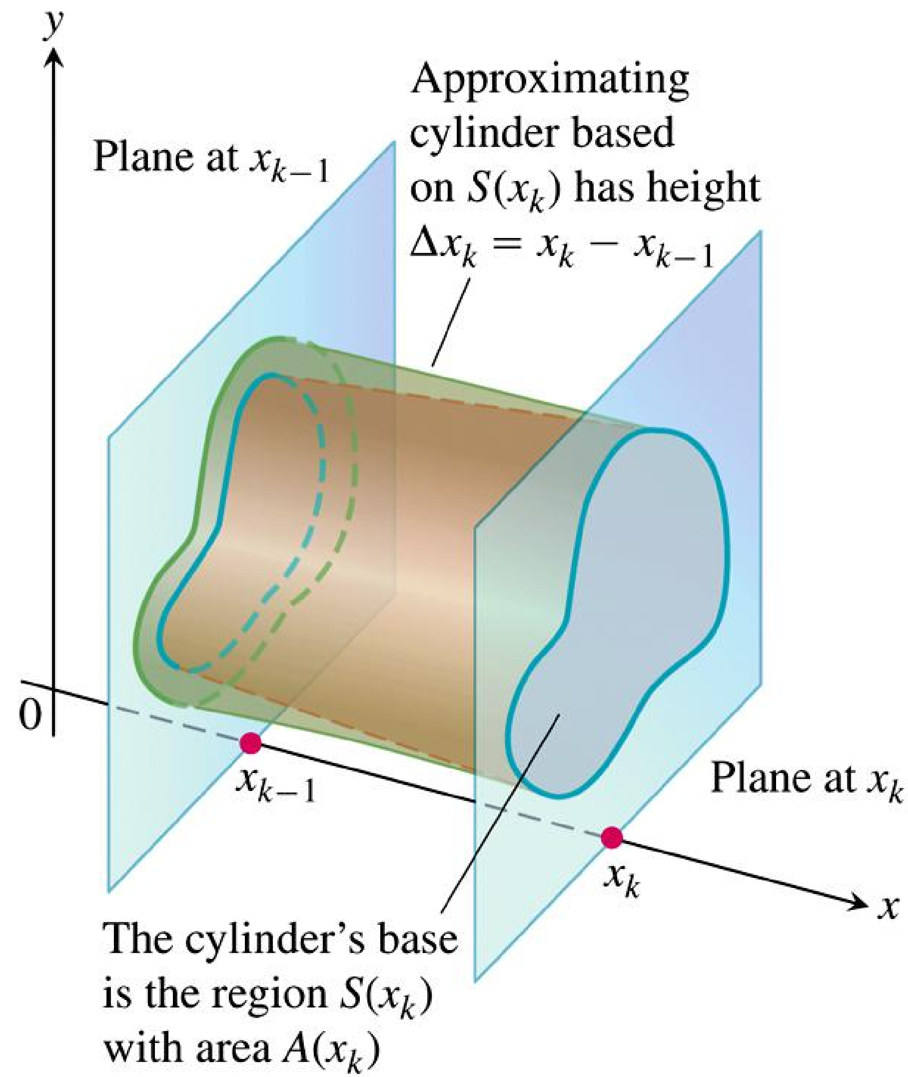
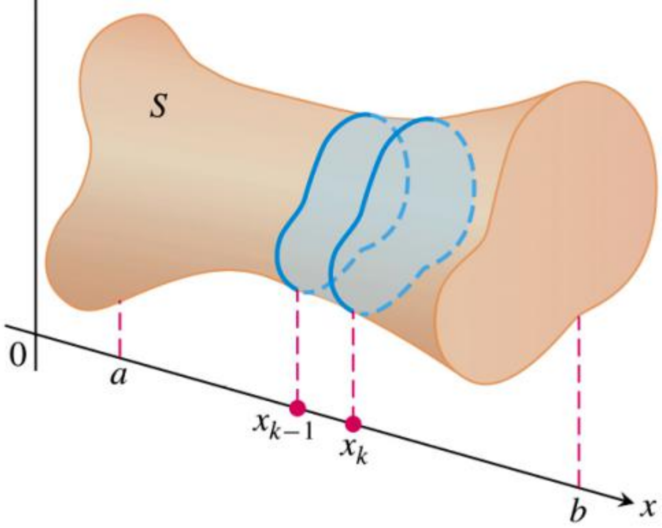


FIGURE 6.4 The solid thin slab in Figure 6.3 is shown enlarged here. It is approximated by the cylindrical solid with base $S(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

Volume of the kth slab $\approx V_k = A(x_k)\,\Delta x_k$.

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\,\Delta x_k.$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k = \int_a^b A(x)\,dx.$$



DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

This definition applies whenever A(x) is integrable,
and in particular when A(x) is continuous.

To apply this definition to calculate the volume of a solid using cross-sections perpendicular to the x-axis, take the following steps:

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ to find the volume.

EXAMPLE 1 A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution

1. A sketch. We draw the pyramid with its altitude along the x-axis and its vertex at the origin and include a typical cross-section (Figure 6.5). Note that by positioning the pyramid in this way, we have vertical cross-sections that are squares, whose areas are easy to calculate.

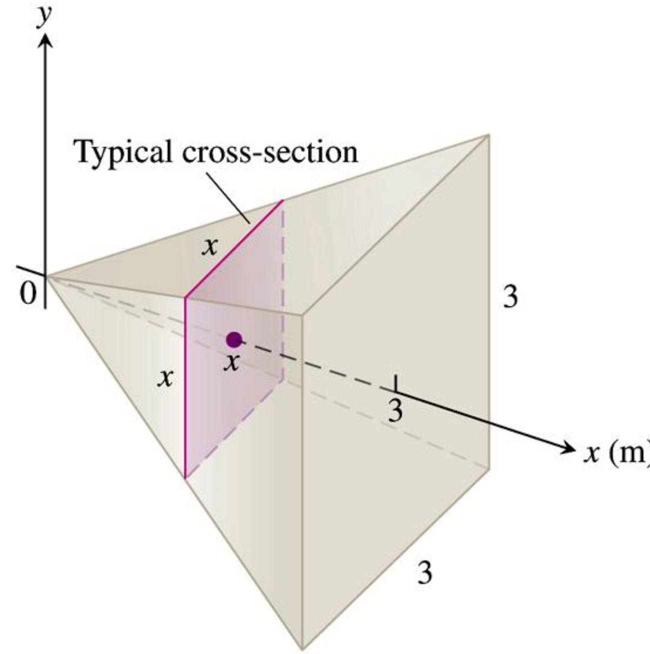


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

2. A formula for A(x). The cross-section at x is a square x meters on a side, so its area is $A(x) = x^2.$

3. The limits of integration. The squares lie on the planes from x = 0 to x = 3.

4. Integrate to find the volume:

$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\bigg]_0^3 = 9\text{ m}^3.$$

EXAMPLE 2 A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution We draw the wedge and sketch a typical cross-section perpendicular to the x-axis (Figure 6.6).

$$A(x) = (\text{height})(\text{width}) = (x)\left(2\sqrt{9 - x^2}\right)$$
$$= 2x\sqrt{9 - x^2}.$$

The rectangles run from x = 0 to x = 3, so we have

$$V = \int_a^b A(x)\,dx = \int_0^3 2x\sqrt{9 - x^2}\,dx$$

Let $u = 9 - x^2$, $du = -2x\,dx$, integrate,

$$= -\frac{2}{3}(9 - x^2)^{3/2}\Big]_0^3$$

$$= 0 + \frac{2}{3}(9)^{3/2}$$

$$= 18.$$

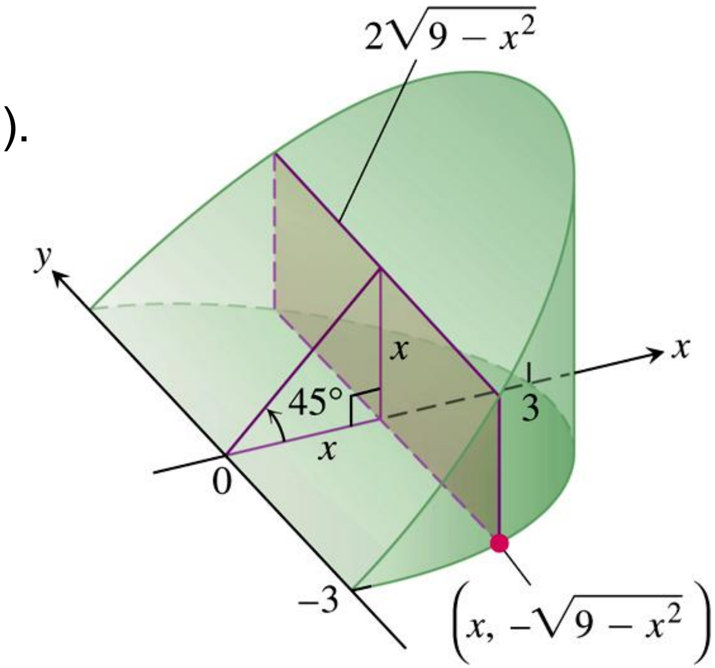
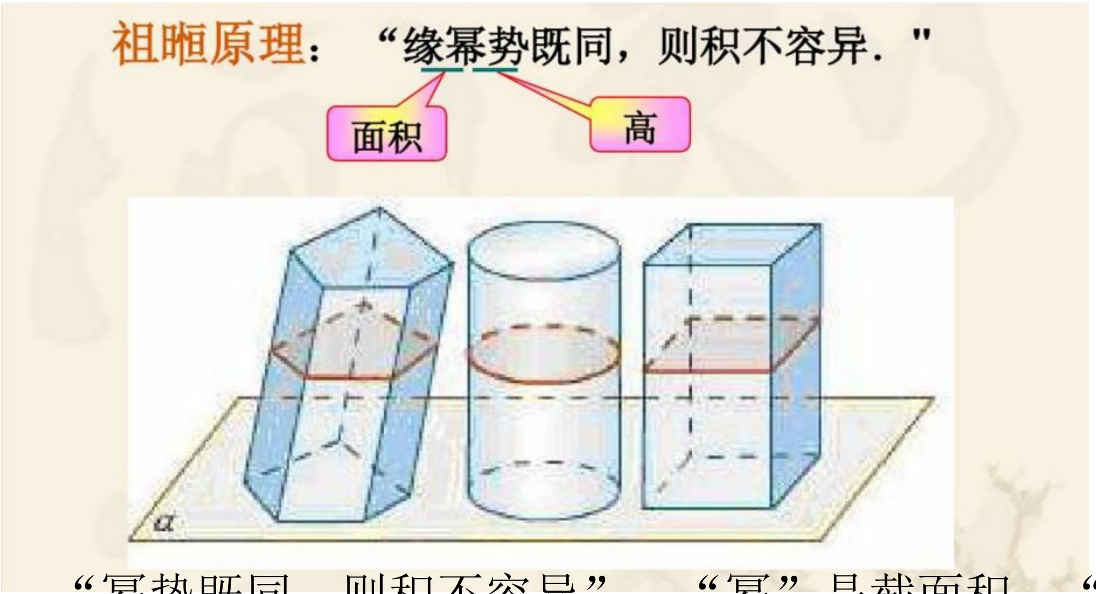


FIGURE 6.6 The wedge of Example 2, sliced perpendicular to the x-axis. The cross-sections are rectangles.

EXAMPLE 3 Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume (Figure 6.7). This follows immediately from the definition of volume, because the cross-sectional area function A(x) and the interval [a, b] are the same for both solids.

Bonaventura cavalieri(1598–1647)
卡瓦利埃里原则，等幂等积定理
祖暅(gèng 太阳的光晕)原理



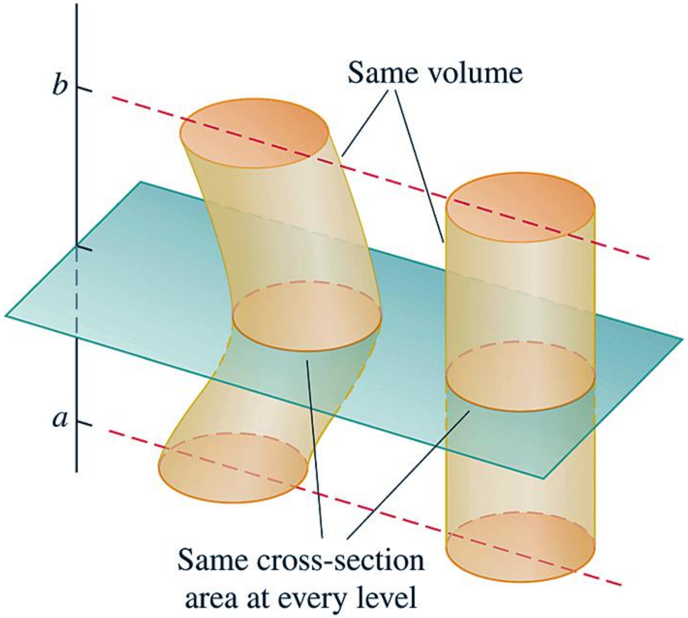
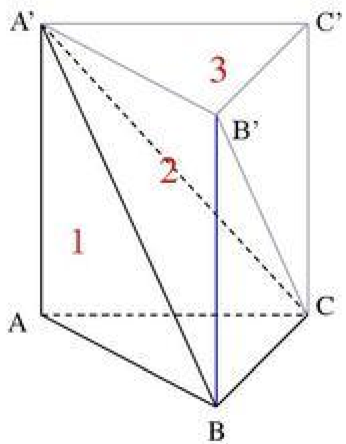


FIGURE 6.7 *Cavalieri's principle:* These solids have the same volume (imagine each solid as a stack of coins).

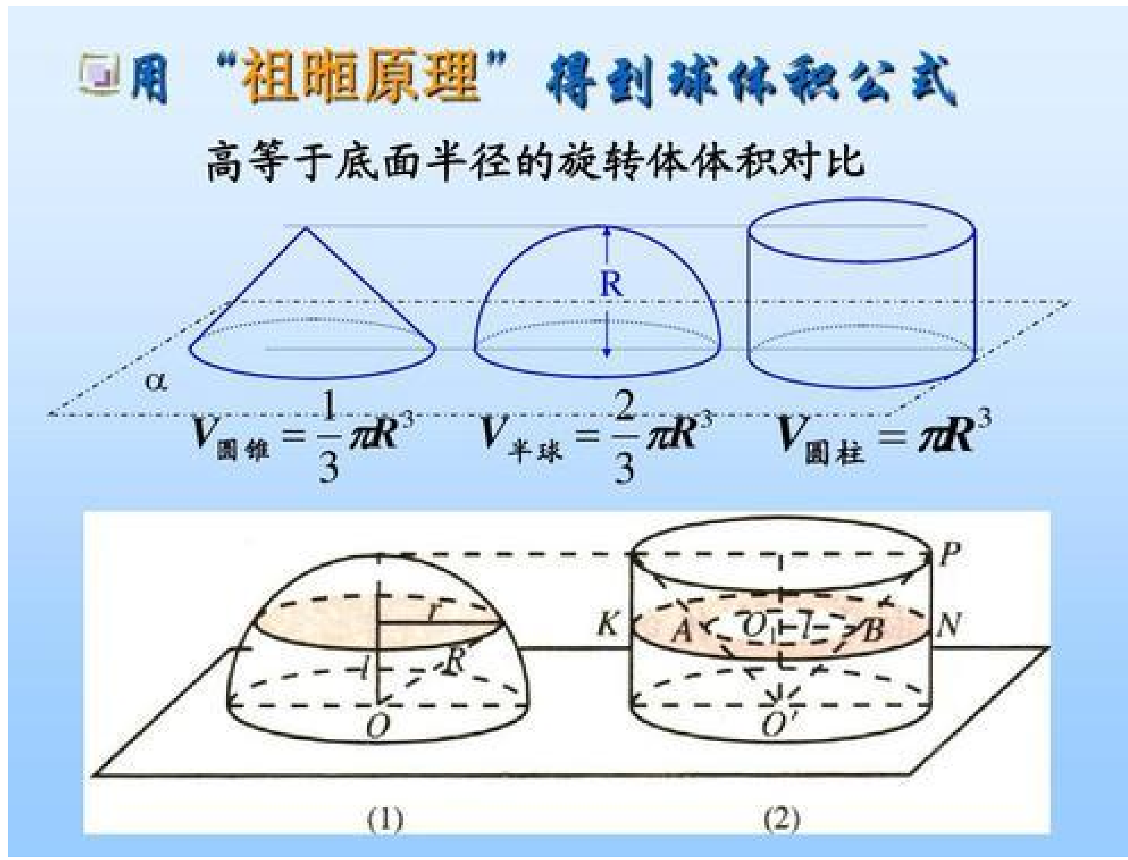
“幂势既同，则积不容异”。“幂”是截面积，“势”是立体的高。意思是两个同高的立体，如在等高处的截面积相等，则体积相等。更详细点说就是，界于两个平行平面之间的两个立体，被任一平行于这两个平面的平面所截，如果两个截面的面积相等，则这两个立体的体积相等。上述原理在中国被称为祖暅原理，国外则一般称之为卡瓦列利原理。

如果三棱锥的底面积是S，高是h，那么它的体积是 $V_{三棱锥}=\frac{1}{3}Sh$



连接B’C,然后把这个三棱柱分割成三个三棱锥。

就是三棱锥1和另两个三棱锥2、3。



Three different methods:

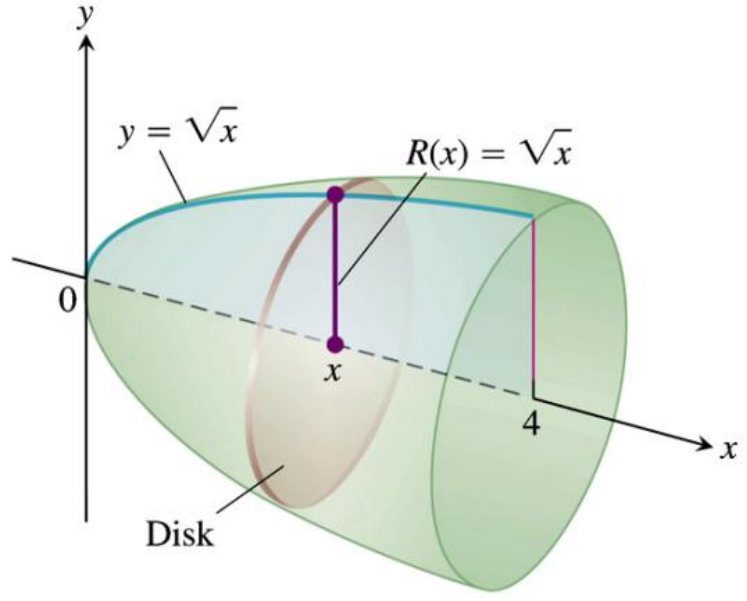
1. the method of slicing,
2. the disk method,
3. the washer method.

Solids of Revolution: The Disk Method

The solid generated by rotating (or revolving) a planar region about an axis in its plane is called a solid of revolution.

Solids of Revolution旋转成的体

revolve vi.旋转,自转



The cross-sectional area A(x) is the area of a disk of radius R(x), where R(x) is the distance from the axis of revolution to the planar region's boundary. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2$$

disk：圆盘 （2维）
circle: 圆 （1维）

Volume by Disks for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

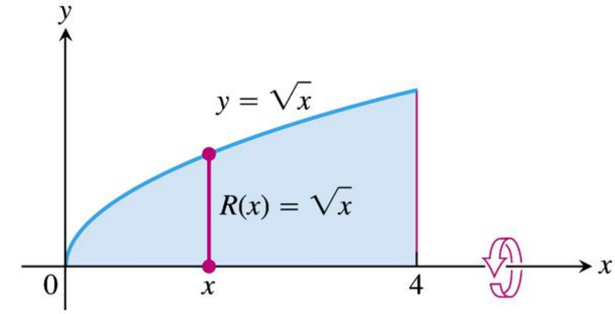
This method for calculating the volume of a solid of revolution is often called the disk method because a cross-section is a circular disk of radius R(x).

EXAMPLE 4 The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

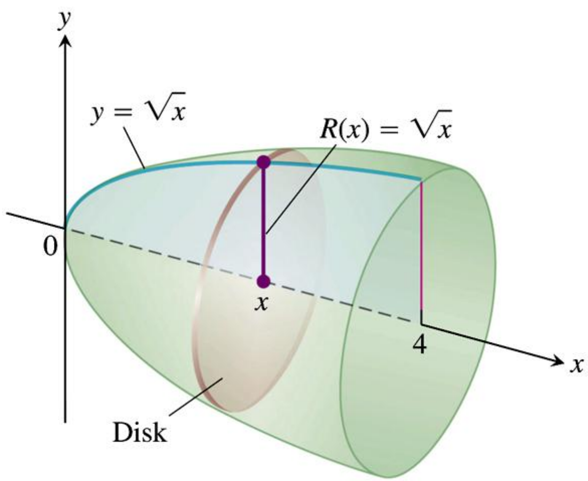
$$V = \int_a^b \pi [R(x)]^2\, dx$$

$$= \int_0^4 \pi [\sqrt{x}]^2\, dx$$

$$= \pi \int_0^4 x\, dx = \pi \frac{x^2}{2}\Big]_0^4 = \pi \frac{(4)^2}{2} = 8\pi.$$



(a)



(b)

EXAMPLE 5 The circle

$$x^2 + y^2 = a^2$$

is rotated about the x-axis to generate a sphere. Find its volume.

Solution： The cross-sectional area at a typical point x between -a and a is

$R(x) = \sqrt{a^2 - x^2}$ for rotation around x-axis.

$$A(x) = \pi y^2 = \pi(a^2 - x^2).$$

$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx$$

$$= \pi\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

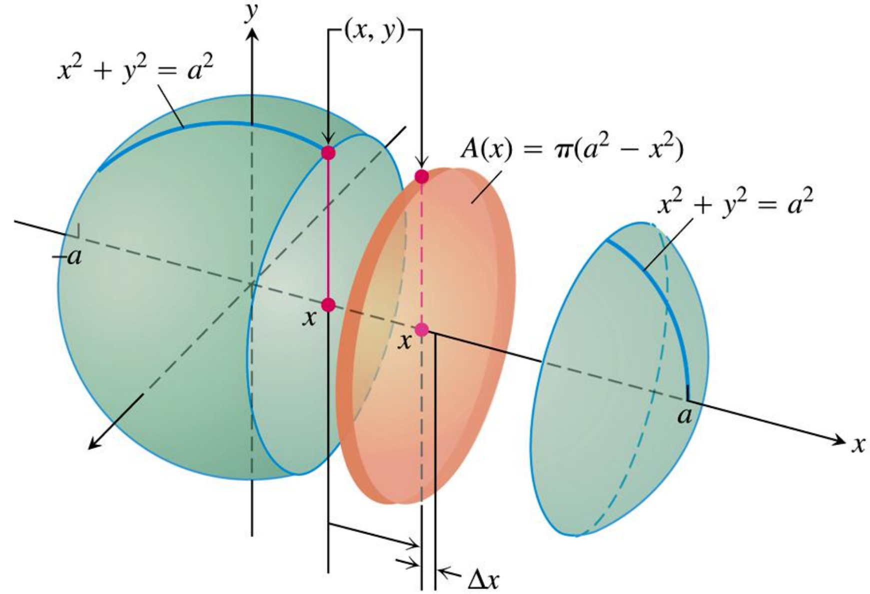


FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x-axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5).

The axis of revolution in the next example is not the x-axis

EXAMPLE 6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

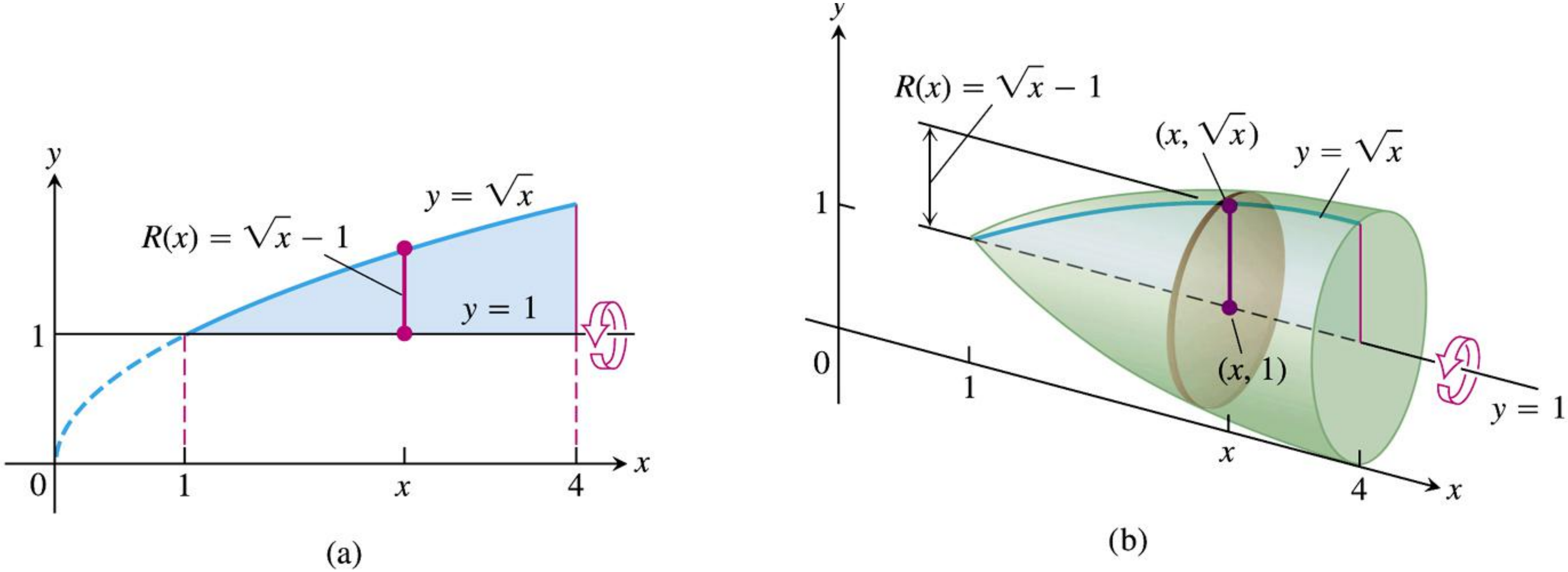


FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6.

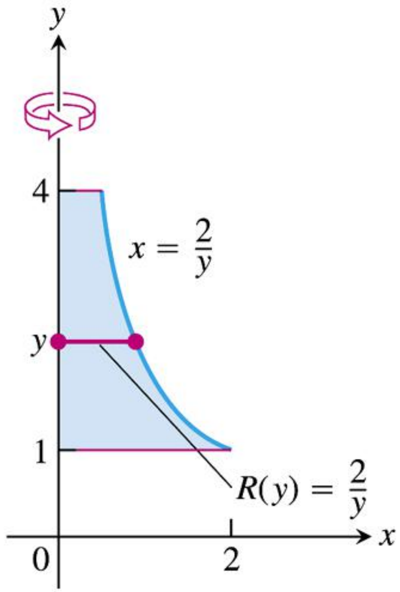
Integrate $\pi(\text{radius})^2$ between appropriate limits.

$$V = \int_1^4 \pi[R(x)]^2\,dx = \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx = \pi\int_1^4 \left[x - 2\sqrt{x} + 1\right] dx$$

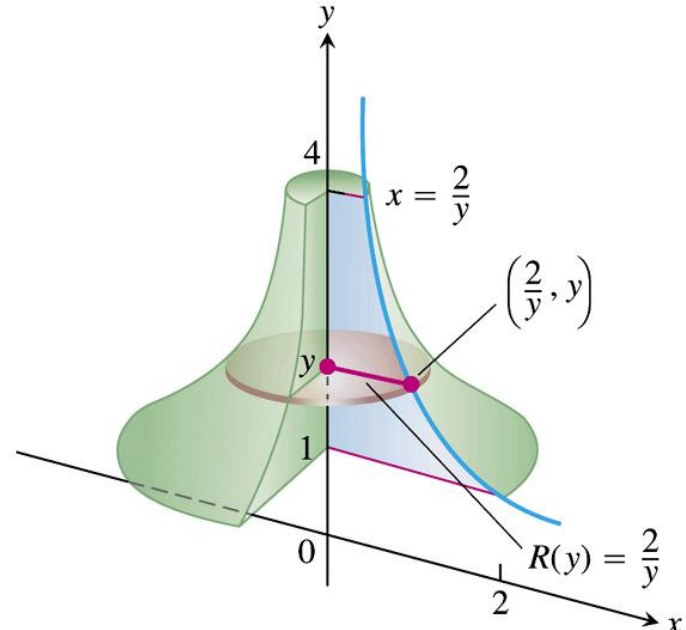
$$= \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}.$$

To find the volume of a solid generated by revolving a region between the y-axis and a curve $x = R(y)$, $c \le y \le d$, about the y-axis, we use the same method with x replaced by y. In this case, the area of the circular cross-section is

$$A(y) = \pi[\text{radius}]^2 = \pi[R(y)]^2,$$



(a)



(b)

Volume by Disks for Rotation About the *y*-axis

$$V = \int_c^d A(y)\,dy = \int_c^d \pi[R(y)]^2\,dy.$$

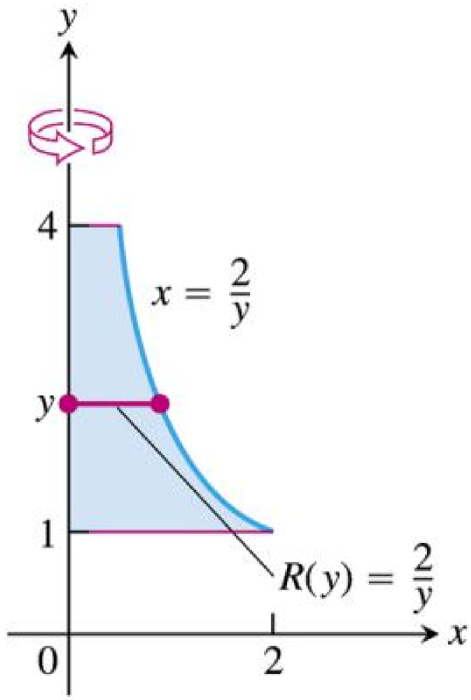
EXAMPLE 7 Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

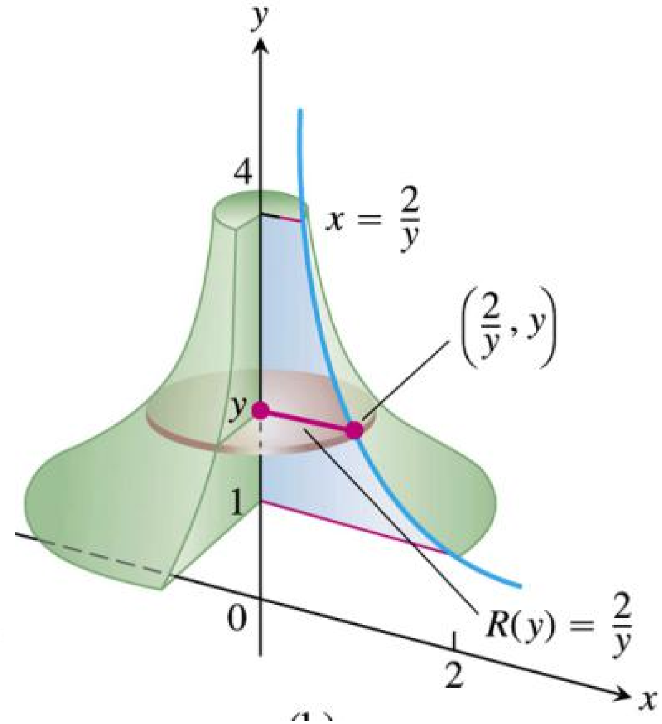
$$V = \int_1^4 \pi [R(y)]^2 \, dy$$

$$= \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy$$

$$= \pi \int_1^4 \frac{4}{y^2} \, dy$$

$$= 4\pi \left[-\frac{1}{y}\right]_1^4 = 4\pi \left[\frac{3}{4}\right] = 3\pi.$$





EXAMPLE 8 Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

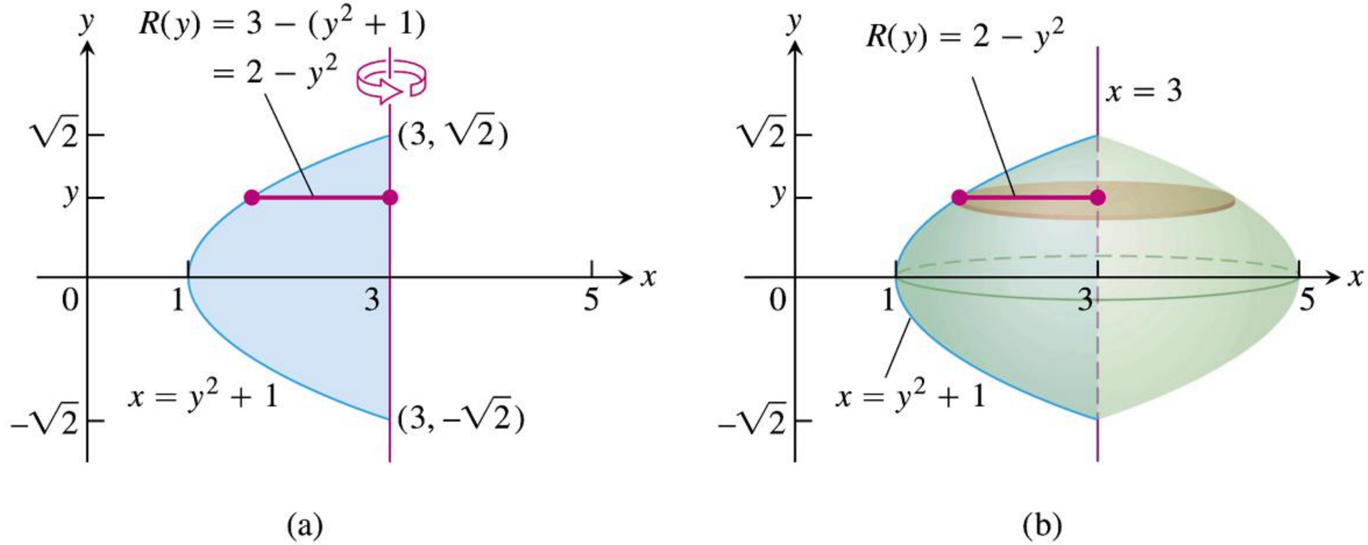


FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8.

from $y = -\sqrt{2}$ to $y = \sqrt{2}$.

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi \left[R(y)\right]^2 dy = \int_{-\sqrt{2}}^{\sqrt{2}} \pi \left[2 - y^2\right]^2 dy = \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left[4 - 4y^2 + y^4\right] dy$$

$$= \pi \left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{64\pi\sqrt{2}}{15}.$$

Three different methods:

1. the method of slicing,
2. the disk method,
3. the washer method.

Solids of Revolution: The Washer Method （螺絲帽下）墊圈，墊片

If the region we revolve to generate a solid does not border on or cross the axis of revolution, then the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are washers (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

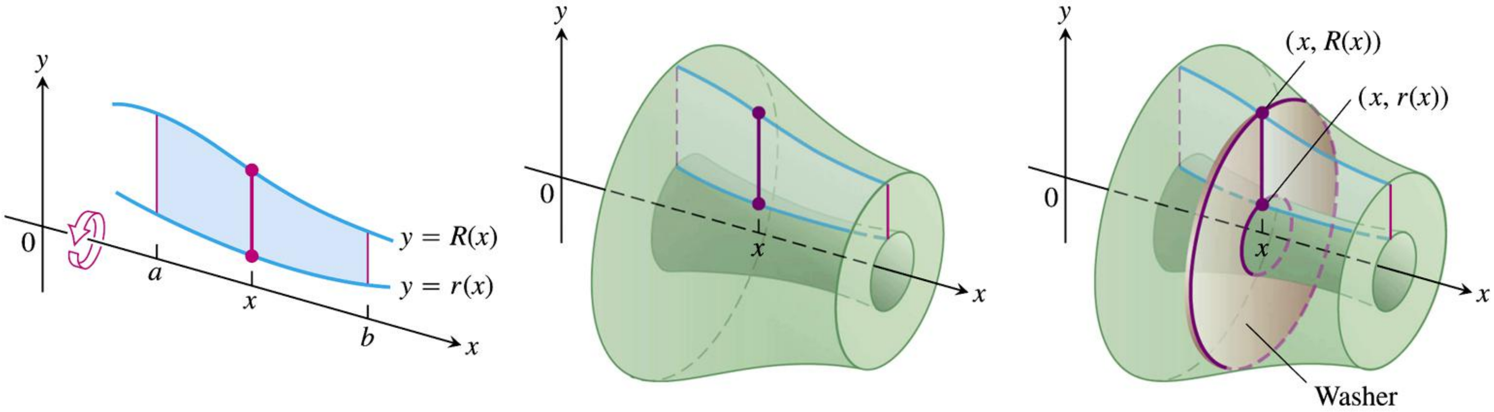


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

Outer radius: R(x)　　　　Inner radius: r(x)

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi\left([R(x)]^2 - [r(x)]^2\right).$$

Volume by Washers for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right)dx.$$

Volume by Washers for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right)dx.$$

This method is called the washer method because a thin slab of the solid resembles a circular washer with outer radius R(x) and inner radius r(x).

EXAMPLE 9 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Draw the region

Outer radius: $R(x) = -x + 3$

Inner radius: $r(x) = x^2 + 1$

Find the limits of integration

$$x^2 + 1 = -x + 3$$

Limits of integration
x = -2, x = 1

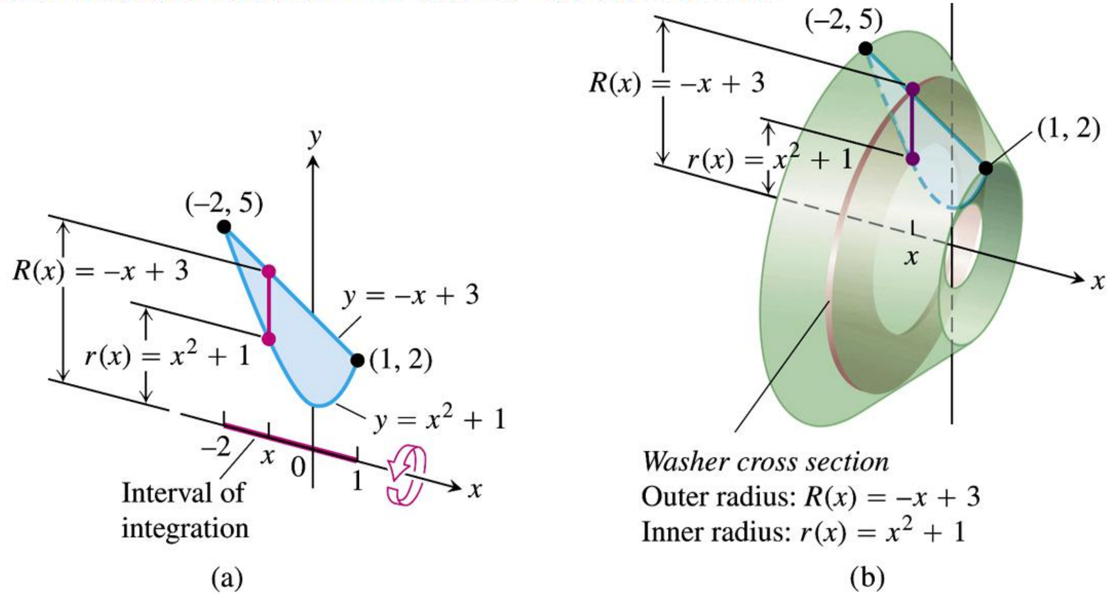


FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x-axis, the line segment generates a washer.

$$V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$$

$$= \int_{-2}^{1} \pi\left((-x + 3)^2 - (x^2 + 1)^2\right) dx \qquad = \pi \int_{-2}^{1} (8 - 6x - x^2 - x^4)\, dx$$

$$= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5}$$

To find the volume of a solid formed by revolving a region about the y-axis, we use the same procedure as in Example 9, but integrate with respect to y instead of x.

EXAMPLE 10 The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

$$R(y) = \sqrt{y},\ r(y) = y/2$$

$$V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy$$

$$= \int_0^4 \pi\left([\sqrt{y}]^2 - \left[\frac{y}{2}\right]^2\right) dy$$

$$= \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy$$

$$= \pi \left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi$$

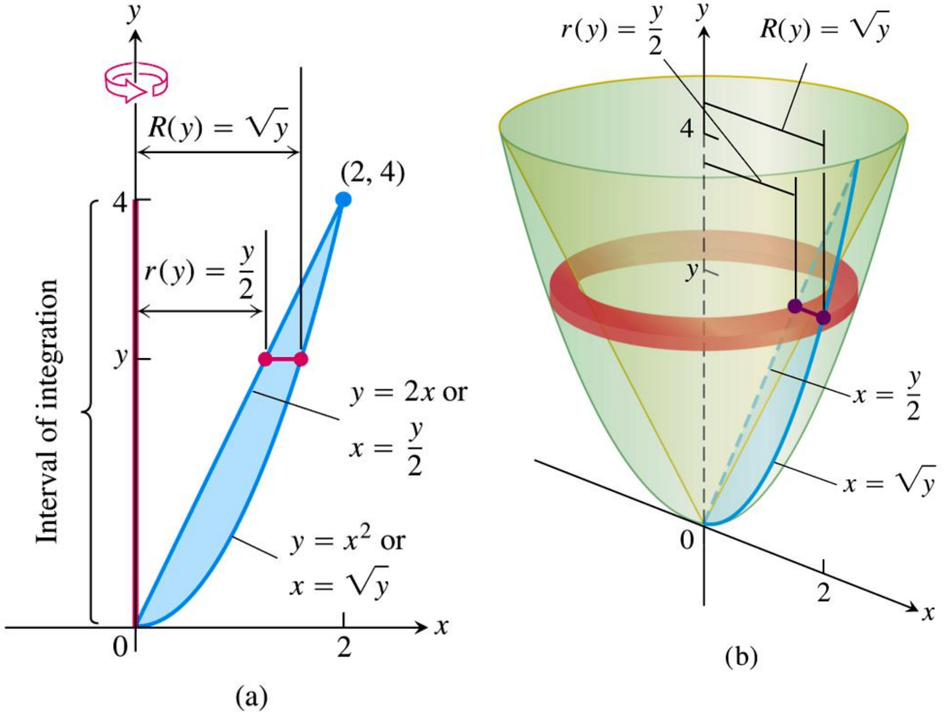
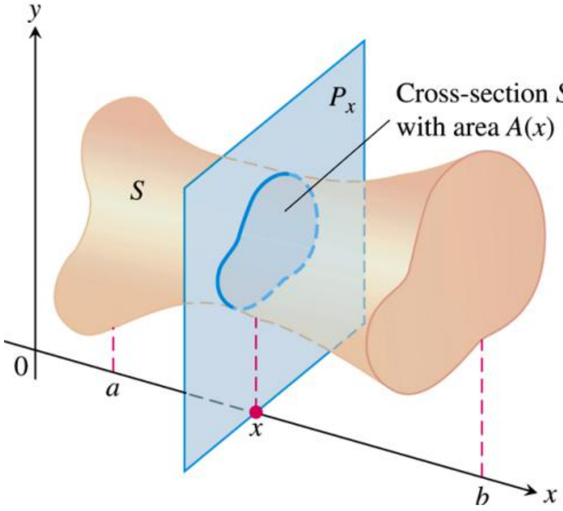
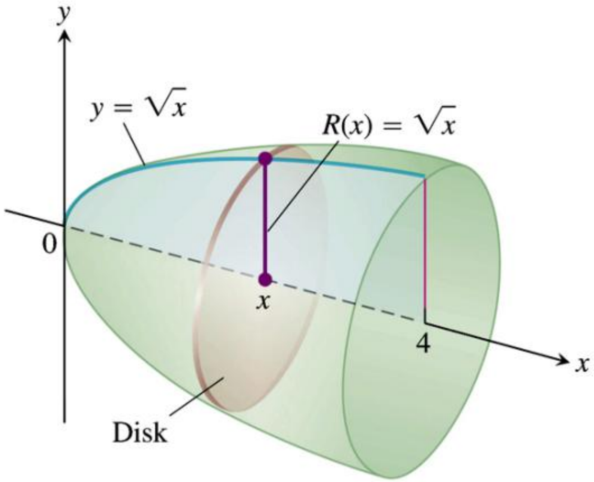
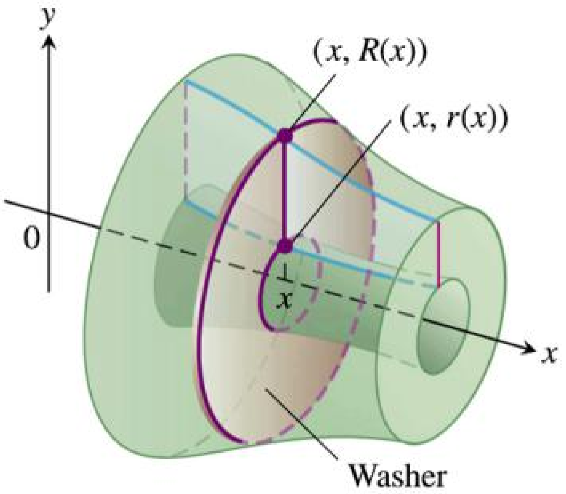


FIGURE 6.15 (a) The region being rotated about the y-axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).







$$V = \int_a^b A(x)\, dx.$$

$$= \int_a^b A(x)\, dx = \int_a^b \pi [R(x)]^2\, dx.$$

$$\int_a^b \pi \left([R(x)]^2 - [r(x)]^2 \right) dx.$$

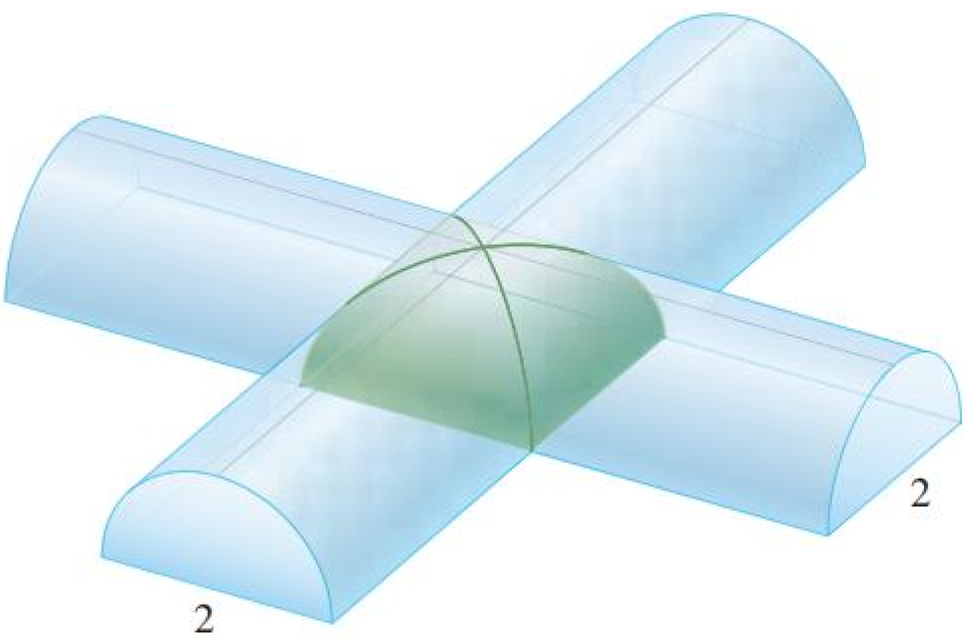
1.the method of slicing, Most General
2.the disk method,
3.the washer method.

Homework

Volumes by Slicing

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

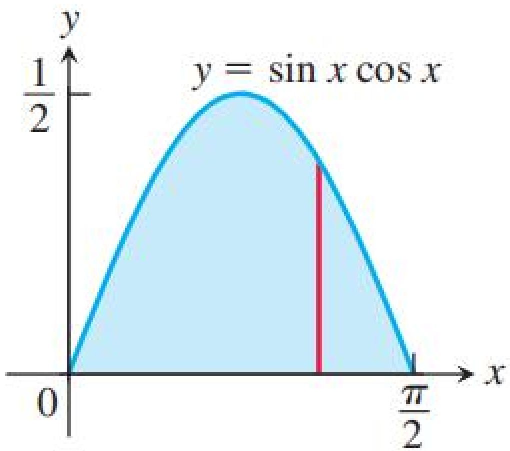
15. Intersection of two half-cylinders Two half-cylinders of diameter 2 meet at a right angle in the accompanying figure. Find the volume of the solid region common to both half-cylinders. (*Hint*: Consider slices parallel to the base of the solid.)



Volumes by the Disk Method

In Exercises 17–20, find the volume of the solid generated by revolving the shaded region about the given axis.

20. About the x-axis



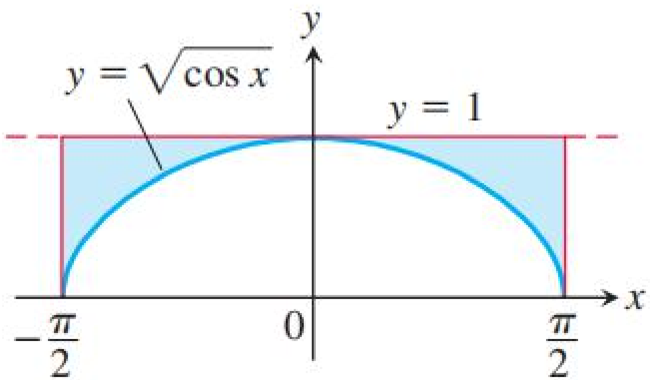
Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 29–34 about the y-axis.

29. The region enclosed by $x = \sqrt{5}y^2, \quad x = 0, \quad y = -1, \quad y = 1$

Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 35 and 36 about the indicated axes.

35. The x-axis **3**



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 37–42 about the x-axis.

40. $y = 4 - x^2, \quad y = 2 - x$

Volumes of Solids of Revolution

50. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

a. the line $x = 1$. **b.** the line $x = 2$.

Theory and Applications

53. The volume of a torus The disk $x^2 + y^2 \leq a^2$ is revolved about the line $x = b$ $(b > a)$ to generate a solid shaped like a doughnut and called a *torus*. Find its volume. (*Hint:* $\int_{-a}^{a} \sqrt{a^2 - y^2}\, dy = \pi a^2/2$, since it is the area of a semicircle of radius a.)

57. Volume of a hemisphere Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross-sections with the cross-sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed, as suggested by the accompanying figure.